Equilibrium strategy for a multi-period weighted mean-variance portfolio selection in a Markov regime-switching market with uncertain time-horizon and a stochastic cash flow

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Abstract

This paper considers a multi-period weighted mean-variance portfolio selection problem with uncertain time-horizon and a stochastic cash flow in a Markov regime-switching market. The random returns of risky assets and amount of the cash flow all depend on the states of a stochastic market which are assumed to follow a discrete-time Markov chain. Based on the conditional distribution of uncertain time-horizon caused by exogenous factors, we construct a more general mean-variance investment model. Within a game theoretic framework, we derive the equilibrium strategy and equilibrium value function in closed-form by applying backward induction approach. In addition, we show the equilibrium efficient frontier and discuss some degenerate cases. Finally, some numerical examples and sensitivity analysis are presented to illustrate equilibrium efficient frontiers and the effects of uncertain time-horizon on the equilibrium strategy and equilibrium efficient frontier as well as regime-switching and stochastic cash flow on the equilibrium efficient frontier.

Keywords: Uncertain time-horizon; Markov regime-switching; Stochastic cash flow; Multi-period weighted mean-variance portfolio selection; Equilibrium strategy; Equilibrium efficient frontier

1. Introduction

Markowitz (1952) pioneered the mean-variance model, which first laid the portfolio selection problem under the return-risk framework and paved the foundation of modern portfolio theory. But due to the non-separability of variance in the sense of dynamic programming, the extension

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of the static single-period model by Markowitz (1952) to dynamic (multi-period or cotinuous-time) models has taken a long time. Until recently, Li and Ng (2000) and Zhou and Li (2000) derived the analytical optimal solutions to multi-period and continuous-time mean-variance models by using an embedding technique, respectively. After that, the study of dynamic mean-variance portfolio models under various realistic conditions has been developed rapidly. For example, Zhu et al. (2004) and Bielecki et al. (2005) considered a multi-period and a continuous-time mean-variance portfolio model with risk control over bankruptcy, respectively. Cui et al. (2014) investigated the optimal strategy of multi-period mean-variance portfolio selection under no short-selling constraint.

The investment time-horizon of most portfolio selection problems is assumed to be predetermined at the beginning of the investment, either finite or infinite. In the real world, however, investors never know exactly when exit the financial market at the initial time. They might be forced to exit the market before the time they planned due to some exogenous or endogenous factors. Exogenous factors include accidental death, sudden need of money, and so on. One possible example of endogenous factors is: When investors' portfolios are underperforming or even in bankruptcy, investors may decide to withdraw from the market ahead of time in order to avoid greater losses. Therefore, it is of great theoretical and practical significance to relax the restrictive assumption of deterministic time-horizon to uncertain time-horizon for the study of portfolio selection problems. Research on this subject is traced back to Yaari (1965), who considered an optimal consumption, investment and life insurance problem with uncertain lifetime. Then, Merton (1971) investigated a continuous-time optimal investment and consumption problem with uncertain exit time following a Poisson process. In recent years, the research with uncertain time-horizon has been paid much attention. For example, Martellini and Urošević (2006) considered a static portfolio selection problem in cases with exogenously and endogenously uncertain exit times which either are independent of or dependent on returns of assets. Yi et al. (2008) studied a multi-period asset-liability management problem with uncertain exit time, in which the random exit time is determined by exogenous factors. Wu and Li (2011) investigated a multi-period mean-variance portfolio optimization problem with uncertain exit time and regime switching market environment. Landriault et al. (2018) studied equilibrium feedback strategies for the mean-variance investment problem over a random horizon under both discrete-time and continuous-time frameworks.

Some related studies in finance do not consider the possibility that the market state changes over time, that is, it is assumed that there is only one state or mode in the underlying market. But a large number of empirical studies show that returns of assets depend on the states of the financial market that are comprehensive reflection of current economic situations, national policies, moods of investors in the market, and other economic factors. A most common model used to characterize stochastic market environment is the Markov jump model, where the number of the market states is assumed to be finite, such as “bullish” and “bearish” in the stock market, and the switching process of the market states follows a Markov chain. This is also known as regime switching. All the key market parameters, such as the bank interest rate, stocks appreciation rates and volatility rates, may change according to different market modes. In recent years, regime-switching models have gradually become popular in finance. For example, Zhou and Yin (2003) investigated a continuous-time mean-variance portfolio selection problem with regime switching while Çakmak and Özekici (2006) considered a discrete-time version. Wu et al. (2014) considered a multi-period mean-variance portfolio selection model with uncertain time-horizon in a regime-switching market, where the conditional distribution of the time-horizon was assumed to be stochastic and depended on the market states.

The current research on many financial issues does not pay substantial attention to stochastic cash flows, that is, these related models are self-financing. In reality, however, the investors might be faced with the situation of fund injection or withdrawal during the investment process. For example, in order to gain greater returns, the investors tend to invest more on some risky assets in the bull market. In contrast, they may reduce the investment amount on risky assets to avoid risk when the market is depressing. For a company, it may unexpectedly receive government subsidies for its product development or be fined for breaking the rules. Recently, there has been a growing interest on this subject. For instance, using the expected utility maximization model, Munk and Sørensen (2010) studied a continous-time asset allocation problem with stochastic income and interest rates. Wu and Li (2012) investigated a multi-period mean-variance portfolio selection problem with regime switching and a stochastic cash flow. Yao et al. (2013) considered a multi-period mean-variance asset-liability management problem with uncontrolled cash flow and uncertain time-horizon. Yao et al. (2016) further studied the asset-liability management problem with stochastic cash flows in a Markov regime-switching market, in which the stochastic cash flows depend on the market states occurred simultaneously in both wealth and liability processes.

It is now accepted that time consistency should be a required condition for multi-period portfolio selection problems. But it is well known that the variance operator lacks iterated-expectation property and hence the dynamic mean-variance portfolio problem is time-inconsistent. Early studies of time inconsistency began with Strotz (1955), which pointed out three different ways to deal with time inconsistency. If the optimal strategy at the initial time is promised to implement by decision makers at any time after the initial time in order to achieve the optimal goal at the initial time, regardless of whether the strategy is currently optimal, it is called a pre-commitment strategy. In view that investment psychology and tastes of people are changing over time, the pre-commitment strategy has been criticized for lacking rationality in recent years. Hence, it is becoming more and more popular to look for the time-consistent strategy of a time-inconsistent problem. Based on this consideration, the game theoretic framework is adopted in many recent related work. Björk and Murgoci (2010) gave a general approach to handle time-inconsistent problems, formally defined the equilibrium strategy, and derived the extended Hamilton-Jacobi-Bellman (HJB) equation and its verification theorem for a very general objective function. Further, Björk et al. (2014) discussed a realistic continuous-time mean-variance portfolio model, in which the risk aversion factor depends dynamically on the current wealth. Wu and Zeng (2015) investigated an equilibrium investment strategy for defined-contribution pension schemes with generalized mean-variance criterion and mortality risk. Xiao et al. (2020) studied several multi-period mean-variance portfolio optimization problems with the serially correlated returns, and derived the analytical expressions of the time-consistent strategies. Readers may refer to Li et al. (2012), Czichowsky (2013), Bensoussan et al. (2014), Wu and Chen (2015), Zhou et al. (2016), Hu and Wang (2018) etc. for more research on time-consistent strategies.

As far as we know, except for Wu and Zeng (2015) and Landriault et al. (2018), the existing multi-period mean-variance portfolio selection problems with uncertain time-horizon only consider pre-commitment strategies, which are time-inconsistent. However, the problem considered by Wu and Zeng (2015) is a DC pension optimization problem, not a general discrete-time mean-variance portfolio problem. And the problem's uncertain exit time is caused by the special background – mortality risk. It does not take into account other possible uncontrollable factors that lead investors to exit the market ahead of schedule, such as unexpected major expenditure of individual investors, or a dim view of the market's prospects. Moreover, Wu and Zeng (2015) only considered a simple case with one market state and with one risk-free asset and one risky asset. In addition, although Wu and Zeng (2015) considered stochastic nonnegative salary, which can be regarded as a special case of stochastic cash flow, they neglected the possibility of withdrawal of funds and the affection of market states on cash inflows or outflows. While Landriault et al. (2018) considered the equilibrium strategies for the mean-variance investment problem over a random horizon within both discrete-time and continuous-time frameworks. Therefore, this paper incorporates Markov regime-switching and stochastic cash flow into a general multi-period mean-variance portfolio selection model with uncertain time-horizon, in a market with one risk-free asset and multiple risky assets, and try to derive the explicit expression for the time-consistent equilibrium investment strategy within the game theoretic framework. In our model, the uncertain time-horizon refers to the termination of investment behavior, exiting the market ahead of time is due to any exogenous factors, and the conditional probability distribution of the exit time can be determined according to the specific circumstances. In particular, our objective function is a weighted sum of linear combinations of the conditional expectation and variance of the wealth at the times of exiting the market. This objective function can be seen as a special form of the objective function in Costa and Nabholz (2007), which not only considered the conditional expectation and variance of terminal wealth, but also the intermediate restrictions on portfolios. Therefore, our model is a more general mean-variance investment model. Mathematically speaking, the uncertain time-horizon portfolio selection problem becomes more complicated after incorporating Markov regime-switching and stochastic cash flow. All these factors drastically increase the difficulty to derive the closed-form equilibrium strategy to the problem under our model. In this paper, we will apply the backward induction approach to solve the problem.

The remainder of this paper is organized as follows. We present the problem formulation and give the assumptions and primal notations in Section 2. In Section 3, the equilibrium investment strategy and equilibrium value function are derived explicitly by backward induction. In addition, the equilibrium efficient frontier is also provided. Some degenerate cases and a property of our equilibrium strategy are presented in Section 4. Section 5 provides some numerical examples to illustrate our results. The paper is concluded in Section 6. Proofs of the lemmas, propositions and theorem are given in Appendices A-H.

2. Model formulation and notations

Assume that an investor, who enters a financial market that consists of one risk-free asset and N risky assets at time 0 with initial wealth x_0, plans to invest her wealth in the market within T consecutive time periods. Suppose that there are L states in the market; the set of all market states is denoted by $\mathbf{S} = \{1, 2, \cdots, L\}$ and the state of the market at period $n+1$ (i.e. the time interval $[n, n+1)$) by $\xi_n, \xi_n \in \mathbf{S}, n = 0, 1, \cdots, T-1$. Let $\{\xi_0, \xi_1, \cdots, \xi_T\}$ be a time-homogeneous Markov chain with transition matrix $Q = (q_{ij})_{L\times L}$, where $q_{ij} = Pr(\xi_{k+1} = j \mid \xi_k = i) \geq 0$, $\sum_{j=1}^{L} q_{ij} = 1$, for $k = 0, 1, \cdots, T-1$ and $i, j \in \mathbf{S}$, and $Pr(\cdot)$ is the probability measure. Denote the return of risk-free asset by a positive constant r_f, and the returns of the risky assets in period $n+1$ with market state $\xi_n \in \mathbf{S}$ by a vector $R_n(\xi_n) = (R_{n,1}(\xi_n), R_{n,2}(\xi_n), \cdots, R_{n,N}(\xi_n))'$, where $R_{n,k}(\xi_n)$ is the random return of the kth risky asset in period $n+1$ with its probability distribution depending on ξ_n. In this paper, the superscript $'$ stands for the transpose of a matrix or vector.

Although the investor originally plans to invest in T consecutive time periods, she may be forced to exit the market at an uncertain time τ for some uncontrollable reasons during the investment process. Suppose that τ is a positive exogenous random variable and takes integer values $1, 2, \cdots$. Therefore, the actual exit time of the investor is $T \wedge \tau = \min\{T, \tau\}$. Let $p_{n,m}$ be the conditional probability that the investor exits the market at time $m (n+1 \leq m \leq T)$ under the condition of staying in the market at time n, i.e.,

$$p_{n,m} := Pr(T \wedge \tau = m \mid T \wedge \tau > n)$$

where $n = 0, 1, \cdots, T-1$; $m = n+1, n+2, \cdots, T$. Then the conditional probability that the investor exits the market as planned at terminal time T

$$p_{n,T} = 1 - \sum_{m=n+1}^{T-1} p_{n,m}, \quad n = 0, 1, \cdots, T-1. \tag{2.1}$$

Note that $p_{n,m}$, the conditional probability of exiting the market, is related to not only the exit time m, but also the initial time n. The definition mechanism of the exit probability is quite different from most current literature with uncertain exit time where the exit probabilities only depend on the exit times and initial time 0.

Assume that transactions are carried out at the beginning of every period, and transaction costs and taxes are not taken into account. Let π_n^k be the amount invested in the kth risky asset at time n, and $\pi_n = (\pi_n^1, \pi_n^2, \cdots, \pi_n^N)'$ be the portfolio at time n. Now, we define $\pi_{n+} := \{\pi_n, \pi_{n+1}, \cdots, \pi_{T-1}\}$ as the investment strategy from time n on, and X_n^π as the wealth at time n under the strategy π, $n = 0, 1, \cdots, T$. Then the amount invested in the risk-free asset is $X_n^\pi - \sum_{k=1}^N \pi_n^k$. As mentioned in the introduction, there would be a random cash inflow or outflow during the investment process. This uncontrolled cash flow may directly affect the investor's tradable wealth level, and thus affect the selection of investment strategies. Let $c_n(\xi_n)$ be the stochastic cash flow at time n on state ξ_n. When $c_n(\xi_n) > 0$, it says that the inflow of stochastic cash is greater than the outflow, in which the difference can be used for additional investment; when $c_n(\xi_n) < 0$, it means that the cash outflow exceeds the inflow, and the amount of money invested in financial market is correspondingly reduced; when $c_n(\xi_n) = 0$, investment will not be affected. Then the dynamic wealth process can be formulated as

$$X_{n+1}^{\pi} = (X_n^{\pi} - \sum_{k=1}^{N} \pi_n^k) r_f + R_n(\xi_n)' \pi_n + c_n(\xi_n) = X_n^{\pi} r_f + R_n^e(\xi_n)' \pi_n + c_n(\xi_n), \;\; n = 0, 1, \cdots, T-1, \tag{2.2}$$

where $R_n^e(\xi_n) = (R_{n,1}^e(\xi_n), R_{n,2}^e(\xi_n), \cdots, R_{n,N}^e(\xi_n))'$,and $R_{n,k}^e(\xi_n) = R_{n,k}(\xi_n) - r_f$ is the excess return of the kth risky asset given state ξ_n in period $n+1$.

Suppose that the investor can observe directly the current wealth and market state. Denoted by $\mathcal{F}_n := \sigma\{(X_k^{\pi}, \xi_k) | 0 \le k \le n\}$ a σ-field, representing the information (including the wealth levels and the market states) available to the investor up to time n. In view of the Markov property, we have $E(\cdot|\mathcal{F}_n) = E(\cdot|(X_n^{\pi}, \xi_n)) := E_n(\cdot)$. An investment strategy $\pi_{n+} = \{\pi_m; m = n, n+1, \cdots, T-1\}$ is time-n admissible if π_m is adapted to $\mathcal{F}_m$ for all $m = n, n+1, \cdots, T-1$. Let $\mathcal{A}_n$ be the collection of all time-n admissible investment strategies.

