The following publication Wang, J., Feng, J., Lou, Y., & Chen, G. (2020). Synchronization of networked harmonic oscillators via quantized sampled velocity feedback. IEEE Transactions on Automatic Control, 66(7), 3267-3273 is available at https://doi.org/10.1109/TAC.2020.3014905

Synchronization of Networked Harmonic Oscillators via Quantized Sampled Velocity Feedback

Jingyi Wang, Jianwen Feng, Yijun Lou, Guanrong Chen, *Fellow, IEEE*

***Abstract*—In this technical note, we propose a practicable quantized sampled velocity data coupling protocol for synchronization of a set of harmonic oscillators. The coupling protocol is designed in a quantized way via interconnecting the velocities encoded by a uniform quantizer with a zooming parameter in either a fixed or an adjustable form over a directed communication network. We establish sufficient conditions for the networked harmonic oscillators to converge to a bounded neighborhood of the synchronized orbits with a fixed zooming parameter. We ensure the oscillators to achieve synchronization by designing the quantized coupling protocol with an adjustable zooming parameter. Finally, we show two numerical examples to illustrate the effectiveness of the proposed coupling protocol.**

***Index Terms*—Synchronization, networked harmonic oscillators, quantized control, sampled velocity data**

I. Introduction

SYNCHRONIZATION phenomena are common in nature and society. Understanding, describing and controlling synchronization have been an active research field in various academic disciplines, as surveyed by [1], [2], [3]. In general, synchronization is a process in which the state of network-interconnected subsystems converge to the same orbit driven by a designed coupling or control protocol [1].

Synchronization of networked harmonic oscillators provides a basic model for studying the dynamics and control problems of complex dynamical networks, with significant practical applications, such as mobile robots [4] and electrical networks [5]. In the past decade, some effective coupling protocols have been presented from different perspectives. For instance, the oscillators with interactions in a continuous-time setting over fixed or switching network topologies were investigated in [4], and the interactions with the sampled-data were investigated in [6], while the discrete-time setting was studied and applied to synchronization control of multiple mobile robots in [7], [8], and their instantaneous interactions under fixed or switching topologies with presence or absence of leaders were considered in [9]. A distributed protocol was proposed in an impulsive form by using the relative position information between the oscillator and its neighbors in [8]. Synchronization can also be reached by directly utilizing delayed position states in [10]. Recently, the synchronization problem was solved even by using noisy sampled-data in [11], [12]. Nonlinear diffusive coupling can also achieve synchronization [13].

On the other hand, almost all of the control systems are implemented digitally today, from large computer systems to small embedded processors, for which sampling and quantization are fundamental tools. Actually, sampling is the reduction of a continuous-time signal to a discrete-time signal at the sampling times and quantization is a kind of mapping from continuous signals to discrete sets by the prescribed rules [14]. In particular, in the study of communications and coupling protocols of oscillators, transmitting data can be obtained via continuous, periodic or aperiodic sampling, which could be then sent and received in a quantized form. With this motivation, stabilization of linear systems via quantized control was studied in the continuous-time [15], discrete-time [16], [14] and switching [17] settings, which may be subject to external disturbances [18] or in nonlinear systems [19]. Moreover, quantization techniques can be used to deal with the stabilization of systems with limited measurement information [20], [21]. The consensus of fixed [22] or switched [23] networks of multi-agent systems was studied based on quantized relative state information, as well as quantization of the absolute state information [24] and sampled data [25].

To the best of our knowledge, few studies have been conducted on the synchronization problem of harmonic oscillators over a directed communication network via quantized feedback coupling by using only sampled velocity data. The objective of this technical note is to present some novel coupling protocols using quantized sampled control to achieve synchronization of a network of harmonic oscillators. First, a quantized coupling protocol with a fixed zooming parameter is proposed, which can guarantee the boundedness of the synchronization errors at periodic discrete-time instants. Then, its modified version with an adjustable zooming parameter is designed to achieve complete synchronization for any initial values of the oscillators.

Notations: Let $\mathbf{1}_n = [1, 1, \ldots, 1]^\top \in \mathbb{R}^n$, $\mathbf{0}_n = [0, 0, \ldots, 0]^\top \in \mathbb{R}^n$, $\mathrm{O}_n = [0] \in \mathbb{R}^{n\times n}$, $\mathrm{O}_{m,n} = [0] \in \mathbb{R}^{m\times n}$ and I_n be the $n-$dimensional identity matrix. Use superscript $\top$ $(*)$ to denote the transpose (conjugate transpose) of a vector or matrix. For $x \in \mathbb{R}$, the ceiling function $\lceil x \rceil = \min\{k \in \mathbb{Z} : k \geq x\}$ is the smallest integer not less than x, and the floor function $\lfloor x \rfloor = \max\{k \in \mathbb{Z} : k \leq x\}$ is the largest integer not greater than x. Let $\mathbf{i} = \sqrt{-1}$ denote the imaginary unit. For a complex number x, $|x|$ denotes its modulus, and $\Re(x)$ and $\Im(x)$ represent its real part and imaginary part,

This work was supported in part by the National Natural Science Foundation of China under Grants 61603260, 61873171 and 61872429, in part by the Natural Science Foundation of Guangdong Province.

J. Wang and J. Feng are with the College of Mathematics and Statistics, Shenzhen University, Shenzhen 518060, China (e-mail: wangjingyi@szu.edu.cn, fengjw@szu.edu.cn)

Y. Lou is with the Department of Applied Mathematics, The Hong Kong Polytechnic University, Hong Kong SAR, China (e-mail: yijun.lou@polyu.edu.hk)

G. Chen is with the Department of Electrical Engineering, City University of Hong Kong, Hong Kong SAR, China (e-mail: eegchen@cityu.edu.hk)

respectively. For a vector $x \in \mathbb{C}^n$, the maximum norm (l_∞-norm) is denoted by $\|x\|_\infty = \max_i |x_i|$, while the l_2-norm is denoted by $\|x\|_2 = \sqrt{x^*x}$. For a square matrix $X \in \mathbb{C}^{n\times n}$, the spectrum of X is denoted by $\sigma(X)$, that is, the set of eigenvalues of X, while the spectral radius of X is denoted by $\rho(X) = \max\{|\lambda| : \lambda \in \sigma(X)\}$. For a matrix $X \in \mathbb{C}^{m\times n}$, the maximum row-sum matrix norm, induced by the l_∞-norm, is defined by $\|X\|_\infty = \max_i \sum_{j=1}^n |x_{ij}|$, while the spectral norm, induced by the l_2-norm, is defined by $\|X\|_2 = \max\{\sqrt{\lambda} : \lambda \in \sigma(X^*X)\}$, that is, the largest singular value of X.

II. Preliminaries and Problem Statement

A. Preliminaries

To investigate the synchronization behavior, the oscillators are interconnected over a communication network described by a directed graph without loops. Let $\mathscr{G} = (\mathscr{V}, \mathscr{E}, \mathscr{A})$ be a directed and connected graph without loops for a set of oscillators $\mathscr{V} = \{1, 2, \ldots, n\}$, a set of edges $\mathscr{E} \subseteq \mathscr{V} \times \mathscr{V}$ and the adjacency matrix $\mathscr{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$, in which a_{ij} is a positive weight for all $(i,j) \in \mathscr{E}$ if and only if oscillator i can access or receive the information from oscillator j and $a_{ij} = 0$ for all $(i,j) \notin \mathscr{E}$. Define the Laplacian matrix $L = [l_{ij}] \in \mathbb{R}^{n\times n}$ associated with $\mathscr{G}$ by $l_{ij} = -a_{ij}$ for $i \neq j$ and $l_{ii} = \sum_{i=1, j\neq i}^n a_{ij}$ for $i = j$. A sequence of edges (i_1, i_2), (i_2, i_3), …, with $i_j \in \mathscr{V}$, is a directed path in $\mathscr{G}$. A directed graph is strongly connected if and only if any two distinct nodes of the graph can be connected via a directed path.

