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OPTIMAL CONTROL OF FISH-FEEDING IN A THREE-DIMENSIONAL CALM FRESHWATER POND CONSIDERING ENVIRONMENTAL CONCERN

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Abstract. This paper describes the optimal fish-feeding in a three-dimensional calm freshwater pond based on the concentrations of seven water quality variables. A certain number of baby fishes are inserted into the pond simultaneously. They are then taken out of the pond simultaneously for harvest after having gone through a feeding program. This feeding program creates additional loads of water quality variables in the pond, which becomes pollutants. Thus, an optimal fish-feeding problem is formulated to maximize the final weight of the fishes, subject to the restrictions that the fishes are not under-fed and over-fed and the concentrations of the pollutants created by the fish-feeding program are not too large. A computational scheme using the finite element Galerkin scheme for the three-dimensional cubic domain and the control parameterization method is developed for solving the problem. Finally, a numerical example is solved.

1. **Introduction.** The environmental impacts of aqua-cultures, such as fish-feeding, have been widely studied in [1], [2], [3], [4], [6], [7], [12], [19], [18], and [23]. Mathematical models on the interactions between aqua-cultures and water pollution have been developed in [4] and [12]. A mathematical model for predicting tidal current and nitrogen levels for a fish-farm configuration in a bay off the Eire coastline in the United Kingdom was developed in [4]. A two-dimensional hydrodynamic model

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involving tidal flows, and salt transport flows was developed in [19] to calculate the water level, velocity, and salinity in each grid cell within the culture area. A biogeochemical model that accounts for the effects of sediment-nutrient interactions on multiple components of phytoplankton metabolism dynamics, carbonate speciation, dissolved oxygen, and biochemical oxygen demands was developed in [23]. The ecological effects of coastal aquaculture wastes from the perspective of environmental protection were studied in [6]. The environmental impact of marine fish farming was studied in [18]. More recently, the environmental sustainability via the maintenance of an ecosystem characteristic diversity, productivity, and biochemical cycling in salmon aquaculture was considered in [1] in 2018. The production of fish feed in an inline recirculating system for urban sites which efficiently combines waste and environmental service concepts in one production system was developed in [7] in 2021. The environmental impacts of the aquaculture farm activities were assessed in [2] in 2021; these assessments are almost based on physical-chemical measures and/or sediment characteristics. Lastly, an overview of the main factors of ecological concern within marine fin-fish aquaculture together with their interactions with the environment was provided in [3] in 2021.

Research work in [19] shows that the deteriorating effect of water pollution created by aquaculture can cause a lot of damages, such as physical obstruction and modification of water movement and sedimentation. Thus, to sustain the future development of aqua-cultures, it is necessary to reduce the number of pollution loading and stock density below environmental capacity, while still producing enough aquatic products for commercial purposes. However, no optimal fish-feeding program has so far been developed to minimize the deteriorating effect of water pollution.

A three-dimensional integrated mathematical model consisting of two main submodels was used in [12] to evaluate the environmental effects on coastal waters due to mainly the non-utilized fish food originating from the aqua-cultures. Based on the concentrations of seven different water quality variables (i.e., Phytoplankton (PHY), Organic Nitrogen (NOR), Ammonia (NAM), Nitrates (NIT), Organic Phosphorous (POR), Inorganic Phosphorous (PIN), and Suspended Solid (SS)), the fish growth and nitrogen and phosphorus transformation (FG-NPT) sub-model in [12] uses a system of time-dependent algebraic equations to calculate the effect of the fish-feeding program on the fishes' weight and the concentrations of pollutants due to nitrogen and phosphorus transformation, whereas the water quality (QUAL-3DL) sub-model incorporates the source terms (the terms arising from the chemical processes) and the pollution terms (the terms arsing from the generated pollution load of the FG-NPT sub-model) into the system of basic diffusion-convention partial differential equations to calculate the concentration of the water quality variables in the enclosed three-dimensional coastal region.

The freshwater pond model considered in this paper is similar to the three-dimensional integrated mathematical model of [12] such that the region of the aquaculture in this paper is a calm water pond, instead of a windy pond. However, we modify the mathematical model of [12] by introducing optimal control of fish-feeding into the model and develop a concrete method for solving the optimal control problem.

The optimal fish-feeding problem that we formula is to maximize the final weight of the fishes, subject to the restrictions that the fishes are not under-fed and over-fed and the concentrations of the pollutants created by the fish-feeding program are not too large. (In this way, both the fishes' health requirement and the environmental protection requirement are not violated.) The formulation of the objective function (i.e., the final weight of the fishes) of the problem simply requires setting up an ordinary differential equation connecting the instantaneous weight of the fishes (i.e. the state variable) and the fish feeding rate (i.e., the control variable). The formulation of the fish-feeding rate constraints (i.e., fishes' health requirement constraints) simply requires setting up a lower bound and an upper bound for the fishes' feeding rate. However, the formulation of the environmental protection requirement constraints requires setting up a Fish-Feeding Water Pollution (FFWP) sub-model and a No-Fish-Feeding Water Pollution (NFFWP) sub-model so that the instantaneous differences in concentrations of the water quality variable between these two models can be calculated.

A Galerkin scheme for the three-dimensional cubic domain is used to convert all the partial differential equations into ordinary differential equations of the above optimal control problem. Hence, an approximated constrained optimal control problem involving lumped systems only is obtained. This constrained optimal control problem is then solved by the well-known control parameterization method. (See [9], [11], [13], [14], [17], [21], and [22] for details.) Thus, based on the parameters given in [12], the optimal fishes' feeding rate and the optimal fishes' weight for harvest are obtained. The corresponding instantaneous concentration of each water quality variable at each grid point of the pond is also obtained.

The contribution of this paper is twofold. From the practical point of view, based on seven water quality variables, this paper analysis the optimal fish-feeding process in a pond at rest, by coupling a system of algebraic equations for fish growth and nitrogen and phosphorus transformation with a standard diffusion (no convection) water quality model from the existing literature. In this way, we optimize the economic benefits of fish feeding, without violating the environmental protection requirement. From a mathematical point of view, this paper develops a concrete Galerkin scheme for solving partial differential equations with three-dimensional cubic domains.

The organization of this paper is as follows. In Section 2, by setting a differential equation relating the instantaneous fishes' weight and the instantaneous fishes' feeding rate, we modified the FG-NPT sub-model of [12] to a new model, called the Fish-Feeding (FF) sub-model, in such a way that all the static forms of the Nitrogen and Phosphorous pollution loads are converted into dynamic forms. In Section 3, we use the chemical processes given in [12] and the various dynamic forms of Nitrogen and Phosphorous loads obtained in Section 2 to explicitly express all the source terms and the input pollution loads as functions of their respective arguments, respectively, and then insert them into the basic diffusion equations to obtain a Fish-Feeding Water Pollution (FFWP) sub-model. In Section 4, we first describe a No-Fish-Feeding Water Pollution (NFFWP) sub-model, which is obtained by deleting the pollution terms from the FFWP sub-model. We then formulate the objective function, the fishes' health requirement constraint (i.e., the fish-feeding rate constraint), and the environmental protection constraints to obtain an optimal control fish-feeding problem involving distributed parameter systems. In Section 5, we use the Galerkin scheme to convert the distributed parameter systems into lumped parameter systems to obtain an approximated optimal control problem. In Section 6, we modify the state equations to handle the non-negativity requirements of the instantaneous concentrations of the water quality variables. Hence we obtain a transformed approximated optimal control problem. In Section 7, we solve the transformed approximated optimal control problem described in Section 6 to obtain the optimal instantaneous fishes' feeding rate, the optimal fishes' weight for harvest, and the corresponding instantaneous concentration of each water quality variable at each grid point of the pond. Concluding remarks are given in Section 8.

2. **The fish-feeding sub-model.** In this section, we describe the fish-feeding (FF) sub-model, which is modified from that of the FG-NPT sub-model of [12]. More precisely, by setting a differential equation involving the instantaneous fishes' weight $x_{weight}(t)$, and the instantaneous fishes' feeding rate $u_{feed}(t)$, we can convert all the static forms of the Nitrogen and Phosphorous pollution loads in [12] into dynamic forms which are functions of $x_{weight}(t)$, $u_{feed}(t)$, and time t only. The FF sub-model can be described as follows:

Similar to the NPT sub-model of [12], the FF sub-model also consists of 7 water quality variables in the pond, namely, Phytoplankton (PHY), Organic Nitrogen (NOR), Ammonia (NAM), Nitrates (NIT), Organic Phosphorous (POR), Inorganic Phosphorous (PIN) and Suspended Solid (SS), which are denoted by W_i $(i = 1, ..., 7)$.

The fish-feeding program lasts for about one year. Within this year, the fishes mature from baby fishes to young adults in such a way that there is no reproduction and no mortality occurs. Thus, the number of fishes in the pond remains unchanged throughout the entire time horizon of the fish-feeding program.

Let n_{fish} be the number of fishes in the pond. Let t_0 be the initial time of the fish-feeding program. Let the instantaneous fishes' weight, $x_{weight}(t)$, be expressed in kg and the instantaneous fishes' feeding rate, $u_{feed}(t)$, be expressed in % kg food/kg fish day. Then, from [5], whenever $u_{feed}(t) \geq 0.9$, the differential equation for $x_{weight}(t)$ can be written as follows:

$$\dot{x}_{weight}(t) = SGR(u_{feed}(t)) \times x_{weight}(t), \tag{1}$$

$$x_{weight}(t_0) = \hat{x}_{weight}(t), \tag{2}$$

where

$$SGR(u_{feed}(t)) = \big(-0.1268 \times (u_{feed}(t))^2 + 1.4390 \times u_{feed}(t) - 1.1270\big) \times 0.65 \tag{3}$$

is the specific growth rate of the fishes (expressed in kg increase in fish weight per kg increase in fish food) corresponding to the fish-feeding rate $u_{feed}(t)$ in the freshwater pond.

Let $Temp(t)$ be the temperature of the water in $^\circ C$ at time t as defined in the Photosynthesis process in Table A1 in the Appendix. Then, from the data given in [12] and (1) - (2), all the static forms of the Nitrogen and Phosphorous pollution loads in [12] are converted into dynamic forms as follows:

Various Forms of Nitrogen Pollution load

$$\text{In Food: } N_{food} = 7.68 \times 10^{-4} n_{fish} x_{weight}(t) u_{feed}(t). \tag{4}$$

$$\text{In Fishes: } N_{fish} = 1.77 \times 10^{-5} n_{fish}. \tag{5}$$

$$\text{In Feces particulate: } N_{fep} = 7.68 \times 10^{-5} n_{fish} x_{weight}(t) u_{feed}(t). \tag{6}$$

$$\text{In Feces diluted: } N_{fed} = 5.38 \times 10^{-5} n_{fish} x_{weight}(t) u_{feed}(t). \tag{7}$$

$$\text{In Wastes: } N_{waste} = 3.84 \times 10^{-5} n_{fish} x_{weight}(t) u_{feed}(t). \tag{8}$$

$$\begin{aligned}
\text{In Excretion: } N_{ex} &= N_{food} - N_{fish} - N_{fed} - N_{fep} - N_{waste} \\
&= 5.99 \times 10^{-4} n_{fish} x_{weight}(t) u_{feed}(t) - 1.77 \times 10^{-5} n_{fish}. \qquad (9)
\end{aligned}$$

$$\text{In Excretion Organic: } N_{exon} = 1.2 \times 10^{-5} n_{fish} x_{weight}(t) u_{feed}(t) - 3.5 \times 10^{-7} n_{fish}. \qquad (10)$$

$$\begin{aligned}
\text{In Excretion Urea: } N_{exur} &= (10^{-3} Temp(t)^2 - 4 \times 10^{-2} Temp(t) + 0.49) \\
&\times (5.99 \times 10^{-4} n_{fish} x_{weight}(t) u_{feed}(t) - 1.77 \times 10^{-5} n_{fish}). \qquad (11)
\end{aligned}$$

$$\begin{aligned}
\text{In Excretion Ammonia: } N_{exam} &= N_{ex} - N_{exon} - N_{exur} \\
&= 5.87 \times 10^{-4} n_{fish} x_{weight}(t) u_{feed}(t) - 1.77 \times 10^{-5} n_{fish} \\
&- (10^{-3} Temp(t)^2 - 4 \times 10^{-2} Temp(t) + 0.49) \\
&\times (5.99 \times 10^{-4} n_{fish} x_{weight}(t) u_{feed}(t) - 1.77 \times 10^{-5} n_{fish}). \qquad (12)
\end{aligned}$$

