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Optimal Control for Discrete-time NCSs with Input Delay and Markovian Packet Losses: Hold-Input Case ⋆

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Abstract

This paper is concerned with the linear-quadratic optimal control problem for networked systems simultaneously with input delay and Markovian packet losses under hold-input compensation strategy, which is different from the literature. Necessary and sufficient conditions for the solvability of optimal control problem over a finite horizon are given by coupled difference Riccati-type equations. Moreover, the networked control system is mean-square stability if and only if coupled algebraic Riccati-type equations have a particular solution. Due to input delay and Markovian packet dropout, it leads to the failure of the separation principle, which is a fundamental obstacle. The key technique in this paper is to resolve forward and backward difference equations by decoupling to overcome the difficulty.

Key words: Input delay; Markovian packet losses; Hold-input; Stabilization.

1 Introduction

A network control system (NCS, for short), known as a communication and control system, is a fully distributed and networked real-time feedback control system. Since the concept of NCSs was proposed in the early 1990s, it has attracted attention. For example, Guo et al [3] discussed the networked control problems for discrete linear systems whose network mediums for the actuators are constrained. Using the Lyapunov-Krasovskii functional method, Yue et al [17] considered the disturbance attenuation problem for NCSs. At the same time, it raised new challenges to traditional control theory and applications. In an NCS, multiple network nodes share a network channel. Due to the limited network bandwidth and irregular changes in data traffic in the network, data collisions and network congestion often occur when multiple nodes exchange data through the network. Therefore, packet losses and time delay will inevitably occur.

Generally, there are two kinds of packet losses. When the process of packet transmission is mutually independent, the packet loss is modeled as a Bernoulli process. Liang and Xu [8] focused on the optimal control problems for NCSs, which are simultaneously controlled by the remote controller and the local controller. Lu et al [10] considered the NCSs with Bernoulli packet losses using an improved switching hold compensation strategy in which the too old held signal is deleted. Based on the characteristic of independent and identical distribution, as said in [19], it is easy to verify that NCSs with Bernoulli packet losses can be regarded as a special case of system with multiplicative white noise. However, if the current packet cases affect future packet cases, a Markov process can represent such effect rather than a Bernoulli process. In fact, it is more involved to be dealt with than Bernoulli packet losses due to the temporal correlation caused by Markovian characteristics. Wang et al [13] considered the H_∞-controller design for NCSs with Markovian packet losses. Xie and Xie [15] presented the necessary and sufficient condition for the mean-square stability of sampled-data networked linear systems with Markovian packet losses.

For the packet loss case, some strategies are usually adopted to compensate in NCSs. The zero-input (i.e., zero value is directly adopted by the actuator input) and the hold-input (i.e., the latest available control signal

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stored in the actuator buffer is used) are two common compensation strategies. For zero-input case, Imer et al [5] discussed the optimal control problem for the linear system with packet losses under TCP and UDP protocols. Sufficient conditions for stability of network communication models with packet losses were studied by Montestruque and Antsaklis [11]. It must be pointed out that, as said in [3] and [10], the zero-input strategy is mainly for mathematical convenience as it gives simpler equations than hold-input strategy, rather than for performance considerations. However, in most practical applications, it is necessary to consider the performance. Moreover, using the latest control input stored in the actuator buffer provides better performance than using zero input, especially during transients, because the true current optimal control input is likely to be close to the previous value. Indeed, the hold-input strategy was studied in many previous works due to its universality in the practical field. For example, Hristu-Varsakelis [4] analyzed the structure properties (e.g., observability and controllability) of the NCSs in which a zero-order hold is included.

Note that most of the aforementioned work only considered packet losses in NCSs [8,10,13,15,18]. Few works focus on the simultaneous occurrence of time-delay. Actually, when transmission delay occurs in NCSs with packet losses, the current controller will be designed with past information, leading to the controller's adaptability problem. Therefore, some of the previous methods (e.g., the Smith predictor method [14] which is only useful for the determinate or additional system) are no longer applicable. Although the time-delay system can be converted to a delay-free system by state augmentation [2], it leads to the design of optimal controller feedback by a high-dimensional gain matrix and a large amount of calculation. What's more, it cannot reflect the influence of time delay on the optimal control problem in essence.

From the above-mentioned analysis, it is not easy to deal with the optimal control problem for NCSs with both input delay and Markovian packet losses. The fundamental difficulty of this problem can be attributed to the failure of the separation principle caused by simultaneous input delay and Markovian packet dropout. In this case, state prediction is required to design a controller, and the feedback gain matrix cannot be given by general Riccati equations. It is a fundamental challenge problem. Indeed, some studies have focused on the stabilization problem in the presence of both data packet dropout and delay due to its theoretical value and practical background. However, methods proposed in previous literature (e.g., predictive control method [9] and Lyapunov-Krasivskii functional approach [17] etc.) are mainly based on linear matrix inequalities. Only sufficient stabilizing conditions are available. The complete solution (i.e., sufficient and necessary conditions for the stabilization problem) are not derived at present. Hence, it is a fundamental challenge problem and has not been solved thoroughly.

In this paper, we consider the optimal control problem for NCSs with both input delay and Markovian packet losses under hold-input strategy. To reduce the computational complexity, the system under hold-input strategy is firstly converted to the linear system with Markovian jump (MJLS), which is another important topic [1,6,7,12,16]. For example, Li and Zhou [6] and Li et al. [7] considered the indefinite stochastic optimal control problems for the MJLS over a finite time horizon and an infinite time horizon, respectively. Also, Costa et al. [1] studied discrete-time Markovian jump linear systems and their applications. In view of these, the key point in this paper is how to deal with the forward and backward stochastic difference equations (FBSDEs), which are derived by the stochastic maximum principle. Inspired by [19] and [20] in which the FBSDEs have made substantial progress in optimal LQ control problem for linear systems, the main results in this paper are derived and can be summarized as follows. First, the necessary and sufficient conditions for the solvability of optimal control problem over a finite horizon are presented by the coupled difference Riccati-type equations (CDREs). Second, the existence of the solution to the coupled algebraic Riccati-type equations (CAREs) is proved. Moreover, the optimal controller and optimal cost functional over an infinite horizon are derived. Finally, the necessary and sufficient conditions for the stabilization of the NCSs are established using the CAREs.

The rest of this article is structured as follows. Section 2 gives the problem statement. Section 3 solves the optimal control problem over a finite horizon and the stabilization problems for the infinite horizon case. A numerical example is presented to verify the obtained results in Section 4. A summary is presented in Section 5. Proofs for some results can be found in the Appendix.

Notation : $\mathbb{R}^n$ is the n-dimensional Euclidean space and $\mathbb{R}^{m\times n}$ the norm bounded linear space of all $m \times n$ matrices. Y' is the transposition of Y and $Y \geq 0 (Y > 0)$ means that $Y \in \mathbb{R}^{n\times n}$ is symmetric positive semi-definite (positive definite). Let $(\Omega, \mathcal{F}, \mathcal{F}_k, \mathcal{P})$ be a complete probability space with the natural filtration $\{\mathcal{F}_k\}_{k\geq 0}$ generated by $\{\theta_0, \cdots, \theta_k\}$. $\mathbb{E}[\cdot|\mathcal{F}_k]$ means the conditional expectation with respect to $\mathcal{F}_k$ and $\mathcal{F}_{-1}$ is understood as $\{\emptyset, \Omega\}$.

2 Problem Statement and Preliminaries

Consider the following discrete-time system:

$$x_{k+1} = Ax_k + Bu^a_{k-d}, \tag{1}$$

$$\begin{cases} u^a_{k-d} = \theta_k u^c_{k-d} + (1-\theta_k)u^a_{k-d-1}, \ k \geq 0, \\ u^a_i = u^c_i, \ i = -d, \cdots, -1, \end{cases} \tag{2}$$

where $x_k \in \mathbb{R}^n$ is the state. $u_k^a \in \mathbb{R}^m$ denotes the control input to the actuator and $u_k^c \in \mathbb{R}^m$ is the desired control input computed by the controller. The stochastic variable θ_k is the packet loss modeled as a two state Markov chain $\theta_k \in \{0,1\}$ with transition probability $\xi_{ij} = \mathrm{P}(\theta_{k+1} = j|\theta_k = i)(i,j=0,1)$ between the controller and the actuator: Take $u_k^a = u_k^c$, if the packet is correctly delivered; otherwise, take $u_k^a = u_{k-1}^a$, if the packet is lost. The initial values are $x_0, u_{-d}^c, \cdots, u_{-1}^c$. In addition, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ are constant matrices. By state augmentation, we have

$$\begin{bmatrix} x_{k+1} \\ u_{k-d}^a \end{bmatrix} = \begin{bmatrix} A & (1-\theta_k)B \\ 0 & (1-\theta_k)I \end{bmatrix} \begin{bmatrix} x_k \\ u_{k-d-1}^a \end{bmatrix} + \begin{bmatrix} \theta_k B \\ \theta_k I \end{bmatrix} u_{k-d}^c. \quad (3)$$

Let $z_{k+1} = \begin{bmatrix} x_{k+1} \\ u_{k-d}^a \end{bmatrix}$, $\bar{A}_{\theta_k} = \begin{bmatrix} A & (1-\theta_k)B \\ 0 & (1-\theta_k)I \end{bmatrix}$, $\bar{B}_{\theta_k} = \begin{bmatrix} \theta_k B \\ \theta_k I \end{bmatrix}$, then (3) can be rewritten as

$$z_{k+1} = \bar{A}_{\theta_k} z_k + \bar{B}_{\theta_k} u_{k-d}^c. \quad (4)$$

Define the following cost functional over an infinite horizon as

$$J = \mathbb{E}\left[\sum_{k=0}^{\infty}\Big(z_k' Q z_k + (u_{k-d}^c)' R u_{k-d}^c\Big)\right], \quad (5)$$

where weighting matrices $Q \in \mathbb{R}^{(n+m)\times(n+m)}, R \in \mathbb{R}^{m\times m}$.

Problem 1 *Find a $\mathcal{F}_{k-1}$-measurable controller u_k^c to stabilize system (1)-(2) and minimize cost functional (5).*

3 Main Results

For discussion, this section will follow two steps. The LQ optimal control problem over a finite horizon will be first considered. On this basis, Problem 1 will be resolved.