Lemma 1. [26] If a directed graph $\mathscr{G}$ is strongly connected, for which the associated Laplacian matrix is L, then

1) $\mathbf{1}_n$ is a right eigenvector of L associated with the eigenvalue $\lambda_1 = 0$ of multiplicity 1, and all the other right eigenvalues $\lambda_2, \ldots, \lambda_n$ have positive real parts;
2) if $\xi = [\xi_1, \xi_2, \ldots, \xi_n]^\top$ is a left eigenvector of L associated with the eigenvalue 0 (i.e. $\xi^\top L = 0$), then $\xi_i > 0$ for all $i = 1, 2, \ldots, n$, and ξ has multiplicity 1;
3) there exists a nonsingular matrix P, in which the first column is $\mathbf{1}_n$ and the first row of P^{-1} is $\xi^\top$, such that $L = PJP^{-1}$ is the Jordan decomposition of L, where $J = \mathrm{diag}\{0, \hat{J}\}$ and $\hat{J}$ is the Jordan upper diagonal block matrix corresponding to the nonzero eigenvalues λ_r ($r = 2, \ldots, n$) of the matrix L.

For the above mentioned claim, assume that $\sum_{i=1}^n \xi_i = 1$, and denote $\bar{\xi} = \max \xi_i$, $P = [p_1, p_2, \ldots, p_n] \in \mathbb{C}^{n\times n}$, where $p_i \in \mathbb{C}^{n\times 1}$, and $P^{-1} = [\bar{p}_1^\top, \bar{p}_2^\top, \ldots, \bar{p}_n^\top]^\top \in \mathbb{C}^{n\times n}$, where $\bar{p}_i \in \mathbb{C}^{1\times n}$. Obviously, $p_1 = \mathbf{1}_n$, $\bar{p}_1 = \xi^\top$, $\bar{p}_i p_i = 1$ and $\bar{p}_i p_j = 0$ for all $i \neq j$. Let $\Xi = p_1\bar{p}_1 \in \mathbb{R}^{n\times n}$, $\hat{P} = [p_2, \ldots, p_n] \in \mathbb{C}^{n\times(n-1)}$, $\hat{P}^\dagger = [\bar{p}_2^\top, \ldots, \bar{p}_n^\top]^\top \in \mathbb{C}^{(n-1)\times n}$, so that $\hat{P}^\dagger\hat{P} = I_{n-1}$ and $\hat{P}^\dagger L\hat{P} = \hat{J}$, where $\hat{J} = \mathrm{diag}\{\hat{J}_{n_1}, \hat{J}_{n_2}, \ldots, \hat{J}_{n_r}\} \in \mathbb{C}^{(n-1)\times(n-1)}$, in which $\hat{\mathcal{J}}_{n_k}$ is an $n_k \times n_k$ Jordan block corresponding to eigenvalue λ_k with geometric multiplicity n_k and $n_1 + n_2 + \cdots + n_r = n-1$. Denote $\mathcal{P} = \mathrm{diag}\{P, P\}$, $\mathcal{J} = \mathrm{diag}\{J, J\}$, $\hat{\mathcal{J}} = \mathrm{diag}\{\hat{J}, \hat{J}\}$, $\hat{\mathcal{P}} = \begin{bmatrix} \hat{P} & \mathrm{O}_{n,n-1} \\ \mathrm{O}_{n,n-1} & \hat{P} \end{bmatrix}$ and $\hat{\mathcal{P}}^\dagger = \begin{bmatrix} \hat{P}^\dagger & \mathrm{O}_{n-1,n} \\ \mathrm{O}_{n-1,n} & \hat{P}^\dagger \end{bmatrix}$.

Lemma 2. [27] For given $A \in \mathbb{C}^{n\times n}$ and $\varepsilon > 0$, there exists a matrix norm $\|\cdot\|$ such that $\rho(A) \leq \|A\| \leq \rho(A) + \varepsilon$.

Lemma 3. [27] Let $A \in \mathbb{C}^{n\times n}$. Then, $\lim_{k\to\infty} A^k = \mathrm{O}_n$ if and only if $\rho(A) < 1$.

Lemma 4. [28] Given a complex-coefficient polynomial of order two, $g(s) = s^2 + (a + b\mathbf{i})s + (c + d\mathbf{i})$, where a, b, c, and d are real constants. Then, $g(s)$ is stable, that is, all roots of $g(s) = 0$ have negative real parts, if and only if $a > 0$ and $abd + a^2c - d^2 > 0$.

B. Quantizer

In this technical note, the class of quantizers proposed in [15], [16] is adopted. Let $\mathcal{Q}$ be a finite subset of $\mathbb{R}$. A quantizer is a piecewise constant function $q : \mathbb{R} \to \mathcal{Q}$. This implies geometrically that $\mathbb{R}$ is divided into a finite number of quantized regions $\{y \in \mathbb{R} : q(y) = y_i, y_i \in \mathcal{Q}\}$. For a quantizer q, there exist positive numbers M and Δ with $M > \Delta$ such that

(i) if $|y| \leq M$, then $|q(y) - y| \leq \Delta$;
(ii) if $|y| > M$, then $|q(y)| > M - \Delta$.

Condition (i) gives an upper bound for the quantization error when the quantizer does not saturate, and condition (ii) is used for detecting quantizer saturation. Here, M and Δ are referred to as the range of q and the quantization error, respectively. To achieve complete synchronization of a network of oscillators, a quantizer can be designed with a suitable parameter $\mu > 0$, with $q(y) := q_\mu(y) = \mu q(y/\mu)$. The parameter μ is regarded as a zooming variable, and the measurement capability and precision accuracy of the quantizer can be adjusted by the zoom-in and zoom-out operations. Fig. 1 demonstrates a quantizer with some different values of parameter μ, in which $q(y)$ is the rounding function, and the dashed line $y + \Delta$ and the dotted line $y - \Delta$, with parameters $\mu = 1$ and $\Delta = 0.5$.

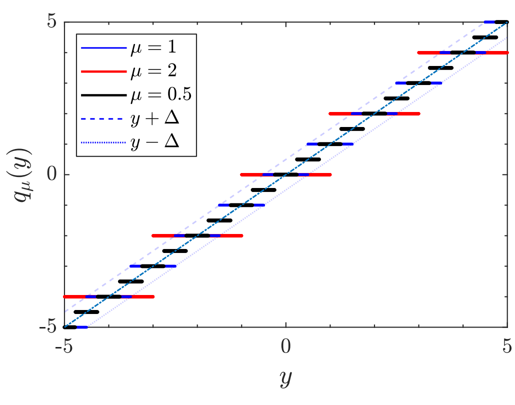


Fig. 1. A quantizer demo

C. Model description

Consider n harmonic oscillators with the control input in the following form:

$$\begin{cases} \dot{r}_i(t) = v_i(t), \\ \dot{v}_i(t) = -\omega^2 r_i(t) + u_i(t), \end{cases} \tag{1}$$

where $r_i(t)$, $v_i(t)$ and $u_i(t) \in \mathbb{R}$ are the position, velocity, and control input of oscillator i, respectively, ω is a positive gain, and the network topology is to be specified.

The objective is to design a coupling protocol for each oscillator such that the networked oscillators can achieve synchronization in the sense that $\lim_{t\to\infty} ||r_i(t)-r_j(t)|| = 0$ and $\lim_{t\to\infty} ||v_i(t)-v_j(t)|| = 0$ for any initial values $r_i(0), v_i(0) \in \mathbb{R}$ with a desired norm $\|\cdot\|$. In this note, the following coupling protocol, using the quantization of the absolute sampled velocity data, is designed

$$u_i(t) = -\sum_{j=1}^{n} a_{ij}(q_\mu(v_i(t_k)) - q_\mu(v_j(t_k))),$$
$$t \in [t_k, t_{k+1}),\ i = 1, 2, \ldots, n, \quad (2)$$

where $v_j(t_k)$ is the sampled velocity of oscillator j, obtained at the sampling instants t_k, $k = 0, 1, 2, \ldots$, satisfying $t_k = k\tau$, where $\tau > 0$ is a fixed sampling period and q_μ is a quantizer. In this note, both fixed and adjustable zooming parameters are investigated for quantized control, as follows:

Case 1 $q_\mu(v_i(t_k)) := \mu q(\frac{v_i(t_k)}{\mu})$, where μ is a fixed zooming parameter;

Case 2 $q_\mu(v_i(t_k)) := q_{\mu(t_k)}(v_i(t_k)) = \mu(t_k) q(\frac{v_i(t_k)}{\mu(t_k)})$, where $\mu(t_k)$ is an adjustable zooming parameter.