Various Forms of Phosphorous Pollution load:

$$\text{In Food: } Ph_{food} = 1.5 \times 10^{-4} n_{fish} x_{weight}(t) u_{feed}(t). \qquad (13)$$

$$\text{In Fishes: } Ph_{fish} = 4.2 \times 10^{-6} n_{fish}. \qquad (14)$$

$$\text{In Feces particulate: } Ph_{fep} = 6.6 \times 10^{-5} n_{fish} x_{weight}(t) u_{feed}(t). \qquad (15)$$

$$\text{In Feces diluted: } Ph_{fed} = 1.05 \times 10^{-5} n_{fish} x_{weight}(t) u_{feed}(t). \qquad (16)$$

$$\text{In Waste: } Ph_{waste} = 7.5 \times 10^{-6} n_{fish} x_{weight}(t) u_{feed}(t). \qquad (17)$$

$$\begin{aligned}
\text{In Excretion: } Ph_{ex} &= Ph_{food} - Ph_{fish} - Ph_{fep} - Ph_{fed} - Ph_{waste} \\
&= 6.6 \times 10^{-5} n_{fish} x_{weight}(t) u_{feed}(t) - 4.2 \times 10^{-6} n_{fish}. \qquad (18)
\end{aligned}$$

After we have converted all the static forms of the Nitrogen and Phosphorous pollution loads into dynamic forms, we can explicitly express the pollution term of each of the i^{th} water quality variables (i.e., $P_{W_i}(X_1, ..., X_7, t)$) as a function of its arguments, which will then inputted into the basic diffusion-convection model to form the Fish-Feeding Water Pollution (FFWP) sub-model in the next section. This completes the description of the FF sub-model.

3. **Three-dimensional fish-feeding water pollution sub-model.** Suppose the three-dimensional FFWP sub-model is located on the cubic pond:
$\{(x, y, z) \ : \ x \in [0, \hat{x}], \ y \in [0, \hat{y}], \ z \in [0, \hat{z}]\}$,
where $\hat{x}$, $\hat{y}$ are, respectively, the x and y coordinates of the pond, $\hat{z}$ is the depth of the water. Let t_0 and t_f be, respectively, the initial time and the final time of the fish-feeding program. Let n_{fish}, $x_{weight}(t)$, $u_{feed}(t)$, and W_i $(i = 1, ..., 7)$ be defined in the same way as in the fish-feeding model. Then the partial differential equation governing this model is as follows:

$$\begin{aligned}
\frac{\partial X_i}{\partial t} &= N_h \frac{\partial^2 X_i}{\partial x^2} + N_h \frac{\partial^2 X_i}{\partial y^2} + N_v \frac{\partial^2 X_i}{\partial z^2} + S_{W_i} + P_{W_i}, \\
& x \in [0, \hat{x}], \ y \in [0, \hat{y}], \ z \in [0, \hat{z}], \ t \in [t_0, t_f], \qquad (19)
\end{aligned}$$

where $X_i = X_i(x, y, z, t)$ stands for the concentration of the i^{th} water quality variables in the pond; N_h and N_v are given constants; W_i stands for the i^{th} water quality variable; $S_{W_i} = S_{W_i}(X_1, ..., X_7, t)$ stands for the source term of the i^{th} water quality variable, which can be obtained from the chemical process given in [12]

(See Tables A1 and A2 for details.); $P_{W_i} = P_{W_i}(x_{weight}(t), u_{feed}(t), t)$ stands for the instantaneous input pollution term of i^{th} water quality variable W_i , which can be calculated from the dynamic forms of the Nitrogen and Phosphorous pollution loads in Section 2.

We first describe the source terms of the water quality variables which are generated from 8 chemical processes given in Tables A1 and A2 in the Appendix. For the sake of ease of understanding of these chemical processes, we need to modify the notation of the concentrations of the water quality variables as follows:
$X_{PHY} = X_1$ =concentration of Phytoplankton (PHY) ;
$X_{NOR} = X_2$ =concentration of Organic Nitrogen (NOR);
$X_{NAM} = X_3$ =concentration of Ammonia (NAM);
$X_{NIT} = X_4$ =concentration of Nitrates (NIT);
$X_{POR} = X_5$ =concentration of Organic Phosphorous (POR);
$X_{PIN} = X_6$ =concentration of Inorganic Phosphorous (PIN);
$X_{SS} = X_7$ =concentration of Suspended Solid (SS).

Using the same notation as those given in Tables A1 and A2 for these chemical processes, we can explicitly express all the source terms $S_{W_i}(X_1, ..., X_7, t)$ as functions of their respective arguments. The detail is as follows:
Source term of each water quality variable:

1. $S_{PHY}\left(X_{PHY}, t\right) = \left(k_{g1}(t) + k_{r1}(t) + k_{d1} + k_{Set1}\right) X_{PHY}$
$$\Leftrightarrow S_{W_1}\left(X_1, t\right) = \left(k_{g1}(t) + k_{r1}(t) + k_{d1} + k_{Set1}\right) X_1, \tag{20}$$
2. $S_{NOR}\left(X_{PHY}, X_{NOR}, t\right) = \left(k_{r2}(t) + k_{d2}\right) X_{PHY} + \left(k_{Set2} + k_{Am2}(t)\right) X_{NOR}$
$$\Leftrightarrow S_{W_2}\left(X_1, X_2, t\right) = \left(k_{r2}(t) + k_{d2}\right) X_1 + \left(k_{Set2} + k_{Am2}(t)\right) X_2, \tag{21}$$
3. $S_{NAM}\left(X_{PHY}, X_{NOR}, X_{NAM}, X_{SS}, t\right)$
$= \left(k_{g3}(t) + k_{r3}(t)\right) X_{PHY} + k_{A3}X_{SS} + k_{Am3}(t)X_{NOR} + k_{Nit3}(t)X_{NAM}$
$\Leftrightarrow S_{W_3}\left(X_1, X_2, X_3, X_7, t\right)$
$$= \left(k_{g3}(t) + k_{r3}(t)\right) X_1 + k_{A3}X_7 + k_{Am3}(t)X_2 + k_{Nit3}(t)X_3, \tag{22}$$
4. $S_{NIT}\left(X_{PHY}, X_{NAM}, t\right) = k_{g4}(t)X_{PHY} + k_{Nit4}(t)X_{NAM}$
$$\Leftrightarrow S_{W_4}\left(X_1, X_3, t\right) = k_{g4}(t)X_1 + k_{Nit4}(t)X_3 \tag{23}$$
5. $S_{POR}\left(X_{PHY}, X_{POR}, t\right) = \left(k_{r5}(t) + k_{d5}\right) X_{PHY} + \left(k_{Set5} + k_{Min5}(t)\right) X_{POR}$
$$\Leftrightarrow S_{W_5}\left(X_1, X_5, t\right) \left(k_{r5}(t) + k_{d5}\right) X_1 + \left(k_{set5} + k_{Min5}(t)\right) X_5, \tag{24}$$
6. $S_{PIN}\left(X_{PHY}, X_{SS}, t\right) = \left(k_{g6}(t) + k_{r6}(t)\right) X_{PHY} + k_{A6}X_{SS} + k_{Min6}X_{POR}$
$$\Leftrightarrow S_{W_6}\left(X_1, X_5X_7, t\right) = \left(k_{g6}(t) + k_{r6}(t)\right) X_1 + k_{A6}X_7 + k_{Min6}X_5, \tag{25}$$
7. $S_{SS}\left(X_{PHY}, X_{SS}\right) = k_{d7}X_{PHY} + k_{Set7}X_{SS}$
$$\Leftrightarrow S_{W_7}\left(X_1, X_7\right) = k_{d7}X_1 + k_{Set7}X_7, \tag{26}$$

where in (20) – (26), the constants $k_{g,i}$ $(i = 1, 3, 4, 6)$, $k_{r,i}$ $(i = 1, 2, 3, 5, 6)$, $k_{d,i}$ $(i = 1, 2, 5, 7)$, $k_{Set,i}$ $(i = 1, 2, 5, 7)$, $k_{A,i}$ $(i = 3, 6)$, $k_{Am,i}$ $(i = 2, 3)$, $k_{Nit,i}$ $(i = 3, 4)$, $k_{Min,i}$ $(i = 5, 6)$, represent the growth or decay constant of the i^{th} water quality variable due to the chemical processes Photosynthesis, Endogenous Respiration, Decay, Settling, Adsorption, Ammonification, Nitrification, Mineralization of Phosphorous, respectively.

We now use the dynamic forms of the Nitrogen and Phosphorous pollution loads in Section 2 to calculate the pollution terms $P_{W_i}(x_{weight}(t), u_{feed}(t), t)$ as functions

of their respective arguments as follows: (There is no input pollution load for the first water quality variable PHY and the fourth water quality variable NAM.)
Input Pollution Load for each water quality
1. Organic Nitrogen
From (7), (10), and (11),

$$\begin{aligned}
&P_{NOR}(x_{weight}(t), u_{feed}(t), t) = N_{fed} + N_{exon} + N_{exur} \\
&= \left(6.57 \times 10^{-5} n_{fish} x_{weight}(t) u_{feed}(t) - 3.54 \times 10^{-7} n_{fish}\right) \\
&\quad + \left(10^{-3} Temp(t)^2 - 4 \times 10^{-2} Temp(t) + 0.49\right) \\
&\times \left(5.99 \times 10^{-4} n_{fish} x_{weight}(t) u_{feed}(t) - 1.77 \times 10^{-5} n_{fish}\right).
\end{aligned} \tag{27}$$

2. Ammonia
From (12),

$$\begin{aligned}
&P_{NAM}(x_{fish}(t), u_{feed}(t), t) = N_{exam} \\
&= (5.87 \times 10^{-4} n_{fish} x_{weight}(t) u_{feed}(t) - 1.73 \times 10^{-5} n_{fish}) \\
&- \left(10^{-3} Temp(t)^2 - 4 \times 10^{-2} Temp(t) + 0.49\right) \\
&\times \left(5.99 \times 10^{-4} n_{fish} x_{\text{weight}}\,(t) u_{feed}(t) - 1.77 \times 10^{-5} n_{fish}\right).
\end{aligned} \tag{28}$$

3. Organic Phosphorous
From (16),

$$P_{POR}(x_{weight}(t), u_{feed}(t)) = Ph_{feed} = 1.05 \times 10^{-5} n_{fish} x_{weight}(t) u_{feed}(t). \tag{29}$$

4. Inorganic Phosphorous
From (18),

$$\begin{aligned}
&P_{PIN}(x_{weight}(t), u_{feed}(t)) = Ph_{ex} \\
&= 6.6 \times 10^{-5} n_{fish} x_{weight}(t) u_{feed}(t) - 4.2 \times 10^{-6} n_{fish}.
\end{aligned} \tag{30}$$

5. Suspended Solids
From equation (29) given in [12],

$$P_{SS}(x_{weight}(t), u_{feed}(t)) = (1.82 \times 10^{-4} n_{fish} x_{weight}(t) u_{feed}(t) + 1.9 \times 10^{-3} u_{feed}(t)). \tag{31}$$

Now, we state the boundary conditions and the initial condition for the three-dimensional FFWP sub-model as follows:
For each W_i, $i = 1, ...7$, we have

$$\frac{\partial X_i(0, y, z, t)}{\partial x} = 0, \ \forall y \in [0, \hat{y}], \forall z \in [0, \hat{z}], \text{ and } \forall t \in [t_0, t_f], \tag{32}$$

$$\frac{\partial X_i(\hat{x}, y, z, t)}{\partial x} = 0, \ \forall y \in [0, \hat{y}], \forall z \in [0, \hat{z}], \text{ and } \forall t \in [t_0, t_f], \tag{33}$$

$$\frac{\partial X_i(x, 0, z, t)}{\partial y} = 0, \ \forall x \in [0, \hat{x}], \forall z \in [0, \hat{z}], \text{ and } \forall t \in [t_0, t_f], \tag{34}$$

$$\frac{\partial X_i(x, \hat{y}, z, t)}{\partial y} = 0, \ \forall x \in [0, \hat{x}], \forall z \in [0, \hat{z}], \text{ and } \forall t \in [t_0, t_f], \tag{35}$$

$$\frac{\partial X_i(x, y, 0, t)}{\partial z} = 0, \ \forall x \in [0, \hat{x}], \forall y \in [0, \hat{y}], \text{ and } \forall t \in [t_0, t_f], \tag{36}$$

$$\frac{\partial X_i(x, y, \hat{z}, t)}{\partial z} = -\bar{k} X_i(x, y, \hat{z}, t), \ \forall x \in [0, \hat{x}], \forall y \in [0, \hat{y}], \text{ and } \forall t \in [t_0, t_f], \tag{37}$$

$$X_i(x, y, z, t_0) = X_{i,0}(x, y, z), \ \forall x \in [0, \hat{x}], \forall y \in [0, \hat{y}], \text{ and } \forall z \in [0, \hat{z}], \tag{38}$$

where $\bar{k}$ is a given constant and $X_{i,0}$ is a given function. Inequality (37) implies that the gradient of the outlet concentration of each water variable at the water surface is proportional to the concentration itself but is decreasing. The situation for the boundary conditions is plotted in Figure 1.