3.1 LQ Optimal Control in Finite Horizon Case

Consider the following cost functional:

$$J_N = \mathbb{E}\left[\sum_{k=0}^{N}\Big(z_k' Q z_k + (u_{k-d}^c)' R u_{k-d}^c\Big) + z_{N+1}' \bar{P}_{N+1} z_{N+1}\right], \quad (6)$$

where weighting matrices $Q \in \mathbb{R}^{(n+m)\times(n+m)}, R \in \mathbb{R}^{m\times m}$ and terminal value $\bar{P}_{N+1}$ are positive semi-definite.

Problem 2 *Find a $\mathcal{F}_{k-d-1}$-measurable controller u_{k-d}^c such that cost functional (6) is minimized subject to (4).*

Applying the maximum principle to Problem 2, the following FBSDEs are obtained

$$\begin{cases} 0 = \mathbb{E}_{k-d-1}\big[\bar{B}_{\theta_k}' \lambda_k\big] + R u_{k-d}^c, \\ \lambda_{k-1} = Q z_k + \mathbb{E}_{k-1}\big[\bar{A}_{\theta_k}' \lambda_k\big], \\ \lambda_N = \bar{P}_{N+1} z_{N+1}, \\ z_{k+1} = \bar{A}_{\theta_k} z_k + \bar{B}_{\theta_k} u_{k-d}^c. \end{cases} \quad (7)$$

Remark 3 *Due to the Markovian jump and input delay, which gives rise to the fundamental difficulty about the adaptability of the controller and the temporal correlation, our problem is important and challenging compared with [19] and [20]. The key technique in this paper is to tackle FBSDEs (7).*

For any $d \le k \le N$, define the following recursive sequence,

$$\bar{P}_{\theta_{k-1}}(k) = Q + \mathbb{E}_{k-1}\big[(\bar{A}_{\theta_k})' \bar{P}_{\theta_k}(k+1) \bar{A}_{\theta_k} - (M_{\theta_{k-1}}^0)' \Gamma_{\theta_{k-1}}^{-1} M_{\theta_{k-1}}^0\big], \quad (8)$$

where

$$\Gamma_{\theta_{k-d-1}} = R + \mathbb{E}_{k-d-1}\left[(\bar{B}_{\theta_k})' \bar{P}_{\theta_k}(k+1) \bar{B}_{\theta_k} - \sum_{i=0}^{d-1} (M_{\theta_{k-d+i}}^{i+1})' \Gamma_{\theta_{k-d+i}}^{-1} M_{\theta_{k-d+i}}^{i+1}\right], \quad (9)$$

$$M_{\theta_{k-d-1}}^0 = \mathbb{E}_{k-d-1}\left[(\tilde{S}_{\theta_{k-1}}^1)' \prod_{j=1}^{d} \bar{A}_{\theta_{k-j}} - \sum_{i=0}^{d-1}\Big((M_{\theta_{k-d+i}}^{i+1})' \times \Gamma_{\theta_{k-d+i}}^{-1} M_{\theta_{k-d+i}}^0 \prod_{s=0}^{i} \bar{A}_{\theta_{k-d+s}}\Big)\right], \quad (10)$$

$$M_{\theta_{k-d-1}}^i = \mathbb{E}_{k-d-1}\left[(\tilde{S}_{\theta_{k-1}}^1)' \prod_{j=1}^{i-1} \bar{A}_{\theta_{k-j}} \bar{B}_{\theta_{k-i}} - \sum_{s=0}^{d-1} (M_{\theta_{k-d+s}}^{s+1})' \times \Gamma_{\theta_{k-d+s}}^{-1} M_{\theta_{k-d+s}}^{i+s+1}\right], \quad i = 1, \cdots, d, \quad (11)$$

$$M_{\theta_{k-d-1}}^i = M_{\theta_{k-d-1}}^0 \mathbb{E}_{k-d-1}\left[\prod_{j=1}^{i-d-1} \bar{A}_{\theta_{k-d-j}} \bar{B}_{\theta_{k-i}}\right], \quad i \ge d+1, \quad (12)$$

$$M_{\theta_{N-s-1}}^i = 0, \quad i \ge 0, \quad s \le d-1, \quad (13)$$

$$\tilde{S}_{\theta_{k-1}}^1 = \mathbb{E}_{k-1}\big[(\bar{A}_{\theta_k})' \bar{P}_{\theta_k}(k+1) \bar{B}_{\theta_k}\big], \quad (14)$$

$$\tilde{S}_{\theta_{k-1}}^j = \mathbb{E}_{k-1}\big[(\tilde{S}_{\theta_k}^{j-1})' \bar{A}_{\theta_k}\big], \quad (15)$$

with terminal value $\bar{P}_{\theta_N}(N+1) = \bar{P}_{N+1}$. And equations (8)-(15) are termed the CDREs.

Remark 4 *According to Markov property (that is, the evolution of the Markov process in the future depends* *only on the present state and does not depend on past history), the conditional expectation of the sequence f_{θ_k} can be expressed as*

$$\mathbb{E}_{k-j}[f_{\theta_k}]=\sum_{\theta_{k-j+1}=0}^{1}\xi_{\theta_{k-j+1}\theta_{k-j}}\left\{\sum_{\theta_{k-j+2}=0}^{1}\xi_{\theta_{k-j+2}\theta_{k-j+1}}\cdots\right.$$
$$\left.\times\left[\sum_{\theta_{k-1}=0}^{1}\xi_{\theta_{k-1}\theta_{k-2}}(\sum_{\theta_k=0}^{1}\xi_{\theta_k\theta_{k-1}})f_{\theta_k}\right]\right\}. \quad (16)$$

Remark 5 *For convenience, set $\Gamma_{\theta_{k-d-1}}(k)\triangleq\Gamma_{\theta_{k-d-1}}$, $\bar{P}_{\theta_k}(k+1)\triangleq\bar{P}_{\theta_k}$, $M^i_{\theta_{k-d-1}}(k)\triangleq M^i_{\theta_{k-d-1}}$, $\tilde{S}^1_{\theta_{k-1}}(k)\triangleq\tilde{S}^1_{\theta_{k-1}}$.*

Lemma 6 *From CDREs (8)-(15), the following relationships can be given*

$$M^0_{\theta_{k-d-1}}=\mathbb{E}_{k-d-1}\left[(F^1_{\theta_{k-1}})'\prod_{j=1}^{d}\bar{A}_{\theta_{k-j}}\right], \quad (17)$$
$$M^i_{\theta_{k-d-1}}=\mathbb{E}_{k-d-1}\left[(F^i_{\theta_{k-i}})'\bar{B}_{\theta_{k-i}}\right], \quad i\geq 1, \quad (18)$$
$$F^i_{\theta_{k-1}}=\mathbb{E}_{k-1}\left[\prod_{j=0}^{i-2}(\bar{A}_{\theta_{k+j}})'F^1_{\theta_{k+i-2}}\right], \quad i\geq 2, \quad (19)$$

with

$$F^1_{\theta_{k-1}}=\tilde{S}^1_{\theta_{k-1}}-\sum_{i=0}^{d}F^{i+2}_{\theta_{k-1}}\Gamma^{-1}_{\theta_{k-d+i}}M^{i+1}_{\theta_{k-d+i}}, \quad (20)$$
$$F^i_{\theta_{k-1}}=\mathbb{E}_{k-1}[\bar{A}'_{\theta_k}F^{i-1}_{\theta_k}], \quad (21)$$
$$F^{N-k+1}_{\theta_{k-1}}=(\tilde{S}^{N-k+1}_{\theta_{k-1}})'. \quad (22)$$

Proof. From (9)-(15) and via mathematical induction, (17)-(19) can be simply calculated, here, we omit the proof.
Based on the preliminaries, the results of Problem 2 can be obtained in this section.

Theorem 7 *There exists a unique solution to Problem 2 if and only if $\Gamma_{\theta_{k-d-1}}$ in (8) is positive definite. In this case, the optimal controller can be given as*

$$u^c_{k-d}=-\Gamma^{-1}_{\theta_{k-d-1}}\left(M^0_{\theta_{k-d-1}}z_{k-d}+\sum_{i=1}^{d}M^i_{\theta_{k-d-1}}u^c_{k-d-i}\right), \quad (23)$$

and the optimal cost functional is

$$J_N=\mathbb{E}\left\{\sum_{k=0}^{d-1}\left[z'_kQz_k+(u^c_{k-d})'Ru^c_{k-d}\right]+z'_d\bar{P}_{\theta_{d-1}}z_d\right.$$
$$-z'_d\sum_{s=0}^{d-1}\left(F^{s+1}_{\theta_{d-1}}\Gamma^{-1}_{\theta_{s-1}}M^0_{\theta_{s-1}}z_s\right)$$
$$\left.-z'_d\sum_{s=0}^{d}\left(F^{s+1}_{\theta_{d-1}}\Gamma^{-1}_{\theta_{s-1}}\sum_{i=s+1}^{d}M^i_{\theta_{s-1}}u^c_{s-i}\right)\right\}. \quad (24)$$

Moreover, the solution of the FBSDEs can be given

$$\lambda_{k-1}=\bar{P}_{\theta_{k-1}}z_k-\sum_{s=0}^{d-1}\left(F^{s+1}_{\theta_{k-1}}\Gamma^{-1}_{\theta_{k-d-1+s}}M^0_{\theta_{k-d-1+s}}z_{k-d+s}\right)$$
$$-\sum_{s=0}^{d-1}\left(F^{s+1}_{\theta_{k-1}}\Gamma^{-1}_{\theta_{k-d-1+s}}\sum_{i=s+1}^{d}M^i_{\theta_{k-d-1+s}}u^c_{k-d-i+s}\right). \quad (25)$$

Proof. See Appendix A.

Remark 8 *For convenience, the notation has been written for short, i.e., $F^i_{\theta_{k-1}}(k)\triangleq F^i_{\theta_{k-1}}$.*

Remark 9 *For the delay-free case, i.e., d=0 in systems (1)-(2), then the result of Theorem 7 can be rewritten as follows. The optimal controller (23) and the solution of the FBSDEs (25) can be re-expressed as*

$$u^c_k=-\Gamma^{-1}_{\theta_{k-1}}M^0_{\theta_{k-1}}z_k, \quad (26)$$
$$\lambda_{k-1}=\bar{P}_{\theta_{k-1}}z_k, \quad (27)$$

where

$$\bar{P}_{\theta_{k-1}}=Q+\mathbb{E}_{k-1}\left[(\bar{A}_{\theta_k})'\bar{P}_{\theta_k}\bar{A}_{\theta_k}-(M^0_{\theta_{k-1}})'\Gamma^{-1}_{\theta_{k-1}}M^0_{\theta_{k-1}}\right], \quad (28)$$
$$\Gamma_{\theta_{k-1}}=R+\mathbb{E}_{k-1}[(\bar{B}_{\theta_k})'\bar{P}_{\theta_k}\bar{B}_{\theta_k}], \quad (29)$$
$$M^0_{\theta_{k-1}}=\mathbb{E}_{k-1}[(\bar{B}_{\theta_k})'\bar{P}_{\theta_k}\bar{A}_{\theta_k}], \quad (30)$$

which are parallel to the results of standard case for MJLSs [1].