Remark 1. The orbit of a simple harmonic oscillator, when the control input is not applied, is an ellipse and the energy function $V(t) = r_i^2(t) + v_i^2(t)/\omega^2$ is a constant for $t \geq 0$ ($V'(t) = 0$). If $\|[r_i(0), v_i(0)]^\top\|_\infty < M_0$, one has $\|[r_i(t), v_i(t)]^\top\|_\infty \leq \sqrt{1+\bar\omega^2}M_0$ for $t \geq 0$, where $\bar\omega = \max\{\omega, 1/\omega\}$ and M_0 is a constant.

III. Main Results

Let $r(t) = [r_1(t), r_2(t), \ldots, r_n(t)]^\top$, $v(t) = [v_1(t), v_2(t), \ldots, v_n(t)]^\top$, $X(t) = [r(t)^\top, v(t)^\top]^\top$, $\mathbf{q}_\mu(r(t)) = [q_\mu(v_1(t)), q_\mu(v_2(t)), \ldots, q_\mu(v_n(t))]^\top$, $\mathcal{B} = \mathrm{diag}\{\mathrm{O}_n, -L\}$,

$$\mathcal{A} = \begin{bmatrix} \mathrm{O}_n & I_n \\ -\omega^2 I_n & \mathrm{O}_n \end{bmatrix}, \mathcal{C}(t_k) = \begin{bmatrix} \mathbf{0}_n \\ -L(\mathbf{q}_\mu(v(t_k)) - v(t_k)) \end{bmatrix},$$

$E = \exp(\mathcal{A}\tau) + \int_0^\tau \exp(\mathcal{A}s)\mathcal{B}ds$, $F = \int_0^\tau \exp(\mathcal{A}s)ds$, $\mathcal{E} = \mathcal{P}^{-1}E\mathcal{P}$, and $\hat{\mathcal{E}} = \hat{\mathcal{P}}^\dagger E\hat{\mathcal{P}}$.

Based on Lemma 2, the following lemma is derived for constructing a matrix norm $\|\cdot\|_\epsilon$ such that $\|\hat{\mathcal{E}}\|_\epsilon \leq 1$, which plays a key role in proving the main theorem..

Lemma 5. If $\rho(\hat{\mathcal{E}}) < 1$, then there exists a constant $\epsilon > 0$ and a matrix norm $\|\cdot\|_\epsilon$ on $\mathbb{C}^{(2n-2)\times(2n-2)}$, such that $\|\hat{\mathcal{E}}\|_\epsilon \leq \rho(\hat{\mathcal{E}}) + \epsilon < 1$. In particle, the matrix norm $\|\cdot\|_\epsilon$ on $\mathbb{C}^{(2n-2)\times(2n-2)}$ induced by a vector norm $\|\cdot\|_\epsilon$ on $\mathbb{C}^{2n-2}$, as follows:

$$\begin{cases} \text{the vector norm } \|\cdot\|_\epsilon = \|\mathcal{D}_\epsilon\mathcal{U}\cdot\|_\infty, \\ \text{the matrix norm } \|\cdot\|_\epsilon = \|\mathcal{D}_\epsilon\mathcal{U}\cdot\mathcal{U}^{-1}\mathcal{D}_\epsilon^{-1}\|_\infty, \end{cases} \quad (3)$$

where $\mathcal{U} \in \mathbb{C}^{(2n-2)\times(2n-2)}$ is a nonsingular matrix such that $\mathcal{U}^{-1}\hat{\mathcal{E}}\mathcal{U} = \mathrm{diag}\{\bar{\mathcal{J}}_1, \bar{\mathcal{J}}_2, \ldots, \bar{\mathcal{J}}_r\}$ is a Jordan matrix, in which $\bar{\mathcal{J}}_k \in \mathbb{C}^{n_k\times n_k}$ are Jordan blocks with $n_1 + n_2 + \cdots + n_r = 2n-2$, and $\mathcal{D}_\epsilon = \mathrm{diag}\{\mathcal{D}_{\epsilon,1}, \mathcal{D}_{\epsilon,2}, \ldots, \mathcal{D}_{\epsilon,r}\}$ with $\mathcal{D}_{\epsilon,k} = \mathrm{diag}\{1, 1/\epsilon, 1/\epsilon^2, \ldots, 1/\epsilon^{n_k-1}\}$.

Proof. Becasue $\rho(\hat{\mathcal{E}}) < 1$, there exists a positive constant ϵ such that $\rho(\hat{\mathcal{E}}) + \epsilon < 1$. Based on the property of the l_∞-norm, one can obtain that $\|\hat{\mathcal{E}}\|_\epsilon = \max_{\|x\|_\epsilon=1}\|\hat{\mathcal{E}}x\|_\epsilon = \max_{\|\mathcal{D}_\epsilon\mathcal{U}x\|_\infty=1}\|\mathcal{D}_\epsilon\mathcal{U}\hat{\mathcal{E}}x\|_\infty = \|\mathcal{D}_\epsilon\mathcal{U}\hat{\mathcal{E}}\mathcal{U}^{-1}\mathcal{D}_\epsilon^{-1}\|_\infty$. So, the matrix norm $\|\cdot\|_\epsilon$ on $\mathbb{C}^{(2n-2)\times(2n-2)}$ is induced by the vector norm $\|\cdot\|_\epsilon$ on $\mathbb{C}^{2n-2}$. By the Jordan canonical, one obtains that $\mathcal{U}^{-1}\hat{\mathcal{E}}\mathcal{U} = \mathrm{diag}\{\bar{\mathcal{J}}_1, \bar{\mathcal{J}}_2, \ldots, \bar{\mathcal{J}}_r\}$. Based on the definition of $\mathcal{D}_\epsilon$, by some elementary computation, one obtains that $\|\hat{\mathcal{E}}\|_\epsilon = \|\mathcal{D}_\epsilon\mathcal{U}\hat{\mathcal{E}}\mathcal{U}^{-1}\mathcal{D}_\epsilon^{-1}\|_\infty \leq \rho(\hat{\mathcal{E}}) + \epsilon < 1$. □

A. *Quantized feedback coupling with a fixed zooming parameter*

In this subsection, consider the convergence of system (1) under control (2) with a fixed zooming parameter μ.

Theorem 1. Assume that the directed graph $\mathscr{G}$ is strongly connected, with a sufficient small $\varepsilon > 0$ and a large enough M compared to Δ such that

$$M > \frac{\|\hat{\mathcal{P}}\mathcal{U}^{-1}\mathcal{D}_\epsilon^{-1}\|_\infty\|\mathcal{D}_\epsilon\mathcal{U}\hat{\mathcal{P}}^\dagger F\mathcal{B}\|_\infty}{(1-\bar\xi)(1-\|\hat{\mathcal{E}}\|_\epsilon)}\Delta,$$

and that the sampling period satisfies

$$\tau \in \{\tau : \cot(\omega\tau/2) > \phi_i, i = 2, \ldots, n\}, \quad (4)$$

where

$$\phi_i = \frac{\Re(\lambda_i)\Im(\lambda_i)^2 + \Re(\lambda_i)^3}{2\omega\Re(\lambda_i)^2} + \frac{\sqrt{\left(\Re(\lambda_i)\Im(\lambda_i)^2 + \Re(\lambda_i)^3\right)^2 + 4\omega^2\Re(\lambda_i)^2\Im(\lambda_i)^2}}{2\omega\Re(\lambda_i)^2}. \quad (5)$$