In our problem, the investor does not know exactly when she will exit the market, and this leads to the difficulty of expressing the conditional expectation and conditional variance of terminal wealth in the traditional mean-variance model. Given the initial time n and the information that $\xi_n = i$ and $X_n^{\pi} = x_n$, instead of considering the conditional expectation and conditional variance of terminal wealth $X_{T\wedge\tau}^{\pi}$, we consider the weighted average conditional expectation and weighted average conditional variance of the wealth after time n, denoted by $E_n^U(X_{T\wedge\tau}^{\pi})$ and $Var_n^U(X_{T\wedge\tau}^{\pi})$ respectively, that is

$$E_n^U(X_{T\wedge\tau}^{\pi}) := \sum_{m=n+1}^{T} p_{n,m} E_n(X_m^{\pi}), \tag{2.3}$$

$$Var_n^U(X_{T\wedge\tau}^{\pi}) := \sum_{m=n+1}^{T} p_{n,m} Var_n(X_m^{\pi}). \tag{2.4}$$

where the weights are the conditional probabilities that the investor exists the market after the initial time n and their sum equals 1, and $Var_n(\cdot) = Var(\cdot|\mathcal{F}_n) = Var(\cdot \mid X_n^{\pi} = x_n, \xi_n = i)$.

We aim to find an optimal investment strategy to the so called multi-period weighted mean-variance portfolio selection problem

$$\max_{\pi_{n+}} E_n^U(X_{T\wedge\tau}^{\pi}) - \omega Var_n^U(X_{T\wedge\tau}^{\pi}) \quad \text{s.t. (2.2)},$$

for $n = 0, 1, \cdots, T-1$, where $\omega > 0$ is the risk aversion coefficient of the investor.

The objective function of the above problem can be equivalently rewritten as

$$J_n(x_n, i; \pi) := \sum_{m=n+1}^{T} p_{n,m} [E_n(X_m^{\pi}) - \omega Var_n(X_m^{\pi})] \tag{2.5}$$

according to (2.3) and (2.4). The objective function is a weighted sum of a series of linear combinations of the expectation and variance of the wealth at exit times conditional on the information on time n.

Remark 1. A similar form of objective function (2.5) first appeared in Costa and Nabholz (2007). In the paper, a multi-period mean-variance optimization model with intermediate restrictions on the portfolio was proposed, which can control the intermediate behavior of the portfolio's expected returns or variances more efficiently. The biggest difference between the above paper and our paper is the decision-making mechanism of the strategy. It focuses on the pre-commitment strategy, which is time-inconsistent, while our paper pursues the time-consistent equilibrium strategy. Furthermore, Zhou et al. (2016), based on the model of Costa and Nabholz (2007), considered a stochastic cash flow and derived the pre-commitment and time-consistent strategies, respectively. While we consider the equilibrium strategy for a multi-period weighted mean-variance portfolio selection problem in a Markov regime-switching market with uncertain time-horizon and a stochastic cash flow that depends on the market states.

Remark 2. Our model is time-inconsistent in the sense that Bellman optimality principle does not hold, that is, the optimal strategy at time n will no longer be optimal for the optimization problem at some time t with $n+1 \leq t \leq T-1$. According to the discussion of time inconsistency in Björk and Murgoci (2010), the time inconsistency of our model results from two factors: one is the mean-variance utility, which is known as one of time-inconsistent utilities, and the other is the inconsistent viewpoints on the exit conditional probabilities when the investor is at different initial times. That is to say, the exit condition probabilities incorporate the information of the initial moment to the objective function, and hence the objective function of the optimization problem changes with initial times, which obviously increases the time inconsistency of the model. Here, we will adopt the game theoretic framework to deal with the time inconsistence.

In Björk and Murgoci (2010), the decision process is regarded as a non-cooperative game. It is assumed that at each point n in time there is one player, who is called "player number n", and the rule is that player number n can only choose her optimal control π_n, not the control of anyone else. As time goes by, the information faced by the player has changed or different preference has been generated. These players who have different informations or preferences at different times in the future are regarded as different future incarnations of player number n. If player number n knows that all players coming after her will choose their optimal controls, then she only needs to choose her own optimal control to obtain a so-called subgame perfect Nash equilibrium strategy. According to Definition 2.2 in Björk and Murgoci (2010), the equilibrium strategy for model (2.5) can be defined as follows.

Definition 2.1. *Consider a given strategy $\hat{\pi}_{n+} := (\hat{\pi}_n, \hat{\pi}_{n+1}, \cdots, \hat{\pi}_{T-1})$. For any $n(n = 0, 1, \cdots, T-1)$, define the strategy*

$$\bar{\pi}_{n+} = (\pi_n, \hat{\pi}_{n+1}, \cdots, \hat{\pi}_{T-1}),$$

where π_n is an arbitrarily portfolio at time n. Then $\hat{\pi}_{n+}$ is said to be a subgame perfect Nash equilibrium strategy (shortly equilibrium strategy) if for all $n < T$, the following condition holds

$$\max_{\pi_n} J_n(x_n, i; \bar{\pi}_{n+}) = J_n(x_n, i; \hat{\pi}_{n+}).$$

If equilibrium strategy $\hat{\pi}_{n+}$ exists, the equilibrium value function is defined as

$$Vn(x_n, i) = J_n(x_n, i; \hat{\pi}_{n+}).$$

For the sake of simplicity, $\hat{\pi}$ represents the equilibrium strategy $\hat{\pi}_{n+}$ in the following paragraphs.

Given wealth level x_n and state $\xi_n = i$ at time n, our objective is to identify the equilibrium investment strategy $\hat{\pi}_{n+}$ in the sense of Definition 2.1 for the following problem:

$$V_n(x_n, i) = \max_{\pi_n} J_n(x_n, i; (\pi_n, \hat{\pi}_{n+1}, \cdots, \hat{\pi}_{T-1})) \quad \text{s.t. } (2.2) \tag{2.6}$$

Throughout this paper, we introduce the following assumptions and notations.

Assumption 2.1. *The covariance matrix $Var(R^e_n(i)) = Var(R_n(i))$ is positive definite for all $n = 0, 1, \cdots, T-1, i \in \mathbf{S}$.*

Assumption 2.2. *Random series $\Upsilon_k(\xi_k) = (R_k(\xi_k)', c_k(\xi_k))$, $k = 0, 1, \cdots, T-1$, are statistically independent, i.e., $\Upsilon_n(i)$ and $\Upsilon_m(j)$ are independent for all $n, m = 0, 1, \cdots, T-1$, $n \neq m$, $i, j \in \mathbf{S}$.*

Assumption 2.3. *$R_n(i)$ and $c_n(i)$ are independent for all $n = 0, 1, \cdots, T-1, i \in \mathbf{S}$.*

In reality, for example, unexpected expenditure of individual investors, changes in contributions of a pension fund, claims encountered by the insurers and dividend payments of firms, which are uncontrolled cash flow, are not affected by the returns of risky assets.

Assumption 2.4. *Given $\xi_n = i \in \mathbf{S}$, $c_0(i), c_1(i), \cdots, c_{T-1}(i)$ are identically distributed.*

Assumption 2.5. *Short selling is permitted for all risky assets in all periods. Unlimited borrowing and lending are allowed for the riskless asset at the current rate of return of the risk-free asset.*

Notation 2.1. *$E(R^e_n(i)) = r^e_n(i)$, which is assumed to be nonzero for $n = 0, 1, \cdots, T-1$.*

Notation 2.2. *$E(c_n(i)) = \mu_1(i)$, $Var(c_n(i)) = \mu_2(i)$, $n = 0, 1, \cdots, T-1$.*

Notation 2.3. *$Q(i)$ is the ith row of matrix Q, $a(i)$ is the ith element of vector a, $diag(a)$ is the diagonal matrix whose diagonal elements are the components of a.*

Notation 2.4. *For any matrix $A_{L\times L}$ and a L-dimension column vector a, define matrix $A_a = Adiag(a)$, $\bar{A} = A\mathbf{1}_{L\times 1}$, $\mathbf{1}_{L\times 1} = (1, 1, \cdots, 1)'$. Specially, $\overline{Q_a} = Qa$.*

Notation 2.5. *For L-dimension column vectors a, b, define $a\cdot b$ as a column vector with $(a\cdot b)(i) = a(i)b(i)$. In particular, a^2 is a column vector with $(a^2)(i) = (a(i))^2$.*

Notation 2.6. *Q^m is the m-step transition probability matrix of the Markov chain, which can be obtained as the mth power of Q. Q^0 is defined as an identity matrix.*

Notation 2.7. *$\sum_{k=m}^{n} u_k = 0$, if $n < m$ for any sequence $\{u_k\}$.*

Notation 2.8. *g_n is a L-dimension column vector whose ith component is $g_n(i) = r^e_n(i)'Var^{-1}(R^e_n(i))r^e_n(i)$, where $Var^{-1}(\cdot)$ is the inverse matrix of $Var(\cdot)$.*

Notation 2.9. *For $k = 0, 1, \cdots$, $l = 1, 2, \cdots$ and $m = 0, 1, \cdots, l-1$, $\gamma_k, \eta_k, \delta_{m,l}$ and $\rho_{m,l}$ are L-dimension column vectors whose ith components are, respectively,*

$$\gamma_k(i) := Var_n(\overline{Q^k_{\mu_1}}(\xi_{n+1})) = \overline{Q_{(\overline{Q^k_{\mu_1}})^2}}(i) - (\overline{Q_{(\overline{Q^k_{\mu_1}})}})^2(i), \tag{2.7}$$

$$\eta_k(i) := Var_n(\overline{Q^k_{g_{n+1+k}}}(\xi_{n+1})) = \overline{Q_{(\overline{Q^k_{g_{n+1+k}}})^2}}(i) - (\overline{Q_{(\overline{Q^k_{g_{n+1+k}}})}})^2(i), \tag{2.8}$$

$$\delta_{m,l}(i) := Cov_n(\overline{Q^m_{\mu_1}}(\xi_{n+1}), \overline{Q^l_{\mu_1}}(\xi_{n+1})) = \overline{Q_{(\overline{Q^m_{\mu_1}}\cdot\overline{Q^l_{\mu_1}})}}(i) - (\overline{Q^{m+1}_{\mu_1}})(i)(\overline{Q^{l+1}_{\mu_1}})(i), \tag{2.9}$$

$$\rho_{m,l}(i) := Cov_n(\overline{Q^m_{g_{n+1+m}}}(\xi_{n+1}), \overline{Q^l_{g_{n+1+l}}}(\xi_{n+1})) = \overline{Q_{(\overline{Q^m_{g_{n+1+m}}}\cdot\overline{Q^l_{g_{n+1+l}}})}}(i) - (\overline{Q^{m+1}_{g_{n+1+m}}})(i)(\overline{Q^{l+1}_{g_{n+1+l}}})(i). \tag{2.10}$$

The second equality in (2.7)-(2.10) can be obtained by Lemma 2.1 below.

Lemma 2.1. *Given $\xi_n = i$, for any L-dimension column vector a and $n = 0, 1, \cdots, T-1$, $k = 0, 1, \cdots$,*

$$E_n[\overline{Q^k_a}(\xi_{n+1})] = \overline{Q^{k+1}_a}(i). \tag{2.11}$$

3. Equilibrium strategy and efficient frontier

In order to obtain the equilibrium strategy, first we give a recursive formula for equilibrium value function. Denote $q_{n+1} := Pr(T \wedge \tau > n+1 \mid T \wedge \tau > n) = 1 - p_{n,n+1},\ n = 0, 1, \cdots, T-2$. According to Definition 2.1 and (2.5), the recursive formula of the equilibrium value function $V_n(x_n, i)$ is provided in the following proposition.

Proposition 3.1.

$$\begin{aligned} V_n(x_n,i) =& \max_{\pi_n}\{q_{n+1}E_n[V_{n+1}(X_{n+1}^{\pi_n},\xi_{n+1})] + p_{n,n+1}[E_n(X_{n+1}^{\pi_n}) - \omega Var_n(X_{n+1}^{\pi_n})] \\ & - \omega \sum_{m=n+2}^{T} p_{n,m} Var_n[h_{n+1,m}(X_{n+1}^{\pi_n},\xi_{n+1})],\ n = 0,1,\cdots,T-2, \end{aligned} \tag{3.1}$$

$$V_{T-1}(x_{T-1},i) = \max_{\pi_{T-1}} \left[E_{T-1}(X_T^{\pi_{T-1}}) - \omega Var_{T-1}(X_T^{\pi_{T-1}})\right], \tag{3.2}$$

where

$$\begin{aligned} h_{n,m}(x_n,i) :=& E_n(X_m^{\hat{\pi}}) = E_n[h_{n+1,m}(X_{n+1}^{\hat{\pi}_n},\xi_{n+1})], \\ & n = 0,1,\cdots,T-1,\ m = n+1, n+2, \cdots, T, \end{aligned} \tag{3.3}$$

$$h_{n,n}(x_n,i) = x_n,\ n = 0,1,\cdots,T. \tag{3.4}$$

Given $\xi_n = i \in \mathbf{S}$, for later use, we make some notations as follows:

$$\lambda_{n,k} = \sum_{m=k+1}^{T} p_{n,m} r_f^{m-1-k}, \tag{3.5}$$

$$\theta_{n,k} = \sum_{m=k+1}^{T} p_{n,m} (r_f^2)^{m-1-k}, \tag{3.6}$$

$$\varpi_n = \frac{\lambda_{n,n}}{\theta_{n,n}} = \frac{\sum_{m=n+1}^{T} p_{n,m} r_f^{m-1-n}}{\sum_{m=n+1}^{T} p_{n,m} (r_f^2)^{m-1-n}}, \tag{3.7}$$

$$\begin{aligned} A_n(i) =& \sum_{k=n}^{T-1} \theta_{n,n+T-1-k} \overline{Q_{\mu_2}^{T-1-k}}(i) + \sum_{j=0}^{T-2-n} \sum_{k=n+1+j}^{T-1} \theta_{n,n+j+T-k} \overline{Q_{\gamma_j}^{T-1-k}}(i) \\ & + 2 \sum_{j=1}^{T-2-n} \sum_{l=0}^{j-1} \sum_{k=n+1+j}^{T-1} r_f^{j-l} \theta_{n,n+j+T-k} \overline{Q_{\delta_{l,j}}^{T-1-k}}(i), \end{aligned} \tag{3.8}$$

$$\alpha_n(i) = \sum_{k=n}^{T-1} \varpi_{n+T-1-k} \lambda_{n,n+T-1-k} \overline{Q_{g_{n+T-1-k}}^{T-1-k}}(i), \tag{3.9}$$

$$\begin{aligned} B_n(i) =& \sum_{j=0}^{T-2-n} \sum_{k=n+1+j}^{T-1} \varpi_{n+j+T-k}^2 \theta_{n,n+j+T-k} \overline{Q_{\eta_j}^{T-1-k}}(i) \\ & + 2 \sum_{j=1}^{T-2-n} \sum_{l=0}^{j-1} \sum_{k=n+1+j}^{T-1} r_f^{j-l} \varpi_{n+j+T-k} \varpi_{n+l+T-k} \theta_{n,n+j+T-k} \overline{Q_{\rho_{l,j}}^{T-1-k}}(i), \end{aligned} \tag{3.10}$$

for $n=0,1,\cdots,T-1,\ k=n,n+1,\cdots,T-1$ and $A_T(i)=\alpha_T(i)=B_T(i)=0$.