3.2 Optimal Control in Infinite Horizon Case

Definition 10 *The following stochastic system:*

$$z_{k+1}=\bar{A}_{\theta_k}z_k, \quad y_k=Q^{1/2}z_k, \quad \theta_k=0,1$$

is said to be exact observable (or $(\bar{A},Q^{1/2})$ is called exact observable where $\bar{A}=(\bar{A}_0,\bar{A}_1)$, for short), if for any $N\geq 0$

$$y_k\equiv 0, a.s.\ \forall 0\leq k\leq N\Longrightarrow z_0=0.$$

In the sequence, the following assumptions will be made.

Assumption 11 *The state weighting matrix Q is positive semi-definite and the control weighting matrix R is strictly positive definite.*

Assumption 12 *$(\bar{A},Q^{\frac{1}{2}})$ is exactly observable.*

Define the following CAREs for $l_j = 0, 1, 0 \le j \le d$,

$$\bar{P}_{l_d} = Q + \mathbb{E}_{l_d}\left[(\bar{A}_{l_{d+1}})'\bar{P}_{l_{d+1}}\bar{A}_{l_{d+1}} - (M^0_{l_d})'\Gamma^{-1}_{l_d}M^0_{l_d}\right], \tag{31}$$

where

$$\Gamma_{l_0} = R + \mathbb{E}_{l_0}\left[(\bar{B}_{l_{d+1}})'\bar{P}_{l_{d+1}}\bar{B}_{l_{d+1}} - \sum_{i=0}^{d-1}(M^{i+1}_{l_{i+1}})'\Gamma^{-1}_{l_{i+1}}M^{i+1}_{l_{i+1}}\right], \tag{32}$$

$$M^0_{l_0} = \mathbb{E}_{l_0}\left[(\tilde{S}^1_{l_d})'\prod_{j=1}^{d}\bar{A}_{l_{d+1-j}} - \sum_{i=0}^{d-1}\left[(M^{i+1}_{l_{i+1}})'\Gamma^{-1}_{l_{i+1}}M^0_{l_{i+1}} \times \prod_{s=0}^{i}\bar{A}_{l_{s+1}}\right]\right], \tag{33}$$

$$M^i_{l_0} = \mathbb{E}_{l_0}\left[(\tilde{S}^1_{l_d})'\prod_{j=1}^{i-1}\bar{A}_{l_{d+1-j}}\bar{B}_{l_{d+1-i}} - \sum_{s=0}^{d-1}(M^{s+1}_{l_{s+1}})'\Gamma^{-1}_{l_{s+1}} \times M^{i+s+1}_{l_{s+1}}\right], \quad i = 1, \cdots, d, \tag{34}$$

$$M^s_{l_1} = M^0_{l_1}\mathbb{E}_{l_1}\left[\prod_{j=1}^{s-d-1}\bar{A}_{l_j}\bar{B}_{l_{s-d}}\right], \quad s \ge d+1, \tag{35}$$

$$\tilde{S}^1_{l_d} = \mathbb{E}_{l_d}\left[(\bar{A}_{l_{d+1}})'\bar{P}_{l_{d+1}}\bar{B}_{l_{d+1}}\right], \tag{36}$$

$$\tilde{S}^j_{l_d} = \mathbb{E}_{l_d}\left[(\bar{A}_{l_{d+1}})'\tilde{S}^{j-1}_{l_{d+1}}\right]. \tag{37}$$

The main result will be presented next.

Theorem 13 *Under Assumption 11 and 12, the system (1) is stabilizable in the mean-square sense if and only if CAREs (31)-(36) have a solution such that*

$$\bar{P}_{l_d} - \sum_{s=0}^{d-1}\left[(F^{s+1}_{l_d})'\Gamma^{-1}_{l_{s-1}}F^{s+1}_{l_d}\right] > 0, \tag{38}$$

in which

$$F^1_{l_d} = \tilde{S}^1_{l_d} - \sum_{i=0}^{d}F^{i+2}_{l_d}\Gamma^{-1}_{l_{i+1}}M^{i+1}_{l_{i+1}}, \tag{39}$$

$$F^i_{l_d} = \mathbb{E}_{l_d}\left[(\bar{A}_{l_{d+1}})'F^{i-1}_{l_{d+1}}\right], \quad F^d_{l_d} = (\tilde{S}^d_{l_d})', \tag{40}$$

$l_i \in \{0, 1\}$, $i = 0, 1, \cdots, d+1$. *Moreover, for $k \ge d$ the optimal controller can be given as*

$$u^c_{k-d} = -\Gamma^{-1}_{l_0}\left(M^0_{l_0}z_{k-d} + \sum_{i=1}^{d}M^i_{l_0}u^c_{k-d-i}\right). \tag{41}$$

The corresponding cost index is presented by

$$J^* = \mathbb{E}\left[z_0'\bar{P}_{l_d}z_0 + \sum_{k=0}^{d-1}\left(u^c_{k-d} + \Gamma^{-1}_iM^0_iz_{k-d} + \Gamma^{-1}_i\sum_{s=1}^{d}(M^s_iu^c_{k-d-s})\right)' \times\Gamma_i\left(u^c_{k-d} + \Gamma^{-1}_iM^0_iz_{k-d} + \Gamma^{-1}_i\sum_{s=1}^{d}(M^s_iu^c_{k-d-s})\right)\right], \tag{42}$$

where $\bar{P}_i, \Gamma_i, M^s_i, i = 0, 1$ satisfy CAREs (31)-(36).

Proof. See Appendix B.

Remark 14 *The optimal control for the MJLS without delay has been well studied in the literature and using the state augmentation method. We resolve the presented problem in this paper. However, it will bring a large amount of calculation, especially for the high dimension system or the large delay.*

Remark 15 *Compared with the previous works only considered either delay or packet loss or under zero-input strategy in NCSs ([5], [10] and so on), the necessary and sufficient conditions for the stabilization of the NCSs including both input delay and Markovian dropout under hold-input strategy are established. To the best of our knowledge, the above necessary and sufficient conditions are firstly presented.*

Remark 16 *Consider delay-free case, i.e., $d = 0$, then Theorem 13 can be degenerated as:*
Under Assumption 11 and 12, the system (1) is stabilizable in the mean-square sense if and only if

$$\bar{P}_i > 0, \quad i = 0, 1, \tag{43}$$

in which

$$\bar{P}_i = Q + \sum_{j=0}^{1}(\xi_{ij}\bar{A}_j'\bar{P}_j\bar{A}_j) - (M^0_i)'\Gamma^{-1}_iM^0_i, \tag{44}$$

with

$$M^0_i = \sum_{j=0}^{1}(\xi_{ij}\bar{B}_j'\bar{P}_j\bar{A}_j), \quad \Gamma_i = R + \sum_{j=0}^{1}(\xi_{ij}\bar{B}_j'\bar{P}_j\bar{B}_j). \tag{45}$$

Moreover, for $k \ge 0$ the optimal controller can be given as $u^c_k = -\Gamma^{-1}_iM^0_iz_k$. The corresponding cost index is presented by $J^* = \mathbb{E}[z_0'\bar{P}_iz_0]$. Thus, it can be seen that equation (38) is a standard assumption and consistent with Lyapunov criterion.

4 Numerical Examples

4.1 Example 1

Consider system (1) with $A = 1, B = 15$, and the initial values $x_0 = 0.001, u^c_{-1} = -0.2$, let the transition probability $\xi_{00} = 0.9, \xi_{11} = 0.7$ and cost functional (6) with $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $R = 10$. Therefore, $\bar{A}_0 = \begin{bmatrix} 1 & 15 \\ 0 & 1 \end{bmatrix}$, $\bar{A}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\bar{B}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\bar{B}_1 = \begin{bmatrix} 15 \\ 1 \end{bmatrix}$. In this case, a sample path of the Markov chain θ_k is shown in Fig. 1.
When d=1:
In view of (31)-(36), the following results can be obtained:

$$P_0 = \begin{bmatrix} 3.7383 & 251.49 \\ 251.49 & 69116.31 \end{bmatrix}, P_1 = \begin{bmatrix} 3.8049 & 88.0818 \\ 88.0818 & 23233.77 \end{bmatrix},$$
$$\Gamma_0 = 3006.02, \Gamma_1 = 11877.83,$$
$$M_0^0 = \begin{bmatrix} 55.01 & 1383.48 \end{bmatrix}, M_1^0 = \begin{bmatrix} 107.89 & 461.16 \end{bmatrix},$$
$$M_0^1 = -1834.41, M_1^1 = -5058.20.$$

Hence, (38) can be calculated

$$\bar{P}_0 - (F_0^1)'\Gamma_i^{-1}F_0^1 = \begin{bmatrix} 4 & 251 \\ 251 & 69116 \end{bmatrix} > 0, i = 0, 1, \quad (46)$$

$$\bar{P}_1 - (F_1^1)'\Gamma_0^{-1}F_1^1 = \begin{bmatrix} 0.37 & 88 \\ 88 & 23234 \end{bmatrix} > 0, \quad (47)$$

$$\bar{P}_1 - (F_1^1)'\Gamma_1^{-1}F_1^1 = \begin{bmatrix} 3 & 88 \\ 88 & 23234 \end{bmatrix} > 0. \quad (48)$$

According to Theorem 13, the optimal controller can be expressed as

$$u_k^c = -\begin{bmatrix} 0.0183 & 0.4602 \end{bmatrix} z_k + 0.6102 u_{k-1}^c, l_0 = 0,$$
$$u_k^c = -\begin{bmatrix} 0.0091 & 0.0388 \end{bmatrix} z_k + 0.4259 u_{k-1}^c, l_0 = 1.$$

When d=0:
In view of (44)-(45), the following results can be obtained:

$$P_0 = \begin{bmatrix} 19.1924 & 2590.97 \\ 2590.97 & 738437 \end{bmatrix}, P_1 = \begin{bmatrix} 13.3152 & 865.513 \\ 865.513 & 246174 \end{bmatrix},$$
$$\Gamma_0 = 27524, \Gamma_1 = 192600,$$
$$M_0^0 = \begin{bmatrix} 106.5241 & 0 \end{bmatrix}, M_1^0 = \begin{bmatrix} 745.6687 & 0 \end{bmatrix}.$$

It is easy to check that $P_i > 0, i = 0, 1$. Hence, the optimal controller can be expressed as

$$u_k^c = -\begin{bmatrix} 0.003870 & 0 \end{bmatrix} z_k, l_0 = 0,$$
$$u_k^c = -\begin{bmatrix} 0.003872 & 0 \end{bmatrix} z_k, l_0 = 1.$$

Simulations of optimal controller u_k^c with $d = 1$ and $d = 0$ are shown in Fig. 2 (a) and (b), respectively.
It is easy to see that system (1)-(2) is mean-square stabilizable with $d = 1$ and $d = 0$ from Fig. 3 (a) and (b), respectively.
Meanwhile, it can be seen the influence of time-delay from simulations.