Then, the solutions $(r(t), v(t))$ of system (1) under control (2) start from $(r(0), v(0))$ inside the set $\mathcal{S}_1(\mu)$ will enter into the set $\mathcal{S}_2(\mu)$ in finite time

$$T = \left\lceil \log_{\|\hat{\mathcal{E}}\|_\epsilon} \frac{\|\hat{\mathcal{P}}\mathcal{U}^{-1}\mathcal{D}_\epsilon^{-1}\|_\infty\|\mathcal{D}_\epsilon\mathcal{U}\hat{\mathcal{P}}^\dagger F\mathcal{B}\|_\infty\Delta\varepsilon}{(1-\bar\xi)(1-\|\hat{\mathcal{E}}\|_\epsilon)M} \right\rceil \tau, \quad (6)$$

where

$$\mathcal{S}_1(\mu) = \Big\{(r,v) : \|\hat{\mathcal{P}}^\dagger([r^\top, v^\top]^\top - [\gamma, \nu]^\top \otimes \mathbf{1}_n)\|_\epsilon \leq \frac{(1-\bar\xi)\mu M}{\|\hat{\mathcal{P}}\mathcal{U}^{-1}\mathcal{D}_\epsilon^{-1}\|_\infty}\Big\}, \quad (7)$$

$$\mathcal{S}_2(\mu) = \Big\{(r,v) : \|\hat{\mathcal{P}}^\dagger([r^\top, v^\top]^\top - [\gamma, \nu]^\top \otimes \mathbf{1}_n)\|_\epsilon \leq \frac{\|\mathcal{D}_\epsilon\mathcal{U}\hat{\mathcal{P}}^\dagger F\mathcal{B}\|_\infty\Delta\mu(1+\varepsilon)}{1-\|\hat{\mathcal{E}}\|_\epsilon}\Big\}, \quad (8)$$

$$\gamma(t) = \cos(\omega t)\xi^\top r(0) + \sin(\omega t)\xi^\top v(0)/\omega, \quad (9)$$

$$\nu(t) = -\omega\sin(\omega t)\xi^\top r(0) + \cos(\omega t)\xi^\top v(0), \quad (10)$$

$\bar\omega = \max\{\omega, 1/\omega\}$, $\lambda_2, \lambda_3, \ldots, \lambda_n$ are non-zero eigenvalues of the Laplacian matrix L of $\mathscr{G}$, $\xi = [\xi_1, \xi_2, \ldots, \xi_n]^\top$ is a left eigenvector of L associated with the zero eigenvalue λ_1.

Proof: By the properties of the quantizer q and the Laplacian matrix L of $\mathscr{G}$, system (1) with control (2) together can be written as

$$\dot X(t) = \mathcal{A}X(t) + \mathcal{B}X(t_k) + \mathcal{C}(t_k), \quad t \in [t_k, t_{k+1}). \quad (11)$$

And, note that $[\gamma(t),\nu(t)]^\top$ in (9) and (10) is the solution of the following equation:

$$\begin{cases}\dot{\gamma}(t)=\nu(t),\\ \dot{\nu}(t)=-\omega^2\gamma(t)\end{cases} \tag{12}$$

with initial value $[\xi^\top r(0),\xi^\top v(0)]^\top$. Simple computation yields that $\gamma^2(t)+\nu^2(t)/\omega^2=(\xi^\top r(0))^2+(\xi^\top v(0))^2/\omega^2=M_0^2/\omega^2$. Obviously, $\mathcal{B}X(t_k)$ and $\mathcal{C}(t_k)$ are constant vectors. For $t\in[t_k,t_{k+1})$, integrating both sides of equation (11) from t_k to t, one obtains

$$X(t)=E(t,t_k)X(t_k)+F(t,t_k)\mathcal{C}(t_k),$$

where $E(t,t_k)=\exp(\mathcal{A}(t-t_k))+\int_{t_k}^t\exp(\mathcal{A}(t-s))\mathcal{B}ds$ and $F(t,t_k)=\int_{t_k}^t\exp(\mathcal{A}(t-s))ds$. Furthermore, for $t=t_{k+1}$, one has

$$X(t_{k+1})=E(t_{k+1},t_k)X(t_k)+F(t_{k+1},t_k)\mathcal{C}(t_k). \tag{13}$$

Notice that $t_{k+1}-t_k=\tau$. So, $E(t_{k+1},t_k)=E$ and $F(t_{k+1},t_k)=F$ for all $k=0,1,2,\ldots$.

Next, in order to implement quantization, the condition $\|X(t_k)\|_\infty<\mu M$ must be satisfied for all $k=1,2,\ldots$. First, when $t=t_1$, one gets

$$X(t_1)=EX(t_0)+F\mathcal{C}(t_0).$$

Moreover, letting $\mathcal{X}(t)=\mathcal{P}^{-1}X(t)=[\mathcal{X}_1(t),\ldots,\mathcal{X}_{2n}(t)]^\top\in\mathbb{R}^{2n}$ gives

$$\mathcal{X}(t_1)=\mathcal{E}\mathcal{X}(t_0)+\mathcal{P}^{-1}F\mathcal{C}(t_0), \tag{14}$$

where $\mathcal{E}=\mathcal{P}^{-1}E\mathcal{P}$. Denote $\bar{\mathcal{X}}(t)=[\mathcal{X}_1(t),\mathcal{X}_{n+1}(t)]^\top\in\mathbb{R}^2$ and $\hat{\mathcal{X}}(t)=[\mathcal{X}_2(t),\ldots,\mathcal{X}_n(t),\mathcal{X}_{n+2}(t),\ldots,\mathcal{X}_{2n}(t)]^\top\in\mathbb{R}^{2n-2}$. By using the property of P^{-1} (the first row of P^{-1} is $\xi^\top$), one gets $\bar{p}_1L(q_\mu(v(t_0))-v(t_0))=0$, so that equation (14) can be written as

$$\begin{cases}\bar{\mathcal{X}}(t_1)=[\gamma(t_1),\nu(t_1)]^\top, & (15a)\\ \hat{\mathcal{X}}(t_1)=\hat{\mathcal{E}}\hat{\mathcal{X}}(t_0)+\hat{\mathcal{P}}^\dagger F\mathcal{C}(t_0). & (15b)\end{cases}$$

By the definition of $\mathcal{P}$ ($p_1=\mathbf{1}_n$), one obtains

$$X(t_1)=\mathcal{P}\mathcal{X}(t_1)=[\gamma(t_1),\nu(t_1)]^\top\otimes\mathbf{1}_n+\hat{\mathcal{P}}\hat{\mathcal{X}}(t_1).$$

By using $\|X(t_0)\|_\infty\le M\mu$, $M_0\le\bar{\xi}\mu M$, and the definitions of $\mathcal{S}_1(\mu)$ and $\|\cdot\|_\epsilon$, it follows that

$$\begin{aligned}\|X(t_1)\|_\infty=&\|[\gamma(t_1),\nu(t_1)]^\top\otimes\mathbf{1}_n+\hat{\mathcal{P}}\hat{\mathcal{X}}(t_1)\|_\infty\\ \le&M_0+\|\hat{\mathcal{P}}\mathcal{U}^{-1}\mathcal{D}_\epsilon^{-1}\mathcal{D}_\epsilon\mathcal{U}\hat{\mathcal{X}}(t_1)\|_\infty\\ \le&\bar{\xi}\mu M+\|\hat{\mathcal{P}}\mathcal{U}^{-1}\mathcal{D}_\epsilon^{-1}\|_\infty\|\hat{\mathcal{X}}(t_0)\|_\epsilon\\ \le&\mu M.\end{aligned}$$

Meanwhile, based on the definition of $\mathcal{S}_2(\mu)$, from equation (15b) and condition (4), it follows that

$$\begin{aligned}&\|\hat{\mathcal{X}}(t_1)\|_\epsilon-\|\hat{\mathcal{X}}(t_0)\|_\epsilon\\ \le&(\|\hat{\mathcal{E}}\|_\epsilon-1)\|\hat{\mathcal{X}}(t_0)\|_\epsilon+\|\hat{\mathcal{P}}^\dagger F\mathcal{C}(t_0)\|_\epsilon\\ <&0.\end{aligned}$$

In the same way, by mathematical induction and the definition of $\mathcal{S}_1(\mu)$ and $\mathcal{S}_2(\mu)$, and Lemma 5, one can show that when

$$\frac{(1-\bar{\xi})\mu M}{\|\hat{\mathcal{P}}\mathcal{U}^{-1}\mathcal{D}_\epsilon^{-1}\|_\infty}\ge\|\hat{\mathcal{X}}(t_k)\|_\epsilon\ge\frac{\|\mathcal{D}_\epsilon\mathcal{U}\hat{\mathcal{P}}^\dagger F\mathcal{B}\|_\infty\Delta\mu}{1-\|\hat{\mathcal{E}}\|_\epsilon},$$

the inequality $\|X(t_k)\|_\infty<\mu M$ holds and the sequence $\{\|\hat{\mathcal{X}}(t_k)\|_\epsilon\}$ decreases.