Lemma 3.1.

$$Var_n\Big[\sum_{k=n+1}^{m-1} r_f^{k-1-n}\varpi_{n+m-k}\overline{Q_{g_{n+m-k}}^{m-1-k}}(\xi_{n+1})\Big]=\sum_{k=0}^{m-n-2} r_f^{2(m-n-2-k)}\varpi_{n+1+k}^2\eta_k(i)$$
$$+2\sum_{j=1}^{m-n-2}\sum_{l=0}^{j-1} r_f^{2(m-n-2)-j-l}\varpi_{n+1+j}\varpi_{n+1+l}\rho_{l,j}(i), \tag{3.11}$$
$$Var_n\Big[\sum_{k=n+1}^{m-1} r_f^{k-1-n}\overline{Q_{\mu_1}^{m-1-k}}(\xi_{n+1})\Big]=\sum_{k=0}^{m-n-2} r_f^{2(m-n-2-k)}\gamma_k(i)+2\sum_{j=1}^{m-n-2}\sum_{l=0}^{j-1} r_f^{2(m-n-2)-j-l}\delta_{l,j}(i). \tag{3.12}$$

Lemma 3.2.

$$\lambda_{n,n}\mu_1(i)+\sum_{k=n+1}^{T-1}\lambda_{n,n+T-k}\overline{Q_{\mu_1}^{T-k}}(i)=\sum_{k=n}^{T-1}\lambda_{n,n+T-1-k}\overline{Q_{\mu_1}^{T-1-k}}(i),$$
$$\theta_{n,n}\mu_2(i)+\sum_{k=n+1}^{T-1}\theta_{n,n+T-k}\overline{Q_{\mu_2}^{T-k}}(i)=\sum_{k=n}^{T-1}\theta_{n,n+T-1-k}\overline{Q_{\mu_2}^{T-1-k}}(i).$$

Lemma 3.3.

$$\sum_{j=0}^{T-3-n}\sum_{k=n+2+j}^{T-1}\theta_{n,n+1+j+T-k}\overline{Q_{\gamma_j}^{T-k}}(i)+\sum_{m=n+2}^{T}\sum_{k=0}^{m-n-2}p_{n,m}(r_f^2)^{m-n-2-k}\gamma_k(i)$$
$$=\sum_{j=0}^{T-2-n}\sum_{k=n+1+j}^{T-1}\theta_{n,n+j+T-k}\overline{Q_{\gamma_j}^{T-1-k}}(i),$$
$$\sum_{j=0}^{T-3-n}\sum_{k=n+2+j}^{T-1}\varpi_{n+1+j+T-k}^2\theta_{n,n+1+j+T-k}\overline{Q_{\eta_j}^{T-k}}(i)+\sum_{m=n+2}^{T}\sum_{k=0}^{m-n-2}p_{n,m}(r_f^2)^{m-n-2-k}\varpi_{n+1+k}^2\eta_k(i)$$
$$=\sum_{j=0}^{T-2-n}\sum_{k=n+1+j}^{T-1}\varpi_{n+j+T-k}^2\theta_{n,n+j+T-k}\overline{Q_{\eta_j}^{T-1-k}}(i).$$

Lemma 3.4.

$$\sum_{j=1}^{T-3-n}\sum_{l=0}^{j-1}\sum_{k=n+2+j}^{T-1}r_f^{j-l}\theta_{n,n+1+j+T-k}\overline{Q_{\delta_{l,j}}^{T-k}}(i)+\sum_{m=n+2}^{T}\sum_{j=1}^{m-n-2}\sum_{l=0}^{j-1}p_{n,m}r_f^{2(m-n-2)-j-l}\delta_{l,j}(i)$$
$$=\sum_{j=1}^{T-2-n}\sum_{l=0}^{j-1}\sum_{k=n+1+j}^{T-1}r_f^{j-l}\theta_{n,n+j+T-k}\overline{Q_{\delta_{l,j}}^{T-1-k}}(i),$$
$$\sum_{j=1}^{T-3-n}\sum_{l=0}^{j-1}\sum_{k=n+2+j}^{T-1}r_f^{j-l}\varpi_{n+1+j+T-k}\varpi_{n+1+l+T-k}\theta_{n,n+1+j+T-k}\overline{Q_{\rho_{l,j}}^{T-k}}(i)$$
$$+\sum_{m=n+2}^{T}\sum_{j=1}^{m-n-2}\sum_{l=0}^{j-1}p_{n,m}r_f^{2(m-n-2)-j-l}\varpi_{n+1+j}\varpi_{n+1+l}\rho_{l,j}(i)$$
$$=\sum_{j=1}^{T-2-n}\sum_{l=0}^{j-1}\sum_{k=n+1+j}^{T-1}r_f^{j-l}\varpi_{n+j+T-k}\varpi_{n+l+T-k}\theta_{n,n+j+T-k}\overline{Q_{\rho_{l,j}}^{T-1-k}}(i).$$

3.1. Equilibrium strategy and equilibrium value function

According to Lemmas 2.1-3.4 and Proposition 3.1, using the previous notations, we can obtain the equilibrium strategy and equilibrium value function as given in the following theorem.

Theorem 3.1. *For Problem (2.6), the equilibrium strategy is given by*

$$\hat{\pi}_n(i) = \frac{1}{2\omega}\varpi_n Var^{-1}(R_n^e(i))r_n^e(i), \ n = 0, 1, \cdots, T-1, \tag{3.13}$$

the equilibrium value function is given by

$$V_n(x_n, i) = r_f\lambda_{n,n}x_n + \sum_{k=n}^{T-1}\lambda_{n,n+T-1-k}\overline{Q_{\mu_1}^{T-1-k}}(i) - \omega A_n(i) + \frac{1}{4\omega}\alpha_n(i) - \frac{1}{4\omega}B_n(i),$$
$$n = 0, 1, \cdots, T-1, \tag{3.14}$$

and we have

$$h_{n,m}(x_n, i) = r_f^{m-n}x_n + \sum_{k=n}^{m-1} r_f^{k-n}\overline{Q_{\mu_1}^{m-1-k}}(i) + \frac{1}{2\omega}\sum_{k=n}^{m-1} r_f^{k-n}\varpi_{n+m-1-k}\overline{Q_{g_{n+m-1-k}}^{m-1-k}}(i),$$
$$n = 0, 1, \cdots, T-1, \ m = n+1, n+2, \cdots, T. \tag{3.15}$$

From (3.13), we can obtain some properties of the equilibrium strategy:

(a) At any time n, the amount invested in risky assets decreases in the risk aversion coefficient ω and increases in the term $Var^{-1}(R_n^e(i))r_n^e(i)$ of the risky assets. In other words, the investor will invest more wealth in risky assets as ω becomes smaller or $Var^{-1}(R_n^e(i))r_n^e(i)$ becomes larger.

(b) The equilibrium strategy $\hat{\pi}_n$ is independent of the current wealth x_n and stochastic cash flow $c_n(i)$, but is dependent on the return of the risk-free asset r_f and the exit conditional probabilities at future time $n+1, n+2, \cdots, T$.

For the equilibrium value function, we have:

(a) The equilibrium value function $V_n(x_n, i)$ is a linear function of the current wealth x_n.

(b) Further, $V_n(x_n, i)$ can be divided into four parts

$$V_n(x_n, i) = W_n^f(i) - \omega A_n(i) + \frac{1}{4\omega}\alpha_n(i) - \frac{1}{4\omega}B_n(i), \tag{3.16}$$

where

$$
\begin{aligned}
W_n^f(i) =& r_f \lambda_{n,n} x_n + \sum_{k=n}^{T-1} \lambda_{n,n+T-1-k} \overline{Q_{\mu_1}^{T-1-k}}(i) \\
=& \sum_{m=n+1}^{T} p_{n,m} x_n r_f^{m-n} + \sum_{k=n}^{T-1} \sum_{m=n+T-k}^{T} p_{n,m} \overline{Q_{\mu_1}^{T-1-k}}(i) r_f^{m-n-T+k}
\end{aligned}
\tag{3.17}
$$

can be regarded as a weighted sum of the returns by investing the current wealth x_n and expected cash flows in the risk-free asset from time n to time $T-1$. By (3.8), $-\omega A_n(i)$ can be seen as the loss of the value function resulting from the stochastic cash flows. By (3.10),$-\frac{1}{4\omega}B_n(i)$ can be viewed as the loss of the value function resulting from the risky investment. By (3.9) and (G.11) in Appendix G, we have

$$
\begin{aligned}
\frac{1}{4\omega}\alpha_n(i) =& \frac{1}{4\omega} q_{n+1} E_n[\alpha_{n+1}(\xi_{n+1})] + E_n\left[\sum_{m=n+2}^{T} p_{n,m} r_f^{m-n-1} R_n^e(i)'\hat{\pi}_n\right] \\
&+ p_{n,n+1}[E_n(R_n^e(i)'\hat{\pi}_n) - \omega Var_n(R_n^e(i)'\hat{\pi}_n)] - \omega \sum_{m=n+2}^{T} p_{n,m} Var_n[r_f^{m-n-1} R_n^e(i)'\hat{\pi}_n] \\
=& \frac{1}{4\omega} q_{n+1} E_n[\alpha_{n+1}(\xi_{n+1})] + \sum_{m=n+1}^{T} p_{n,m}[E_n\big(r_f^{m-n-1} R_n^e(i)'\hat{\pi}_n\big) - \omega Var_n\big(r_f^{m-n-1} R_n^e(i)'\hat{\pi}_n\big)],
\end{aligned}
$$

and

$$
\frac{1}{4\omega}\alpha_{T-1}(i) = \frac{1}{4\omega} g_{T-1}(i) = E\big(R_{T-1}^e(i)'\hat{\pi}_{T-1}\big) - \omega Var\big(R_{T-1}^e(i)'\hat{\pi}_{T-1}\big).
$$

Since $r_f^{m-n-1} R_n^e(i)'\hat{\pi}_n$ can be taken as the total wealth obtained by investing the random excess returns of the risky assets in the risk-free asset until the time the investor exits the market, $\frac{1}{4\omega}\alpha_n(i)$ can be regarded as the accumulated trade off between the expectation and the variance of the excess return of the risky investment from time n to time $T-1$.

3.2. Equilibrium efficient frontier

In this subsection, we will derive the explicit expressions for the expectation and variance of the terminal wealth with uncertain exit time, and then give the equilibrium efficient frontier.

By (2.3) and (2.4), under the equilibrium strategy, the weighted average expectation and weighted average variance of the terminal wealth at time n are, respectively,

$$
\begin{aligned}
E_n^U(X_{T\wedge\tau}^{\hat{\pi}}) &= \sum_{m=n+1}^{T} p_{n,m} E_n(X_m^{\hat{\pi}}), \\
Var_n^U(X_{T\wedge\tau}^{\hat{\pi}}) &= \sum_{m=n+1}^{T} p_{n,m} Var_n(X_m^{\hat{\pi}}).
\end{aligned}
$$

Given $X_n^{\hat{\pi}} = x_n,\ \xi_n = i$, by (2.2), we have

$$X_m^{\hat{\pi}} = r_f^{m-n} x_n + \sum_{k=n}^{m-1} r_f^{m-1-k} R_k^e(\xi_k)'\hat{\pi}_k + \sum_{k=n}^{m-1} r_f^{m-1-k} c_k(\xi_k),\ m = n+1, n+2, \cdots, T.$$

By (3.13), we further have

$$\begin{aligned} E_n(X_m^{\hat{\pi}}) &= E_n\left[r_f^{m-n} x_n + \frac{1}{2\omega}\sum_{k=n}^{m-1} r_f^{m-1-k}\varpi_k R_k^e(\xi_k)' Var^{-1}(R_k^e(\xi_k)) r_k^e(\xi_k) + \sum_{k=n}^{m-1} r_f^{m-1-k} c_k(\xi_k)\right] \\ &= r_f^{m-n} x_n + \frac{1}{2\omega}\sum_{k=n}^{m-1} r_f^{m-1-k}\varpi_k \overline{Q_{g_k}^{k-n}}(i) + \sum_{k=n}^{m-1} r_f^{m-1-k}\overline{Q_{\mu_1}^{k-n}}(i). \end{aligned}$$

Combining the expression (2.3) of $E_n^U(X_{T\wedge\tau}^{\hat{\pi}})$, we obtain

$$\begin{aligned} E_n^U(X_{T\wedge\tau}^{\hat{\pi}}) &= \sum_{m=n+1}^{T} p_{n,m}\left[r_f^{m-n} x_n + \frac{1}{2\omega}\sum_{k=n}^{m-1} r_f^{m-1-k}\varpi_k \overline{Q_{g_k}^{k-n}}(i) + \sum_{k=n}^{m-1} r_f^{m-1-k}\overline{Q_{\mu_1}^{k-n}}(i)\right] \\ &= \sum_{m=n+1}^{T} p_{n,m} r_f^{m-n} x_n + \sum_{m=n+1}^{T}\sum_{k=n}^{m-1} p_{n,m} r_f^{m-1-k}\overline{Q_{\mu_1}^{k-n}}(i) \\ &\quad + \frac{1}{2\omega}\sum_{m=n+1}^{T}\sum_{k=n}^{m-1} p_{n,m} r_f^{m-1-k}\varpi_k \overline{Q_{g_k}^{k-n}}(i) \\ &= r_f\lambda_{n,n} x_n + \sum_{k=n}^{T-1}\sum_{m=n+T-k}^{T} p_{n,m} r_f^{m-n-T+k}\overline{Q_{\mu_1}^{T-1-k}}(i) \\ &\quad + \frac{1}{2\omega}\sum_{k=n}^{T-1}\varpi_{n+T-1-k}\sum_{m=n+T-k}^{T} p_{n,m} r_f^{m-n-T+k}\overline{Q_{g_{n+T-1-k}}^{T-1-k}}(i) \\ &= r_f\lambda_{n,n} x_n + \sum_{k=n}^{T-1}\lambda_{n,n+T-1-k}\overline{Q_{\mu_1}^{T-1-k}}(i) + \frac{1}{2\omega}\sum_{k=n}^{T-1}\varpi_{n+T-1-k}\lambda_{n,n+T-1-k}\overline{Q_{g_{n+T-1-k}}^{T-1-k}}(i). \end{aligned}$$

In view of (3.17) and (3.9), it follows that

$$E_n^U(X_{T\wedge\tau}^{\hat{\pi}}) = W_n^f(i) + \frac{1}{2\omega}\alpha_n(i). \tag{3.18}$$

By (3.18), (3.16) and noting that $V_n(x_n, i) = E_n^U(X_{T\wedge\tau}^{\hat{\pi}}) - \omega Var_n^U(X_{T\wedge\tau}^{\hat{\pi}})$, we obtain

$$Var_n^U(X_{T\wedge\tau}^{\hat{\pi}}) = \frac{E_n^U(X_{T\wedge\tau}^{\hat{\pi}}) - V_n(x_n, i)}{\omega} = \frac{1}{4\omega^2}\alpha_n(i) + A_n(i) + \frac{1}{4\omega^2}B_n(i). \tag{3.19}$$

The following proposition gives the equilibrium efficient frontier, i.e., the relationship between the expectation and variance of the terminal wealth for Problem (2.6).

Proposition 3.2. *Given $X_n^{\hat{\pi}} = x_n,\ \xi_n = i \in \mathbf{S}$, the equilibrium efficient frontier at time n for Problem (2.6) is*

$$Var_n^U(X_{T\wedge\tau}^{\hat{\pi}}) = \frac{\alpha_n(i) + B_n(i)}{(\alpha_n(i))^2}[E_n^U(X_{T\wedge\tau}^{\hat{\pi}}) - W_n^f(i)]^2 + A_n(i), \tag{3.20}$$

where $W_n^f(i), A_n(i), \alpha_n(i)$ and $B_n(i)$ are given by (3.17),(3.8), (3.9) and (3.10), respectively.

It is known from Proposition 3.2 that the risks in our uncertain time-horizon model cannot be completely eliminated even if the investor invests all of her wealth in the risk-free asset. The reason for this phenomenon is that the risks generated by stochastic cash flow and uncertain exit time cannot be fully hedged by investing in the financial market.

