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OPEN-LOOP SOLVABILITY FOR MEAN-FIELD STOCHASTIC LINEAR QUADRATIC OPTIMAL CONTROL PROBLEMS OF MARKOV REGIME-SWITCHING SYSTEM

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Abstract. This paper investigates the mean-field stochastic linear quadratic optimal control problem of Markov regime switching system (M-MF-SLQ, for short). The representation of the cost functional for the M-MF-SLQ is derived using the technique of operators. It is shown that the convexity of the cost functional is necessary for the finiteness of the M-MF-SLQ problem, whereas uniform convexity of the cost functional is sufficient for the open-loop solvability of the problem. By considering a family of uniformly convex cost functionals, a characterization of the finiteness of the problem is derived and a minimizing sequence, whose convergence is equivalent to the open-loop solvability of the problem, is constructed. We demonstrate with a few examples that our results can be employed for tackling some financial problems such as mean-variance portfolio selection problem.

Linear-quadratic (LQ, for short) optimal control problem plays important roles in control theory. It is a classical and fundamental problem in the field of control theory. In the past few decades, both deterministic and stochastic linear quadratic control problems are widely studied. Stochastic linear quadratic (SLQ, for short) optimal control problem was first considered by Kushner [12] using dynamic programming method. Later, Wonham [23] studied the generalized version of the matrix Riccati equation arose in the problems of stochastic control and filtering. Using functional analysis techniques, Bismut [2] proved the existence of the Riccati equation and derived the existence of the optimal control in a random feedback form for stochastic LQ optimal control with random coefficients.

One extension to SLQ problems is to involve the mean-field term in the state system and the cost functional, which is called mean-field stochastic linear-quadratic optimal control problem (MF-SLQ, for short). The theory of the mean-field stochastic differential equation (MF-SDE, for short) can be traced back to Kac [11], who proposed the McKean-Vlasov stochastic differential equation motivated by a

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stochastic toy model for the Vlasov kinetic equation of plasma. Since then, research on related topics and their applications has been extensively studied in the areas of applied probability and stochastic controls, particularly in financial engineering. For instance, Buckdahn-Djehiche-Li-Peng [4] formulated mean-field backward stochastic differential equations, Andersson-Djehiche [1], and Buckdahn-Djehiche-Li [3] established stochastic maximum principles for a class of mean-field stochastic controls, Huang-Li-Yong [10] and Yong [28] studied MF-SLQ controls by decoupled Riccati equations, Cui-Li-Li [5] applied mean-field formulations to study optimal multi-period mean-variance portfolio selection, Elliott-Li-Ni [9] established necessary and sufficient conditions for the solvability of discrete-time MF-SLQ problems.

Another extension to SLQ problems is to involve random jumps in the state systems, such as Poisson jumps or the regime switching jumps. Wu and Wang [24] was the first to consider the SLQ problems with Poisson jumps and obtain the existence and uniqueness of the deterministic Riccati equation. Existence and uniqueness of the stochastic Riccati equation with jumps and connections between the stochastic Riccati equation with jumps and the associated Hamilton systems of stochastic LQ optimal control problem were also presented. Yu [29] investigated a kind of infinite horizon backward stochastic LQ optimal control problems and differential game problems under the jump-diffusion model state system. Li et al. [15] solved the indefinite SLQ problem with Poisson jumps.

The stochastic control problems involving regime switching jumps are of interest and of practical importance in various fields such as science, engineering, financial management and economics. The regime-switching models and related topics have been extensively studied in the areas of applied probability and stochastic controls. More recently, there has been dramatically increasing interest in studying this family of stochastic control problems as well as their financial applications. For instance, Li-Zhou [13] and Li-Zhou-Ait Rami [15] introduced indefinite stochastic LQ controls with Markovian jumps, Liu-Yin-Zhou [17] considered near optimal controls of regime-switching LQ problems with indefinite control weight costs, Donnelly [6] analyzed the stochastic maximum principle for the optimal control of a regime-switching diffusion model, Tao-Wu [22] investigated the stochastic maximum principle for optimal control problems of forward-backward regime-switching systems. In the finance field, investors could face two market regimes, one of which represents a bull market with price increase, while the other regime represents a bear market with price drop. Therefore, the regime-switching type portfolio selection problem is of great interest and importance in financial investment. Typical examples that are applicable include, but are not limited to, those presented in Yiu-Liu-Siu-Ching [26], Donnelly-Heunis [7] and etc.

Recently, Sun [21] investigated the open-loop solvability for MF-SLQ problem. It was shown in [21] that the open-loop solvability is equivalence to the existence of an adapted solution to a forward-backward stochastic differential equation (FBSDE, for short) with constraint. As a continuation work of [21], Li et al. [14] studied the closed-loop solvability for MF-SLQ problems. Moreover, the equivalence between the strongly regular solvability of the Riccati equation and the uniform convexity of the cost functional is established.