Furthermore, equation (13) implies

$$\begin{aligned}X(t_k)=&E^kX(t_0)+E^{k-1}F\mathcal{C}(t_0)+E^{k-2}F\mathcal{C}(t_1)\\ &+\cdots+EF\mathcal{C}(t_{k-2})+F\mathcal{C}(t_{k-1}).\end{aligned}$$

and

$$\begin{aligned}\mathcal{X}(t_k)=&\mathcal{E}^k\mathcal{X}(t_0)+\mathcal{E}^{k-1}\mathcal{P}^{-1}F\mathcal{C}(t_0)+\mathcal{E}^{k-2}\mathcal{P}^{-1}F\mathcal{C}(t_1)\\ &+\cdots+\mathcal{E}\mathcal{P}^{-1}F\mathcal{C}(t_{k-2})+\mathcal{P}^{-1}F\mathcal{C}(t_{k-1}).\end{aligned} \tag{16}$$

By using the property of P^{-1} (the first row of P^{-1} is $\xi^\top$), one can get $\bar{p}_1L(q_\mu(v(t_k))-v(t_k))=0$, so that equation (16) can be written as

$$\begin{cases}\bar{\mathcal{X}}(t_k)=[\gamma(t_k),\nu(t_k)]^\top,\\ \hat{\mathcal{X}}(t_k)=\hat{\mathcal{E}}^k\hat{\mathcal{X}}(t_0)+\hat{\mathcal{E}}^{k-1}\hat{\mathcal{P}}^\dagger F\mathcal{C}(t_0)+\hat{\mathcal{E}}^{k-2}\hat{\mathcal{P}}^\dagger F\mathcal{C}(t_1)\\ \qquad+\cdots+\hat{\mathcal{E}}\hat{\mathcal{P}}^\dagger F\mathcal{C}(t_{k-2})+\hat{\mathcal{P}}^\dagger F\mathcal{C}(t_{k-1}),\end{cases} \tag{17}$$

where $[\gamma(t),\nu(t)]^\top$ is defined by equation (12). From Lemma 5, Theorems 5.4.10 and 5.6.15 in [27], and equation (17), one can prove that the vector sequence $\{\hat{\mathcal{X}}(t_k)\}$ converges with respect to any norm if $\rho(\hat{\mathcal{E}})<1$.

Hereinafter, we consider the case of $\rho(\hat{\mathcal{E}})<1$. In this scenario, there exists a sufficiently small $\epsilon>0$ such that $\rho(\hat{\mathcal{E}})+\epsilon<1$ by Lemma 5. Using the properties of the quantizer $|q_\mu(v_i(t))-v_i(t)|\le\Delta\mu$, one has $\|\hat{\mathcal{P}}^\dagger F\mathcal{C}(t_k)\|_\epsilon\le\|\mathcal{D}_\epsilon\mathcal{U}\hat{\mathcal{P}}^\dagger F\mathcal{B}\|_\infty\Delta\mu$ for $k=1,2,\ldots$, and

$$\begin{aligned}\|\hat{\mathcal{X}}(t_k)\|_\epsilon\le&\|\hat{\mathcal{E}}\|_\epsilon^k\|\hat{\mathcal{X}}(t_0)\|_\epsilon+\|\hat{\mathcal{E}}\|_\epsilon^{k-1}\|\hat{\mathcal{P}}^\dagger F\mathcal{C}(t_0)\|_\epsilon\\ &+\|\hat{\mathcal{E}}\|_\epsilon^{k-2}\|\hat{\mathcal{P}}^\dagger F\mathcal{C}(t_1)\|_\epsilon+\cdots\\ &+\|\hat{\mathcal{E}}\|_\epsilon\|\hat{\mathcal{P}}^\dagger F\mathcal{C}(t_{k-2})\|_\epsilon+\|\hat{\mathcal{P}}^\dagger F\mathcal{C}(t_{k-1})\|_\epsilon\\ \le&\|\hat{\mathcal{E}}\|_\epsilon^k\|\hat{\mathcal{X}}(t_0)\|_\epsilon+\frac{\|\mathcal{D}_\epsilon\mathcal{U}\hat{\mathcal{P}}^\dagger F\mathcal{B}\|_\infty\Delta\mu(1-\|\hat{\mathcal{E}}\|_\epsilon^k)}{1-\|\hat{\mathcal{E}}\|_\epsilon}\\ \le&\|\hat{\mathcal{E}}\|_\epsilon^k\|\hat{\mathcal{X}}(t_0)\|_\epsilon+\frac{\|\mathcal{D}_\epsilon\mathcal{U}\hat{\mathcal{P}}^\dagger F\mathcal{B}\|_\infty\Delta\mu}{1-\|\hat{\mathcal{E}}\|_\epsilon}.\end{aligned} \tag{18}$$

By $\|\hat{\mathcal{E}}\|_\epsilon<1$, we further obtain

$$\limsup_{k\to\infty}\|\hat{\mathcal{X}}(t_k)\|_\epsilon\le\frac{\|\mathcal{D}_\epsilon\mathcal{U}\hat{\mathcal{P}}^\dagger F\mathcal{B}\|_\infty\Delta\mu}{1-\|\hat{\mathcal{E}}\|_\epsilon}. \tag{19}$$

Using the definitions of $\mathcal{S}_1(\mu)$ and $\mathcal{S}_2(\mu)$, and inequality (18), the solutions $r(t)$ and $v(t)$ of system (1) under control (2) go to $\mathcal{S}_2(\mu)$ from $\mathcal{S}_1(\mu)$ by time T, which was defined in (6).

Finally, sufficient conditions are derived to ensure that $\rho(\hat{\mathcal{E}})<1$. The characteristic polynomial of $\hat{\mathcal{E}}$ is $p(x)=\Pi_{i=2}^np_i(x)$, where

$$p_i(x)=\det\begin{bmatrix}\cos(\omega\tau)-x & \frac{1}{\omega}\sin(\omega\tau)+\frac{\lambda_i}{\omega^2}(\cos(\omega\tau)-1)\\ -\omega\sin(\omega\tau) & \cos(\omega\tau)-\frac{\lambda_i}{\omega}\sin(\omega\tau)-x\end{bmatrix}.$$

So, $\rho(\hat{\mathcal{E}})<1$ if and only if $|x|<1$, where x is the root of $p_i(x)=0$ for all $i=2,3,\ldots,n$. By simply calculations, $p_i(x)=0$ can be rewritten as

$$p_i(x) = x^2 + \big(\sin(\omega\tau)\lambda_i/\omega - 2\cos(\omega\tau)\big)x + \big(1 - \sin(\omega\tau)\lambda_i/\omega\big) = 0. \quad (20)$$

If $\sin(\omega\tau) = 0$, then $|x| = 1$, so $\tau \neq \frac{k\pi}{\omega}$ for all $k \in \mathbb{N}$, that is, $\sin(\omega\tau) \neq 0$ and $\cos(\omega\tau) = \pm 1$. Let $x = (s+1)/(s-1)$. Then, equation (20) is transformed into

$$s^2 + \frac{\lambda_i}{\omega}\cot(\omega\tau/2)s + \cot^2(\omega\tau/2) - \frac{\lambda_i}{\omega}\cot(\omega\tau/2) = 0. \quad (21)$$

It is easy to see from the property of the bilinear transformation that $|x| < 1$ in equation (20) holds if and only if $\Re(s) < 0$ in equation (21) holds. By Lemma 4, one can get that $\Re(s) < 0$ if and only if

$$\frac{1}{\omega}\cot(\omega\tau/2)\Re(\lambda_i) > 0$$

and

$$\omega\Re(\lambda_i)^2\cot^2(\omega\tau/2) - \big(\Re(\lambda_i)\Im(\lambda_i)^2 + \Re(\lambda_i)^3\big) \times \cot(\omega\tau/2) - \omega\Im(\lambda_i)^2 > 0. \quad (22)$$

Recall that $\Re(\lambda_i) > 0$ for all $i = 2, 3, \ldots, n$, and inequality (22) holds by equation (5). So, if condition (5) holds, then $\rho(\hat{\mathcal{E}}) < 1$. This completes the proof of Theorem 1. ■

Remark 2. From Lemma 3, the matrix sequence $\{\hat{\mathcal{E}}^k\}$ converges to O_{2n-2} if and only if $\rho(\hat{\mathcal{E}}) < 1$, and from Lemma 2, one can see that $\rho(\hat{\mathcal{E}}) < 1$ is a necessary condition for $\|\hat{\mathcal{E}}\| \leq \rho(\hat{\mathcal{E}}) + \epsilon < 1$ to guarantee the convergence of the series $\sum_{k=0}^{\infty}\hat{\mathcal{E}}^k\mathcal{P}^{-1}F\mathcal{C}(t_i)$. So, the case of $\rho(\hat{\mathcal{E}}) < 1$ is only needed to be considered in the proof of Theorem 1. At the same time, the condition $\rho(\hat{\mathcal{E}}) < 1$ is a necessary and sufficient condition to achieve synchronization for system (1) under control (2), in which $q(v(t_k)) = v(t_k)$. This can be proved in the same way as the proof of Theorem 1 [6].