4. Some degenerated cases

Case 1. There is no cash flow, that is, $c_n(i) = 0$ for all $n = 0, 1, \cdots, T-1,\ i \in \mathbf{S}$. Then the equilibrium strategy $\hat{\pi}_n(i)$ does not change, and the equilibrium value function (3.14) is reduced to

$$V_n(x_n, i) = r_f \lambda_{n,n} x_n + \frac{1}{4\omega}\alpha_n(i) - \frac{1}{4\omega}B_n(i). \tag{4.1}$$

By (3.18) and (3.19), the expectation and variance of the terminal wealth are respectively reduced to

$$E_n^U(X_{T\wedge\tau}^{\hat{\pi}}) = r_f \lambda_{n,n} x_n + \frac{1}{2\omega}\alpha_n(i),$$
$$Var_n^U(X_{T\wedge\tau}^{\hat{\pi}}) = \frac{1}{4\omega^2}\alpha_n(i) + \frac{1}{4\omega^2}B_n(i),$$

and the equilibrium efficient frontier at time n is simplified to

$$Var_n^U(X_{T\wedge\tau}^{\hat{\pi}}) = \frac{\alpha_n(i) + B_n(i)}{(\alpha_n(i))^2}[E_n^U(X_{T\wedge\tau}^{\hat{\pi}}) - r_f \lambda_{n,n} x_n]^2. \tag{4.2}$$

Case 2. The exit time is deterministic, i.e., $q_{n,T} = 1,\ n = 0, 1, \cdots, T-1$. Then the equilibrium strategy (3.13) is reduced to

$$\hat{\pi}_n^c(i) = \frac{Var^{-1}(R_n^e(i))r_n^e(i)}{2\omega r_f^{T-1-n}}, \tag{4.3}$$

and the equilibrium value function (3.14) is reduced to

$$\begin{aligned} V_n(x_n, i) =& r_f^{T-n} x_n + \sum_{k=n}^{T-1} r_f^{k-n} \overline{Q_{\mu_1}^{T-1-k}}(i) \\ & - \omega \Bigg[\sum_{k=n}^{T-1} (r_f^2)^{k-n} \overline{Q_{\mu_2}^{T-1-k}}(i) + \sum_{j=0}^{T-2-n} \sum_{k=n+1+j}^{T-1} (r_f^2)^{k-n-1-j} \overline{Q_{\gamma_j}^{T-1-k}}(i) \\ & + 2 \sum_{j=1}^{T-2-n} \sum_{l=0}^{j-1} \sum_{k=n+1+j}^{T-1} r_f^{2(k-1-n)-j-l} \overline{Q_{\delta_{l,j}}^{T-1-k}}(i) \Bigg] + \frac{1}{4\omega} \sum_{k=n}^{T-1} \overline{Q_{g_{n+T-1-k}}^{T-1-k}}(i) \\ & - \frac{1}{4\omega} \Bigg[\sum_{j=0}^{T-2-n} \sum_{k=n+1+j}^{T-1} \overline{Q_{\eta_j}^{T-1-k}}(i) + 2 \sum_{j=1}^{T-2-n} \sum_{l=0}^{j-1} \sum_{k=n+1+j}^{T-1} \overline{Q_{\rho_{l,j}}^{T-1-k}}(i) \Bigg]. \end{aligned} \tag{4.4}$$

The expectation and variance of the terminal wealth are respectively reduced to

$$
\begin{aligned}
E_n^U(X_{T\wedge\tau}^{\hat{\pi}}) =& E_n(X_T^{\hat{\pi}}) = r_f^{T-n}x_n + \sum_{k=n}^{T-1} r_f^{k-n}\overline{Q_{\mu_1}^{T-1-k}}(i) + \frac{1}{2\omega}\sum_{k=n}^{T-1}\overline{Q_{g_{n+T-1-k}}^{T-1-k}}(i),\\
Var_n^U(X_{T\wedge\tau}^{\hat{\pi}}) =& Var_n(X_T^{\hat{\pi}}) = \frac{1}{4\omega^2}\sum_{k=n}^{T-1}\overline{Q_{g_{n+T-1-k}}^{T-1-k}}(i) + \Bigg[\sum_{k=n}^{T-1} r_f^{2(k-n)}\overline{Q_{\mu_2}^{T-1-k}}(i)\\
&+ \sum_{j=0}^{T-2-n}\sum_{k=n+1+j}^{T-1} r_f^{2(k-1-n-j)}\overline{Q_{\gamma_j}^{T-1-k}}(i)\\
&+ 2\sum_{j=1}^{T-2-n}\sum_{l=0}^{j-1}\sum_{k=n+1+j}^{T-1} r_f^{2(k-n-1)-j-l}\overline{Q_{\delta_{l,j}}^{T-1-k}}(i)\Bigg]\\
&+ \frac{1}{4\omega^2}\Bigg[\sum_{j=0}^{T-2-n}\sum_{k=n+1+j}^{T-1}\overline{Q_{\eta_j}^{T-1-k}}(i) + 2\sum_{j=1}^{T-2-n}\sum_{l=0}^{j-1}\sum_{k=n+1+j}^{T-1}\overline{Q_{\rho_{l,j}}^{T-1-k}}(i)\Bigg],
\end{aligned}
$$

and the equilibrium efficient frontier at time n is simplified to

$$
\begin{aligned}
Var_n^U(X_{T\wedge\tau}^{\hat{\pi}}) =& \frac{\left(E_n^U(X_{T\wedge\tau}^{\hat{\pi}}) - r_f^{T-n}x_n - \sum_{k=n}^{T-1} r_f^{k-n}\overline{Q_{\mu_1}^{T-1-k}}(i)\right)^2}{\sum_{k=n}^{T-1}\overline{Q_{g_{n+T-1-k}}^{T-1-k}}(i)}\Bigg[1\\
&+ \frac{\sum_{j=0}^{T-2-n}\sum_{k=n+1+j}^{T-1}\overline{Q_{\eta_j}^{T-1-k}}(i)}{\sum_{k=n}^{T-1}\overline{Q_{g_{n+T-1-k}}^{T-1-k}}(i)} + 2\frac{\sum_{j=1}^{T-2-n}\sum_{l=0}^{j-1}\sum_{k=n+1+j}^{T-1}\overline{Q_{\rho_{l,j}}^{T-1-k}}(i)}{\sum_{k=n}^{T-1}\overline{Q_{g_{n+T-1-k}}^{T-1-k}}(i)}\Bigg]\\
&+ \Bigg[\sum_{k=n}^{T-1} r_f^{2(k-n)}\overline{Q_{\mu_2}^{T-1-k}}(i) + \sum_{j=0}^{T-2-n}\sum_{k=n+1+j}^{T-1} r_f^{2(k-1-n-j)}\overline{Q_{\gamma_j}^{T-1-k}}(i)\\
&+ 2\sum_{j=1}^{T-2-n}\sum_{l=0}^{j-1}\sum_{k=n+1+j}^{T-1} r_f^{2(k-1-n)-j-l}\overline{Q_{\delta_{l,j}}^{T-1-k}}(i)\Bigg].
\end{aligned}
\tag{4.5}
$$

Case 3. When the exit time is deterministic and there is no cash flow, then the equilibrium strategy $\hat{\pi}_n(i) = \hat{\pi}_n^c(i)$, and the equilibrium value function (3.14) is reduced to

$$
\begin{aligned}
V_n(x_n, i) =& r_f^{T-n}x_n + \frac{1}{4\omega}\sum_{k=n}^{T-1}\overline{Q_{g_{n+T-1-k}}^{T-1-k}}(i)\\
&- \frac{1}{4\omega}\Bigg[\sum_{j=0}^{T-2-n}\sum_{k=n+1+j}^{T-1}\overline{Q_{\eta_j}^{T-1-k}}(i) + 2\sum_{j=1}^{T-2-n}\sum_{l=0}^{j-1}\sum_{k=n+1+j}^{T-1}\overline{Q_{\rho_{l,j}}^{T-1-k}}(i)\Bigg],
\end{aligned}
\tag{4.6}
$$

which is consistent with that in Wu and Chen (2015) (let $\omega(i) = \omega$). The expectation and variance of the terminal wealth are respectively reduced to

$$
E_n^U(X_{T\wedge\tau}^{\hat{\pi}}) = r_f^{T-n}x_n + \frac{1}{2\omega}\sum_{k=n}^{T-1}\overline{Q_{g_{n+T-1-k}}^{T-1-k}}(i),
$$

$$
\begin{aligned}
Var_n^U(X_{T\wedge\tau}^{\hat{\pi}}) =& \frac{1}{4\omega^2}\sum_{k=n}^{T-1}\overline{Q_{g_{n+T-1-k}}^{T-1-k}}(i) + \frac{1}{4\omega^2}\Bigg[\sum_{j=0}^{T-2-n}\sum_{k=n+1+j}^{T-1}\overline{Q_{\eta_j}^{T-1-k}}(i) \\
&+ 2\sum_{j=1}^{T-2-n}\sum_{l=0}^{j-1}\sum_{k=n+1+j}^{T-1}\overline{Q_{\rho_{l,j}}^{T-1-k}}(i)\Bigg],
\end{aligned}
$$

and the equilibrium efficient frontier at time n is simplified to

$$
\begin{aligned}
Var_n^U(X_{T\wedge\tau}^{\hat{\pi}}) =& \frac{(E_n^U(X_{T\wedge\tau}^{\hat{\pi}}) - r_f^{T-n}x_n)^2}{\sum_{k=n}^{T-1}\overline{Q_{g_{n+T-1-k}}^{T-1-k}}(i)}\Bigg[1 + \frac{\sum_{j=0}^{T-2-n}\sum_{k=n+1+j}^{T-1}\overline{Q_{\eta_j}^{T-1-k}}(i)}{\sum_{k=n}^{T-1}\overline{Q_{g_{n+T-1-k}}^{T-1-k}}(i)} \\
&+ 2\frac{\sum_{j=1}^{T-2-n}\sum_{l=0}^{j-1}\sum_{k=n+1+j}^{T-1}\overline{Q_{\rho_{l,j}}^{T-1-k}}(i)}{\sum_{k=n}^{T-1}\overline{Q_{g_{n+T-1-k}}^{T-1-k}}(i)}\Bigg].
\end{aligned}
\tag{4.7}
$$

Case 4. When the exit time is deterministic, and there are no cash flow and no regime-switching in the market, then the equilibrium strategy (3.13) is reduced to

$$
\hat{\pi}_n = \frac{1}{2\omega r_f^{T-1-n}} Var^{-1}(R_n^e) r_n^e, \tag{4.8}
$$

and the equilibrium value function (3.14) is reduced to

$$
V_n(x_n) = r_f^{T-n}x_n + \frac{1}{4\omega}\sum_{k=n}^{T-1}(r_k^e)' Var^{-1}(R_k^e) r_k^e, \tag{4.9}
$$

which is consistent with that in Wu (2013) (let $r_k^f = r_f$ for any k). The expectation and variance of the terminal wealth are respectively reduced to

$$
\begin{aligned}
E_n^U(X_{T\wedge\tau}^{\hat{\pi}}) &= r_f^{T-n}x_n + \frac{1}{2\omega}\sum_{k=n}^{T-1}(r_k^e)' Var^{-1}(R_k^e) r_k^e, \\
Var_n^U(X_{T\wedge\tau}^{\hat{\pi}}) &= \frac{1}{4\omega^2}\sum_{k=n}^{T-1}(r_k^e)' Var^{-1}(R_k^e) r_k^e,
\end{aligned}
$$

and the equilibrium efficient frontier at time n is simplified to

$$
Var_n^U(X_{T\wedge\tau}^{\hat{\pi}}) = \frac{(E_n^U(X_{T\wedge\tau}^{\hat{\pi}}) - r_f^{T-n}x_n)^2}{\sum_{k=n}^{T-1}(r_k^e)' Var^{-1}(R_k^e) r_k^e}. \tag{4.10}
$$

Furthermore, we propose a property of the equilibrium strategy. Due to the form of ϖ_n in the equilibrium strategy (3.13), the following proposition further reveals the effect of the exit conditional probability on the equilibrium strategy, and gives the relationship between $\hat{\pi}_n(i)$ and $\hat{\pi}_n^c(i)$.

Proposition 4.1. *Given $\xi_n = i \in \mathbf{S}$, $\hat{\pi}_n(i)$ is increasing in $p_{n,m}$ for $m = n+1, n+2, \cdots, T-1$. In particular, $\hat{\pi}_n^c(i) \le \hat{\pi}_n(i) \le \frac{Var^{-1}(R_n^e(i)) r_n^e(i)}{2\omega}$, where $\hat{\pi}_n^c(i)$ is given by (4.3).*

The proof of Proposition 4.1 is omitted, becasue it is similar to that of Proposition 3.2 in Wu and Zeng (2015), which the reader can refer to.

Proposition 4.1 shows that under the equilibrium strategy, the greater the exit conditional probabilities at time $m = n+1, n+2, \cdots, T-1$, the greater the amount invested in the risky assets at the current time n. Specifically, given $\xi_n = i$, if the investor will certainly exit the market at time $n+1 \leq T$, then $\varpi_n = 1$, and $\hat{\pi}_n(i) = \frac{Var^{-1}(R_n^e(i))r_n^e(i)}{2\omega}$ which reaches a maximum. If the investor decides to exit the market at time T, from Case 2 we know $\hat{\pi}_n(i) = \hat{\pi}_n^c(i)$, which reaches a minimum. Therefore, Proposition 4.1 also indicates that in general the amount invested in the risky assets is between the amounts in the above two extreme cases.

5. Numerical analysis

In this section, we provide some numerical examples to illustrate our results obtained in the previous sections. First of all, we idealize the market states to only two regimes: $i = 1$ is bearish and $i = 2$ is bullish. Suppose that the investor enters the financial market at time 0 with initial wealth $x_0 = 1$, initial state $\xi_0 = 1$ and risk aversion $\omega = 1$, and makes an investment plan for $T = 5$ periods. Assume that there are 4 assets in the market, including a risk-free asset and three risky assets. We use part of the data in the simulations of Yao et al. (2016), and three risky assets are Cisco Systems, Forest City Enterprises and Tandy Brands Accessories in the American market.

Suppose that the risk-free return $r_f = 1.0017$, and the market parameters of three risky assets with different market regimes as well as the state transition probability matrix are listed as follows:

$$r_n^e(1) = (-0.0475, -0.0734, -0.0756)', \qquad r_n^e(2) = (0.0753, 0.0627, 0.0647)'$$

$$Var(R_n(1)) = \begin{pmatrix} 0.0155 & -0.0005 & 0.0008 \\ -0.0005 & 0.0110 & -0.0004 \\ 0.0008 & -0.0004 & 0.0143 \end{pmatrix}, \qquad Var(R_n(2)) = \begin{pmatrix} 0.0280 & 0.0015 & 0.0027 \\ 0.0015 & 0.0068 & 0 \\ 0.0027 & 0 & 0.0147 \end{pmatrix},$$

$$n = 0, 1, \cdots, 4, \qquad Q = \begin{pmatrix} 0.4615 & 0.5385 \\ 0.5385 & 0.4615 \end{pmatrix}.$$

Assume that the expectations and variances of stochastic cash flows on different market states are $\mu_1(1) = 0.4513, \mu_1(2) = 0.4054$; $\mu_2(1) = 0.5055, \mu_2(2) = 0.3975$. Based on the fact that people always tend to complete the whole investment plan in actual investment and usually do not exit the market too early in normal circumstances, we assume that the increase of the exit conditional

Table 1: Values of the exit conditional probabilities $p_{n,m}$.

n \ m	1	2	3	4	5
0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$
1	-	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{2}{3}$
2	-	-	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{3}{4}$
3	-	-	-	$\frac{1}{5}$	$\frac{4}{5}$
4	-	-	-	-	1

probability $p_{n,m}$ about the exit time m is reasonable. For this reason, assume that $p_{n,m}$ are as shown in Table 1.

5.1. Equilibrium efficient frontier at time 0

This subsection illustrates the results of the equilibrium efficient frontier at time 0 and its several degenerated cases as above.