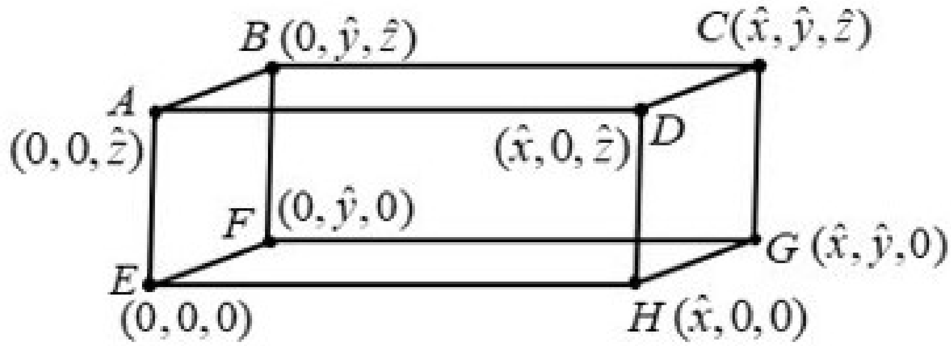

FIGURE 1. Boundary conditions of the water pollution model

Plane $ABFE$: $\frac{\partial X_i(0,y,z,t)}{\partial x} = 0$
Plane $DCGH$: $\frac{\partial X_i(\hat{x},y,z,t)}{\partial x} = 0$
Plane $ADHE$: $\frac{\partial X_i(x,0,z,t)}{\partial y} = 0$
Plane $BCGF$: $\frac{\partial X_i(x,\hat{y},z,t)}{\partial y} = 0$
Plane $FGHE$: $\frac{\partial X_i(x,y,0,t)}{\partial z} = 0$
Plane $BCDA$: $\frac{\partial X_i(x,y,\hat{z},t)}{\partial z} = -\bar{k}X_i(x,y,0,t)$

Lastly, we are in a position to provide a theorem concerning the existence and uniqueness of the solution of each water quality variable of the FFWP model, which can be stated as follows:

Theorem 3.1. *For any piecewise control $u_{feed}(t)$, there exists a unique classical solution for each water qualify variable $X_i(x,y,z,t)$ (i=1,...,7) $\in L_\infty(Q)$ satisfying the partial differential equations and the boundary conditions given by the FFWP sub-model, where Q =[0,$\hat{x}$]×[0,$\hat{y}$]×[0,$\hat{z}$]×$[t_0,t_f]$, and*

$$\|X_i\|_{L_\infty(Q)} = \sup\{X_i(x,y,z,t) : \ (x,y,z,t) \in Q\}. \tag{39}$$

Proof. From (20) – (26), it is clear that for any time $t \in [t_0,t_f]$, S_{W_1}, S_{W_2}, S_{W_3}, S_{W_4}, S_{W_5}, S_{W_6}, S_{W_7} in (19) are, respectively, linear functions of X_1, X_1 and X_2, X_1 and X_2 and X_3 and X_7, X_1 and X_3, X_1 and X_5, X_1 and X_5 and X_7, X_1 and X_7. Moreover, from Table A2 in the Appendix, it is clear that all the time functions or constants (i.e., $k_{g,i}$ $(i = 1,3,4,6)$, $k_{r,i}$ $(i = 1,2,3,5,6)$, $k_{d,i}$ $(i = 1,2,5,7)$, $k_{Set,i}$ $(i = 1,2,5,7)$, $k_{A,i}$ $(i = 3,6)$, $k_{Am,i}$ $(i = 2,3)$, $k_{Nit,i}$ $(i = 3,4)$, $k_{Min,i}$ $(i = 5,6)$ associated with X_1, X_2, X_3, X_4, X_5, X_6, X_7 are bounded on $[t_0,t_f]$.

Furthermore, for any piecewise control $u_{feed}(t)$, it is clear from (27) – (31) and (1) – (3) that $P_{W_i}(x_{weight}(t), u_{feed}(t), t), (i = 2,3,5,6,7)$ in (19) are bounded for all $t \in [t_0,t_f]$.

Thus, for any piecewise control $u_{feed}(t)$, the distributed parameter system (1) together with the boundary conditions (32) – (38) constitute a linear parabola system with mixed linear Dirichlet and Neumann boundary conditions. Thus, the proof of this theorem follows easily from the theory of linear parabola systems with linear boundary conditions. (See [10] for details.) □

This completes the description of the NFFWP sub-model.

4. **The optimal control fish-feeding problem with distributed parameter system.** Before we can formulate the optimal control fish-feeding problem with distributed parameter system, we first need to formulate the three-dimensional NFFWP sub-model.

4.1. **Three-dimensional no-fish-feeding water pollution sub-model.** The three-dimensional NFFWP sub-model is similar to the three-dimensional FFWP sub-model described in Section 3, except that there is no fish-feeding in the pond. Thus, the partial differential equations of this NFFWP sub-model are obtained by deleting the pollution term P_{W_i} from the partial differential equations (19) of the FFWP sub-model. The full details of this NFFWP sub-model are as follows:

Let the initial time, the final time, the dimensions of the pond be the same as those given in Section 3. In view of (19), the partial differential equation governing this NFFWP sub-model is

$$\frac{\partial \bar{X}_i}{\partial t} = N_h \frac{\partial^2 \bar{X}_i}{\partial x^2} + N_h \frac{\partial^2 \bar{X}_i}{\partial y^2} + N_v \frac{\partial^2 \bar{X}_i}{\partial z^2} + S_{W_i},$$
$$\forall x \in [0, \hat{x}], \forall y \in [0, \hat{y}], \forall z \in [0, \hat{z}], \text{ and } \forall t \in [t_0, t_f]\,, \tag{40}$$

where $\bar{X}_i = \bar{X}_i(x, y, z, t)$ stands for the concentration of the various water quality variables in the pond of this NFFWP sub-model; N_h, N_v, W_i, and S_{W_i} are as defined in (19) of the FFWP sub-model. In view of (32) − (38), the boundary conditions of this sub-model and the initial condition of this sub-model, which are the same as those in the FFWP sub-model, can be stated as follows:

$$\frac{\partial \bar{X}_i(0, y, z, t)}{\partial x} = 0, \forall y \in [0, \hat{y}], \forall z \in [0, \hat{z}], \text{ and } \forall t \in [t_0, t_f]\,, \tag{41}$$

$$\frac{\partial \bar{X}_i(\hat{x}, y, z, t)}{\partial x} = 0, \forall y \in [0, \hat{y}], \forall z \in [0, \hat{z}], \text{ and } \forall t \in [t_0, t_f]\,, \tag{42}$$

$$\frac{\partial \bar{X}_i(x, 0, z, t)}{\partial y} = 0, \forall x \in [0, \hat{x}], \forall z \in [0, \hat{z}], \text{ and } \forall t \in [t_0, t_f]\,, \tag{43}$$

$$\frac{\partial \bar{X}_i(x, \hat{y}, z, t)}{\partial y} = 0, \forall x \in [0, \hat{x}], \forall z \in [0, \hat{z}], \text{ and } \forall t \in [t_0, t_f]\,, \tag{44}$$

$$\frac{\partial \bar{X}_i(x, y, 0, t)}{\partial z} = 0, \forall x \in [0, \hat{x}], \forall y \in [0, \hat{y}], \text{ and } \forall t \in [t_0, t_f]\,, \tag{45}$$

$$\frac{\partial \bar{X}_i(x, y, \hat{z}, t)}{\partial z} = -\bar{k}\bar{X}_i(x, y, \hat{z}, t), \forall x \in [0, \hat{x}], \forall y \in [0, \hat{y}], \text{ and } \forall t \in [t_0, t_f]\,, \tag{46}$$

$$\bar{X}_i\,(x, y, z, t_0) = \bar{X}_{i,0}(x, y, z) = X_{i,0}(x, y, z), \forall x \in [0, \hat{x}], \forall y \in [0, \hat{y}], \text{ and } \forall z \in [0, \hat{z}]. \tag{47}$$

We are in a position to provide a theorem concerning the existence and uniqueness of the solution of each water quality variable of the NFFWP model, which can be stated as follows:

Theorem 4.1. *There exists a unique classical solution for each water quality variable* $\bar{X}_i(x, y, z, t)$ *(i=1,...,7)* $\in L_\infty(Q)$ *satisfying the partial differential equations and the boundary conditions given by the NFFWP model, where* $Q = [0,\hat{x}]\times[0,\hat{y}]\times[0,\hat{z}] \times [t_0, t_f]$*, and* $\|\bar{X}_i\|_{L_\infty(Q)}$ *is as defined as in* (39).

Proof. Since the system of the partial differential equation (40) with boundary conditions (41) – (47) of the NFFWP model is obtained from that of the FFWP model (i.e, equation (19) and (32) – (38)) by replacing the input pollution term P_{W_i} in equation (19) of the FFWP model by zero, the proof of this theory is the same as that given for Theorem 3.1. □

4.2. **Formation of the optimal control problem with distributed parameter system.** After having formulated the three-dimensional FFWP sub-model and the three-dimensional NFFWP sub-model, we can formulate an optimal control fish-feeding problem. For this purpose, let t be the time in days measured from midnight of 1st January of any year. Let $t_0, t_f, u_{feed}(t), \bar{X}_i(x, y, z, t)$ be as defined in Section 2 and Section 3 of this paper, where $u_{feed}(t)$ is now the control function of our problem. Then, $u_{feed}(t)$ is chosen to be a piecewise continuous function. For each $i = 1, ..., 7$, let $x_{weight}(t|u_{feed})$ and $X_i(x, y, z, t|u_{feed})$ be, respectively, the instantaneous weight of the fishes and the instantaneous concentration of W_i of the NFFWP model at time t, when the control function is equal to $u_{feed}(t)$. Then, our objective, which is to find an optimal control $u_{feed}(t)$ that maximizes the final weight of the fishes, is given as follows:

$$\text{Max } J(u_{feed}) = x_{weight}(t_f|u_{feed}). \tag{48}$$

Due to the health requirement of the fishes, we need to impose the upper bound and the lower bound on the fishes' feeding rate (i.e., the fishes' feeding rate constraint) as follows:

$$\underline{u}_{feed} \le u_{feed}(t) \le \overline{u}_{feed},\ t \in [t_0, t_f], \tag{49}$$

where $\underline{u}_{feed}(t)$ and $\overline{u}_{feed}(t)$ are given constants. Furthermore, due to the requirement for environmental concern, we need to ensure that the increase in the average concentration of each water quality variable W_i ($i = 2, ..., 7$) created by the feeding program at any time t, $t \in [t_0, t_f]$ should not exceed M_i, where M_i is a given number. (i.e, the differences in average concentrations of W_i between the FFWP model and the NFFWP model should not exceed M_i.) Thus, we need to introduce the following all-time water quality variable concentration constraints (i.e. environment protection requirement constraints).

$$\int_0^{\hat{z}}\int_0^{\hat{y}}\int_0^{\hat{x}} X_i\left(x, y, z, t \mid u_{feed}(t)\right) dxdydz - \int_0^{\hat{z}}\int_0^{\hat{y}}\int_0^{\hat{x}} \bar{X}_i(x, y, z, t) dxdydz \le M_i,$$
$$\forall i = 1, ..7, \quad t \in [t_0, t_f]. \tag{50}$$

(Note that any fish-feeding program does not create an additional load of W_1 (Phytoplankton) in the pond. In other words, irrespective of the control variable $u_{feed}(t)$, the first constraint of (50)

$$\int_0^{\hat{z}}\int_0^{\hat{y}}\int_0^{\hat{x}} X_1\left(x, y, z, t \mid u_{feed}(t)\right) dxdydz - \int_0^{\hat{z}}\int_0^{\hat{y}}\int_0^{\hat{x}} \bar{X}_1(x, y, z, t) dxdydz \le M_1,$$
$$\forall t \in [t_0, t_f],$$

is always satisfied.)