Remark 3. Obviously, for any $S \in \mathbb{C}^{(2n-2)\times(2n-2)}$, one has $\max_{\|x\|_\epsilon=1}\|Sx\|_\epsilon = \max_{\|\mathcal{D}_\epsilon\mathcal{U}x\|_\infty=1}\|\mathcal{D}_\epsilon\mathcal{U}Sx\|_\infty = \max_{\|D_\epsilon\mathcal{U}x\|_\infty=1}\|\mathcal{D}_\epsilon\mathcal{U}S\mathcal{U}^{-1}D_\epsilon^{-1}D_\epsilon\mathcal{U}x\|_\infty = \|S\|_\epsilon$, so the matrix norm $\|\cdot\|_\epsilon$ can be induced by the vector norm $\|\cdot\|_\epsilon$ in Theorem 1. For a finite-dimensional real or complex vector space, all norms are equivalent. Consequently, the convergence of a sequence of vectors in a finite-dimensional space is independent of the norm (see, e.g., Corollary 5.4.6. in [27]).

Remark 4. In Theorem 1, the set $\mathcal{S}_2$ is not an invariant region of system (1), but $(r(t), v(t)) \in \mathcal{S}_2$, where $t = k\tau$ for all $k \geq T/\tau$ and $k \in \mathbb{N}$. By some elementary computations, one can verify that, for $t \in (t_k, t_{k+1})$, $X(t) = \mathcal{P}\mathcal{X}(t) = [\gamma(t), \nu(t)]^\top \otimes \mathbf{1}_n + \hat{\mathcal{P}}\hat{\mathcal{X}}(t)$, $\hat{\mathcal{X}}(t) = \hat{\mathcal{E}}(t - t_k)\hat{\mathcal{X}}(t_k) + \hat{\mathcal{P}}^\dagger F(t, t_k)\mathcal{C}(t_k)$ and $\|[r(t)^\top, v(t)^\top]^\top - [\gamma(t), \nu(t)]^\top \otimes \mathbf{1}_n\|_\infty \leq \|\hat{\mathcal{E}}(t - t_k)\|_\epsilon\|\hat{\mathcal{E}}\|_\epsilon^k\|\hat{\mathcal{X}}(t_0)\|_\epsilon + \|\hat{\mathcal{E}}(t - t_k)\|_\epsilon\|\mathcal{D}_\epsilon\mathcal{U}\hat{\mathcal{P}}^\dagger F\mathcal{B}\|_\infty\Delta\mu(1 - \|\hat{\mathcal{E}}\|_\epsilon^k)/(1 - \|\hat{\mathcal{E}}\|_\epsilon) + \|\mathcal{D}_\epsilon\mathcal{U}\hat{\mathcal{P}}^\dagger F(t, t_k)\mathcal{B}\|_\infty\Delta\mu$. Obviously, the norms $\|\hat{\mathcal{E}}(t - t_k)\|_\epsilon$ and $\|F(t, t_k)\|_\epsilon$ are time-varying but bounded, and the set $\mathcal{S}_2$ is more important and useful than the invariant region of system (1). From equation (19), we can show that the norm $\|[r(t_k)^\top, v(t_k)^\top]^\top - \mathbf{1}_n \otimes [\gamma(t_k), \nu(t_k)]^\top\|_\infty$ is bounded as $k \to \infty$, that is, the states of the synchronized oscillators converge to a bounded region of the orbits.

If the interconnected network is undirected, one has the following corollary.

Corollary 1. Assume that the graph $\mathscr{G}$ is undirected and connected with an arbitrary $\varepsilon > 0$, and large enough M compared to Δ such that

$$M > \frac{\|\hat{\mathcal{P}}\mathcal{U}^{-1}\mathcal{D}_\epsilon^{-1}\|_\infty\|\mathcal{D}_\epsilon\mathcal{U}\hat{\mathcal{P}}^\dagger F\mathcal{B}\|_\infty}{(1 - \bar{\xi})(1 - \|\hat{\mathcal{E}}\|_\epsilon)}\Delta,$$

for $\tau \in \{\tau : \cot(\omega\tau/2) > \lambda_i/\omega, i = 2, 3, \ldots, n\}$. Then, the solutions $(r(t), v(t))$ of system (1) under control (2) starting form $(r(0), v(0))$ inside the set $\mathcal{S}_1(\mu)$ will enter into the set $\mathcal{S}_2(\mu)$ in finite time.

Remark 5. When the graph $\mathscr{G}$ is undirected, the Laplacian matrix L of $\mathscr{G}$ is symmetric and λ_i are real ($\Im(\lambda_i) = 0$), so $\phi_i = \lambda_i/\omega$.

B. Quantized feedback coupling with an adjustable zooming parameter

In this subsection, we consider the convergence of equation (1) with an adjustable zooming parameter $\mu(t)$ at sampling instants.

Theorem 2. Assume that the directed graph $\mathscr{G}$ is strongly connected with an arbitrarily small $\varepsilon > 0$, and that M is large enough compared with Δ such that

$$M > \max\Big\{2\Delta, \frac{\|\hat{\mathcal{P}}\mathcal{U}^{-1}\mathcal{D}_\epsilon^{-1}\|_\infty\|\mathcal{D}_\epsilon\mathcal{U}\hat{\mathcal{P}}^\dagger F\mathcal{B}\|_\infty}{(1 - \bar{\xi})(1 - \|\hat{\mathcal{E}}\|_\epsilon)}\Delta\Big\}.$$

If $\cot(\omega\tau/2) > \phi_i$ for all $i = 2, \ldots, n$, where ϕ_i is defined by equation (5). Then, there exists a right-continuous and piecewise-constant function $\mu(t)$ such that the solutions $[r_i(t), v_i(t)]^\top$ of system (1) under control (2) exponentially converge to $[\gamma(t), \nu(t)]^\top$ in equations (9) and (10).

Proof: To construct the adjustable zooming parameter μ, we divide the proof into two steps.

Step 1. The zooming-out stage, in which one can increase μ to obtain a larger quantization range such that the quantizer can capture the output.

Set the control law $u_i(t) = 0$. Let $\mu(t_0) = 0$ and $\mu(t) = k\Delta$ for $t \in [t_k, t_{k+1})$. Then, there exists a $k_0 \in \mathbb{N}$ such that

$$\left|\frac{v_i(t_{k_0})}{\mu(t_{k_0})}\right| \leq M - 2\Delta, \text{ for } i = 1, 2, \ldots, n, \quad (23)$$

where $t_{k_0} = k_0\tau$. This implies, by condition (i), that

$$\begin{cases} \left|q\big(\frac{v_i(t_{k_0})}{\mu(t_{k_0})}\big)\right| \leq M - \Delta, \\ \left|q_{\mu(t_{k_0})}(v_i(t_{k_0})) - \Delta\mu(t_{k_0})\right| \leq M\mu(t_{k_0}). \end{cases}$$

In view of conditions (i) and (ii), one has $(r(t_{k_0}), v(t_{k_0})) \in \mathcal{S}_1(\mu(t_{k_0}))$.

Step 2: The zooming-in stage, in which one can decrease μ to obtain a smaller quantization error such that the solution $(r_i(t), v_i(t))$ of the system converges to $(\gamma(t), \nu(t))$.

Let $u_i(t)$ be as in equation (2) in Case 2, where $\mu(t) = \mu(t_{k_0}) = k_0\Delta$ for $t = [t_{k_0}, t_{k_0} + T)$, and T is given by equation (6). By using Theorem 1, we have $(r(t_{k_0}+T), v(t_{k_0}+T)) \in \mathcal{S}_2(\mu(t_{k_0}))$.