According to Proposition 3.2, we can obtain the equilibrium efficient frontier at time 0

$$Var_0^U(X_{T\wedge\tau}^{\hat{\pi}}) = \frac{\alpha_0(i)+B_0(i)}{(\alpha_0(i))^2}[E_0^U(X_{T\wedge\tau}^{\hat{\pi}}) - W_0^f(i)]^2 + A_0(i). \tag{5.1}$$

Substituting the aforementioned data into Eqs.(3.5)-(3.7) yields

$$\begin{cases} \overrightarrow{\lambda_0} = (1.0052, 0.9411, 0.8771, 0.7509, 0.5000), \\ \lambda_{1,1} = 1.0041, \lambda_{2,2} = 1.0029, \lambda_{3,3} = 1.0014, \lambda_{4,4} = 1, \\ \overrightarrow{\theta_0} = (1.0105, 0.9448, 0.8793, 0.7517, 0.5000), \\ \theta_{1,1} = 1.0082, \theta_{2,2} = 1.0058, \theta_{3,3} = 1.0027, \theta_{4,4} = 1, \\ \overrightarrow{\varpi} = (0.9948, 0.9959, 0.9971, 0.9987, 1), \end{cases} \tag{5.2}$$

where $\overrightarrow{\lambda_0} = (\lambda_{0,0}, \lambda_{0,1}, \cdots, \lambda_{0,4})$, $\overrightarrow{\theta_0} = (\theta_{0,0}, \theta_{0,1}, \cdots, \theta_{0,4})$, $\overrightarrow{\varpi} = (\varpi_0, \varpi_1, \cdots, \varpi_4)$.

Further calculations yield

$$\begin{cases} \overline{Q_{\mu_1}}(1) = 0.4266, \overline{Q_{\mu_1}^2}(1) = 0.4285, \overline{Q_{\mu_1}^3}(1) = 0.4283, \overline{Q_{\mu_1}^4}(1) = 0.4284, \\ \overline{Q_{\mu_2}}(1) = 0.4473, \overline{Q_{\mu_2}^2}(1) = 0.4518, \overline{Q_{\mu_2}^3}(1) = \overline{Q_{\mu_2}^4}(1) = 0.4515, \\ \gamma_0(1) = \gamma_0(2) = \overline{Q_{\gamma_0}^{4-k}}(1) = 0.0005, k = 1, 2, 3, \\ \overline{Q_{\gamma_j}^{4-k}}(1) = 0, j = 1, 2, 3, k = j+1, j+2, \cdots, 4, \\ \overline{Q_{\delta_{l,j}}^{4-k}}(1) = 0, j = 1, 2, 3, l = 0, 1, \cdots, j-1, k = j+1, j+2, \cdots, 4, \end{cases} \tag{5.3}$$

$$\begin{cases} g_0(1) = 1.0590, g_0(2) = 0.9534, \overline{Q_{g_1}}(1) = 1.0021, \\ \overline{Q^2_{g_2}}(1) = 1.0065, \overline{Q^3_{g_3}}(1) = \overline{Q^4_{g_4}}(1) = 1.0062, \\ \eta_0(1) = 0.0028, \eta_0(2) = 0.0027, \overline{Q_{\eta_0}}(1) = 0.0027, \\ \overline{Q^2_{\eta_0}}(1) = 0.0028, \overline{Q^3_{\eta_0}}(1) = 0.0027, \\ \overline{Q^{4-k}_{\eta_j}}(1) = 0, j = 1, 2, 3, k = j+1, j+2, \cdots, 4, \\ \overline{Q^{4-k}_{\rho_{l,j}}}(1) = 0, j = 1, 2, 3, l = 0, 1, \cdots, j-1, k = j+1, j+2, \cdots, 4. \end{cases} \tag{5.4}$$

Substituting Eqs.(5.3),(5.4)into Eqs.(3.17), (3.8)-(3.10)(Let $n = 0, i = 1$) yields $W_0^f(1) = 2.7737$, $A_0(1) = 1.8974$, $\alpha_0(1) = 4.1361$ and $B_0(1) = 0.0084$. By (5.1)(Let $i = 1$), we obtain the equilibrium efficient frontier at time 0

$$Var_0^U(X^{\hat{\pi}}_{T\wedge\tau}) = 0.2423(E_0^U(X^{\hat{\pi}}_{T\wedge\tau}) - 2.7737)^2 + 1.8974.$$

Subsequently, we discuss the degenerated cases.

Case 1. There is no cash flow, that is, $c_n(i) = 0,\ n = 0, 1, \cdots, T-1,\ i \in \mathbf{S}$. By (4.2), the equilibrium efficient frontier is

$$Var_0^U(X^{\hat{\pi}}_{T\wedge\tau}) = 0.2423(E_0^U(X^{\hat{\pi}}_{T\wedge\tau}) - 1.0069)^2.$$

Case 2. The exit time is deterministic, i.e., $q_{n,T} = 1,\ n = 0, 1, \cdots, T-1$. Then by (4.5), (5.3) and (5.4), the equilibrium efficient frontier is

$$Var_0^U(X^{\hat{\pi}}_{T\wedge\tau}) = 0.1973(E_0^U(X^{\hat{\pi}}_{T\wedge\tau}) - 3.1791)^2 + 2.3257.$$

Case 3. When the exit time is deterministic and there is no cash flow, by (4.7),(5.3) and (5.4), the equilibrium efficient frontier is

$$Var_0^U(X^{\hat{\pi}}_{T\wedge\tau}) = 0.1973(E_0^U(X^{\hat{\pi}}_{T\wedge\tau}) - 1.0085)^2.$$

Case 4. When the exit time is deterministic, and there are no cash flow and no regime-switching in the market, by (4.10), the equilibrium efficient frontier is

$$Var_0^U(X^{\hat{\pi}}_{T\wedge\tau}) = 0.1889(E_0^U(X^{\hat{\pi}}_{T\wedge\tau}) - 1.0085)^2.$$

Figure 1 presents equilibrium efficient frontiers corresponding to the general case in Problem (2.6) and degenerated cases 1-4. From Figure 1, compared with the general case we find that the equilibrium efficient frontier for Case 1 is relatively flat, which implies that when the expected terminal wealth is less than a critical value (about 5), there is less risk in the case of no cash flow. Otherwise, the general case will make investors face less risk. Contrary to Case 1, the equilibrium efficient frontier for Case 2 is relatively steep. The equilibrium efficient frontier for Case 3 dominates the general case and Case 4 dominates Case 3. This shows that when the exit time is deterministic and there is no cash flow, for a given expected terminal wealth, the risk faced by investors is lower than that for the general case, and if there is no regime-switching in the market at the same time, the risk is the lowest.

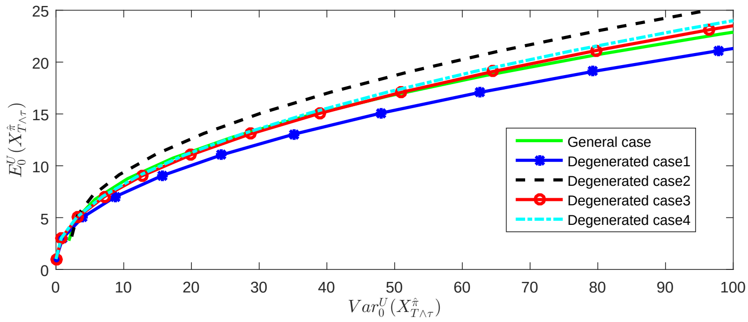


Figure 1: Equilibrium efficient frontiers for General case and Degenerated cases 1-4.

5.2. Impact of uncertain time-horizon

In the following, we further examine the impacts of uncertain time-horizon on the equilibrium strategy and the equilibrium efficient frontier at time 0. To this end, we consider six cases about uncertain time-horizon, which are represented by different conditional probability distributions of exit time at initial time 0:

$$\begin{cases} P_1 = (0,0,0,0,1), P_2 = (\frac{1}{40}, \frac{1}{40}, \frac{1}{40}, \frac{1}{40}, \frac{9}{10}), \\ P_3 = (\frac{1}{20}, \frac{1}{20}, \frac{1}{20}, \frac{1}{20}, \frac{4}{5}), P_4 = (\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{2}), \\ P_5 = (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}), P_6 = (\frac{3}{10}, \frac{3}{10}, \frac{3}{10}, 0, \frac{1}{10}). \end{cases}$$

It is worth pointing out that P_1 correspond to Case 2, that is, the exit time is deterministic. The exit mechanisms of the cases corresponding to P_2 to P_5 are defined as follows. For the investor who enters the market at time $n(\neq T-1)$, assume that the exit conditional probability at terminal time T is always equal to that of the investor who enters the market at initial time 0, and the exit conditional probabilities at other times $n+1, n+2, \cdots, T-1$ are equal. The case correspond to P_6 also meets the above uncertain exit mechanism except that the exit conditional probability at time $m = 4$ is 0.

Table 2: Values of ϖ_n in six cases of uncertain time-horizon.

	P_1	P_2	P_3	P_4	P_5	P_6
ϖ_0	0.9881	0.9889	0.9895	0.9917	0.9940	0.9961
ϖ_1	0.9911	0.9916	0.9922	0.9940	0.9958	0.9978
ϖ_2	0.9940	0.9946	0.9950	0.9963	0.9976	0.9994
ϖ_3	0.9970	0.9973	0.9976	0.9985	0.9994	0.9970
ϖ_4	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

According to the equilibrium strategy (3.13) and the assumption that the market parameters of the risky assets are independent with time, given state i, the values of $\hat{\pi}_n(i)$ depend entirely on the values of $\varpi_n(n = 0, 1, \cdots, 4)$. Table 2 shows the values of $\varpi_n(n = 0, 1, \cdots, 4)$ in six cases. In the cases of P_1 to P_5, $\varpi_n(n = 0, 1, \cdots, 4)$ increases along with $p_{n,m}(n = 0, 1, \cdots, T-2; m = n+1, n+2, \cdots, T-1)$. For P_5 and P_6, $\varpi_n(n = 0, 1, 2)$ increases as $p_{n,m}(n = 0, 1, 2; m = n+1, n+2, \cdots, 3)$ increases, and ϖ_3 decreases as $p_{n,4}(n = 0, 1, 2, 3)$ decreases. Consequently, the equilibrium strategy increases in $p_{n,m}$. Moreover, since $\frac{1}{r_f^{4-n}} \leq \varpi_n \leq 1$, $\hat{\pi}_n^c(i) \leq \hat{\pi}_n(i) \leq \frac{Var^{-1}(R_n^e(i))r_n^e(i)}{2\omega}$. Specifically, in the case corresponding to P_1, $p_{n,m} = 0(n = 0, 1, 2, 3; m = n+1, n+2, \cdots, 4)$ results in less wealth being invested in the risky assets, in other words, the amount of investment in the risky assets with uncertain exit time is not less than that with fixed exit time. When the investor enters the market at time $n = 4$, whatever the uncertain time-horizon is, she is sure to exit the market at next time $T = 5$. Accordingly, $\varpi_4 = 1$ and $\hat{\pi}_4(i)$ reaches the maximum, that is, the amount of money invested in the risky assets achieves the maximum, which means that in the case of only one investment opportunity at time $T-1$, a higher exit conditional probability or a shorter remaining time horizon leads to more wealth being invested in the risky assets in order to obtain the optimal mean-variance utility. At this point, all cases of uncertain exit time are equivalent to Case 2, and so the equilibrium investment strategies at the final time period in our example are all equal, i.e., $\hat{\pi}_4(i) = \hat{\pi}_4^c(i)$, which is consistent with the conclusion of Proposition 4.1.

To examine the impact of uncertain time-horizon on the equilibrium efficient frontier, we draw the equilibrium efficient frontiers for these six different cases in Figure 2. We conclude that the uncertain time-horizon has the clockwise rotated equilibrium efficient frontiers, and the more the uncertainty is, the more the rotation is. A possible explanation for this phenomenon is as follows. When the given expectation $E_0^U(X_{T\wedge\tau}^{\hat{\pi}})$ of the terminal wealth with uncertain exit time is small, short-term investment is easier to achieve the objective and can avoid the risk of financial market volatility due to long-term investment. Therefore, the greater the exit conditional probability $P_{n,m}$, the smaller the variance $Var_0^U(X_{T\wedge\tau}^{\hat{\pi}})$ of the terminal wealth, and the graphs of the corresponding equilibrium efficient frontiers from P_6 to P_1 are from left to right. On the other hand, when $E_0^U(X_{T\wedge\tau}^{\hat{\pi}})$ is high, short-term investment is difficult to achieve the goal and is more risky, and long-term and robust investment will be more beneficial to the investor at this time. Consequently, the graphs of the corresponding equilibrium efficient frontiers from P_6 to P_1 present the opposite trend from right to left.

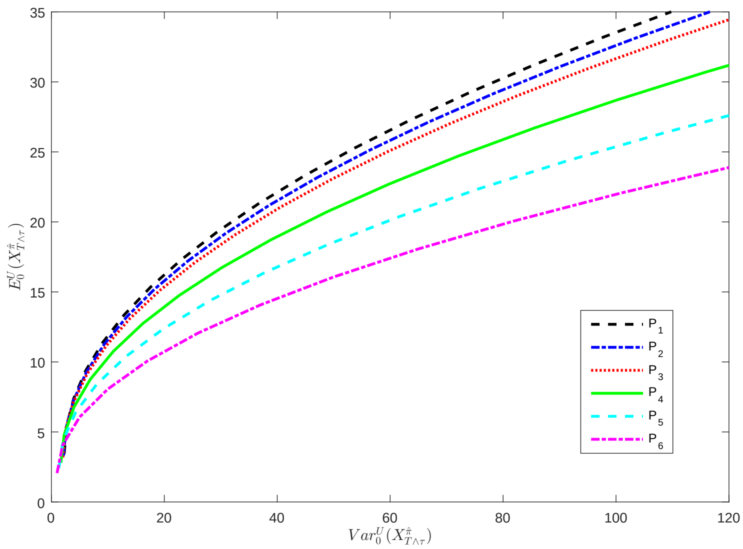


Figure 2: Impact of uncertain time-horizon on the equilibrium efficient frontier.

In the following numerical sensitivity analysis, we all assume that P_4 is the exit probability of the original Problem (2.6) at initial time 0, and its corresponding exit mechanism is as described above.

5.3. Impacts of regime switching on the equilibrium efficient frontier

In this subsection, we examine the impacts of regime switching on the equilibrium efficient frontier. First, to focus on the difference of the equilibrium efficient frontiers with different starting market modes, we depict the equilibrium efficient frontiers with the initial market states of bear market and bull market, respectively, in Figure 3. We see that the equilibrium efficient frontier with the bullish initial state ($\xi_0 = 2$) lies in the upper left of that with the bearish initial state ($\xi_0 = 1$), which means that if the investor starts at a bull market, she can face less risk than beginning at a bear market for a given expected return $E_0^U(X_{T\wedge\tau}^{\hat{\pi}})$. Therefore, it is reasonable to start our investment when the market is bullish.

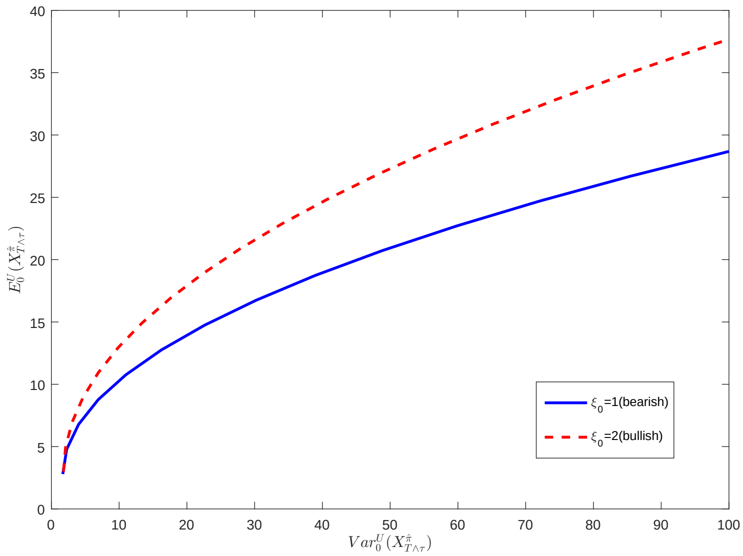


Figure 3: Equilibrium efficient frontiers with different initial market states.