Now, our optimal control problem $(P1)$ can be stated as follows:

Problem $(P1)$: Subject to the distributed system (19) – (38) for the FFWP model, the distributed system (40) – (47) for the NFFWP model, the fishes' weight equations (1) – (2), the feeding rate constraint (49) and the all-time water quality variable concentration constraints (50), find a piecewise continuous function $u_{feed}(t)$ which maximizes $J(u_{feed})$ given by (48).

We are in a position to provide a theorem concerning the existence of optimal control of the problem $(P1)$, which can be stated as follows:

Theorem 4.2. *The optimal control problem $(P1)$ has an optimal control.*

Proof. Let $(\overline{P}1)$ be the optimal control problem obtained from $(P1)$ by deleting the all-time water quality variable concentration constraints (50) from $(P1)$. Then $(\overline{P}1)$ is a simplified version of the optimal control problem described in [20] such that all the smoothness assumptions imposed in [20] are well satisfied by the given functions in $(\overline{P}1)$. Thus, from Theorem 6.1 of [20], we know that optimal control $(P1)$ has an optimal solution. From the fact that $(\overline{P}1)$ has an optimal control and $(P1)$ has at least one control which satisfies constraints (50), (When $u_{feed}(t) = 0$ for all $t \in [t_0, t_f]$, then constraints (50) becomes $0 \leq M_i$, $\forall i = 1, ..., 7,\ t \in [t_0, t_f]$, which is obviously true) we conclude that $(P1)$ also has an optimal control. □

5. Formulation of an approximated optimal control problem with lumped parameter system by using the galerkin scheme.

5.1. Converting the system of partial differential equations in the fish-feeding water pollution model into ordinary differential equations.

In this sub-section, we first convert the system of partial differential equations (19) and (32) – (38) of the FFWP model into ordinary differential equations by using the Galerkin Scheme. The method is as follows: We first divide the domain $\Omega = [0, \hat{x}] \times [0, \hat{y}] \times [0, \hat{z}]$ into a finite number of sub-regions, which are cubic tanks. Points at the corner of each sub-region are called grid points. For the sake of constructing root functions, we denote the grid points by (x_i, y_j, z_k), for $i = 0, 1, 2, 3$, $j = 0, 1, 2$, and $k = 0, 1, 2$. For illustrative purposes, we divide the domain into 12 equal cubic sub-regions $S_{i,j,k}$ for $i = 1, 2, 3$, $j = 1, 2$, and $k = 1, 2$, where

$$S_{i,j,k} = \{(x, y, z) : x_{i-1} \leq x \leq x_i, y_{j-1} \leq y \leq y_j, z_{k-1} \leq z \leq z_k\},$$

with

$$x_i - x_{i-1} = \frac{\hat{x}}{3} \text{ for } i = 1, 2, 3,\ y_j - y_{j-1} = \frac{\hat{y}}{2} \text{ for } j = 1, 2,\ z_k - z_{k-1} = \frac{\hat{z}}{2} \text{ for } k = 1, 2,$$

and

$$(x_0, y_0, z_0) = (0, 0, 0).$$

Then the root function corresponding to the grid point (x_i, y_j, z_k), for $i = 0, 1, 2, 3$, $j = 0, 1, 2$, and $k = 0, 1, 2$, can be constructed as follows:

$$R_{i,j,k}(x,y,z) = \begin{cases} \frac{12(x-x_i)(y-y_j)(z-z_k)}{\hat{x}\hat{y}\hat{z}}, & \text{if } (x,y,z) \in \text{ closure of } S_{i+1,j+1,k+1}, \\ \frac{12(x-x_{i-1})(y-y_j)(z-z_k)}{\hat{x}\hat{y}\hat{z}}, & \text{if } (x,y,z) \in \text{ closure of } S_{i,j+1,k+1}, \\ \frac{12(x-x_i)(y-y_{j-1})(z-z_k)}{\hat{x}\hat{y}\hat{z}}, & \text{if } (x,y,z) \in \text{ closure of } S_{i+1,j,k+1}, \\ \frac{12(x-x_i)(y-y_j)(z-z_{k-1})}{\hat{x}\hat{y}\hat{z}}, & \text{if } (x,y,z) \in \text{ closure of } S_{i+1,j+1,k}, \\ \frac{12(x-x_{i-1})(y-y_{j-1})(z-z_k)}{\hat{x}\hat{y}\hat{z}}, & \text{if } (x,y,z) \in \text{ closure of } S_{i,j,k+1}, \\ \frac{12(x-x_{i-1})(y-y_j)(z-z_{k-1})}{\hat{x}\hat{y}\hat{z}}, & \text{if } (x,y,z) \in \text{ closure of } S_{i,j+1,k}, \\ \frac{12(x-x_i)(y-y_{j-1})(z-z_{k-1})}{\hat{x}\hat{y}\hat{z}}, & \text{if } (x,y,z) \in \text{ closure of } S_{i+1,j,k}, \\ \frac{12(x-x_{i-1})(y-y_{j-1})(z-z_{k-1})}{\hat{x}\hat{y}\hat{z}}, & \text{if } (x,y,z) \in \text{ closure of } S_{i,j,k}, \\ 0, & \text{otherwise.} \end{cases} \tag{51}$$

The situation is depicted in Figure 2.

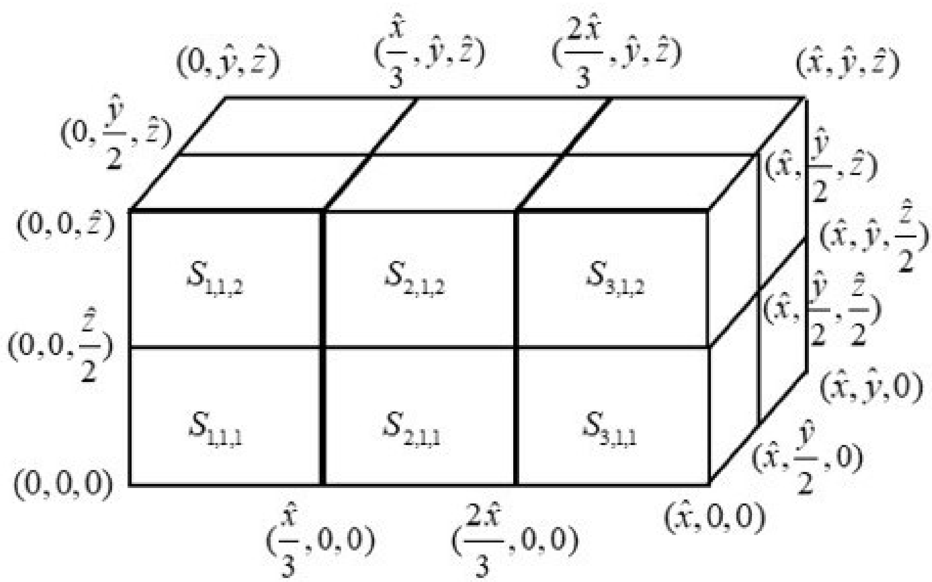

FIGURE 2. The Front Part of the Fish Pond

Remark 5.1. From (51), it is clear that

$$R_{i,j,k}(x,y,z) = \begin{cases} 1, & \text{at grid point } (x_i,y_j,z_k), \\ 0, & \text{at the other grid points }. \end{cases} \tag{52}$$

Next, to simplify our notation, we replace $R_{i,j,k}(x,y,z)$ by $R_l(x,y,z)$, where $l = i + 4j + 12k + 1$ for $i = 1, 2, 3$, $j = 0, 1, 2$ and $k = 0, 1, 2$. Then $R_l(x,y,z)$ represents the l^{th} global grid point of the whole domain, where all the global grid points are as depicted in Figure 3.

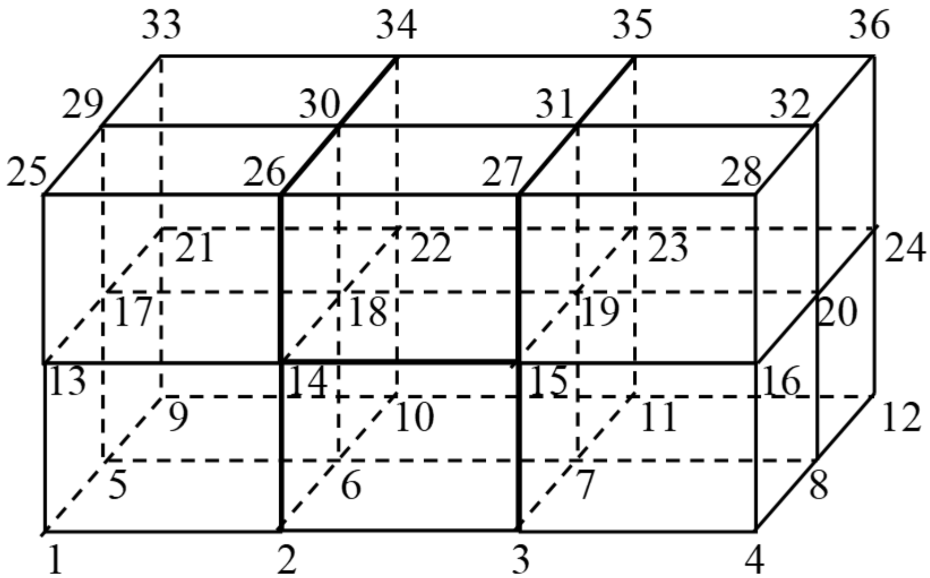

FIGURE 3. The global node point of the fish pond

Now, we approximate the concentration of the i^{th} water quality variable ($i = 1, ..., 7$) by

$$X_i^{36}(x, y, z, t) = \sum_{l=1}^{36} T_{i,l}(t) R_l(x, y, z). \tag{53}$$

From (53), the initial condition (38) becomes

$$T_{i,l}(t_0) = X_{i,l,0},\ i = 1, ..., 7,\ l = 1, ..., 36, \tag{54}$$

where $X_{i,l,0}$ is the initial concentration of the i^{th} water quality variable at the l^{th} global net-point.