For $t = [t_{k_0} + T, t_{k_0} + 2T)$, let $\mu(t) = \mu(t_{k_0} + T) = \theta\mu(t_{k_0}) = \theta k_0\Delta$, where

$$\theta = \frac{\|\hat{\mathcal{P}}\mathcal{U}^{-1}\mathcal{D}_\epsilon^{-1}\|_\infty\|\mathcal{D}_\epsilon\mathcal{U}\hat{\mathcal{P}}^\dagger F\mathcal{B}\|_\infty(1 + \varepsilon)\Delta}{(1 - \bar{\xi})(1 - \|\hat{\mathcal{E}}\|_\epsilon)M}.$$

Obviously, $\theta < 1$ by equation (2), and $\mathcal{S}_1(\mu(t_{k_0}+T)) = \mathcal{S}_2(\mu(t_{k_0}))$ by equations (7) and (8). From Theorem 1, one has $(r(t_{k_0}+2T), v(t_{k_0}+2T)) \in \mathcal{S}_2(\mu(t_{k_0}+T))$.

By mathematical induction, for $t = [t_{k_0}+kT, t_{k_0}+(k+1)T)$, letting $\mu(t) = \mu(t_{k_0}+kT) = \theta^k \mu(t_{k_0}) = \theta^k k_0 \Delta$, we obtain $(r(t_{k_0}+(k+1)T), v(t_{k_0}+(k+1)T)) \in \mathcal{S}_2(\mu(t_{k_0}+kT))$ and $\mathcal{S}_2(\mu(t_{k_0}+kT)) = \mathcal{S}_1(\mu(t_{k_0}+(k+1)T))$.

Based on the above analysis, the boundedness of $\|\hat{\mathcal{E}}(t-t_k)\|_\epsilon$ and $\|F(t,t_k)\|_\epsilon$ for $t \in [t_k, t_{k+1})$, imply that $\mu(t) \to 0$ and $\|[r(t)^\top, v(t)^\top]^\top - \mathbf{1}_n \otimes [\gamma(t), \nu(t)]^\top\|_\infty \to 0$ as $t \to \infty$. This completes the proof of Theorem 2. ■

Remark 6. By the proof of Theorem 2, one obtains the piecewise control law $u_i(t)$ and the zooming parameter $\mu(t)$ as follows:

$$\begin{cases} u_i(t) = 0 \text{ and } \mu(t) = \lceil t/\tau \rceil \Delta, & \text{if } t < k_0\tau, \\ u_i(t) \text{ in (2) and } \mu(t) = \theta^{\lfloor \frac{t-\tau k_0}{T} \rfloor} k_0 \Delta, & \text{if } t \geq k_0\tau, \end{cases} \tag{24}$$

where k_0 is dependent on the initial values and determined by equation (23). The parameter $\mu(t)$ can be preset, which is different from the adaptive control, where the parameter changes with the state of the system [29], [30], [31].

Remark 7. In this note, the quantization is the absolute information $q(v_i)$ and the data is transmitted in a quantizer form, which is different from [22], where the data is transmitted in a real form and the relative information is quantized as $q(v_i - v_j)$. If the feedback is designed without quantization $q(y) = y$, the synchronization can be completely achieved [6].

IV. Numerical Simulations

In this section, two numerical simulations are presented to demonstrate the effectiveness of the theorems established in the previous section.

Consider a directed network of 10 agents moving in the one-dimensional Euclidean space with the topology $\mathscr{G}$ as shown in Fig. 2.

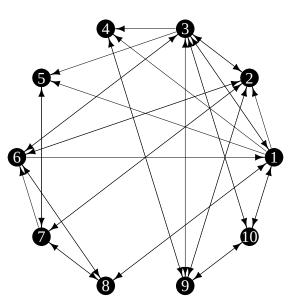


Fig. 2. Directed graph $\mathscr{G}$, where $a_{ij} = 1$ for all $(j,i) \in \mathscr{E}$ and $a_{ij} = 0$ otherwise.

First, consider the system under protocol (2) with a fixed zooming parameter (Case 1). Let $\omega = \sqrt{\pi/2}$, thus $\tau \in (k\pi/\omega, k\pi/\omega + \tau_0)$ for all $k = 0, 1, 2, \ldots$, where $\tau_0 = \min \operatorname{arccot}(\phi_i)$. The relationship between λ_i and ϕ_i is shown in Table I. Choose $\tau = 0.1$, $M = 10$ and $\Delta = 0.5$. Direct computation yields $\rho(\hat{\mathcal{E}}) = 0.9747$, so that the conditions of Theorem 1 are satisfied. Define the synchronization error by $E(t) = [r(t)^\top, v(t)^\top]^\top - [\gamma(t), \nu(t)]^\top \otimes \mathbf{1}_n$. The evolutions of the oscillators are shown in Fig. 3 for the chosen initial values $(r_i(0), v_i(0))$, $i = 1, 2, \ldots, 10$. One can observe from Fig. 3 that complete synchronization cannot been reached, although the states converge to a bounded region of the synchronized orbits.

TABLE I
COMPUTATION RESULTS.

i	λ_i	ϕ_i	$\operatorname{arccot}(\phi_i)$
2	1.5594	1.2442	0.6770
3	$6.3182 + 0.0706\mathbf{i}$	5.0419	0.1958
4	$6.3182 - 0.0706\mathbf{i}$	5.0419	0.1958
5	2.9473	2.3516	0.4021
6	$3.4893 + 0.2867\mathbf{i}$	2.8052	0.3424
7	$3.4893 - 0.2867\mathbf{i}$	2.8052	0.3424
8	5.1342	4.0965	0.2394
9	4.7440	3.7852	0.2583
10	3.0000	2.3937	0.3957

Next, consider the system under protocol (2) with an adjustable zooming parameter (Case 2), where $u_i(t)$ and $\mu(t)$ are defined in equation (24). By computation, one can obtain $T = 5.9267$ from equation (6). Fig. 4 shows the evolutions of the oscillators for the same initial values. It is clear from Fig. 4 that the complete synchronization is achieved.

Comparing Fig. 3 with Fig. 4, one can see that the quantization error Δ affects the synchronization error $E(t)$, and the synchronization error remains in a range by using a fixed zooming parameter (see the enlargement from 280 to 300 in Fig. 3), and the asymptotic synchronization can be achieved by using an adjustable zooming parameter (see the enlargement from 261.91 to 262.01 in Fig. 4).

V. Conclusion

In this technical note, an effective quantized sampled-data feedback coupling protocol has been designed and evaluated for synchronizing networked harmonic oscillators. The quantizer with a fixed or an adjustable zooming parameter has also been designed by using only sampled velocity data. Some sufficient conditions have been established under which the networked harmonic oscillators could achieve complete synchronization.

Future studies may include the synchronization of some general complex dynamical systems via quantized control, and the synchronization by designing coupling and control protocols with logarithmic quantizers.

Acknowledgment

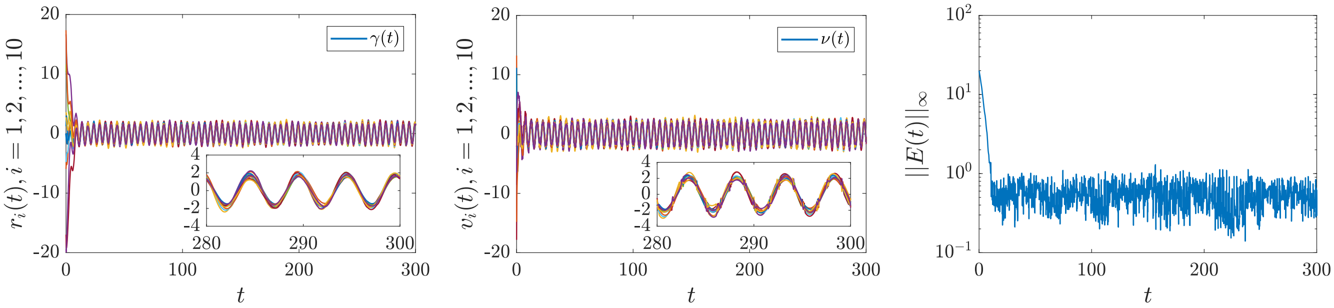
The authors would like to thank the associate editor and the anonymous reviewers for their insightful suggestions.