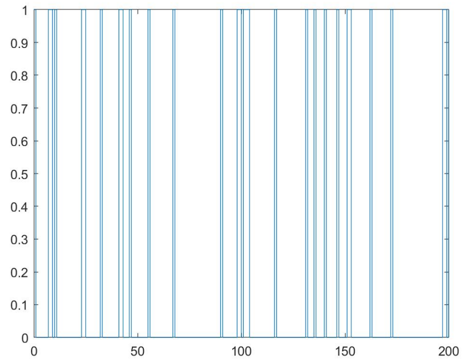
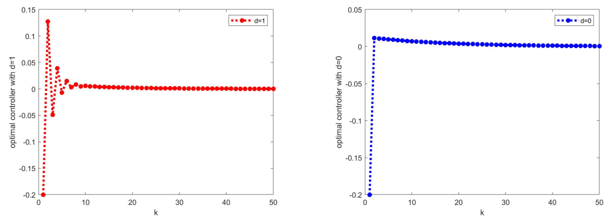
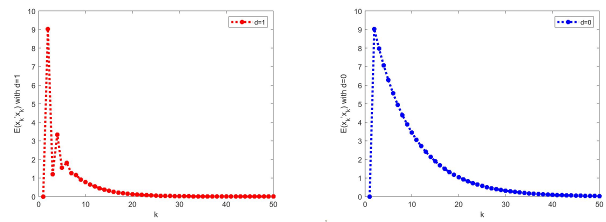


Fig. 1. A sample path with q=0.9 and p=0.7



(a) d=1 (b) d=0

Fig. 2. Optimal Controller



(a) d=1 (b) d=0

Fig. 3. Dynamic Behavior of $\mathbb{E}[x_k' x_k]$

4.2 Example 2

In this section, we will discuss the case for $d = 10$. Consider system (1) with $A = 1, B = 0.1$, and the initial values $x_0 = 0.001, u_{-1}^c = -0.2$, the transition probability $\xi_{00} = 0.9, \xi_{11} = 0.7$ and cost functional (6) with $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, R = 10$. Therefore, $\bar{A}_0 = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix}, \bar{A}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \bar{B}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \bar{B}_1 = \begin{bmatrix} 0.1 \\ 1 \end{bmatrix}$. In view of (31)-(36), the following results can be obtained:

$$P_0=\begin{bmatrix}41.4159 & 30.9107\\ 30.9107 & 66.0722\end{bmatrix}, P_1=\begin{bmatrix}43.7072 & 10.4716\\ 10.4716 & 22.7943\end{bmatrix},$$
$\Gamma_0=9.9961, \Gamma_1=35.2531,$
$M_0^0=\begin{bmatrix}3.5052 & 1.8147\end{bmatrix}, M_1^0=\begin{bmatrix}3.3197 & 0.4685\end{bmatrix}$

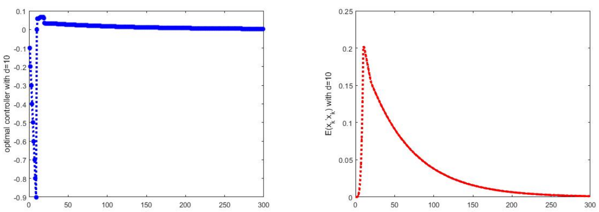
and values of (34) can be shown in the following table, i.e., Table 1.

Table 1 Values of M_i^j.

M_0^1	-0.0096	M_1^1	0.9392
M_0^2	0.0720	M_1^2	0.4717
M_0^3	0.0626	M_1^3	0.4062
M_0^4	0.0606	M_1^4	0.3922
M_0^5	0.0627	M_1^5	0.4065
M_0^6	0.0668	M_1^6	0.4356
M_0^7	0.0720	M_1^7	0.4714
M_0^8	0.0774	M_1^8	0.5094
M_0^9	0.0828	M_1^9	0.5472
M_0^{10}	0.0879	M_1^{10}	0.5833

Hence, (38) can be calculated and the positiveness can be judged by the function 'chol()' in MATLAB.

According to Table 1 and Theorem 13, the optimal controller can be expressed, either. Simulation of optimal controller u_k^c with $d=10$ is shown in Fig. 4 (a). Moreover, it can be seen that system (1)-(2) is mean-square stabilizable with $d=10$ from Fig. 4 (b).



(a) optimal controller (b) stabilization

Fig. 4. Simulation for d=10.

Compared with the state augmentation approach which needs to calculate $(d+2)$-dimensional matrices, only 2-dimensional matrices needs to be computed by the proposed algorithm for any larger delay. In view of this, the proposed algorithm avoids the calculation of high-dimensional matrices.

5 Conclusion

In this paper, the optimal LQ control problem for NCSs simultaneously with input delay and Markovian dropout is discussed. Compared with the literature results, we mainly consider the hold-input strategy, which is much more computationally complicated than zero-input strategy. Necessary and sufficient conditions for the solvability of optimal control problem over a finite horizon are presented by the CDREs. Moreover, the NCS is mean-square stability if and only if the CAREs have a particular solution. The key technique in this paper is to tackle the FBSDEs, which are more difficult to be dealt with, due to the adaptability of the controller and the temporal correlation caused by simultaneous input delay and Markovian jump.

A Proof of Theorem 7

Proof. ($\Longrightarrow$) $\Gamma_{\theta_{k-d-1}}>0$ will be proved by mathematical induction. Denote

$$\tilde{J}_k=\mathbb{E}\left\{\sum_{i=k}^{N}(z_i'Qz_i+(u_{i-d}^c)'Ru_{i-d}^c)+z_{N+1}'\bar{P}_{N+1}z_{N+1}\right\}. \quad (49)$$

Let $k=N$ in (49), according to Assumption 11, we know that $\tilde{J}_N\geq 0$ for any z_N, u_{N-d}^c. For considering $\Gamma_{\theta_{k-d-1}}$, let $z_N=0$. We have

$$\begin{aligned}\tilde{J}_N&=\mathbb{E}\big[(u_{N-d}^c)'(R+\bar{B}_{\theta_N}'\bar{P}_{N+1}\bar{B}_{\theta_N})u_{N-d}^c\big]\\ &=\mathbb{E}\big[(u_{N-d}^c)'\Gamma_{\theta_{N-d-1}}u_{N-d}^c\big].\end{aligned} \quad (50)$$

Obviously, $u_{N-d}^c=0$ can minimize $\tilde{J}_N$, i.e., $\tilde{J}_N=0$. Thus, due to the uniqueness of the solution to Problem 2, we have $\tilde{J}_N>0$ for any nonzero u_{N-d}^c. Therefore, from (50) and the arbitrariness of nonzero u_{N-d}^c, $\Gamma_{\theta_{N-d-1}}>0$ is derived. In this case, it follows from (7) that u_{N-d}^c and λ_{N-1} can be calculated as follows

$$\begin{aligned}0=&\mathbb{E}_{N-d-1}\left[\bar{B}_{\theta_N}'\bar{P}_{N+1}(\bar{A}_{\theta_N}z_N+\bar{B}_{\theta_N}u_{N-d}^c)+Ru_{N-d}^c)\right]\\ =&\mathbb{E}_{N-d-1}\left[\bar{B}_{\theta_N}'\bar{P}_{N+1}\bar{A}_{\theta_N}z_N+(R+\bar{B}_{\theta_N}'\bar{P}_{N+1}\bar{B}_{\theta_N})u_{N-d}^c\right]\\ =&\mathbb{E}_{N-d-1}\left[(\tilde{S}_{\theta_{N-1}}^1)'\prod_{j=1}^{d}\bar{A}_{\theta_{N-j}}\right]z_{N-d}+\sum_{i=1}^{d}\mathbb{E}_{N-d-1}\Bigg[(\tilde{S}_{\theta_{N-1}}^1)'\\ &\times\prod_{j=1}^{i-1}\bar{A}_{\theta_{N-j}}\bar{B}_{\theta_{N-i}}\Bigg]u_{N-d-i}^c+\Gamma_{\theta_{N-d-1}}u_{N-d}^c,\end{aligned} \quad (51)$$

and

$$\begin{aligned}\lambda_{N-1}=&\mathbb{E}_{N-1}\left[Q+\bar{A}_{\theta_N}'\bar{P}_{N+1}\bar{A}_{\theta_N}\right]z_N+\tilde{S}_{\theta_{N-1}}^1u_{N-d}^c\\ =&\bar{P}_{\theta_{N-1}}z_N-\tilde{S}_{\theta_{N-1}}^1\Gamma_{\theta_{N-d-1}}^{-1}\Bigg(M_{\theta_{N-d-1}}^0z_{N-d}\\ &+\sum_{i=1}^{d}M_{\theta_{N-d-1}}^iu_{N-d-i}^c\Bigg),\end{aligned} \quad (52)$$

which hold for (23) and (25) in case of $k=N$. Taking $d\leq l\leq N$, assume that $\Gamma_{\theta_{k-d-1}}(k)>0$, we have (23) and (25) for $k\geq l+1$. Finally, we prove $\Gamma_{\theta_{l-d-1}}(l)>0$. From (7), we have

$$\mathbb{E}\big[z_k'\lambda_{k-1} - z_{k+1}'\lambda_k\big]{=}\mathbb{E}\big[z_k'Qz_k + (u_{k-d}^c)'Ru_{k-d}^c\big]. \tag{53}$$