Table 3: Different transition probabilities.

q_{11}	0.1	0.3	0.5	0.7	0.8
q_{22}	0.9	0.7	0.5	0.3	0.2

Next, we study how different regime switching probabilities affect the equilibrium efficient frontier. Table 3 shows five different transition probabilities to indicate that the market becomes "more bearish" when q_{11} increases from 0.1 to 0.8 and q_{22} decreases from 0.9 to 0.2. Under the assumption that the investor always enters the market at bearish time, i.e., $\xi_0 = 1$, Figure 4 presents the equilibrium efficient frontiers corresponding to these transition probabilities. From Figure 4, we know that when an expected return $E_0^U(X^{\hat{\pi}}_{T\wedge\tau})$ is given, with the change of q_{11} from 0.1 to 0.8, the corresponding equilibrium efficient frontier moves to the lower right. This indicates an increasing investment risk during this shift. Therefore, we conclude that it is not suitable to invest when the market presents a downward trend or continues to decline.

5.4. Impact of stochastic cash flow on the equilibrium efficient frontier

To clarify the impact of stochastic cash flow on the equilibrium efficient frontier, we describe the equilibrium efficient frontiers corresponding to three different cash flows in Figure 5(a), where negative cash flow $c_n(1) = -0.6$, $c_n(2) = -0.2$, $n = 0, 1, \cdots, 4$ indicates that the corresponding

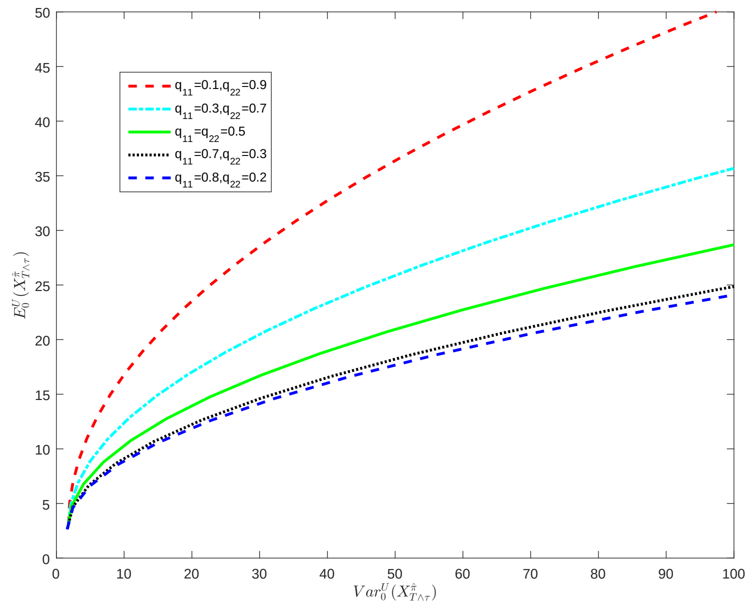
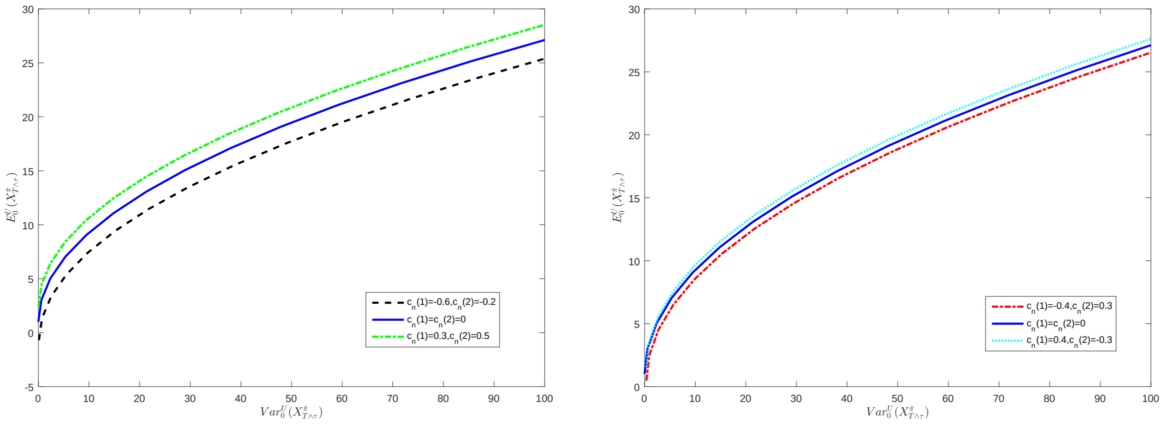


Figure 4: Equilibrium efficient frontiers with different regime switching probabilities.

funds are extracted under different market states to reduce investment, no cash flow $c_n(1) = c_n(2) = 0$, $n = 0, 1, \cdots, 4$ corresponds to Case 1 in Subsection 5.1, and positive cash flow $c_n(1) = 0.3$, $c_n(2) = 0.5$, $n = 0, 1, \cdots, 4$ indicates that the corresponding funds are added under different market states to increase investment.Figure 5(a) shows that the equilibrium efficient frontier with positive cash flow dominates that without cash flow, which in turn dominates that with negative cash flow. This means that the additional investment of capital will reduce the risk faced by the investor, while reducing investment will increase the risk accordingly. As for the more complicated situations, Figure 5(b) presents the equilibrium efficient frontiers corresponding to cash flows with different symbols under different states and without cash flow, where positions of these equilibrium efficient frontiers depend on the values of cash flow.

6. Conclusion

In this paper, we study a generalized multi-period mean-variance portfolio selection problem with uncertain time-horizon and a stochastic cash flow in a Markov regime-switching market. Assume that the financial market contains a limited number of states which are modelled by a discrete-time Markov chain. The investor may be forced to exit the market at some time before the planned investment period for some uncontrollable reasons during the investment process, which results in our optimization problem with an uncertain time horizon. Moreover, the capital injection



(a) (b)

Figure 5: Equilibrium efficient frontiers with different stochastic cash flows.

or withdrawal at each period is allowed, and hence the market is not self-financing. The random returns of risky assets and amount of the cash flow all depend on the states of the stochastic market. By setting the objective function as the weighted sum of a linear combination of the expectation and variance of the wealth at the time of exiting the market, where the weighted coefficients are the corresponding exit conditional probabilities, we construct a more general mean-variance investment model. Within the game theoretic framework, using backward induction approach, we derive the equilibrium strategy, equilibrium value function and equilibrium efficient frontier in closed-form. In addition, some degenerate cases are discussed. Finally, some numerical examples are presented to illustrate equilibrium efficient frontiers, especially to exemplify the impacts of uncertain time-horizon on the equilibrium strategy and equilibrium efficient frontier as well as regime-switching and stochastic cash flow on the equilibrium efficient frontier.

Based on the work in this paper, the following interesting problems deserve further research:

(1) If the market has a bankruptcy state, and when bankruptcy happens, the investor can only retrieve a random fraction of the wealth that she should acquire and must invest it in a risk-free asset until the terminal time, then how to get the equilibrium strategy?

(2) When the conditional distribution of uncertain time-horizon depends on market states, how to obtain the equilibrium strategy of the problem?

(3) With other realistic conditions, such as stochastic interest rate and inflation factor, how to obtain the equilibrium strategy of the portfolio selection with uncertain time-horizon?

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Appendix A

Proof of Lemma 2.1.

$$E_n[\overline{Q_a^k}(\xi_{n+1})] = \sum_{j=1}^{L} q_{ij}\overline{Q_a^k}(j) = (Q \cdot \overline{Q_a^k})(i) = (Q \cdot Q^k a)(i) = (Q^{k+1}a)(i) = \overline{Q_a^{k+1}}(i).$$

Appendix B

Proof of Proposition 3.1. By (2.1), $p_{T-1,T} = 1$, then

$$V_{T-1}(x_{T-1}, i) = \max_{\pi_{T-1}} \left[E_{T-1}(X_T^{\pi_{T-1}}) - \omega Var_{T-1}(X_T^{\pi_{T-1}})\right].$$

For $n = 0, 1, \cdots, T-2$, by (2.5), we have

$$J_n(x_n, i; \pi) = \sum_{m=n+1}^{T} p_{n,m}[E_n(X_m^\pi) - \omega Var_n(X_m^\pi)].$$

Thus,

$$J_{n+1}(X_{n+1}^{\pi_n}, \xi_{n+1}; \pi) = \sum_{m=n+2}^{T} p_{n+1,m}[E_{n+1}(X_m^\pi) - \omega Var_{n+1}(X_m^\pi)].$$

Then we have

$$\begin{aligned}
J_n(x_n, i; \pi) =& q_{n+1}E_n[J_{n+1}(X_{n+1}^{\pi_n}, \xi_{n+1}; \pi)] + J_n(x_n, i; \pi) - q_{n+1}E_n[J_{n+1}(X_{n+1}^{\pi_n}, \xi_{n+1}; \pi)] \\
=& q_{n+1}E_n[J_{n+1}(X_{n+1}^{\pi_n}, \xi_{n+1}; \pi)] + p_{n,n+1}[E_n(X_{n+1}^{\pi_n}) - \omega Var_n(X_{n+1}^{\pi_n})] \\
&+ \sum_{m=n+2}^{T} p_{n,m}[E_n(X_m^\pi) - \omega Var_n(X_m^\pi)] \\
&- q_{n+1}E_n\left\{\sum_{m=n+2}^{T} p_{n+1,m}[E_{n+1}(X_m^\pi) - \omega Var_{n+1}(X_m^\pi)]\right\}
\end{aligned}$$

$$
\begin{aligned}
=&q_{n+1}E_n[J_{n+1}(X_{n+1}^{\pi_n},\xi_{n+1};\pi)]+p_{n,n+1}[E_n(X_{n+1}^{\pi_n})-\omega Var_n(X_{n+1}^{\pi_n})]\\
&+\sum_{m=n+2}^{T}p_{n,m}[E_n(X_m^{\pi})-\omega Var_n(X_m^{\pi})]\\
&-\sum_{m=n+2}^{T}q_{n+1}\cdot p_{n+1,m}E_n[E_{n+1}(X_m^{\pi})-\omega Var_{n+1}(X_m^{\pi})].
\end{aligned}
$$

Since $q_{n+1}\cdot p_{n+1,m}=p_{n,m}$, according to the tower property of conditional expectations, we get

$$
\begin{aligned}
J_n(x_n,i;\pi)=&q_{n+1}E_n[J_{n+1}(X_{n+1}^{\pi_n},\xi_{n+1};\pi)]+p_{n,n+1}[E_n(X_{n+1}^{\pi_n})-\omega Var_n(X_{n+1}^{\pi_n})]\\
&-\omega\sum_{m=n+2}^{T}p_{n,m}\{Var_n(X_m^{\pi})-E_n[Var_{n+1}(X_m^{\pi})]\}.
\end{aligned}
$$

Further, using the variance formula $Var(X)=E(X^2)-[E(X)]^2$ and the tower property of conditional expectations again yields

$$
\begin{aligned}
J_n(x_n,i;\pi)=&q_{n+1}E_n[J_{n+1}(X_{n+1}^{\pi_n},\xi_{n+1};\pi)]+p_{n,n+1}[E_n(X_{n+1}^{\pi_n})-\omega Var_n(X_{n+1}^{\pi_n})]\\
&-\omega\sum_{m=n+2}^{T}p_{n,m}\{E_n[(X_m^{\pi})^2]-E_n^2[E_{n+1}(X_m^{\pi})]\}\\
&+\omega\sum_{m=n+2}^{T}p_{n,m}\{E_n[(X_m^{\pi})^2]-E_n[E_{n+1}^2(X_m^{\pi})]\}\\
=&q_{n+1}E_n[J_{n+1}(X_{n+1}^{\pi_n},\xi_{n+1};\pi)]+p_{n,n+1}[E_n(X_{n+1}^{\pi_n})-\omega Var_n(X_{n+1}^{\pi_n})]\\
&-\omega\sum_{m=n+2}^{T}p_{n,m}\{E_n[E_{n+1}^2(X_m^{\pi})]-E_n^2[E_{n+1}(X_m^{\pi})]\}\\
=&q_{n+1}E_n[J_{n+1}(X_{n+1}^{\pi_n},\xi_{n+1};\pi)]+p_{n,n+1}[E_n(X_{n+1}^{\pi_n})-\omega Var_n(X_{n+1}^{\pi_n})]\\
&-\omega\sum_{m=n+2}^{T}p_{n,m}Var_n[E_{n+1}(X_m^{\pi_n})].
\end{aligned}
$$

Let

$$
\begin{aligned}
h_{n,m}(x_n,i):=&E_n(X_m^{\hat{\pi}})=E_n[h_{n+1,m}(X_{n+1}^{\hat{\pi}_n},\xi_{n+1})],\\
&n=0,1,\cdots,T-1,\ m=n+1,n+2,\cdots,T,\\
h_{n,n}(x_n,i)=&x_n,\ n=0,1,\cdots,T.
\end{aligned}
$$

Then

$$
V_n(x_n,i)=\max_{\pi_n}J_n(x_n,i;(\pi_n,\hat{\pi}_{n+1},\cdots,\hat{\pi}_{T-1}))
$$

$$
\begin{aligned}
&= \max_{\pi_n}\{q_{n+1}E_n[J_{n+1}(X^{\pi_n}_{n+1},\xi_{n+1};\hat{\pi})] + p_{n,n+1}[E_n(X^{\pi_n}_{n+1}) - \omega Var_n(X^{\pi_n}_{n+1})] \\
&\quad - \omega \sum_{m=n+2}^{T} p_{n,m}Var_n[E_{n+1}(X^{\hat{\pi}}_m)]\} \\
&= \max_{\pi_n}\{q_{n+1}E_n[V_{n+1}(X^{\pi_n}_{n+1},\xi_{n+1})] + p_{n,n+1}[E_n(X^{\pi_n}_{n+1}) - \omega Var_n(X^{\pi_n}_{n+1})] \\
&\quad - \omega \sum_{m=n+2}^{T} p_{n,m}Var_n[h_{n+1,m}(X^{\pi_n}_{n+1},\xi_{n+1})]\}, \qquad n = 0,1,\cdots,T-2.
\end{aligned}
$$

The proposition is proved.