References

[1] S. Boccaletti, J. Kurths, G. Osipov, D. L. Valladares, and C. Zhou, "The synchronization of chaotic systems," *Physics Reports*, vol. 366, no. 1-2, pp. 1–101, 2002.

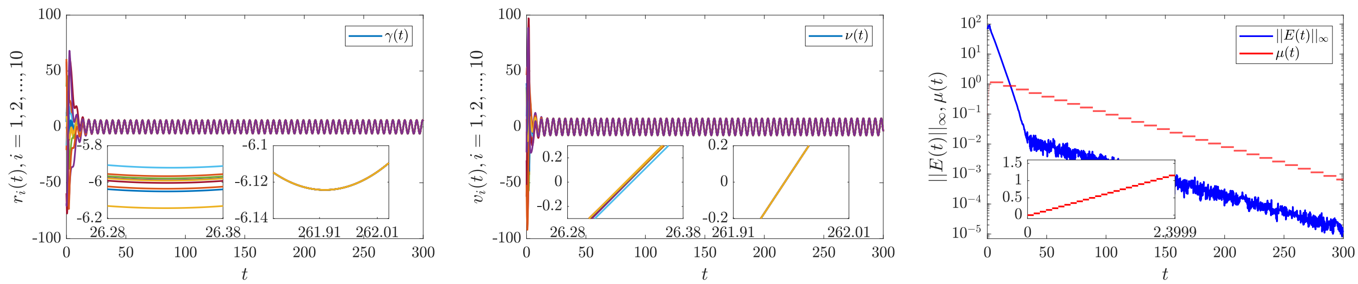
[2] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 215–233, Jan 2007.

[3] F. Dörfler and F. Bullo, "Synchronization in complex networks of phase oscillators: A survey," *Automatica*, vol. 50, no. 6, pp. 1539–1564, 2014.



(a) $r_i(t)$ (b) $v_i(t)$ (c) $||E(t)||_\infty$

Fig. 3. Time evolutions of $r_i(t)$, $v_i(t)$ and $||E(t)||_\infty$ with a fixed zooming parameter μ.



(a) $r_i(t)$ (b) $v_i(t)$ (c) $\|E(t)\|_\infty, \mu(t)$

Fig. 4. Time evolutions of $r_i(t)$, $v_i(t)$ and $||E(t)||_\infty$ with an adjustable zooming parameter $\mu(t)$.

[4] W. Ren, "Synchronization of coupled harmonic oscillators with local interaction," *Automatica*, vol. 44, no. 12, pp. 3195–3200, 2008.

[5] S. E. Tuna, "Synchronization of harmonic oscillators under restorative coupling with applications in electrical networks," *Automatica*, vol. 75, pp. 236–243, 2017.

[6] W. Sun, J. Lü, S. Chen, and X. Yu, "Synchronisation of directed coupled harmonic oscillators with sampled-data," *IET Control Theory Applications*, vol. 8, no. 11, pp. 937–947, July 2014.

[7] L. Ballard, Y. Cao, and W. Ren, "Distributed discrete-time coupled harmonic oscillators with application to synchronised motion coordination," *IET Control Theory & Applications*, vol. 4, no. 5, pp. 806–816, May 2010.

[8] H. Zhang, Q. Wu, and J. Ji, "Synchronization of discretely coupled harmonic oscillators using sampled position states only," *IEEE Transactions on Automatic Control*, vol. 63, no. 11, pp. 3994–3999, Nov 2018.

[9] J. Zhou, H. Zhang, L. Xiang, and Q. Wu, "Synchronization of coupled harmonic oscillators with local instantaneous interaction," *Automatica*, vol. 48, no. 8, pp. 1715–1721, 2012.

[10] Q. Song, W. Yu, J. Cao, and F. Liu, "Reaching synchronization in networked harmonic oscillators with outdated position data," *IEEE Transactions on Cybernetics*, vol. 46, no. 7, pp. 1566–1578, July 2016.

[11] J. Wang, J. Feng, C. Xu, M. Z. Chen, Y. Zhao, and J. Feng, "The synchronization of instantaneously coupled harmonic oscillators using sampled data with measurement noise," *Automatica*, vol. 66, pp. 155–162, 2016.

[12] J. Wang, C. Xu, M. Z. Chen, J. Feng, and G. Chen, "Stochastic feedback coupling synchronization of networked harmonic oscillators," *Automatica*, vol. 87, pp. 404–411, 2018.

[13] X. Liu and T. Iwasaki, "Design of coupled harmonic oscillators for synchronization and coordination," *IEEE Transactions on Automatic Control*, vol. 62, no. 8, pp. 3877–3889, Aug 2017.

[14] M. Zhang, P. Shi, L. Ma, J. Cai, and H. Su, "Quantized feedback control of fuzzy Markov jump systems," *IEEE Transactions on Cybernetics*, vol. 49, no. 9, pp. 3375–3384, Sep. 2019.

[15] R. W. Brockett and D. Liberzon, "Quantized feedback stabilization of linear systems," *IEEE Transactions on Automatic Control*, vol. 45, no. 7, pp. 1279–1289, Jul 2000.

[16] D. Liberzon, "Hybrid feedback stabilization of systems with quantized signals," *Automatica*, vol. 39, no. 9, pp. 1543–1554, 2003.

[17] M. Wakaiki and Y. Yamamoto, "Stabilization of switched linear systems with quantized output and switching delays," *IEEE Transactions on Automatic Control*, vol. 62, no. 6, pp. 2958–2964, June 2017.

[18] Y. Sharon and D. Liberzon, "Input to state stabilizing controller for systems with coarse quantization," *IEEE Transactions on Automatic Control*, vol. 57, no. 4, pp. 830–844, April 2012.

[19] M. Zhang, P. Shi, L. Ma, J. Cai, and H. Su, "Network-based fuzzy control for nonlinear Markov jump systems subject to quantization and dropout compensation," *Fuzzy Sets and Systems*, vol. 371, pp. 96 – 109, 2019.

[20] N. Elia and S. K. Mitter, "Stabilization of linear systems with limited information," *IEEE Transactions on Automatic Control*, vol. 46, no. 9, pp. 1384–1400, Sep 2001.

[21] D. Liberzon, "On stabilization of linear systems with limited information," *IEEE Transactions on Automatic Control*, vol. 48, no. 2, pp. 304–307, Feb 2003.

[22] Y. Xu and J. Wang, "The synchronization of linear systems under quantized measurements," *Systems & Control Letters*, vol. 62, no. 10, pp. 972–980, 2013.

[23] Y. Zhu, Y. Zheng, and Y. Guan, "Consensus of switched multi-agent systems under quantised measurements," *International Journal of Systems Science*, vol. 48, no. 9, pp. 1796–1804, 2017.

[24] F. Ceragioli, C. D. Persis, and P. Frasca, "Discontinuities and hysteresis in quantized average consensus," *Automatica*, vol. 47, no. 9, pp. 1916–1928, 2011.

[25] X. Li, M. Z. Q. Chen, and H. Su, "Quantized consensus of multi-agent networks with sampled data and markovian interaction links," *IEEE Transactions on Cybernetics*, vol. 49, no. 5, pp. 1816–1825, May 2019.

[26] W. Ren and R. W. Beard, "Consensus seeking in multiagent systems under dynamically changing interaction topologies," *IEEE Transactions on Automatic Control*, vol. 50, no. 5, pp. 655–661, May 2005.

[27] R. A. Horn and C. R. Johnson, *Matrix Analysis*, 2nd ed. New York: Cambrigdge University Press, 2012.

[28] P. C. Parks and V. Hahn, *Stability Theory*. New York, USA: Prentice Hall, 1993.

[29] Q. Shen, P. Shi, Y. Shi, and J. Zhang, "Adaptive output consensus with saturation and dead-zone and its application," *IEEE Transactions on Industrial Electronics*, vol. 64, no. 6, pp. 5025–5034, June 2017.

[30] P. Shi and Q. K. Shen, "Observer-based leader-following consensus

of uncertain nonlinear multi-agent systems," *International Journal of Robust and Nonlinear Control*, vol. 27, no. 17, pp. 3794–3811, 2017.

[31] Q. Shen, P. Shi, J. Zhu, and L. Zhang, "Adaptive consensus control of leader-following systems with transmission nonlinearities," *International Journal of Control*, vol. 92, no. 2, pp. 317–328, 2019.