Adding from $k = l+1$ to $k = N$ on both sides of (53), and taking $z_l = 0$, we have

$$\begin{aligned}\tilde{J}_l{=}&\mathbb{E}\Bigg\{(u_{l-d}^c)'Ru_{l-d}^c{+}(u_{l-d}^c)'\bar{B}_{\theta_l}'\Bigg[\bar{P}_{\theta_l}\bar{B}_{\theta_l}u_{l-d}^c\\ &-\sum_{s=0}^{d}\left(F_{\theta_l}^{s+1}\Gamma_{\theta_{l-d+s}}^{-1}M_{\theta_{l-d+s}}^{0}\prod_{i=0}^{s-1}\bar{A}_{\theta_{l-d+i+1}}\right)z_{l-d+1}\\ &-\sum_{i=1}^{d}\left(\sum_{s=0}^{d}F_{\theta_l}^{s+1}\Gamma_{\theta_{l-d+s}}^{-1}M_{\theta_{l-d+s}}^{i+s}\right)u_{l-d-i+1}^c\Bigg]\Bigg\}\\ {=}&\mathbb{E}\Bigg\{(u_{l-d}^c)'\mathbb{E}_{l-d-1}\Bigg[R{+}\bar{B}_{\theta_l}'\bar{P}_{\theta_l}\bar{B}_{\theta_l}{-}\sum_{s=0}^{d}(M_{\theta_{l-d+s}}^{s+1})'\\ &\times\Gamma_{\theta_{l-d+s}}^{-1}M_{\theta_{l-d+s}}^{s+1}\Bigg]u_{l-d}^c\Bigg\}\\ {=}&\mathbb{E}\Big[(u_{l-d}^c)'\Gamma_{\theta_{l-d-1}}u_{l-d}^c\Big].\end{aligned} \tag{54}$$

Due to the uniqueness of the optimal control, for any nonzero $u(l-d)$, we have $\Gamma_{\theta_{l-d-1}} > 0$.
It follows from (7) that we have

$$\begin{aligned}0{=}&\mathbb{E}_{l-d-1}\Bigg\{\bar{B}_{\theta_l}'\Bigg[\bar{P}_{\theta_l}(\bar{A}_{\theta_l}z_l{+}\bar{B}_{\theta_l}u_{l-d}^c){-}\sum_{s=0}^{d}\Big(F_{\theta_l}^{s+1}\Gamma_{\theta_{l-d+s}}^{-1}\\ &\times M_{\theta_{l-d+s}}^{0}\prod_{i=0}^{s-1}\bar{A}_{\theta_{l-d+i+1}}\Big)(\bar{A}_{\theta_{l-d}}z_{l-d}{+}\bar{B}_{\theta_{l-d}}u_{l-2d}^c)\\ &-\sum_{i=1}^{d}\left(\sum_{s=0}^{d}F_{\theta_l}^{s+1}\Gamma_{\theta_{l-d+s}}^{-1}M_{\theta_{l-d+s}}^{i+s}\right)u_{l-d-i+1}^c\Bigg]{+}Ru_{l-d}^c\Bigg\}\\ {=}&\mathbb{E}_{l-d-1}\Bigg\{\Gamma_{\theta_{l-d-1}}u_{l-d}^c{+}\Gamma_{\theta_{l-d-1}}u_{l-d}^c{+}(\tilde{S}_{\theta_l}^1)'z_l\\ &-\sum_{s=0}^{d}\Bigg[(M_{\theta_{l-d+s}}^{s+1})'\Gamma_{\theta_{l-d+s}}^{-1}M_{\theta_{l-d+s}}^{0}\prod_{i=0}^{s}\bar{A}_{\theta_{l-d+i}}\Bigg]z_{l-d}\\ &-\sum_{i=1}^{d}\left(\sum_{s=0}^{d}(M_{\theta_{l-d+s}}^{s+1})'\Gamma_{\theta_{l-d+s}}^{-1}M_{\theta_{l-d+s}}^{i+s+1}\right)u_{l-d-i}^c\Bigg\}\\ {=}&\Gamma_{\theta_{l-d-1}}u_{l-d}^c{+}\mathbb{E}_{l-d-1}\Bigg[(\tilde{S}_{\theta_{l-1}}^1)'\prod_{j=1}^{d}\bar{A}_{\theta_{l-j}}{-}\sum_{i=0}^{d}\Bigg((M_{\theta_{l-d+i}}^{i+1})'\\ &\times\Gamma_{\theta_{l-d+i}}^{-1}M_{\theta_{l-d+i}}^{0}\prod_{s=0}^{i}\bar{A}_{\theta_{l-d+s}}\Bigg)\Bigg]z_{l-d}{-}\mathbb{E}_{l-d-1}\Bigg[(\tilde{S}_{\theta_{l-1}}^1)'\\ &\times\prod_{j=1}^{i-1}\bar{A}_{\theta_{l-j}}\bar{B}_{\theta_{l-i}}{-}\sum_{s=0}^{d}\Big((M_{\theta_{l-d+s}}^{s+1})'\Gamma_{\theta_{l-d+s}}^{-1}M_{\theta_{l-d+s}}^{i+s+1}\Big)\Bigg]u_{l-d-i}^c,\end{aligned} \tag{55}$$

and

$$\begin{aligned}\lambda_{l-1}{=}&\mathbb{E}_{l-1}\Bigg\{Qz_l{+}\bar{A}_{\theta_l}'\Bigg[\bar{P}_{\theta_l}(\bar{A}_{\theta_l}z_l{+}\bar{B}_{\theta_l}u_{l-d}^c){-}\sum_{s=0}^{d}\Bigg(F_{\theta_l}^{s+1}\Gamma_{\theta_{l-d+s}}^{-1}\\ &\times M_{\theta_{l-d+s}}^{0}\prod_{i=0}^{s-1}\bar{A}_{\theta_{l-d+i+1}}\Bigg)(\bar{A}_{\theta_{l-d}}z_{l-d}{+}\bar{B}_{\theta_{l-d}}u_{l-2d}^c)\\ &-\sum_{i=1}^{d}\left(\sum_{s=0}^{d}F_{\theta_l}^{s+1}\Gamma_{\theta_{l-d+s}}^{-1}M_{\theta_{l-d+s}}^{i+s}\right)u_{l-d-i+1}^c\Bigg]\Bigg\}\\ {=}&\bar{P}_{\theta_{l-1}}z_l{+}\Bigg\{\tilde{S}_{\theta_{l-1}}^1{-}\sum_{s=0}^{d}\Big[\mathbb{E}_{l-1}\bar{A}_{\theta_l}'F_{\theta_l}^{s+1}\Gamma_{\theta_{l-d+s}}^{-1}M_{\theta_{l-d+s}}^{s+1}\Big]\Bigg\}u_{l-d}^c\\ &-\mathbb{E}_{l-1}\Bigg\{\sum_{s=0}^{d}\Bigg(\bar{A}_{\theta_l}'F_{\theta_l}^{s+1}\Gamma_{\theta_{l-d+s}}^{-1}M_{\theta_{l-d+s}}^{0}\prod_{i=0}^{s}\bar{A}_{\theta_{l-d+i}}\Bigg)z_{l-d}\\ &+\sum_{i=1}^{d}\left(\sum_{s=0}^{d}\bar{A}_{\theta_l}'F_{\theta_l}^{s+1}\Gamma_{\theta_{l-d+s}}^{-1}M_{\theta_{l-d+s}}^{i+s+1}\right)u_{l-d-i}^c\Bigg\}\\ {=}&\bar{P}_{\theta_{l-1}}z_l{+}\Bigg[\tilde{S}_{\theta_{l-1}}^1{-}\sum_{s=0}^{d}\Big(F_{\theta_{l-1}}^{s+2}\Gamma_{\theta_{l-d+s}}^{-1}M_{\theta_{l-d+s}}^{s+1}\Big)\Bigg]\\ &-\Bigg\{\sum_{s=0}^{d}\Big(F_{\theta_{l-1}}^{s+2}\Gamma_{\theta_{l-d+s}}^{-1}M_{\theta_{l-d+s}}^{0}\prod_{i=0}^{s}\bar{A}_{\theta_{l-d+i}}\Big)z_{l-d}\\ &+\sum_{i=1}^{d}\left(\sum_{s=0}^{d}F_{\theta_{l-1}}^{s+2}\Gamma_{\theta_{l-d+s}}^{-1}M_{\theta_{l-d+s}}^{i+s+1}\right)u_{l-d-i}^c\Bigg]\Bigg\},\end{aligned} \tag{56}$$

i.e., u_{l-d}^c and λ_{l-1} are as (23) and (25) with $k = l$, respectively.
($\Longleftarrow$) When $\Gamma_{\theta_{k-d-1}}(k) > 0$, we will investigate the unique solvability of Problem 2. Define

$$\begin{aligned}&V_N(k)\\ {=}&\mathbb{E}\Bigg\{z_k'\bar{P}_{\theta_{k-1}}z_k{-}z_k'\sum_{s=0}^{d}\Big(F_{\theta_{k-1}}^{s+1}\Gamma_{\theta_{k-d-1+s}}^{-1}M_{\theta_{k-d-1+s}}^{0}\prod_{i=0}^{s-1}\bar{A}_{\theta_{k-d+i}}\Big)\\ &\times z_{k-d}{-}z_k'\sum_{i=1}^{d}\Big(\sum_{s=0}^{d}F_{\theta_{k-1}}^{s+1}\Gamma_{\theta_{k-d-1+s}}^{-1}M_{\theta_{k-d-1+s}}^{i+s}\Big)u_{k-d-i}^c\Bigg\}.\end{aligned} \tag{57}$$

From Remark 4, we have

$$\begin{aligned}&V_N(k) - V_N(k+1)\\ {=}&\mathbb{E}\Bigg\{z_k'[\bar{P}_{\theta_{k-1}}{-}\bar{A}_{\theta_k}'\bar{P}_{\theta_k}\bar{A}_{\theta_k}]z_k{-}z_k'\sum_{s=0}^{d}\Bigg[\bar{A}_{\theta_k}'\bar{P}_{\theta_k}\bar{B}_{\theta_k}{-}F_{\theta_{k-1}}^{s+2}\\ &\times\Gamma_{\theta_{k-d+s}}^{-1}M_{\theta_{k-d+s}}^{s+1}\Bigg]u_{k-d}^c\;{-}(u_{k-d}^c)'\bar{B}_{\theta_k}'\bar{P}_{\theta_k}\bar{A}_{\theta_k}z_k\\ &-z_k'\Bigg[\sum_{s=0}^{d}(F_{\theta_{k-1}}^{s+1}\Gamma_{\theta_{k-d-1+s}}^{-1}M_{\theta_{k-d-1+s}}^{0}\prod_{i=0}^{s-1}\bar{A}_{\theta_{k-d+i}})\\ &-\sum_{s=0}^{d}(F_{\theta_{k-1}}^{s+2}\Gamma_{\theta_{k-d+s}}^{-1}M_{\theta_{k-d+s}}^{0}\prod_{i=0}^{s}\bar{A}_{\theta_{k-d+i}})\Bigg]z_{k-d}{-}(u_{k-d}^c)'\\ &\times\Bigg[\bar{B}_{\theta_k}'\bar{P}_{\theta_k}\bar{B}_{\theta_k}{-}\sum_{s=0}^{d}(M_{\theta_{k-d+s}}^{s+1})'\Gamma_{\theta_{k-d+s}}^{-1}M_{\theta_{k-d+s}}^{s+1}\Bigg]u_{k-d}^c\end{aligned}$$