To convert the system of partial differential (19) into a system of ordinary differential equations, we first let $\beta(x, y, z)$ be an arbitrary function in $C^1(\Omega)$ almost everywhere. Multiply (19) by $\beta(z, y, z)$ and integrate over the region Ω, we get

$$\begin{aligned}
&\int_0^{\hat{z}}\int_0^{\hat{y}}\int_0^{\hat{x}} \beta(x,y,z)\frac{\partial X_i(x,y,z,t)}{\partial t}dxdydz \\
&= \int_0^{\hat{z}}\int_0^{\hat{y}}\int_0^{\hat{x}} N_h\beta(x,y,z)\frac{\partial^2 X_i(x,y,z,t)}{\partial x^2}dxdydz \\
&+ \int_0^{\hat{z}}\int_0^{\hat{y}}\int_0^{\hat{x}} N_h\beta(x,y,z)\frac{\partial^2 X_i(x,y,z,t)}{\partial y^2}dxdydz \\
&+ \int_0^{\hat{z}}\int_0^{\hat{y}}\int_0^{\hat{x}} N_v\beta(x,y,z)\frac{\partial^2 X_i(x,y,z,t)}{\partial z^2}dxdydz \\
&+ \int_0^{\hat{z}}\int_0^{\hat{y}}\int_0^{\hat{x}} \beta(x,y,z)S_{X_i}\left(X_1,\ldots,X_7,t\right)dxdydz \\
&\quad + P_{X_i}\left(x_{weight}(t),u_{feed}(t),t\right) \times \int_0^{\hat{z}}\int_0^{\hat{y}}\int_0^{\hat{x}} \beta(x,y,z)dxdydz. \qquad (55)
\end{aligned}$$

In view of (32) and (33), we have

$$\begin{aligned}
&\int_0^{\hat{z}}\int_0^{\hat{y}}\int_0^{\hat{x}} N_h\beta(x,y,z)\frac{\partial X_i^2(x,y,z,t)}{\partial x^2}dxdydz \\
&= \int_0^{\hat{z}}\int_0^{\hat{y}} N_h\left[\beta(\hat{x},y,z)\frac{\partial X_i(\hat{x},y,z,t)}{\partial x} - \beta(0,y,z)\frac{\partial X_i(0,y,z,t)}{\partial x}\right]dydz \\
&\quad - \int_0^{\hat{z}}\int_0^{\hat{y}}\int_0^{\hat{x}} N_h\beta_x(x,y,z)\frac{\partial X_i(x,y,z,t)}{\partial x}dxdydz \\
&= -\int_0^{\hat{z}}\int_0^{\hat{y}}\int_0^{\hat{x}} N_h\beta_x(x,y,z)\frac{\partial X_i(x,y,z,t)}{\partial x}dxdydz. \qquad (56)
\end{aligned}$$

Similarly, in view of (34) and (35), we have

$$\begin{aligned}
&\int_0^{\hat{z}}\int_0^{\hat{y}}\int_0^{\hat{x}} N_h\beta(x,y,z)\frac{\partial X_i^2(x,y,z,t)}{\partial y^2}dxdydz \\
&= -\int_0^{\hat{z}}\int_0^{\hat{y}}\int_0^{\hat{x}} N_h\beta_y(x,y,z)\frac{\partial X_i(x,y,z,t)}{\partial y}dxdydz. \qquad (57)
\end{aligned}$$

Similarly, in view of (36) and (37), we have

$$\begin{aligned}
&\int_0^{\hat{z}}\int_0^{\hat{y}}\int_0^{\hat{x}} N_v\beta(x,y,z)\frac{\partial X_i^2(x,y,z,t)}{\partial z^2}dxdydz \\
&= -\int_0^{\hat{y}}\int_0^{\hat{x}} \bar{k}N_v\beta(x,y,\hat{z})X_i(x,y,\hat{z},t)dxdy \\
&- \int_0^{\hat{z}}\int_0^{\hat{y}}\int_0^{\hat{x}} N_v\beta_z(x,y,z)\frac{\partial X_i(x,y,z,t)}{\partial z}dxdydz \\
&+ P_{X_i}\left(x_{weight}(t),u_{feed}(t),t\right) \times \int_0^{\hat{z}}\int_0^{\hat{y}}\int_0^{\hat{x}} \beta(x,y,z)dxdydz. \qquad (58)
\end{aligned}$$

Thus, from (55) – (58), we have

$$\begin{aligned}
&\int_0^{\hat{z}}\int_0^{\hat{y}}\int_0^{\hat{x}} \beta(x,y,z)\frac{\partial X_i(x,y,z,t)}{\partial t}dxdydz \\
&= -\int_0^{\hat{z}}\int_0^{\hat{y}}\int_0^{\hat{x}} N_h\beta_x(x,y,z)\frac{\partial X_i(x,y,z,t)}{\partial x}dxdydz \\
&- \int_0^{\hat{z}}\int_0^{\hat{y}}\int_0^{\hat{x}} N_h\beta_y(x,y,z)\frac{\partial X_i(x,y,z,t)}{\partial y}dxdydz \\
&- \int_0^{\hat{y}}\int_0^{\hat{x}} \bar{k}N_v\beta(x,y,\hat{z})X_i(x,y,\hat{z},t)dxdy \\
&- \int_0^{\hat{z}}\int_0^{\hat{y}}\int_0^{\hat{x}} N_v\beta_z(x,y,z)\frac{\partial X_i(x,y,z,t)}{\partial z}dxdydz \\
&+ \int_0^{\hat{z}}\int_0^{\hat{y}}\int_0^{\hat{x}} \beta(x,y,z)S_{X_i}\left(X_1,\dots,X_7,t\right)dxdydz \\
&+ P_{X_i}\left(x_{weight}(t),u_{feed}(t),t\right)\times\int_0^{\hat{z}}\int_0^{\hat{y}}\int_0^{\hat{x}} \beta(x,y,z)dxdydz. \qquad (59)
\end{aligned}$$

Substituting X_i^{36} and $R_m(x,y,z)$ for X_i and β in (59) for all $m \in \{1,...,36\}$, we obtain from (59), (53) and (20) – (26) that

$$\begin{aligned}
&\sum_{i=1}^{36}\dot{T}_{i,l}A_{l,m} = -\sum_{l=1}^{36}T_{i,i}\left(N_h\bar{A}_{l,m}+N_h\tilde{A}_{l,m}+\bar{k}N_vB_{l,m}+N_v\hat{A}_{l,m}\right) \\
&+\sum_{\bar{l}=1}^{7}\left(f_{i,\bar{l}}(t)\sum_{l=1}^{36}T_{\bar{l},l}A_{l,m}\right)+P_{X_i}\left(x_{weight}(t),u_{feed}(t),t\right)\times\bar{B}_m, \qquad (60)
\end{aligned}$$

where

$$A_{l,m} = \int_0^{\hat{z}}\int_0^{\hat{y}}\int_0^{\hat{x}} R_l(x,y,z)R_m(x,y,z)dxdydz, \qquad (61)$$

$$\bar{A}_{l,m} = \int_0^{\hat{z}}\int_0^{\hat{y}}\int_0^{\hat{x}} \frac{\partial R_l(x,y,z)}{\partial x}\frac{\partial R_m(x,y,z)}{\partial x}dxdydz, \qquad (62)$$

$$\tilde{A}_{l,m} = \int_0^{\hat{z}}\int_0^{\hat{y}}\int_0^{\hat{x}} \frac{\partial R_l(x,y,z)}{\partial y}\frac{\partial R_m(x,y,z)}{\partial y}dxdydz, \qquad (63)$$

$$\hat{A}_{l,m} = \int_0^{\hat{z}}\int_0^{\hat{y}}\int_0^{\hat{x}} \frac{\partial R_l(x,y,z)}{\partial z}\frac{\partial R_m(x,y,z)}{\partial z}dxdydz, \qquad (64)$$

$$B_{l,m} = \int_0^{\hat{y}}\int_0^{\hat{x}} R_l(x,y,\hat{z})R_m(x,y,\hat{z})dxdy, \qquad (65)$$

$$\bar{B}_m = \int_0^{\hat{z}}\int_0^{\hat{y}}\int_0^{\hat{x}} R_m(x,y,z)dxdydz, \qquad (66)$$

$$f_{i,\bar{l}}(t) = \begin{cases} (k_{g1}(t) + k_{r1}(t) + k_{d1} + k_{Set1}), \text{ when } i = 1,\ \bar{l} = 1, \\ (k_{r2}(t) + k_{d2}), \text{ when } i = 2,\ \bar{l} = 1, \\ (k_{Set2} + k_{Am2}(t)), \text{ when } i = 2,\ \bar{l} = 2, \\ (k_{g3}(t) + k_{r3}(t)), \text{ when } i = 3,\ \bar{l} = 1, \\ (k_{Am3}(t)), \text{ when } i = 3,\ \bar{l} = 2, \\ (k_{Nit3}(t)), \text{ when } i = 3,\ \bar{l} = 3, \\ (k_{A3}), \text{ when } i = 3,\ \bar{l} = 7, \\ (k_{g4}(t)), \text{ when } i = 4,\ \bar{l} = 1, \\ (k_{Nit4}(t)), \text{ when } i = 4,\ \bar{l} = 3, \\ (k_{r5}(t) + k_{d5}), \text{ when } i = 5,\ \bar{l} = 1, \\ (k_{Set5} + k_{Min5}(t)), \text{ when } i = 5,\ \bar{l} = 5, \\ (k_{g6}(t) + k_{r6}(t)), \text{ when } i = 6,\ \bar{l} = 1, \\ (k_{Min6}(t)), \text{ when } i = 6,\ \bar{l} = 5, \\ (k_{A6}), \text{ when } i = 6,\ \bar{l} = 7, \\ (k_{d7}), \text{ when } i = 7,\ \bar{l} = 1, \\ (k_{Set7}), \text{ when } i = 7,\ \bar{l} = 7, \\ 0, \text{ otherwise.} \end{cases} \tag{67}$$

From (60), we obtain the state equations of the FFWP by the Galerkin scheme as follows:

$$\dot{T}_i(t) = A^{-1}\psi T_i(t) + \sum_{\bar{l}=1}^{7} f_{i,\bar{l}}(t) T_{\bar{l}}(t) + A^{-1}\overline{\overline{B}} \times P_{X_i}\left(x_{weigh}(t), u_{feed}(t), t\right), i = 1, \ldots, 7, \tag{68}$$

where $T_i = (T_{i,1}, \ldots, T_{i,36})^T$, $A = (A_{l,m})_{\substack{l=1,\ldots,36\\ m=1,\ldots,36}}$, $\psi = (\psi_{l,m})_{\substack{l=1,\ldots,36\\ m=1,\ldots,36}}$, $\overline{\overline{B}} = \left(\bar{B}_1, \ldots, \bar{B}_{36}\right)^T$ and

$$\psi_{l,m} = -\left(N_h \bar{A}_{l,m} + N_h \tilde{A}_{l,m} + \bar{k} N_v B_{l,m} + N_v \hat{A}_{l,m}\right), \tag{69}$$

and the initial condition for the above state equation is given by (54).

5.2. **Converting the system of the partial differential equations in the no-fish-feeding water pollution model into ordinary differential equations.** In this sub-section, we first convert the system of partial differential equations (40)– (47) in the NFFWP model into ordinary differential equations by using the Galerkin Scheme. Similar to the FFWP model, we can approximate the instantaneous concentration of the i^{th} water quality variable, $\bar{X}_i(x, y, z, t)$ $(i = 1, \ldots, 7)$, by $\bar{X}_i^{36}(x, y, z, t)$, where

$$\overline{X}_i^{36}(x, y, z, t) = \sum_{l=1}^{36} \overline{T}_{i,l}(t) R_l(x, y, z), \tag{70}$$

and $R_i(x, y, z)$ is as defined after equation (52). From (68), we obtain the state equations of the NFFWP model by the Galerkin scheme as follows:

$$\dot{\overline{T}}_i(t) = A^{-1}\psi \overline{T}_i(t) + \sum_{\bar{l}=1}^{7} f_{i,\bar{l}}(t) \overline{T}_{\bar{l}}(t), \tag{71}$$

where A, ψ and $f_{i,\bar{l}}(t)$ are as defined in (61), (69), and (67) respectively. From (54), the initial condition for the state variables of the NFFWP sub-model is

$$\overline{T}_{i,l}(t_0) = X_{i,l,0},\ i = 1, \ldots, 7,\ l = 1, \ldots, 36, \tag{72}$$

where $X_{i,l,0}$ is the initial concentration of the i^{th} water quality variable at the l^{th} global net-point.

5.3. **Formulation of an approximated optimal control problem with lumped parameter system.** After having converted the systems of partial differential equations of the FFWP model and the NFFWP model into ordinary differential equations, we can formulate the all-time water quality constraints of the approximated optimal control problem. From (50), (53) and (70), the all-time water quality variable concentration constraints are

$$\sum_{l=1}^{36} T_{i,l}(t|u_{feed}(t))D_l - \sum_{l=1}^{36} \overline{T}_{i,l}(t)D_l \leq M_i,\ i = 1, ..., 7, \tag{73}$$

where

$$D_l = \int_0^{\hat{z}} \int_0^{\hat{y}} \int_0^{\hat{x}} R_l(x, y, z)dxdydz. \tag{74}$$

Thus, we can now transform the problem $(P1)$ into an approximated problem $(Q1)$, where $(Q1)$ can be stated as follows:

Problem $(Q1)$: Subject to the differential equation for fishes' weight given by equations (1) and (2), the system (68) and (54) for the FFWP sub-model, the system (71) and (72) for the NFFWP sub-model, the fishes' feeding rate constraint (49), and the water quality variable concentration constraint (73), find a piecewise continuous function $u_{feed}(t)$ which maximizes $J(u_{feed})$ given by (48).