Appendix C

Proof of Lemma 3.1. We only prove the first equality while the second equality can be proved in a similar way.

$$
\begin{aligned}
&Var_n\left[\sum_{k=n+1}^{m-1} r_f^{k-n-1}\varpi_{n+m-k}\overline{Q^{m-1-k}_{g_{n+m-k}}}(\xi_{n+1})\right] = Var_n\left[\sum_{k=0}^{m-n-2} r_f^{m-n-2-k}\varpi_{n+1+k}\overline{Q^{k}_{g_{n+1+k}}}(\xi_{n+1})\right] \\
&\quad = \sum_{k=0}^{m-n-2} Var_n[r_f^{m-n-2-k}\varpi_{n+1+k}\overline{Q^{k}_{g_{n+1+k}}}(\xi_{n+1})] \\
&\quad + 2\sum_{j=1}^{m-n-2}\sum_{l=0}^{j-1} Cov_n[r_f^{m-n-2-j}\varpi_{n+1+j}\overline{Q^{j}_{g_{n+1+j}}}(\xi_{n+1}),\ r_f^{m-n-2-l}\varpi_{n+1+l}\overline{Q^{l}_{g_{n+1+l}}}(\xi_{n+1})].
\end{aligned}
$$

By (2.8) and (2.10), we have

$$
\begin{aligned}
&Var_n\left[\sum_{k=n+1}^{m-1} r_f^{k-n-1}\varpi_{n+m-k}\overline{Q^{m-1-k}_{g_{n+m-k}}}(\xi_{n+1})\right] = \sum_{k=0}^{m-n-2} r_f^{2(m-n-2-k)}\varpi^2_{n+1+k}\eta_k(i) \\
&\quad + 2\sum_{j=1}^{m-n-2}\sum_{l=0}^{j-1} r_f^{2(m-n-2)-j-l}\varpi_{n+1+j}\varpi_{n+1+l}\rho_{l,j}(i).
\end{aligned}
$$

Appendix D

Proof of Lemma 3.2. We only prove the first equality while the second equality can be proved in a similar way.

$$
\begin{aligned}
\lambda_{n,n}\mu_1(i) + \sum_{k=n+1}^{T-1}\lambda_{n,n+T-k}\overline{Q^{T-k}_{\mu_1}}(i) &= \lambda_{n,n}\mu_1(i) + \sum_{k=n}^{T-2}\lambda_{n,n+T-1-k}\overline{Q^{T-1-k}_{\mu_1}}(i) \\
&= \sum_{k=n}^{T-1}\lambda_{n,n+T-1-k}\overline{Q^{T-1-k}_{\mu_1}}(i).
\end{aligned}
$$

Appendix E

Proof of Lemma 3.3. We only prove the second equality while the first equality can be proved in a similar way.

$$\sum_{j=0}^{T-3-n}\sum_{k=n+2+j}^{T-1}\varpi_{n+1+j+T-k}^{2}\theta_{n,n+1+j+T-k}\overline{Q_{\eta_j}^{T-k}}(i)+\sum_{m=n+2}^{T}\sum_{k=0}^{m-n-2}p_{n,m}(r_f^2)^{m-n-2-k}\varpi_{n+1+k}^{2}\eta_k(i)$$
$$=\sum_{j=0}^{T-2-n}\sum_{k=n+1+j}^{T-2}\varpi_{n+j+T-k}^{2}\theta_{n,n+j+T-k}\overline{Q_{\eta_j}^{T-1-k}}(i)$$
$$+\sum_{k=0}^{T-2-n}\sum_{m=n+2+k}^{T}p_{n,m}(r_f^2)^{m-n-2-k}\varpi_{n+1+k}^{2}\eta_k(i)$$
$$=\sum_{j=0}^{T-2-n}\sum_{k=n+1+j}^{T-2}\varpi_{n+j+T-k}^{2}\theta_{n,n+j+T-k}\overline{Q_{\eta_j}^{T-1-k}}(i)+\sum_{j=0}^{T-2-n}\theta_{n,n+1+j}\varpi_{n+1+j}^{2}\eta_j(i)$$
$$=\sum_{j=0}^{T-2-n}\sum_{k=n+1+j}^{T-1}\varpi_{n+j+T-k}^{2}\theta_{n,n+j+T-k}\overline{Q_{\eta_j}^{T-1-k}}(i).$$

Appendix F

Proof of Lemma 3.4. We only prove the second equality while the first equality can be proved in a similar way.

$$\sum_{j=1}^{T-3-n}\sum_{l=0}^{j-1}\sum_{k=n+2+j}^{T-1}r_f^{j-l}\varpi_{n+1+j+T-k}\varpi_{n+1+l+T-k}\theta_{n,n+1+j+T-k}\overline{Q_{\rho_{l,j}}^{T-k}}(i)$$
$$+\sum_{m=n+2}^{T}\sum_{j=1}^{m-n-2}\sum_{l=0}^{j-1}p_{n,m}r_f^{2(m-n-2)-j-l}\varpi_{n+1+j}\varpi_{n+1+l}\rho_{l,j}(i)$$
$$=\sum_{j=1}^{T-2-n}\sum_{l=0}^{j-1}\sum_{k=n+1+j}^{T-2}r_f^{j-l}\varpi_{n+j+T-k}\varpi_{n+l+T-k}\theta_{n,n+j+T-k}\overline{Q_{\rho_{l,j}}^{T-1-k}}(i)$$
$$+\sum_{j=1}^{T-2-n}\sum_{l=0}^{j-1}\sum_{m=n+2+j}^{T}p_{n,m}(r_f^2)^{m-n-2-j}r_f^{j-l}\varpi_{n+1+j}\varpi_{n+1+l}\rho_{l,j}(i)$$
$$=\sum_{j=1}^{T-2-n}\sum_{l=0}^{j-1}\sum_{k=n+1+j}^{T-2}r_f^{j-l}\varpi_{n+j+T-k}\varpi_{n+l+T-k}\theta_{n,n+j+T-k}\overline{Q_{\rho_{l,j}}^{T-1-k}}(i)$$
$$+\sum_{j=1}^{T-2-n}\sum_{l=0}^{j-1}r_f^{j-l}\theta_{n,n+1+j}\varpi_{n+1+j}\varpi_{n+1+l}\rho_{l,j}(i)$$
$$=\sum_{j=1}^{T-2-n}\sum_{l=0}^{j-1}\sum_{k=n+1+j}^{T-1}r_f^{j-l}\varpi_{n+j+T-k}\varpi_{n+l+T-k}\theta_{n,n+j+T-k}\overline{Q_{\rho_{l,j}}^{T-1-k}}(i).$$

Appendix G

Proof of Theorem 3.1. For $n = T-1$, by (3.2), we have

$$
\begin{aligned}
V_{T-1}(x_{T-1}, i) =& \max_{\pi_{T-1}} \left[E_{T-1}(X_T^{\pi_{T-1}}) - \omega Var_{T-1}(X_T^{\pi_{T-1}}) \right] \\
=& \max_{\pi_{T-1}} \left[E_{T-1}(r_f x_{T-1} + R^e_{T-1}(i)'\pi_{T-1} + c_{T-1}(i)) \right. \\
& \left. - \omega Var_{T-1}(r_f x_{T-1} + R^e_{T-1}(i)'\pi_{T-1} + c_{T-1}(i)) \right] \\
=& r_f x_{T-1} + \mu_1(i) - \omega\mu_2(i) + \max_{\pi_{T-1}} [r^e_{T-1}(i)'\pi_{T-1} - \omega\pi'_{T-1} Var(R^e_{T-1}(i))\pi_{T-1}].
\end{aligned}
$$

Using the first-order condition, we have

$$
\hat{\pi}_{T-1} = \frac{1}{2\omega} Var^{-1}(R^e_{T-1}(i)) r^e_{T-1}(i). \tag{G.1}
$$

Substituting (G.1) back into $V_{T-1}(x_{T-1}, i)$ and (3.3), respectively, yields

$$
V_{T-1}(x_{T-1}, i) = r_f x_{T-1} + \mu_1(i) - \omega\mu_2(i) + \frac{1}{4\omega} g_{T-1}(i), \tag{G.2}
$$

$$
h_{T-1,T}(x_{T-1}, i) = E_{T-1}(X_T^{\hat{\pi}}) = r_f x_{T-1} + \mu_1(i) + \frac{1}{2\omega} g_{T-1}(i). \tag{G.3}
$$

Since $\varpi_{T-1} = \lambda_{T-1,T-1} = \theta_{T-1,T-1} = 1$, $A_{T-1}(i) = \mu_2(i)$, $\alpha_{T-1}(i) = g_{T-1}(i)$, $B_{T-1}(i) = 0$, (G.1)-(G.3) indicate that (3.13)-(3.15) hold for $n = T-1$. Now we assume that (3.13)-(3.15) hold for $T-1, T-2, \cdots, n+1$. Then for n, by (3.1) and (3.14), we have

$$
\begin{aligned}
J_n(x_n, i; (\pi_n, \hat{\pi}_{n+1}, \cdots, \hat{\pi}_{T-1})) =& q_{n+1} E_n[V_{n+1}(X_{n+1}^{\pi_n}, \xi_{n+1})] + p_{n,n+1}[E_n(X_{n+1}^{\pi_n}) - \omega Var_n(X_{n+1}^{\pi_n})] \\
& - \omega \sum_{m=n+2}^{T} p_{n,m} Var_n[h_{n+1,m}(X_{n+1}^{\pi_n}, \xi_{n+1})] \\
=& q_{n+1} E_n[r_f \lambda_{n+1,n+1} X_{n+1}^{\pi_n} + \sum_{k=n+1}^{T-1} \lambda_{n+1,n+T-k} \overline{Q_{\mu_1}^{T-1-k}}(\xi_{n+1}) - \omega A_{n+1}(\xi_{n+1}) \\
& + \frac{1}{4\omega}\alpha_{n+1}(\xi_{n+1}) - \frac{1}{4\omega} B_{n+1}(\xi_{n+1})] + p_{n,n+1}[E_n(X_{n+1}^{\pi_n}) - \omega Var_n(X_{n+1}^{\pi_n})] \\
& - \omega \sum_{m=n+2}^{T} p_{n,m} Var_n\Big[r_f^{m-n-1} X_{n+1}^{\pi_n} + \sum_{k=n+1}^{m-1} r_f^{k-n-1} \overline{Q_{\mu_1}^{m-1-k}}(\xi_{n+1}) \\
& + \frac{1}{2\omega} \sum_{k=n+1}^{m-1} r_f^{k-n-1} \varpi_{n+m-k} \overline{Q_{g_{n+m-k}}^{m-1-k}}(\xi_{n+1}) \Big].
\end{aligned}
$$

Substituting (3.5) and (2.2) into the above formula, then taking advantage of Lemma 2.1 for further calculations yields

$$
\begin{aligned}
&J_n(x_n, i;(\pi_n, \hat{\pi}_{n+1}, \cdots, \hat{\pi}_{T-1})) \\
&\quad = q_{n+1}\Bigg\{ r_f \sum_{m=n+2}^{T} p_{n+1,m} r_f^{m-n-2}\Big(r_f x_n + r_n^e(i)'\pi_n + \mu_1(i)\Big) \\
&\quad\quad + \sum_{k=n+1}^{T-1} \sum_{m=n+T-k+1}^{T} p_{n+1,m} r_f^{m-n-T+k-1} \overline{Q_{\mu_1}^{T-k}}(i) - \omega E_n[A_{n+1}(\xi_{n+1})] \\
&\quad\quad + \frac{1}{4\omega} E_n[\alpha_{n+1}(\xi_{n+1})] - \frac{1}{4\omega} E_n[B_{n+1}(\xi_{n+1})] \Bigg\} + p_{n,n+1}\Big[\Big(r_f x_n + r_n^e(i)'\pi_n + \mu_1(i)\Big) \\
&\quad\quad - \omega\Big(\pi_n' Var(R_n^e(i))\pi_n + \mu_2(i)\Big)\Big] - \omega \sum_{m=n+2}^{T} p_{n,m} Var_n\Big[r_f^{m-n-1}\Big(r_f x_n + R_n^e(i)'\pi_n + c_n(i)\Big) \\
&\quad\quad + \sum_{k=n+1}^{m-1} r_f^{k-n-1} \overline{Q_{\mu_1}^{m-1-k}}(\xi_{n+1}) + \frac{1}{2\omega} \sum_{k=n+1}^{m-1} r_f^{k-n-1} \varpi_{n+m-k} \overline{Q_{g_{n+m-k}}^{m-1-k}}(\xi_{n+1})\Big].
\end{aligned}
$$

By means of $q_{n+1} \cdot p_{n+1,m} = p_{n,m}$, Assumptions 2.2-2.3 and Lemma 3.1, we have

$$
\begin{aligned}
&J_n(x_n, i;(\pi_n, \hat{\pi}_{n+1}, \cdots, \hat{\pi}_{T-1})) = \Bigg\{ \sum_{m=n+2}^{T} p_{n,m} r_f^{m-n-1}\Big(r_f x_n + r_n^e(i)'\pi_n + \mu_1(i)\Big) \\
&\quad + \sum_{k=n+1}^{T-1} \sum_{m=n+T-k+1}^{T} p_{n,m} r_f^{m-n-T+k-1} \overline{Q_{\mu_1}^{T-k}}(i) - \omega E_n[q_{n+1} A_{n+1}(\xi_{n+1})] \\
&\quad + \frac{1}{4\omega} E_n[q_{n+1}\alpha_{n+1}(\xi_{n+1})] - \frac{1}{4\omega} E_n[q_{n+1} B_{n+1}(\xi_{n+1})] \Bigg\} + p_{n,n+1}\Big[\Big(r_f x_n + r_n^e(i)'\pi_n + \mu_1(i)\Big) \\
&\quad - \omega\Big(\pi_n' Var(R_n^e(i))\pi_n + \mu_2(i)\Big)\Big] - \omega \sum_{m=n+2}^{T} p_{n,m}\Big[r_f^{2(m-n-1)} \pi_n' Var(R_n^e(i))\pi_n \\
&\quad + r_f^{2(m-n-1)} \mu_2(i) + \sum_{k=0}^{m-n-2} r_f^{2(m-n-2-k)} \gamma_k(i) + 2 \sum_{j=1}^{m-n-2} \sum_{l=0}^{j-1} r_f^{2(m-n-2)-j-l} \delta_{l,j}(i) \\
&\quad + \frac{1}{4\omega^2} \sum_{k=0}^{m-n-2} r_f^{2(m-n-2-k)} \varpi_{n+1+k}^2 \eta_k(i) + \frac{1}{2\omega^2} \sum_{j=1}^{m-n-2} \sum_{l=0}^{j-1} r_f^{2(m-n-2)-j-l} \varpi_{n+1+j}\varpi_{n+1+l}\rho_{l,j}(i)\Big].
\end{aligned}
$$

Merging and rearranging the terms of the above formula, we have

$$
\begin{aligned}
J_n(x_n, i;(\pi_n, \hat{\pi}_{n+1}, \cdots, \hat{\pi}_{T-1})) &= \sum_{m=n+1}^{T} p_{n,m} r_f^{m-n-1} r_f x_n \\
&\quad + \left(\sum_{m=n+1}^{T} p_{n,m} r_f^{m-n-1} \mu_1(i) + \sum_{k=n+1}^{T-1} \sum_{m=n+T-k+1}^{T} p_{n,m} r_f^{m-n-T+k-1} \right) \overline{Q_{\mu_1}^{T-k}}(i)
\end{aligned}
$$

$$
\begin{aligned}
&-\omega\Big\{E_n[q_{n+1}A_{n+1}(\xi_{n+1})]+\sum_{m=n+1}^{T}p_{n,m}(r_f^2)^{m-n-1}\mu_2(i)\\
&+\sum_{m=n+2}^{T}\sum_{k=0}^{m-n-2}p_{n,m}(r_f^2)^{m-n-2-k}\gamma_k(i)\\
&+2\sum_{m=n+2}^{T}\sum_{j=1}^{m-n-2}\sum_{l=0}^{j-1}p_{n,m}r_f^{2(m-n-2)-j-l}\delta_{l,j}(i)\Big\}\\
&-\frac{1}{4\omega}\Big\{E_n[q_{n+1}B_{n+1}(\xi_{n+1})]+\sum_{m=n+2}^{T}\sum_{k=0}^{m-n-2}p_{n,m}(r_f^2)^{m-n-2-k}\varpi_{n+1+k}^2\eta_k(i)\\
&+2\sum_{m=n+2}^{T}\sum_{j=1}^{m-n-2}\sum_{l=0}^{j-1}p_{n,m}r_f^{2(m-n-2)-j-l}\varpi_{n+1+j}\varpi_{n+1+l}\rho_{l,j}(i)\Big\}\\
&+\frac{1}{4\omega}E_n[q_{n+1}\alpha_{n+1}(\xi_{n+1})]+\Big[\sum_{m=n+1}^{T}p_{n,m}r_f^{m-n-1}r_n^e(i)'\pi_n\\
&-\omega\sum_{m=n+1}^{T}p_{n,m}(r_f^2)^{m-n-1}\pi_n'Var(R_n^e(i))\pi_n\Big].
\end{aligned}
$$