(58)

$$
\begin{aligned}
&+(u^c_{k-d})'\sum_{s=0}^{d}\left((M^{s+1}_{\theta_{k-d+s}})'\Gamma^{-1}_{\theta_{k-d+s}}M^0_{\theta_{k-d+s}}\prod_{i=0}^{s}\bar{A}_{\theta_{k-d+i}}\right)z_{k-d}\\
&-z_k'\sum_{i=1}^{d}\left(F^{s+1}_{\theta_{k-1}}\Gamma^{-1}_{\theta_{k-d+s-1}}M^{i+s}_{\theta_{k-d+s-1}}-F^{s+2}_{\theta_{k-1}}\Gamma^{-1}_{\theta_{k-d+s}}M^{i+s+1}_{\theta_{k-d+s}}\right)u^c_{k-d-i}\\
&+(u^c_{k-d})'\sum_{i=1}^{d}\left(\sum_{s=0}^{d}(M^{s+1}_{\theta_{k-d+s}})'\Gamma^{-1}_{\theta_{k-d+s}}M^{i+s+1}_{\theta_{k-d+s}}\right)u^c_{k-d-i}\Bigg\}\\
=&\mathbb{E}\Bigg\{z_k'Qz_k+(u^c_{k-d})'Ru^c_{k-d}-(u^c_{k-d})'\Gamma_{\theta_{k-d-1}}u^c_{k-d}\\
&-(u^c_{k-d})'M^0_{\theta_{k-d-1}}z_{k-d}-(u^c_{k-d})'\sum_{i=1}^{d}M^i_{\theta_{k-d-1}}u^c_{k-d-i}\\
&-z_{k-d}'(M^0_{\theta_{k-d-1}})'u^c_{k-d}-z_{k-d}'(M^0_{\theta_{k-d-1}})'\Gamma^{-1}_{\theta_{k-d-1}}M^0_{\theta_{k-d-1}}z_{k-d}\\
&-z_{k-d}'\sum_{i=1}^{d}(M^0_{\theta_{k-d-1}})'\Gamma^{-1}_{\theta_{k-d-1}}M^i_{\theta_{k-d-1}}u^c_{k-d-i}-\sum_{i=1}^{d}(u^c_{k-d-i})'\\
&\times(M^i_{\theta_{k-d-1}})'u^c_{k-d}-\sum_{i=1}^{d}(u^c_{k-d-i})'(M^i_{\theta_{k-d-1}})'\Gamma^{-1}_{\theta_{k-d-1}}\\
&\times\sum_{i=1}^{d}M^i_{\theta_{k-d-1}}u^c_{k-d-i}\Bigg\}\\
=&\mathbb{E}\Bigg\{z_k'Qz_k+(u^c_{k-d})'Ru^c_{k-d}\Big(u^c_{k-d}+\Gamma^{-1}_{\theta_{k-d}}M^0_{\theta_{k-d-1}}z_{k-d}\\
&+\Gamma^{-1}_{\theta_{k-d-1}}\sum_{i=1}^{d}M^i_{\theta_{k-d-1}}u^c_{k-d-i}\Big)'\Gamma_{\theta_{k-d-1}}\Big(u^c_{k-d}+\Gamma^{-1}_{\theta_{k-d-1}}\\
&\times M^0_{\theta_{k-d-1}}z_{k-d}+\Gamma^{-1}_{\theta_{k-d-1}}\sum_{i=1}^{d}M^i_{\theta_{k-d-1}}u^c_{k-d-i}\Big)\Bigg\}. \qquad (59)
\end{aligned}
$$

Summing up from $k=d$ to $k=N$ on both sides of (59), and in view of $\Gamma_{\theta_{k-d-1}}>0$ for $k\geq d$, the optimal controller and the optimal cost functional can be obtained as (23) and (24), respectively. The proof is completed.

B Proof of Theorem 13

Proof. ($\Longrightarrow$) The existence of solution to CAREs (31)-(36) will be shown. Consider the following delay-free MJLS:

$$
Y_{k+1}=C_{\theta_k}Y_k+Du^c_k, \qquad (60)
$$

where

$$
Y_k=\begin{bmatrix} z_k\\ u^c_{k-1}\\ \vdots\\ u^c_{k-d}\end{bmatrix}, C_{\theta_k}=\begin{bmatrix}\bar{A}_{\theta_k} & 0 & \cdots & 0 & \bar{B}_{\theta_k}\\ 0 & 0 & \cdots & 0 & 0\\ 0 & I & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & I & 0\end{bmatrix}, D=\begin{bmatrix}0\\ I\\ 0\\ \vdots\\ 0\end{bmatrix}.
$$

Also, the cost functional over an infinite horizon is as follows

$$
\mathcal{J}=\sum_{k=0}^{\infty}\mathbb{E}[Y_k'\mathcal{Q}Y_k+(u^c_k)'Ru^c_k], \qquad (61)
$$

where $\mathcal{Q}=\begin{bmatrix}Q & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0\end{bmatrix}$. The corresponding cost functional over a finite horizon is

$$
\mathcal{J}_N=\mathbb{E}\left\{\sum_{k=0}^{N}[Y_k'\mathcal{Q}Y_k+(u^c_k)'Ru^c_k]+Y_{N+1}'\mathcal{P}_{N+1}Y_{N+1}\right\}. \qquad (62)
$$

By maximum principle, the following forward and backward difference equations can be given as

$$
\begin{cases}0=\mathbb{E}_{k-1}[Ru^c_k+D'\xi_k],\\ \xi_{k-1}=\mathbb{E}_{k-1}[\mathcal{Q}Y_k+C_{\theta_k}\xi_k],\\ \xi_N=\mathcal{P}_{N+1}Y_{N+1}.\end{cases} \qquad (63)
$$

It follows from Theorem 7 that we have
(1) the following recursive sequence

$$
\mathcal{P}^{(N)}_{N+1}=\bar{P}^{(N)}_{N+1}, \qquad (64)
$$
$$
\mathcal{P}^{(N)}_{\theta_{k-1}}=\mathbb{E}_{k-1}[\mathcal{Q}+C_{\theta_k}'\mathcal{P}^{(N)}_{\theta_k}C_{\theta_k}-(\mathcal{M}^{(N)}_{\theta_{k-1}})'(\Upsilon^{(N)}_{\theta_{k-1}})^{-1}\mathcal{M}^{(N)}_{\theta_{k-1}}], \qquad (65)
$$

in which

$$
\begin{aligned}
\mathcal{M}^{(N)}_{\theta_{k-1}}=&\mathbb{E}_{k-1}[D'\mathcal{P}^{(N)}_{\theta_k}C_{\theta_k}]\\
=&\mathbb{E}_{k-1}[(\mathcal{P}^{(N)}_{\theta_k}(2,1)\bar{A}_{\theta_k},\mathcal{P}^{(N)}_{\theta_k}(2,3),\cdots,\mathcal{P}^{(N)}_{\theta_k}(2,d+1),\\
&\mathcal{P}^{(N)}_{\theta_k}(2,1)\bar{B}_{\theta_k})], \qquad (66)\\
\Upsilon^{(N)}_{\theta_{k-1}}=&\mathbb{E}_{k-1}[R+D'\mathcal{P}^{(N)}_{\theta_{k-1}}D]=\mathbb{E}_{k-1}[R+\mathcal{P}^{(N)}_{\theta_{k-1}}(2,2)]; \qquad (67)
\end{aligned}
$$

(2) the costate

$$
\xi_{k-1}=\mathcal{P}^{(N)}_{\theta_{k-1}}Y_k; \qquad (68)
$$

(3) the optimal control

$$
\begin{aligned}
u^c_k=&-(\Upsilon^{(N)}_{\theta_{k-1}})^{-1}\mathcal{M}^{(N)}_{\theta_{k-1}}Y_k\\
=&-(R+\mathcal{P}^{(N)}_{\theta_{k-1}}(2,2))^{-1}\big[(\mathcal{P}^{(N)}_{\theta_k}(2,1)\bar{A}_{\theta_k})z_k+\mathcal{P}^{(N)}_{\theta_k}(2,3)\\
&\times u^c_{k-1}+\cdots+\mathcal{P}^{(N)}_{\theta_k}(2,d+1)u^c_{k-d+1}+\mathcal{P}^{(N)}_{\theta_k}(2,1)\bar{B}_{\theta_k}u^c_{k-d}\big]\\
=&-\Gamma^{-1}_{\theta_{k-1}}(k,N)M^0_{\theta_{k-1}}(k,N)z_k-\Gamma^{-1}_{\theta_{k-1}}(k,N)\\
&\times\sum_{i=1}^{d}M^i_{\theta_{k-1}}(k,N)u^c_{k-i}, \qquad (69)
\end{aligned}
$$

where $\mathcal{P}^{(N)}_{\theta_k}(i,j)$ represents block matrix with suitable dimension.