Note that the problem $(Q1)$ consists of 1 control variable, namely, $u_{feed}(t)$ and $14 \times 36 + 1$ state variables, namely, $T_{i,j}(t)$, $\bar{T}_{i,j}(t)$ $(i = 1, ..., 7,\ j = 1, ..., 36)$ and $x_{weight}(t)$.

6. **Modifying the state equations to handle the non-negativity requirement of the instantaneous water quality variable concentrations.** In the section, we use a technique similar to those used in [8], [15], and [16] to handle the non-negativity requirement of the instantaneous water quality variable concentrations.

During the process of solving the problem $(Q1)$ by the control parameterization method, we may encounter a situation where some state variables $T_{i,l}(t)$ generated by the evolution of the state equations (68) and (54) of the FFWP sub-model (respectively, $\bar{T}_{i,l}(t)$ generated by the evolution of the state equations (71) and (72) of the NFFWP sub-model) have negative values. This implies from the properties of the linear spines that the concentration $X_i(x, y, z, t)$ of the FFWP sub-model (respectively $\bar{X}_i(x, y, z, t)$ of the NFFWP sub-model) also takes on negative values, which is physically impossible. In fact, the system governed by (68) and (54) for the FFWP sub-model (respectively, the system governed by (71) and (72) for the NFFWP sub-model) is valid provided that all $T_{i,l}(t)$ (respectively, all $\bar{T}_{i,l}(t)$) remain non-negative all the time.

To handle the non-negativity requirement of $T_{i,l}(t)$, we amend the state equations (68) and (64) of the FFWP sub-model as follows:

By letting

$$\hat{f}(T_i(t), u_{feed}(t), t) = A^{-1}\psi T_i(t) + \sum_{\bar{l}=1}^{7} f_{i,\bar{l}}(t) T_{\bar{l}}(t)$$
$$+ A^{-1}\overline{\overline{B}} \times P_{X_i}(x_{weight}(t), u_{feed}(t), t) \tag{75}$$

in the right-hand sided of (68), we obtain the smooth state equations reflecting the real-life situation of the FFWP sub-model as follows:

$$\dot{T}_{i,l}(t) = \xi_\varepsilon(T_{i,l}(t), (\hat{f}(T_i(t), u_{feed}(t), t)_l), \ i = 1, ..., 7, \ l = 1, ..., 36, \tag{76}$$
$$T_{i,l}(t_0) = X_{x,l,0}, \ i = 1, ...7, \ l = 1, ..., 36, \tag{77}$$

where ε is a small given number and the function $\xi_\varepsilon : R^2 \to R$ is defined by

$$\xi_\varepsilon(y, \bar{y}) = \begin{cases} \bar{y}, & \text{if } y > 0, \\ I_\varepsilon(y) \times \bar{y} + (1 - I_\varepsilon(y)) \times \max_\varepsilon(\bar{y}), & \text{if } -\varepsilon \le y \le 0, \\ \max_\varepsilon(\bar{y}), & \text{if } y < -\varepsilon, \end{cases} \tag{78}$$

and $\max_\varepsilon(\bar{y})$ is the function used for smoothing $\max(\bar{y}, 0)$ defined by

$$\max_\varepsilon(\bar{y}) = \begin{cases} 0, & \text{if } \bar{y} \le -\varepsilon \\ \frac{(\bar{y}+\varepsilon)^2}{4\varepsilon}, & \text{if } -\varepsilon \le \bar{y} \le \varepsilon \ , \\ \bar{y}, & \text{if } \bar{y} > \varepsilon \end{cases} \tag{79}$$

and

$$I_\varepsilon(\bar{y}) = -2\left(\frac{\bar{y}}{\varepsilon}\right)^3 - 3\left(\frac{\bar{y}}{\varepsilon}\right)^2 + 1, \quad -\varepsilon \le y \le 0 \tag{80}$$

is a real number between 0 and 1.

Similarly, to handle the non-negativity requirement of $\bar{T}_{i,l}(t)$, we need to amend the state equations (71) – (72) of the NFFWP as follows:

$$\dot{\bar{T}}_{i,l}(t) = \xi_\varepsilon\left(\bar{T}_{i,l}(t), \left(\hat{\hat{f}}\left(\bar{T}_i(t)\right)\right)_l\right), \ i = 1, ..., 7, \ l = 1, \dots, 36 \tag{81}$$
$$\bar{T}_{i,l}\left(t_0\right) = X_{i,l,0}, \quad i = 1, \dots, 7, \ l = 1, \dots, 36, \tag{82}$$

where

$$\hat{\hat{f}}\left(\bar{T}_i(t)\right) = A^{-1}\psi \bar{T}_i(t) + \sum_{\bar{l}=1}^{7} f_{i,\bar{l}}(t)\bar{T}_{\bar{l}}(t). \tag{83}$$

Thus, by using systems (76) and (77) for finding the concentration of the water quality variables of the FFWP sub-model, and systems (81) and (82) for finding those of the NFFWP sub-model, we obtain the problem $(Q1(\varepsilon))$ as follows:

Problem $(Q1(\varepsilon))$: Subject to the differential equation for fishes' weight given by equations (1) and (2), systems (81) and (82) of the FFWP sub-model, system (76) and (77) of the NFFWP sub-model, the fishes' feeding rate constraint (49), and the all-time water quality variable concentration constraint (73), find a piecewise continuous function $u_{feed}(t)$ which maximizes $J(u_{feed})$ given by (48).

Problem $(Q1(\varepsilon))$ is a standard constrained optimal control problem which can be solved by the control parametrization method.

7. **Numerical result.** In the following numerical example (Example 7.1), the dimensions of the pond are expressed in meter; the concentrations are expressed in kg per cubic meter (or mg per liter); the time t is expressed in the number of days measured from the first of January of a particular year; the weight of the fishes is expressed in kg. Using the above units of measurement, we can describe Example 7.1 as follows:

Example 7.1. Consider the optimal fish-feeding program with the following data: Place where the fish-feeding program takes place: calm freshwater pond. (In other words, the specific growth rate of the fishes is given by equation (3).

Location of the pond $= \{(x, y, z) : x \in [0, \hat{x}], y \in [0, \hat{y}], z \in [0, \hat{z}]\}$, where $\hat{x} = 30$, $\hat{y} = 20$, $\hat{z} = 10$.
Number of Fishes in the pond = 50000.
The initial time of the fish-feeding program: $t_0 = 180$. (i.e., at the end of June)
The final time of the fish-feeding program: $t_f = 550$ (i.e., at the beginning of July of the next year)
The initial weight of each fish: $\hat{x}_{weight} = 0.065$.
The initial concentration of each water quality variable W_i ($i = 1, ..., 7$) at any location $(x, y, z) = 0.5$.
$\bar{k}$ for each water quality variable (i.e., amount of outlet concentration at sea surface/amount of actual concentration at sea level) = 0.02.
The diffusion coefficient N_h in (19) along the O_x and O_y axis = 0.05.
The diffusion coefficient N_v in (19) along the O_z axis = 0.05.
kg increase in fish weight/kg food: $\hat{k} = 0.3$.
Upper bound of the feeding rate: $\overline{u}_{feed}(t) = 1.5$.
Lower bound of the feeding rate: $\underline{u}_{feed}(t) = 0.9$.
The all-time water quality variable concentration constraints are as follows:
Increase in the average concentration of W_3(Ammonia), W_4(Nitrate) and W_6(Inorganic Phosphorous) at any time t during the fish-feeding program is not greater than M_3, M_4 and M_6 respectively, i.e.

$$\int_0^{\hat{z}}\int_0^{\hat{y}}\int_0^{\hat{x}} X_i\left(x, y, z, t \mid u_{feed}(t)\right) dxdydz - \int_0^{\hat{z}}\int_0^{\hat{y}}\int_0^{\hat{x}} \bar{X}_i(x, y, z, t) dxdydz \leq M_i,$$
$$i = 3, 4, 6, \;\; t \in [180, 550], \tag{84}$$

where $M_3 = 3, M_4 = 9$, and $M_6 = 1$.

Remark 7.1. In the above example, we do not need to impose the all-time average concentration constraints for the other water quality variables
(i.e., W_1(Phytoplankton), W_2(Organic Nitrogen), W_5 (Organic Phosphorus), and W_7(Suspended Solid) for the following reasons:

(i) Any fish-feeding program does not create an additional load of W_1 in the pond.
(ii) The differences in the average concentrations of W_2, W_5 and W_7 between the FFWP sub-model and the NFFWP sub-model will be very small, even if the control used is at the upper bound of the fish-feeding rate for all time. (See Figure 5(a), 5(d) and 5(f) given later in this section.)

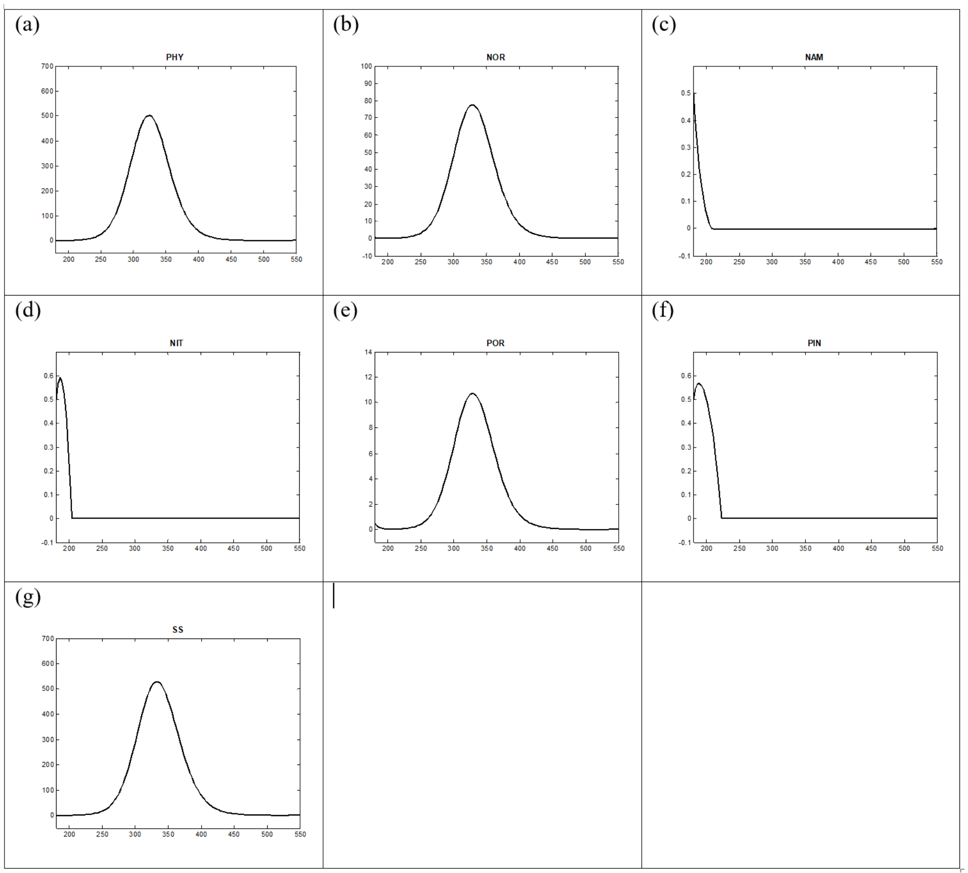

FIGURE 4. Instantaneous average concentration in the No-Fish-Feeding Water Pollution (NFFWP) sub-model

(a) Phytoplankton (PHY) (b) Organic Nitrogen (NOR) (c) Ammonia (NAM) (d) Nitrates (NIT) (e) Organic Phosphorous (POR) (f) Inorganic Phosphorous (PIN) (g) Suspended Solid (SS)

Remark 7.2. From the given data of this numerical example, the initial concentration of each water quality variable is uniformly distributed throughout the pond. This implies that for each $t \in [t_0, t_f]$, the concentration of each water quality variable will be symmetric about the plane $x = 0.5\hat{x}$ and the plane $y = 0.5\hat{y}$. However, the concentrations will not be symmetric about the plane $z = 0.5\hat{z}$ because the boundary condition for the concentration of each water variable at the sea level given by (45) is different from that at the bottom of the pond given by (46). Hence, if we divide the whole domain into 12 sub-regions as that described in Section 4, we can easily reduce the number of state variables in the Galerkin scheme for both the FFWP sub-model and the NFFWP sub-model from 7×36 to 7×12. Thus, the total number of state variables in the optimal fish-feeding problem will be reduced from $14 \times 36 + 1$ to $14 \times 12 + 1$ and the total number of state variables in each of the constraint functions (73) will be reduced from 2×36 to 2×12. Therefore, the computational time required for the execution of one iteration of any optimization routine (such as NLPQL provided by the software Visual MISER in [21]) will be tremendously reduced.