Using (3.5) and (3.6) to simplify the above formula, and by Lemma 3.2, we have

$$
\begin{aligned}
J_n(x_n,i;(\pi_n,\hat{\pi}_{n+1},\cdots,\hat{\pi}_{T-1}))&=r_f\lambda_{n,n}x_n+\sum_{k=n}^{T-1}\lambda_{n,n+T-1-k}\overline{Q_{\mu_1}^{T-1-k}}(i)-\omega\Big\{E_n[q_{n+1}A_{n+1}(\xi_{n+1})]\\
&+\theta_{n,n}\mu_2(i)+\sum_{m=n+2}^{T}\sum_{k=0}^{m-n-2}p_{n,m}(r_f^2)^{m-n-2-k}\gamma_k(i)\\
&+2\sum_{m=n+2}^{T}\sum_{j=1}^{m-n-2}\sum_{l=0}^{j-1}p_{n,m}r_f^{2(m-n-2)-j-l}\delta_{l,j}(i)\Big\}\\
&-\frac{1}{4\omega}\Big\{E_n[q_{n+1}B_{n+1}(\xi_{n+1})]+\sum_{m=n+2}^{T}\sum_{k=0}^{m-n-2}p_{n,m}(r_f^2)^{m-n-2-k}\varpi_{n+1+k}^2\eta_k(i)\\
&+2\sum_{m=n+2}^{T}\sum_{j=1}^{m-n-2}\sum_{l=0}^{j-1}p_{n,m}r_f^{2(m-n-2)-j-l}\varpi_{n+1+j}\varpi_{n+1+l}\rho_{l,j}(i)\Big\}\\
&+\frac{1}{4\omega}E_n[q_{n+1}\alpha_{n+1}(\xi_{n+1})]+\Big[\lambda_{n,n}r_n^e(i)'\pi_n-\omega\theta_{n,n}\pi_n'Var(R_n^e(i))\pi_n\Big].
\end{aligned}
\tag{G.4}
$$

By (3.6)-(3.9), we have

$$
\begin{aligned}
E_n[q_{n+1}A_{n+1}(\xi_{n+1})]=&E_n\Big[q_{n+1}\Big(\sum_{k=n+1}^{T-1}\theta_{n+1,n+T-k}\overline{Q_{\mu_2}^{T-1-k}}(\xi_{n+1})\\
&+\sum_{j=0}^{T-3-n}\sum_{k=n+2+j}^{T-1}\theta_{n+1,n+1+j+T-k}\overline{Q_{\gamma_j}^{T-1-k}}(\xi_{n+1})\\
&+2\sum_{j=1}^{T-3-n}\sum_{l=0}^{j-1}\sum_{k=n+2+j}^{T-1}r_f^{j-l}\theta_{n+1,n+1+j+T-k}\overline{Q_{\delta_{l,j}}^{T-1-k}}(\xi_{n+1})\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=n+1}^{T-1} \theta_{n,n+T-k} \overline{Q_{\mu_2}^{T-k}}(i) + \sum_{j=0}^{T-3-n} \sum_{k=n+2+j}^{T-1} \theta_{n,n+1+j+T-k} \overline{Q_{\gamma_j}^{T-k}}(i) \\
&\quad + 2 \sum_{j=1}^{T-3-n} \sum_{l=0}^{j-1} \sum_{k=n+2+j}^{T-1} r_f^{j-l} \theta_{n,n+1+j+T-k} \overline{Q_{\delta_{l,j}}^{T-k}}(i),
\end{aligned} \tag{G.5}
$$

$$
\begin{aligned}
E_n[q_{n+1} B_{n+1}(\xi_{n+1})] &= E_n\Big[q_{n+1}\Big(\sum_{j=0}^{T-3-n} \sum_{k=n+2+j}^{T-1} \varpi_{n+1+j+T-k}^2 \theta_{n+1,n+1+j+T-k} \overline{Q_{\eta_j}^{T-1-k}}(\xi_{n+1}) \\
&\quad + 2 \sum_{j=1}^{T-3-n} \sum_{l=0}^{j-1} \sum_{k=n+2+j}^{T-1} r_f^{j-l} \varpi_{n+1+j+T-k} \varpi_{n+1+l+T-k} \theta_{n+1,n+1+j+T-k} \overline{Q_{\rho_{l,j}}^{T-1-k}}(\xi_{n+1})\Big)\Big] \\
&= \sum_{j=0}^{T-3-n} \sum_{k=n+2+j}^{T-1} \varpi_{n+1+j+T-k}^2 \theta_{n,n+1+j+T-k} \overline{Q_{\eta_j}^{T-k}}(i) \\
&\quad + 2 \sum_{j=1}^{T-3-n} \sum_{l=0}^{j-1} \sum_{k=n+2+j}^{T-1} r_f^{j-l} \varpi_{n+1+j+T-k} \varpi_{n+1+l+T-k} \theta_{n,n+1+j+T-k} \overline{Q_{\rho_{l,j}}^{T-k}}(i),
\end{aligned} \tag{G.6}
$$

$$
\begin{aligned}
E_n[q_{n+1} \alpha_{n+1}(\xi_{n+1})] &= E_n\Big[q_{n+1} \sum_{k=n+1}^{T-1} \varpi_{n+T-k} \lambda_{n+1,n+T-k} \overline{Q_{g_{n+T-k}}^{T-1-k}}(\xi_{n+1})\Big] \\
&= \sum_{k=n+1}^{T-1} \varpi_{n+T-k} \lambda_{n,n+T-k} \overline{Q_{g_{n+T-k}}^{T-k}}(i).
\end{aligned} \tag{G.7}
$$

In (G.4), substituting (G.5) into the following formula, and according to Lemmas 3.2-3.4, we have

$$
\begin{aligned}
&E_n[q_{n+1} A_{n+1}(\xi_{n+1})] + \theta_{n,n}\mu_2(i) + \sum_{m=n+2}^{T} \sum_{k=0}^{m-n-2} p_{n,m} (r_f^2)^{m-n-2-k} \gamma_k(i) \\
&\quad + 2 \sum_{m=n+2}^{T} \sum_{j=1}^{m-n-2} \sum_{l=0}^{j-1} p_{n,m} r_f^{2(m-n-2)-j-l} \delta_{l,j}(i) \\
&= \Big(\sum_{k=n+1}^{T-1} \theta_{n,n+T-k} \overline{Q_{\mu_2}^{T-k}}(i) + \theta_{n,n}\mu_2(i) \Big) + \Big(\sum_{j=0}^{T-3-n} \sum_{k=n+2+j}^{T-1} \theta_{n,n+1+j+T-k} \overline{Q_{\gamma_j}^{T-k}}(i) \\
&\quad + \sum_{m=n+2}^{T} \sum_{k=0}^{m-n-2} p_{n,m} (r_f^2)^{m-n-2-k} \gamma_k(i) \Big) \\
&\quad + 2\Big(\sum_{j=1}^{T-3-n} \sum_{l=0}^{j-1} \sum_{k=n+2+j}^{T-1} r_f^{j-l} \theta_{n,n+1+j+T-k} \overline{Q_{\delta_{l,j}}^{T-k}}(i) + \sum_{m=n+2}^{T} \sum_{j=1}^{m-n-2} \sum_{l=0}^{j-1} p_{n,m} r_f^{2(m-n-2)-j-l} \delta_{l,j}(i) \Big) \\
&= \sum_{k=n}^{T-1} \theta_{n,n+T-1-k} \overline{Q_{\mu_2}^{T-1-k}}(i) + \sum_{j=0}^{T-2-n} \sum_{k=n+1+j}^{T-1} \theta_{n,n+j+T-k} \overline{Q_{\gamma_j}^{T-1-k}}(i) \\
&\quad + 2 \sum_{j=1}^{T-2-n} \sum_{l=0}^{j-1} \sum_{k=n+1+j}^{T-1} r_f^{j-l} \theta_{n,n+j+T-k} \overline{Q_{\delta_{l,j}}^{T-1-k}}(i) \\
&= A_n(i).
\end{aligned} \tag{G.8}
$$

Similarly, according to (G.6) and Lemmas 3.3-3.4, we have

$$
\begin{aligned}
&E_n[q_{n+1}B_{n+1}(\xi_{n+1})] + \sum_{m=n+2}^{T}\sum_{k=0}^{m-n-2} p_{n,m}(r_f^2)^{m-n-2-k}\varpi_{n+1+k}^2\eta_k(i)\\
&\quad + 2\sum_{m=n+2}^{T}\sum_{j=1}^{m-n-2}\sum_{l=0}^{j-1} p_{n,m}r_f^{2(m-n-2)-j-l}\varpi_{n+1+j}\varpi_{n+1+l}\rho_{l,j}(i)\\
&=\Big(\sum_{j=0}^{T-3-n}\sum_{k=n+2+j}^{T-1}\varpi_{n+1+j+T-k}^2\theta_{n,n+1+j+T-k}\overline{Q_{\eta_j}^{T-k}}(i)\\
&\quad + \sum_{m=n+2}^{T}\sum_{k=0}^{m-n-2} p_{n,m}(r_f^2)^{m-n-2-k}\varpi_{n+1+k}^2\eta_k(i)\\
&\quad + 2\Big(\sum_{j=1}^{T-3-n}\sum_{l=0}^{j-1}\sum_{k=n+2+j}^{T-1} r_f^{j-l}\varpi_{n+1+j+T-k}\varpi_{n+1+l+T-k}\theta_{n,n+1+j+T-k}\overline{Q_{\rho_{l,j}}^{T-k}}(i)\\
&\quad + \sum_{m=n+2}^{T}\sum_{j=1}^{m-n-2}\sum_{l=0}^{j-1} p_{n,m}r_f^{2(m-n-2)-j-l}\varpi_{n+1+j}\varpi_{n+1+l}\rho_{l,j}(i)\Big)\\
&=\sum_{j=0}^{T-2-n}\sum_{k=n+1+j}^{T-1}\varpi_{n+j+T-k}^2\theta_{n,n+j+T-k}\overline{Q_{\eta_j}^{T-1-k}}(i)\\
&\quad + 2\sum_{j=1}^{T-2-n}\sum_{l=0}^{j-1}\sum_{k=n+1+j}^{T-1} r_f^{j-l}\varpi_{n+j+T-k}\varpi_{n+l+T-k}\theta_{n,n+j+T-k}\overline{Q_{\rho_{l,j}}^{T-1-k}}(i)\\
&=B_n(i).
\end{aligned}
\tag{G.9}
$$

By (G.4), the optimal solution to Problem (2.6) obviously exists and is

$$
\hat{\pi}_n(i) = \frac{1}{2\omega}\frac{\lambda_{n,n}}{\theta_{n,n}}Var^{-1}(R_n^e(i))r_n^e(i) = \frac{1}{2\omega}\varpi_n Var^{-1}(R_n^e(i))r_n^e(i). \tag{G.10}
$$

Substituting (G.7)-(G.10) into (G.4), we have

$$
\begin{aligned}
V_n(x_n,i) =& r_f\lambda_{n,n}x_n + \sum_{k=n}^{T-1}\lambda_{n,n+T-1-k}\overline{Q_{\mu_1}^{T-1-k}}(i) - \omega A_n(i) - \frac{1}{4\omega}B_n(i)\\
&+ \frac{1}{4\omega}\sum_{k=n+1}^{T-1}\varpi_{n+T-k}\lambda_{n,n+T-k}\overline{Q_{g_{n+T-k}}^{T-k}}(i) + \frac{1}{4\omega}\frac{\lambda_{n,n}^2}{\theta_{n,n}}r_n^e(i)'Var^{-1}(R_n^e(i))r_n^e(i)\\
=& r_f\lambda_{n,n}x_n + \sum_{k=n}^{T-1}\lambda_{n,n+T-1-k}\overline{Q_{\mu_1}^{T-1-k}}(i) - \omega A_n(i) - \frac{1}{4\omega}B_n(i)\\
&+ \Big(\frac{1}{4\omega}\sum_{k=n}^{T-2}\varpi_{n+T-1-k}\lambda_{n,n+T-1-k}\overline{Q_{g_{n+T-1-k}}^{T-1-k}}(i) + \frac{1}{4\omega}\varpi_n\lambda_{n,n}g_n(i)\Big)
\end{aligned}
$$

$$
\begin{aligned}
=& r_f\lambda_{n,n}x_n + \sum_{k=n}^{T-1}\lambda_{n,n+T-1-k}\overline{Q_{\mu_1}^{T-1-k}}(i) - \omega A_n(i) - \frac{1}{4\omega}B_n(i)\\
&+ \frac{1}{4\omega}\sum_{k=n}^{T-1}\varpi_{n+T-1-k}\lambda_{n,n+T-1-k}\overline{Q_{g_{n+T-1-k}}^{T-1-k}}(i) \qquad \text{(G.11)}\\
=& r_f\lambda_{n,n}x_n + \sum_{k=n}^{T-1}\lambda_{n,n+T-1-k}\overline{Q_{\mu_1}^{T-1-k}}(i) - \omega A_n(i) + \frac{1}{4\omega}\alpha_n(i) - \frac{1}{4\omega}B_n(i).
\end{aligned}
$$

Furthermore, for $m = n+1, n+2, \cdots, T$, by (3.3), (3.13) and (3.15), we have

$$
\begin{aligned}
h_{n,m}(x_n, i) =& E_n[h_{n+1,m}(X_{n+1}^{\hat{\pi}_n}, \xi_{n+1})]\\
=& E_n\Bigg[r_f^{m-n-1}X_{n+1}^{\hat{\pi}_n} + \sum_{k=n+1}^{m-1} r_f^{k-n-1}\overline{Q_{\mu_1}^{m-1-k}}(\xi_{n+1})\\
&+ \frac{1}{2\omega}\sum_{k=n+1}^{m-1} r_f^{k-n-1}\varpi_{n+m-k}\overline{Q_{g_{n+m-k}}^{m-1-k}}(\xi_{n+1})\Bigg]\\
=& r_f^{m-n-1}\Big(r_f x_n + r_n^e(i)'\hat{\pi}_n + \mu_1(i)\Big) + \sum_{k=n+1}^{m-1} r_f^{k-n-1}\overline{Q_{\mu_1}^{m-k}}(i)\\
&+ \frac{1}{2\omega}\sum_{k=n+1}^{m-1} r_f^{k-n-1}\varpi_{n+m-k}\overline{Q_{g_{n+m-k}}^{m-k}}(i)\\
=& r_f^{m-n}x_n + \Big(r_f^{m-n-1}\mu_1(i) + \sum_{k=n}^{m-2} r_f^{k-n}\overline{Q_{\mu_1}^{m-1-k}}(i)\Big)\\
&+ \Big(\frac{1}{2\omega}r_f^{m-n-1}\varpi_n g_n(i) + \frac{1}{2\omega}\sum_{k=n}^{m-2} r_f^{k-n}\varpi_{n+m-1-k}\overline{Q_{g_{n+m-1-k}}^{m-1-k}}(i)\Big)\\
=& r_f^{m-n}x_n + \sum_{k=n}^{m-1} r_f^{k-n}\overline{Q_{\mu_1}^{m-1-k}}(i) + \frac{1}{2\omega}\sum_{k=n}^{m-1} r_f^{k-n}\varpi_{n+m-1-k}\overline{Q_{g_{n+m-1-k}}^{m-1-k}}(i).
\end{aligned}
$$

Therefore, it is clear that (3.13)-(3.15) also hold for n. By the principle of mathematical induction, the theorem is proved.

Appendix H

Proof of Proposition 3.2. By (3.18), we have

$$
\frac{1}{2\omega} = \frac{E_n^U(X_{T\wedge\tau}^{\hat{\pi}}) - W_n^f(i)}{\alpha_n(i)}.
$$

Substituting this formula into (3.19) leads to the proposition.

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