The following relationship can be obtained in view of (23) and (69).

$$\begin{cases} \quad (R+\mathcal{P}^{(N)}_{\theta_{k-1}}(2,2))^{-1}\mathbb{E}_{k-1}[\mathcal{P}^{(N)}_{\theta_k}(2,1)\bar{A}_{\theta_k}] \\ = \Gamma^{-1}_{\theta_{k-1}}(k,N)M^0_{\theta_{k-1}}(k,N) \\ \quad (R+\mathcal{P}^{(N)}_{\theta_{k-1}}(2,2))^{-1}\mathbb{E}_{k-1}[\mathcal{P}^{(N)}_{\theta_k}(2,3)] \\ = \Gamma^{-1}_{\theta_{k-1}}(k,N)M^1_{\theta_{k-1}}(k,N) \\ \vdots \\ \quad (R+\mathcal{P}^{(N)}_{\theta_{k-1}}(2,2))^{-1}\mathbb{E}_{k-1}[\mathcal{P}^{(N)}_{\theta_k}(2,d+1)] \\ = \Gamma^{-1}_{\theta_{k-1}}(k,N)M^{d-1}_{\theta_{k-1}}(k,N) \\ \quad (R+\mathcal{P}^{(N)}_{\theta_{k-1}}(2,2))^{-1}\mathbb{E}_{k-1}[\mathcal{P}^{(N)}_{\theta_k}(2,1)\bar{B}_{\theta_k}] \\ = \Gamma^{-1}_{\theta_{k-1}}(k,N)M^d_{\theta_{k-1}}(k,N). \end{cases}$$

The convergence of $\mathcal{P}^{(N)}_{\theta_{k-1}}$ can be obtained in a similar manner with [1]. On this basis, from (70), $\mathcal{P}^{(N)}_{\theta_{k-1}}$ $\Gamma^{-1}_{\theta_{k-1}}(k,N)M^j_{\theta_{k-1}}(k,N), j=0,1,\cdots,d$ are convergent. Let $\xi_{k-1} = \begin{bmatrix} \xi^0_{k-1} & \xi^1_{k-1} & \cdots & \xi^d_{k-1} \end{bmatrix}'$, and from (68) we know

$$\xi^0_{k-1} = \mathcal{P}^{(N)}_{\theta_{k-1}}(1,1)z_k + \mathcal{P}^{(N)}_{\theta_{k-1}}(1,2)u^c_{k-1} + \cdots \\ +\mathcal{P}^{(N)}_{\theta_{k-1}}(1,d+1)u^c_{k-d}. \tag{70}$$

Further, from (63), we have

$$\xi^0_{k-1} = \mathbb{E}_{k-1}[Qz_k + \bar{A}_{\theta_k}\xi^0_k], \tag{71}$$

comparing with (4) and (71), it is obvious that, if $\xi^0_N = \bar{P}_{N+1}$, the following relationship holds

$$\xi^0_{k-1} = \lambda_{k-1}. \tag{72}$$

Consider (23), (25), (70) and (72), and we can find the following relationship using a direct calculation

$$\mathcal{P}^{(N)}_{\theta_{k-1}}(1,1) = \bar{P}_{\theta_{k-1}}. \tag{73}$$

Therefore, we have

$$\lim_{N\to\infty} \bar{P}_{\theta_{k-1}}(k,N) \triangleq \bar{P}_{l_{d-1}}, \tag{74}$$

where $\theta_{k-1} = l_{d-1}, k \geq d, l_{d-1} = 0,1$. In view of (8), we know that $(M^0_{\theta_{k-1}})'\Gamma^{-1}_{\theta_{k-1}}M^0_{\theta_{k-1}}$ is convergent.
On this basis, it is esay to verify that $\Gamma_{\theta_{k-1}}(k,N)$, $M^i_{\theta_{k-1}}(k,N), i = 0,\cdots,d$, $F^{d-j+1}_{\theta_{k-j-1}}(k-j,N)$ and $\tilde{S}^j_{\theta_{k-1}}(k,N), j = 1,\cdots,d$ are also convergent and CAREs (31)-(36) have a solution.
(2) Now, we will prove inequality (38).
Let $\hat{J}_d(m)$ represent cost functional (6) which starts at d and ends at $m, m \geq N$. In view of Lemma 6, we can obtain the optimal cost value $\hat{J}^*_d(m)$ as follows

$$\begin{aligned} &\hat{J}^*_d(m) \\ =&\mathbb{E}\Big\{z'_d\bar{P}_{\theta_{d-1}}(d,m)z_d - z'_d\sum_{s=0}^{d-1}[F^{s+1}_{\theta_{d-1}}(d,m)\Gamma^{-1}_{\theta_{s-1}}(d+s,m) \\ &\times M^0_{\theta_{s-1}}(d+s)z_s] - z'_d\sum_{s=0}^{d-1}\Big[F^{s+1}_{\theta_{d-1}}(d,m)\Gamma^{-1}_{\theta_{s-1}}(d+s,m) \\ &\times\sum_{i=s+1}^{d}M^i_{\theta_{s-1}}(d+s)u^c_{s-i}\Big]\Big\} \\ =&\mathbb{E}\Big\{z'_d\Big[\bar{P}_{\theta_{d-1}}(d,m) - \sum_{s=0}^{d-1}\Big[F^{s+1}_{\theta_{d-1}}(d,m)\Gamma^{-1}_{\theta_{s-1}}(d+s,m) \\ &\times(F^{s+1}_{\theta_{d-1}}(d,m))'\Big]\Big]z_d\Big\} \\ =&z'_d\Big[\bar{P}_{\theta_{d-1}}(d,m) - \sum_{s=0}^{d-1}\Big[F^{s+1}_{\theta_{d-1}}(d,m)\Gamma^{-1}_{\theta_{s-1}}(d+s,m) \\ &\times(F^{s+1}_{\theta_{d-1}}(d,m))'\Big]\Big]z_d \\ \geq&0. \end{aligned} \tag{75}$$

Since z_d is arbitrary, we have

$$\bar{P}_{\theta_{d-1}}(d,m) - \sum_{s=0}^{d-1}\Big[F^{s+1}_{\theta_{d-1}}(d,m)\Gamma^{-1}_{\theta_{s-1}}(d+s,m) \\ \times(F^{s+1}_{\theta_{d-1}}(d,m))'\Big] \geq 0. \tag{76}$$

For $k \geq d$, let $m = N-k+d$. In view of the time-variance, it yields that

$$\bar{P}_{\theta_{k-1}}(k,N) - \sum_{s=0}^{d-1}\Big[F^{s+1}_{\theta_{k-1}}(k,N)\Gamma^{-1}_{\theta_{k-d+s-1}}(k+s,N) \\ \times(F^{s+1}_{\theta_{k-1}}(k,N))'\Big] \geq 0. \tag{77}$$

By virtue of the convergence, it is easy to derive (38). From Lemma 3 in [19], we can find an integer G such that

$$\bar{P}_{\theta_d}(d,G) - \sum_{s=0}^{d-1}[F^{s+1}_{\theta_d}(d,G)\Gamma^{-1}_{\theta_{s-1}}(d+s,G) \\ \times(F^{s+1}_{\theta_d}(d,G))'] > 0. \tag{78}$$

Also, the monotonicity with respect ro N of (77) deduces that

$$
\begin{aligned}
&\bar{P}_{l_d}-\sum_{s=0}^{d-1}[F_{l_d}^{s+1}\Gamma_{l_{s-1}}^{-1}(F_{l_d}^{s+1})']\\
=&\lim_{N\to\infty}\bigg[\bar{P}_{\theta_d}(d,N)-\sum_{s=0}^{d-1}\Big(F_{\theta_d}^{s+1}(d,N)\Gamma_{\theta_{s-1}}^{-1}(d+s,N)\\
&\times(F_{\theta_d}^{s+1}(d,N))'\Big)\bigg]\\
\geq&\bar{P}_{\theta_d}(d,G)-\sum_{s=0}^{d-1}\Big[F_{\theta_d}^{s+1}(d,G)\Gamma_{\theta_{s-1}}^{-1}(d+s,G)\\
&\times(F_{\theta_d}^{s+1}(d,G))'\Big]>0.
\end{aligned}
$$

The proof is completed.

($\Longleftarrow$) The mean-square stabilization of system (1) will be illustrated. Define

$$
\begin{aligned}
\mathcal{L}(k)=&\mathbb{E}\bigg\{z_k'\bar{P}_{\theta_{k-1}}z_k-z_k'\sum_{s=0}^{d-1}\Big(F_{\theta_{k-1}}^{s+1}\Gamma_{\theta_{k-d-1+s}}^{-1}M_{\theta_{k-d-1+s}}^{0}z_{k-d+s}\Big)\\
&-z_k'\sum_{s=0}^{d-1}\Big(F_{\theta_{k-1}}^{s+1}\Gamma_{\theta_{k-d-1+s}}^{-1}\sum_{i=s+1}^{d}M_{\theta_{k-d-1+s}}^{i}u_{k-d-i+s}^{c}\Big)\bigg\}. \quad (79)
\end{aligned}
$$

In view of (55), we have

$$
\begin{aligned}
&\mathcal{L}(k)-\mathcal{L}(k+1)\\
=&\mathbb{E}\bigg\{z_k'Qz_k+(u_{k-d}^c)'Ru_{k-d}^c-\Big(u_{k-d}^c+\Gamma_i^{-1}M_i^0z_{k-d}+\Gamma_i^{-1}\\
&\times\sum_{s=1}^{d}(M_i^su_{k-d-s}^c)\Big)'\Gamma_i\Big(u_{k-d}^c+\Gamma_i^{-1}M_i^0z_{k-d}\\
&+\Gamma_i^{-1}\sum_{s=1}^{d}(M_i^su_{k-d-s}^c)\Big)\bigg\} \quad (80)\\
=&\mathbb{E}\Big[z_k'Qz_k+(u_{k-d}^c)'Ru_{k-d}^c\Big]\geq 0,\quad k\geq d, \quad (81)
\end{aligned}
$$

with $u_{k-d}^c=-\Gamma_i^{-1}\Big(M_i^0z_{k-d}+\sum_{s=1}^{d}M_i^su_{k-d-s}^c\Big)$. Therefore, $\mathcal{L}(k)$ decreases with respect to k. From (19)-(18), $\mathcal{L}(k)$ can be expressed as

$$
\begin{aligned}
\mathcal{L}(k)=&\mathbb{E}\bigg\{z_k'\bigg[\bar{P}_{\theta_{k-1}}-\sum_{s=0}^{d-1}\Big(F_{\theta_{k-1}}^{s+1}\Gamma_{\theta_{k-d+s-1}}^{-1}(F_{\theta_{k-1}}^{s+1})'\Big)\bigg]z_k\\
&+\sum_{s=0}^{d-1}\Big(F_{\theta_{k-1}}^{s+1}z_k-\mathbb{E}[F_{\theta_{k-1}}^{s+1}z_k|\mathcal{F}_{k-s-1}]\Big)'\Gamma_{\theta_{k-s-1}}^{-1}\\
&\times\Big(F_{\theta_{k-1}}^{s+1}z_k-\mathbb{E}[F_{\theta_{k-1}}^{s+1}z_k|\mathcal{F}_{k-s-1}]\Big)\bigg\}\\
\geq&\mathbb{E}\bigg\{z_k'\bigg[\bar{P}_{\theta_{k-1}}-\sum_{s=0}^{d-1}[F_{\theta_{k-1}}^{s+1}\Gamma_{\theta_{k-d+s-1}}^{-1}(F_{\theta_{k-1}}^{s+1})']\bigg]z_k\bigg\}\\
\geq&0,\quad k\geq d, \quad (82)
\end{aligned}
$$

i.e., $\mathcal{L}(k)$ is bounded. Therefore, $\mathcal{L}(k)$ is convergent.