Remark 7.3. When the size of the pond is equal to 30 meter×20 meter×10 meter as given in this example, the partition of cubic domain into 12 sub-domain is sufficient to ensure that the solutions obtained by Galerkin scheme approximation have high degrees of accuracy.

Thus, by dividing the domain into 12 sub-regions as those described in Section 4, we solve the transformed problem $(Q1(\varepsilon))$ with $\varepsilon = 0.00001$ by the combined Galerkin scheme and the control parametrization method. The layouts of the results are given in Figures 4(a) – Figure 4(g), Figure 5(a) – Figure 5(f), Figure 6, Figures 7(a) – 7(f) and Tables 1 – 2.

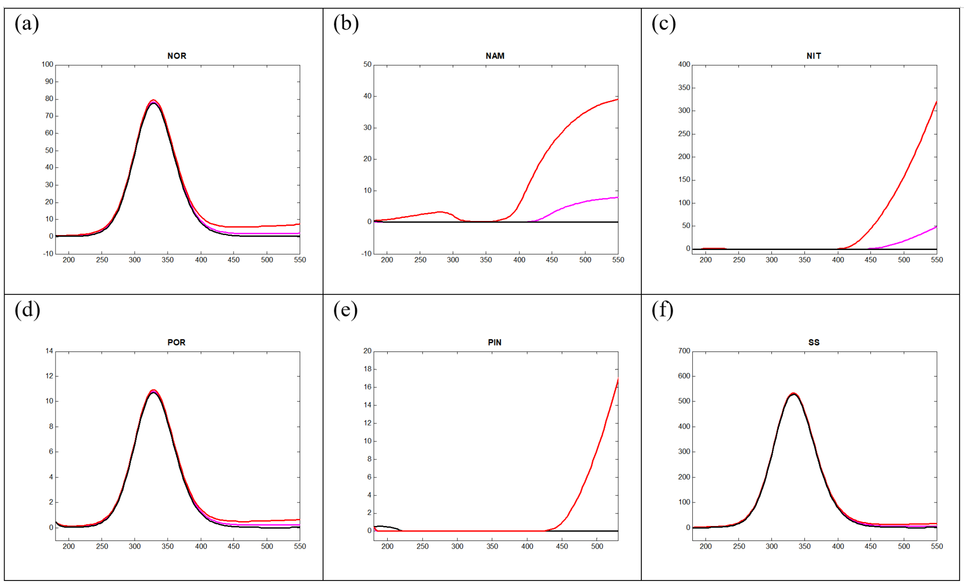

FIGURE 5. Comparison of the instantaneous average concentration between the NFFWP sub-model, the FFWP sub-model with $u(t) = 1.25$ for all $t \in [180, 550]$, and the FFWP sub-model with $u(t) = 1.5$ for all $t \in [180, 550]$

(a) Organic Nitrogen (NOR) (b) Ammonia (NAM) (c) Nitrates (NIT)
(d)Organic Phosphorous(POR) (e)Inorganic Phosphorous(PIN) (f)Suspended Solid(SS)
(The purple and the brown curve in Figure 5(e) almost coincide with each other.)

In Figure 5(a) - Figure 5(f), the black curves represent the instantaneous concentrations of the water quality variables of the No-Fish-Feeding Water Pollution (NFFWP) sub-model, the purple curves represent those of the water quality variables of the Fish-Feeding Water Pollution (NFFWP) sub-model with $u(t) = 1.25$ for all $t \in [180, 550]$, the brown curves represent those of the Fish-Feeding Water Pollution (FFWP) sub-model with $u(t) = 1.5$ for all $t \in [180, 550]$.

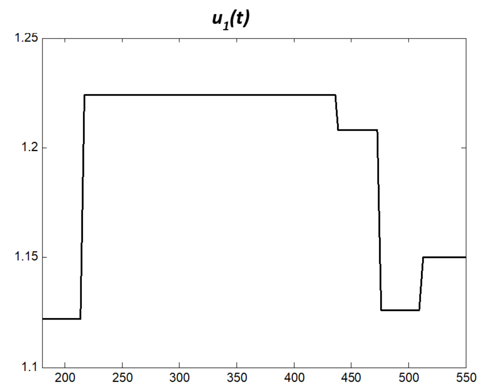

FIGURE 6. The instantaneous optimal control (i.e. the instantaneous optimal fishes' feeding rate)

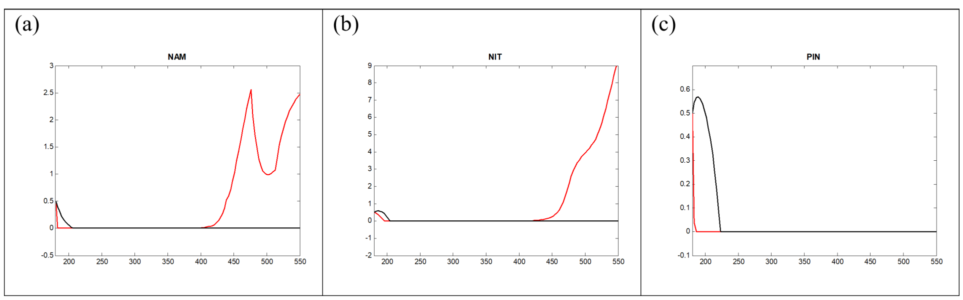

FIGURE 7. Comparison of the instantaneous average concentration between the NFFWP sub-model and the FFWP sub-model obtained by using the optimal control $u^*(t)$.

(a) Ammonia (NAM) (b) Nitrates (NIT) (c) Inorganic Phosphorous (PIN) (In Figure 7(a)-Figure 7(c), the black curves represent the instantaneous concentrations of the water quality variables of the No-Fish-Feeding Water Pollution (NFFWP) sub-model, and the brown curves represent the instantaneous Fish-Feeding Water Pollution (FFWP) sub-model obtained by using the optimal control $u^*(t)$.)

The optimal weight of the fishes at the final time	0.1748
Fishes' weight at the final time obtained by using the maximum allowable feeding rate $\left(\text{i.e. } u(t)=1.5,\ t\in[180,\ 550]\right)$	0.3911

TABLE 1. Weight of the fishes at the final time.

	At $z=0$ meter (the bottom of the pond)				At $z=\hat{z}/2$ meter (midway between sea level and the bottom of the pond)				At $z=\hat{z}$ meter (at sea level)			
(x,y)	(0,0)	$(\hat{x}/3,0)$	$(\hat{y}/2,0)$	$(\hat{x}/3,\hat{y}/2)$	(0,0)	$(\hat{x}/3,0)$	$(\hat{y}/2,0)$	$(\hat{x}/3,\hat{y}/2)$	(0,0)	$(\hat{x}/3,0)$	$(\hat{y}/2,0)$	$(\hat{x}/3,\hat{y}/2)$
PHY	636.68	602.96	411.52	720.83	465.39	487.53	1006.21	217.44	340.24	443.82	91.66	583.88
NOR	99.88	94.59	64.67	113.02	73.09	76.58	157.63	34.35	53.56	69.73	14.71	91.63
NAM	2.66	2.53	2.49	2.63	2.24	2.74	2.54	2.60	2.71	2.36	2.56	2.44
NIT	10.85	10.09	7.55	11.80	5.49	11.58	12.48	8.22	10.94	6.60	7.64	8.39
POR	13.82	13.09	8.96	15.63	10.11	10.60	21.79	4.78	7.43	9.66	2.06	12.68
PIN	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50
SS	671.07	635.53	434.36	759.43	490.89	514.43	1059.35	230.53	359.72	468.38	98.42	615.60

TABLE 2. Maximum concentration at various places of the pond in the fish-feeding water pollution (FFWP) model obtained by using optimal fish-feeding rate

(Note that the concentration of each of the water quality variables is symmetric along the plane $x = 0.5\hat{x}$ and the plane $y = 0.5\hat{y}$.) (Also note that the maximum concentrations of PHY, NOR, POR, and SS are attained at about $t = 330$, i.e. near the end of November, whereas the maximum concentrations of NAM, NIT are attained at $t = 550$, i.e. at the final day of the feeding program and the maximum concentration of PIT is attained at $t = 180$, i.e. at the beginning of the feeding program.)

From the above graphs and tables, we observe the following:

1. Comparison of the graphs of the different water quality variables versus time

From Figures 4(a), 4(b), 4(e), 4(g), 5(a), 5(d), 5(f), 7(a), 7(d), and 7(f), the graphs of the average concentrations of Phytoplankton (PHY), Organic Nitrogen (NOR), Organic Phosphorus (POR), and Suspended Solid (SS) in the NFFWP sub-model and the FFWP sub-model due to any fish-feeding program resemble that of a normal distribution graph, whose maximum concentration occurs at day 330, which is about the end of November. Thus, the average concentrations of these water quality variables at the final time are almost the same as those at the initial time. Moreover, the black, the purple and the brown curves of Figures 5(a), 5(d), and 5(f) have very small separations, which implies that there are no significant differences in the average concentrations of NOR, POR, and SS between the NFFWP sub-model and the FWWP sub-model due to any fish-feeding programs. (As mentioned earlier, any fish-feeding program does not create an additional load of Phytoplankton in the pond.)

On the other hand, from Figures 4(c), 4(d), and 4(f), the concentrations of Ammonia (NAM), Nitrates (NIT), and Inorganic Phosphorus (PIN) in the NFFWP model become zero after a certain period, which occurs at about 160 days, 30 days, and 60 days from the initial time, respectively. (i.e. about the middle of December, the end of July, and the end of August, respectively.) Moreover, the black, the purple, and the brown curves of Figures 5(b), 5(c) have very large separations, which implies that there are large percentage differences in the average concentrations of NAM and NIT between the NFFWP sub-model, the FWWP sub-model with $u(t) = 1.25$, and the FWWP sub-model with $u(t) = 1.5$. Furthermore, the black and the purple curves of Figure 5(e) also have very large separation, but the purple and the brown curves of this figure almost coincide with each other; this implies that there is also a large percentage difference in the average concentration of PIT between the NFFWP sub-model and the FFWP sub-model with $u(t) = 1.25$, but almost no difference in that between the FFWP sub-model with $u(t) = 1.25$, and the FFWP sub-model with $u(t) = 1.5$. Thus, if uncontrolled, the fish-feeding program can create a huge increase in NAM, NIT, and PIN, especially NIT. (More significantly, Figure 5(c) shows that by using the maximum fishes' feeding rate $u(t) = 1.5$, the average concentration of NIT in the pond rises from almost zero at the beginning of the fishes' feeding period to more than 300 at the end of the fishes' feeding program.)

2. Comparison of the optimal feeding rate with the maximum allowable feeding rate; comparison of the optimal fishes' weight at harvest with fishes' weight at harvest obtained by using the maximum allowable feeding rate

From Figure 6, the optimal fishes' feeding rate $u^*(t)$ of this example is between 1.11 and 1.21, which is between 74% and 80% of the maximum allowable fishes' feeding rate $u(t) = 1.5$. From Table 1, the optimal fishes' weight at harvest is about 56.76% less than that at harvest obtained by using the maximum allowable fishes' feeding rate.