For any $l\geq 0$, summing up from $k=l+d$ to $k=l+N$ on both sides of (81), when $l\to\infty$, we can derive that

$$
\begin{aligned}
&\lim_{l\to\infty}\sum_{k=l+d}^{l+N}\mathbb{E}[z_k'Qz_k+(u_{k-d}^c)'Ru_{k-d}^c]\\
=&\lim_{l\to\infty}[\mathcal{L}(l+d)-\mathcal{L}(l+N+1)]=0. \quad (83)
\end{aligned}
$$

Recall that

$$
\begin{aligned}
&\sum_{k=d}^{N}\mathbb{E}[z_k'Qz_k+(u_{k-d}^c)'Ru_{k-d}^c]\\
\geq&\mathbb{E}\bigg\{z_d'\bar{P}_{\theta_{d-1}}z_d-z_d'\sum_{s=0}^{d-1}\Big(F_{\theta_{d-1}}^{s+1}\Gamma_{\theta_{s-1}}^{-1}M_{\theta_{s-1}}^{0}z_s\Big)\\
&-z_d'\sum_{s=0}^{d}\Big(F_{\theta_{d-1}}^{s+1}\Gamma_{\theta_{s-1}}^{-1}\sum_{i=s+1}^{d}M_{\theta_{s-1}}^{i}u_{s-i}^c\Big)\bigg\}.
\end{aligned}
$$

Therefore, the following relationship can be deduced that

$$
\begin{aligned}
&\sum_{k=l+d}^{l+N}\mathbb{E}\Big[z_k'Qz_k+(u_{k-d}^c)'Ru_{k-d}^c\Big]\\
\geq&\mathbb{E}\bigg\{z_{l+d}'\bigg[\bar{P}_{\theta_{l+d-1}}(l+d,l+N)-\sum_{s=0}^{d-1}[F_{\theta_{l+d-1}}^{s+1}(l+d,l+N)\\
&\times\Gamma_{\theta_{l+s-1}}^{-1}(l+s,l+N)(F_{\theta_{l+d-1}}^{s+1}(l+d,l+N))']\bigg]z_{l+d}\bigg\}\\
=&\mathbb{E}\bigg\{z_{l+d}'\bigg[\bar{P}_{\theta_{d-1}}(d,N)-\sum_{s=0}^{d-1}[F_{\theta_{d-1}}^{s+1}(d,N)\Gamma_{\theta_{s-1}}^{-1}(s,N)\\
&\times(F_{\theta_{d-1}}^{s+1}(d,N))']\bigg]z_{l+d}\bigg\}\\
\geq&0.
\end{aligned}
$$

Using (83), we have

$$
\begin{aligned}
&\lim_{l\to\infty}\mathbb{E}\bigg\{z_{l+d}'\bigg[\bar{P}_{\theta_{d-1}}(d,N)-\sum_{s=0}^{d-1}[F_{\theta_{d-1}}^{s+1}(d,N)\\
&\times\Gamma_{\theta_{s-1}}^{-1}(s,N)(F_{\theta_{d-1}}^{s+1}(d,N))']\bigg]z_{l+d}\bigg\}\\
=&0,\quad \forall N\geq d. \quad (84)
\end{aligned}
$$

From (78) and (84), it deduces that $\lim_{l\to\infty}\mathbb{E}[z_{l+d}'z_{l+d}]=0$. Therefore, system (6) is mean-square stabilizable with controller (41).

Finally, the optimal cost functional will be calculated.

Summing up from $k=0$ to $k=N$ on both sides of (80), it yields that

$$
\begin{aligned}
&\mathbb{E}\left\{\sum_{k=0}^{N}[z_k'Qz_k+(u_{k-d}^c)'Ru_{k-d}^c]\right\}\\
&=\mathcal{L}(0)-\mathcal{L}(N+1)+\sum_{k=0}^{N}\mathbb{E}\bigg\{\Big(u_{k-d}^c+\Gamma_i^{-1}M_i^0z_{k-d}\\
&\quad+\Gamma_i^{-1}\sum_{s=1}^{d}(M_i^su_{k-d-s}^c)\Big)'\Gamma_i\Big(u_{k-d}^c+\Gamma_i^{-1}M_i^0z_{k-d}\\
&\quad+\Gamma_i^{-1}\sum_{s=1}^{d}(M_i^su_{k-d-s}^c)\Big)\bigg\}. \qquad (85)
\end{aligned}
$$

Since $0 \leq \mathcal{L}(k) \leq \mathbb{E}[z_k\bar{P}_{\theta_{k-1}}z_k]$ and the system (4) is stabilized in the mean-square sense, we have $\lim\limits_{k\to\infty}\mathbb{E}[z_k\bar{P}_{\theta_{k-1}}z_k]=0$, i.e., $\lim\limits_{k\to\infty}\mathcal{L}(k)=0$.

Let $N\to\infty$ on both sides of (85), then

$$
\begin{aligned}
J=&\mathcal{L}(0)+\sum_{k=d}^{\infty}\mathbb{E}\bigg\{\Big(u_{k-d}^c+\Gamma_i^{-1}M_i^0z_{k-d}+\Gamma_i^{-1}\sum_{s=1}^{d}(M_i^su_{k-d-s}^c)\Big)'\\
&\times\Gamma_i\Big(u_{k-d}^c+\Gamma_i^{-1}M_i^0z_{k-d}+\Gamma_i^{-1}\sum_{s=1}^{d}(M_i^su_{k-d-s}^c)\Big)\bigg\}\\
&+\sum_{k=0}^{d-1}\mathbb{E}\bigg[\Big(u_{k-d}^c+\Gamma_i^{-1}M_i^0z_{k-d}+\Gamma_i^{-1}\sum_{s=1}^{d}(M_i^su_{k-d-s}^c)\Big)'\\
&\times\Gamma_i\Big(u_{k-d}^c+\Gamma_i^{-1}M_i^0z_{k-d}+\Gamma_i^{-1}\sum_{s=1}^{d}(M_i^su_{k-d-s}^c)\Big)\bigg]\bigg\}. \qquad (86)
\end{aligned}
$$

In view of the positive definiteness of Γ_i, in order to minimize (86), we take (41) as the optimal controller. Then, the corresponding optimal cost functional can be expressed as (42). The desired sufficiency is proved.

References

[1] O. Costa, M. Fragoso, and R. Marques (2005). Discrete-Time Markovian Jump Linear Systems. London, U.K.: Springer-Verlag.

[2] M. C. Delfour (1984). Linear optimal control of systems with state and control variable delays. *Automatica*, 20(1), 69–77.

[3] G. Guo, Z. Lu, and P. Shi (2014). Event-driven actuators: to zero or to hold?. *International Journal of Robust Nonlinear Control*, 24(17), 2761–2773.

[4] D. Hristu-Varsakelis (2008). Short-Period Communication and the Role of Zero-Order Holding in Networked Control Systems. *IEEE Transactions on Automatic Control*, 53(5), 1285–1290.

[5] O. C. Imer, S. Yuksel, and T. Basar (2006). Optimal control of LTI systems over unreliable communication links. *Automatica*, 42(9), 1429–1439.

[6] X. Li, and X. Zhou (2002). Indefinite stochastic LQ controls with Markovian jumps in a finite time horizon. *Communications in Information and Systems*, 2(3), 265–282.

[7] X. Li, X. Zhou, and M. A. Rami (2003). Indefinite stochastic linear quadratic control with Markovian jumps in infinite time horizon. *Journal of Global Optimization*, 27, 149–175.

[8] X. Liang, and J. Xu (2018). Control for networked control systems with remote and local controllers over unreliable communication channel. *Automatica*, 98, 86–94.

[9] G. Liu (2010). Predictive controller design of networked systems with communication delays and data loss. *IEEE Transactions on Circuits and Systems–II: Express Briefs*, 57(6), 481–485.

[10] Z. Lu, S. Zhong, and L. Qu (2019). A Switching Approach to Packet Loss Compensation Strategy. *IEEE Access*, 7, 5609–5615.

[11] L. Montestruque, and P. Antsaklis (2004). Stability of model-based networked control systems with time-varying transmission times. *IEEE Transactions on Automatic Control*, 49, 1562–1572.

[12] Y. Tian, H. Yan, et al (2020). Dynamic output-feedback control of linear semi-Markov jump systems with incomplete semi-Markov kernel. *Automatica*, 117, 108997.

[13] D. Wang, J. Wang, and W. Wang (2013). H_∞ controller design of networked control systems with Markov packet dropouts. *IEEE Transactions on Systems, Man and Cybernetics: Systems*, 43(3), 689–697.

[14] K. Watanabe, and M. Ito (1981). A process-model control for linear systems with delay. *IEEE Transactions on Automatic Control*, 26(6), 1261–1269.

[15] L. Xie, and L. Xie (2009). Stability analysis of networked sampled-data linear systems with markovian packet losses. *IEEE Transactions on Automatic Control*, 54(6), 1368–1374.

[16] M. Xue, H. Yan, et al (2020). Event-triggered Guaranteed Cost Controller Design for T-S Fuzzy Markovian Jump Systems with Partly Unknown Transition Probabilities. *IEEE Transactions on Fuzzy Systems*, 1-1, doi:10.1109/TFUZZ.2020.2968866.

[17] D. Yue, Q.-L. Han, and J. Lam (2005). Network-based robust H_∞ control of systems with uncertainty. *Automatica*, 41(6), 999–1007.

[18] X. Zhan, J. Wu, T. Jiang, and X. Jiang (2015). Optimal performance of networked control systems under the packet dropouts and channel noise. *ISA Transactions*, 58, 214–221.

[19] H. Zhang, L. Li, J. Xu, and M. Fu (2015). Linear quadratic regulation and stabilization of discrete-time systems with delay and multiplicative noise. *IEEE Transactions on Automatic Control*, 60(10), 2599–2613.

[20] H. Zhang, and J. Xu (2017). Control for Itô stochastic systems with input delay. *IEEE Transactions on Automatic Control*, 62(1), 350—365.