3. Comparing the tendency of violating the environmental protection requirement of each water quality variable

From Remark 7.1, Figures 5(a), 5(d), and 5(f), the differences in the average concentration of each of these water quality variables PHY, NOR, POR, SS between the FFWP sub-model and the NFFWP sub-model are either zero or very small. Thus, these water quality variables can satisfy the environmental protection requirement very easily. From Figures 7(a), 7(b), and 7(c), the constraints for the average concentrations of NAM and PIN (Constraints (7.1) and (7.3)) are unbinding at the optimal solution, whereas the constraint for the average concentration of NIT (Constraint (7.2)) is binding at the optimal solution when $t = t_f$ (i.e., at the final time of the fishes' feeding program.) This implies that the water quality variables NAM and PIN can also satisfy the environmental protection requirement constraints easily, whereas the water quality variable NIT can barely satisfy the environmental protection requirement.

4. Comparing the concentration of the water quality variables at various places of the pond

From Table 2, for each water quality variable W_i, the average concentration of the FFWP model obtained by using optimal control is largest at $z = 0$ (i.e. at the bottom of the pond), which is slightly larger than that at $z = 0.5\hat{z}$ (i.e. at midway between the bottom of the pond and the sea level). The average concentration at $z = \hat{z}$ (i.e. at the sea level) is much smaller than those at $z = 0.5\hat{z}$ and at $z = 0$. This phenomenon is due to the boundary condition (37), which states that the gradient of the outlet concentration at sea level is proportional to the concentration itself but is decreasing. At each depth $z = 0, z = 0.5\hat{z}$ and $z = \hat{z}$ of the pond, the concentration of each water quality variable W_i is not uniformly distributed in the x-y plane. However, as mentioned in Remark 7.2, since the initial concentration of each water quality variable is uniformly distributed throughout the pond, the concentration of each water quality is always symmetric about the plane $x = 0.5\hat{x}$ and the plane $y = 0.5\hat{y}$.

8. **Conclusion.** Based on seven water quality variables, the optimal fish-feeding process in a pond at rest has been analyzed by coupling an algebraic equations system for fish growth and nitrogen and phosphorus transformation with a standard diffusion (no convection) water quality model from the existing literature. A computational scheme using the finite element Galerkin scheme for the three-dimensional cubic domain and the control parameterization method has been developed for

finding the optimal control of fish-feeding in this calm water pond considering environmental concern.

The optimal fishes' feeding rate and the optimal fishes' weight at harvest are obtained. The instantaneous concentration of each water quality variable of the no-fish-feeding water pollution (NFFWP) sub-model, as well as those obtained by using optimal control of the fish-feeding water pollution (FFWP) sub-model at each grid point of the pond, are also calculated. Intuitive explanations have been given, justifying all the computational results concerning the optimal fishes' feeding rate, the optimal fishes' weight at harvest, and the corresponding instantaneous concentration of each water quality variable at each grid point of the pond. The extension of our method to finding the optimal control of fish-feeding in a windy pond (instead of a calm water pond) will be very challenging research because it requires incorporating a hydrodynamic sub-model to calculate the velocities flow into our water pollution model, which generates more state variables.

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- Funding

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APPENDIX

TABLE A1. Processes described in the Three-Dimensional Water Pollution Model and their equations

D. Decay of PHY (D_1) with releases of NOR (D_2), POR (D_5), and SS (D_7) .
$D_1 = k_{d1} \times X_{PHY}, D_2 = k_{d2} \times X_{PHY}, D_5 = k_{d5} \times X_{PHY}, D_7 = k_{d7} \times X_{PHY}$,
where
(i) $k_{d1} = -k_d, k_{d2} = 0.1761 \times k_d, k_{d5} = 0.1761 \times k_d, k_{d7} = 0.379 \times k_d$,
(ii) k_d is a parameter whose value is given in Table 2 .

G. Photosynthesis (Growth of PHY, G_1), with uptakes of NAM (G_3), NIT(G_4), and PIN(G_6).
$G_1 = k_{g1}(t) \times X_{PHY}$, $G_3 = k_{g3}(t) \times X_{PHY}$, $G_4 = k_{g4}(t) \times X_{PHY}$, $G_6 = k_{g6}(t) \times X_{PHY}$,
where
(i) $k_{g1}(t) = k_{g\max}(t) \times ef(L(t)) \times [\ ef(N) + ef(P)]$
is the growth rate of PHY;
$k_{g\max}(t) = k_{g\max 20} \times 1.047^{Tenp(t)-20}$
is the effect of the temperature on the growth of PHY at temperature $Temp(t)°C$, where $Temp(t)$ is the water temperature in at time t given by
$Temp(t) = Temp_{\min} + \frac{(Temp_{\max} - Temp_{\min})}{2}\left(1 - \cos\frac{360(t-76)}{365} \times \frac{\pi}{180}\right)$,
with $Temp_{\min} = 13.7$ and $Temp_{\max} = 25.9$;
$ef(L(t)) = \frac{L(t)}{L_S}\exp\left(1 - \frac{L(t)}{L_S}\right)$ is the effect of light on the growth rate of PHY, where $L(t)$ is the incident solar radiation (expressed in cal/cm^2) given by

$$L(t) = L_{\min} + \frac{(L_{\max} - L_{\min})}{2}\left(1 - \cos\frac{(t-15) \times 360}{365} \times \frac{\pi}{180}\right)$$

with $L_{\min} = 120\text{cal/cm}^2$, and $L_{\max} = 192\text{cal/cm}^2$;
$ef(N)$ is the effect of nutrients due to the uptake of NAM and NIT,
$ef(P)$ is the effect of nutrients due to the uptake of PIN; their average values are given in Table $A2$; $k_{g\max 20}$ and L_S are parameters, whose values are given in Table A2,
(ii) $k_{g3}(t) = -0.1761 \times P_{NAM} \times k_{g1}(t)$,
$k_{g4}(t) = -0.1761 \times (1 - P_{NAM})\, k_{g1}(t)$, $k_{g6}(t) = -0.1761 \times k_{g1}(t)$,
P_{NAM} is the preference term for NAM whose approximated value is also given in Table A2.

A. Adsorption of NAM (A_3) and PIN (A_6).

$$A_3 = k_{A3} \times X_{SS},\ \ A_6 = k_{A6} \times X_{SS},$$

where
(i) $k_{A3} = -\frac{SV_{SS}}{H} \times a_{NAM}, k_{A6} = -\frac{SV_{SS}}{H} \times a_{PIN}$,
(ii) $SV_{SS}, a_{NAM}, H, a_{PIN}$ are parameters whose values are given in Table 2.

Set. Settling of PHY (S_1), NOR (S_2), POR (S_5), and SS (S_7)

$$Set_1 = k_{Set1} \times X_{PHY},\ Set_2 = k_{Set2} \times X_{NOR},\quad Set_5 = k_{Set5} \times X_{POR},$$
$$Set_7 = k_{Set7} \times X_{SS},$$

where
(i) $k_{Set1} = -\frac{SV_{PHY}}{H},\ k_{Set2} = -\frac{SV_{NOR}}{H}(1 - C_{NOR})$,
$k_{Set5} = -\frac{SV_{POR}}{H}(1 - C_{POR}),\ k_{Set7} = -\frac{SV_{SS}}{H}$,
(ii) $SV_{PHY},\ SV_{SS},\ SV_{NOR},\ SV_{POR},\ C_{NOR},\ C_{POR}$ and H are parameters whose values are given in Table 2.

Am. Ammonification - Mineralization of NOR
$Am_2 = k_{Am2}(t) \times X_{NOR}, Am_3 = k_{Am3}(t) \times X_{NOR}$ where
(i) $k_{Am2}(t) = -k_{Nmin}(t) \times c_{NOR}, k_{Am3}(t) = k_{Nmin}(t) \times c_{NOR}$,
(ii) $k_{Nmin}(t) = k_{Nmin} \times 1.047^{Temp(t)-20}$
is the saturation constant for Nitrogen mineralization at $Temp(t)^oC$, where the formula for $Temp(t)$ is as given in the Photosynthesis process,
(iii) c_{NOR} is a parameter whose value is given in Table 2.

N. Nitrification (Nit_3 and Nit_4).

$$Nit_3 = k_{Nit3}(t) \times X_{NAM}, Nit_4 = k_{Nit4}(t) \times X_{NAM},$$

where
(i) $k_{Nit3}(t) = -k_{Nit}(t), k_{Nit4}(t) = k_{Nit}(t)$,
(ii) $k_{Nit}(t) = k_{Nit20} \times 1.047^{Temp(t)-20}$ is the nitrification rate at $Temp(t)^oC$, where the formula for $Temp(t)$ is as given in the Photosynthesis process.
(iii) k_{Nit20} is a parameter whose value is given in Table A2.

R. Endogenous respiration of PHY (R_1) with the release of NOR (R_2), NAM (R_3), POR (R_5) and PIN (R_6).

$$R_1 = k_{r1}(t) \times X_{PHY},\ R_2 = k_{r2}(t) \times X_{PHY},\ R_3 = k_{r3}(t) \times X_{PHY},$$
$$R_5 = k_{r5}(t) \times X_{PHY},\ R_6 = k_{r6}(t) \times X_{PHY},$$

where
(i) $k_{r1}(t) = -k_r(t)$,
$k_{r2}(t) = 0.1761 \times f_{NOR} \times k_r(t)$,
$k_{r3}(t) = 0.1761 \times (1 - f_{NOR}) \times k_r(t)$
$k_{r5}(t) = 0.1761 \times f_{POR} \times k_r(t)$,
$k_{r6}(t) = 0.1761 \times (1 - f_{POR}) \times k_r(t)$,
(ii) $k_r(t) = k_{r20} \times 1.047^{Temp(t)-20}$ is the respiration rate at $Temp(t)^oC$, where the formula for $Temp(t)$ is as given in the Photosynthesis process,
(iii) k_{r20}, f_{NOR} are parameters whose values are given in Table 2.

P. Mineralization of POR (P_5 and P_6).

$$Min_5 = k_{Min5}(t) \times X_{POR}, Min_6 = k_{Min6}(t) \times X_{POR}$$

where
(i) $k_{min5}(t) = -k_{Pmin}(t) \times c_{POR}, k_{Min6}(t) = k_{Pmin}(t) \times c_{POR}$,
(ii) $k_{Pmin} = k_{Pmin20} \times 1.047^{Temp(t)-20}$
is the mineralization rate of NOR at $Temp(t)^oC$, where the formula for $Temp(t)$ is as given in the Photosynthesis process,
(iii) c_{POR} is a parameter whose value is given in Table A2.

TABLE A2. Parameters of the three-dimensional water pollution model and their values

Symbol	Process	Coefficient	Values
k_d	D	Decay of PHY	0.04
k_{g20}	G	Max growth of PHY at 20°C	1.7
k_{Nmin20}	Am	Mineralization of NOR at 20°C	0.10
k_{Pmin20}	Min	Mineralization of POR at 20°C	0.10
k_{r20}	R	Endogenous respiration of PHY at 20°C	0.18
k_{nit20}	N	Nitrification rate at 20°C	0.10
a_{NAM}	A	Adsorption coefficient for NAM	0.15
a_{PIN}	A	Adsorption coefficient for PIN	0.05
c_{NOR}	S, Am	Fraction of NOR	0.50
c_{POR}	S,P	Fraction of POR	0.50
f_{NOR}	R	Fraction of dead algae recycled to NOR	0.50
f_{POR}	R	Fraction of dead algae recycled to POR	0.50
$ef(N)$	G	Effect of nutrients due to the uptake of NAM and NIT.	0.105
$ef(P)$	G	Effect of nutrients due to the uptake of PIN	0.105
P_{NAM}	G	Preference term for NAM	0.25
L_S	G	Optimal light intensity	250
SV_{PHY}	S	Maximum settling velocity of PHY	0.10
SV_{NOR}	S	Maximum settling velocity of NOR	0.20
SV_{POR}	S	Maximum settling velocity of POR	0.20
SV_{SS}	S, A	Maximum settling velocity of SS	0.10
H	S, A	Constant in the Settling and Adsorption Process	1.00

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