In this paper, we extend the mean-field results of Andersson-Djehiche [1], Buckdahn-Djehiche-Li [3] and Yong [28] to the cases involving random coefficients with regime-switching. To the best of our knowledge, such a problem has never been studied. This can be regarded as an extension of Sun [21] to the case of MF-SLQ problems with regime switching jumps. In particular, we develop the sufficient and necessary condition for the open-loop solution. The solvability and uniqueness of the problem is also discussed. In addition, we derive the minimizing sequence that can converge to the solution of the problem. Meanwhile, our results can be applied to real-world financial investment problems. We illustrate the results with 3 examples in which the mean-variance portfolio selection problem can be considered as a special case of our problem.

The rest of the paper is organized as follows. In Section 2, we introduce some useful notations, summarize some preliminary results and state the M-MF-SLQ problem. In Section 3, we study Problem (M-MF-SLQ) in a Hilbert space and derive necessary and sufficient conditions for the finiteness and open-loop solvability of the problem by considering a family of uniformly convex cost functionals. In Section 4, some examples are considered. We investigate the mean-variance portfolio selection problem and derive the solution. In Section 5, we conclude the results of the paper and suggest some future research extensions.

1. **Preliminaries and model formulation.** Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space on which a standard one-dimensional Brownian motion $W = \{W(t); 0 \leq t < \infty\}$ and a continuous time, finite-state, Markov chain $\alpha = \{\alpha(t); 0 \leq t < \infty\}$ are defined, where $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ is the natural filtration of W and α augmented by all the $\mathbb{P}$-null sets in $\mathcal{F}$. For the rest of our paper, we will use the following notations:

$\mathbb{N}$:	the set of natural numbers;
$\mathbb{R}_+, \overline{\mathbb{R}}_+$:	the sets $[0, \infty)$ and $[0, +\infty]$ respectively;
$\mathbb{R}^n$:	the n-dimensional Euclidean space;
$M^\top$:	the transpose of any vector or matrix M;
$\operatorname{tr}[M]$:	the trace of a square matrix M;
$\mathcal{R}(M)$:	the range of the matrix M;
$\langle \cdot, \cdot \rangle$:	the inner products in possibly different Hilbert spaces;
$M^\dagger$:	the Moore-Penrose pseudo-inverse of the matrix M;
$\mathbb{R}^{n\times m}$:	the space of all $n \times m$ matrices endowed with the inner product $\langle M, N\rangle \mapsto \operatorname{tr}[M^\top N]$ and the norm $\lvert M\rvert = \sqrt{\operatorname{tr}[M^\top M]}$;
$\mathbb{S}^n$:	the set of all $n \times n$ symmetric matrices;
$\overline{\mathbb{S}^n_+}$:	the set of all $n \times n$ positive semi-definite matrices;
$\mathbb{S}^n_+$:	the set of all $n \times n$ positive-definite matrices.

Next, let $T > 0$ be a fixed time horizon. For any $t \in [0, T)$ and Euclidean space $\mathbb{H}$, let

$$
\begin{aligned}
C([t,T];\mathbb{H}) &= \Big\{\varphi : [t,T] \to \mathbb{H} \;\Big|\; \varphi(\cdot) \text{ is continuous}\Big\},\\
L^p(t,T;\mathbb{H}) &= \left\{\varphi : [t,T] \to \mathbb{H} \;\middle|\; \int_t^T |\varphi(s)|^p ds < \infty\right\}, \quad 1 \leq p < \infty,\\
L^\infty(t,T;\mathbb{H}) &= \left\{\varphi : [t,T] \to \mathbb{H} \;\middle|\; \operatorname*{esssup}_{s\in[t,T]} |\varphi(s)| < \infty\right\}.
\end{aligned}
$$

We denote

$$L^2_{\mathcal{F}_T}(\Omega;\mathbb{H}) = \Big\{\xi:\Omega\to\mathbb{H} \;\Big|\; \xi \text{ is } \mathcal{F}_T\text{-measurable}, \ \mathbb{E}|\xi|^2<\infty\Big\},$$

$$L^2_{\mathbb{F}}(t,T;\mathbb{H}) = \Bigg\{\varphi:[t,T]\times\Omega\to\mathbb{H} \;\Big|\; \varphi(\cdot) \text{ is } \mathbb{F}\text{-progressively measurable}, \ \mathbb{E}\int_t^T |\varphi(s)|^2 ds<\infty\Bigg\},$$

$$L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{H})) = \Bigg\{\varphi:[t,T]\times\Omega\to\mathbb{H} \;\Big|\; \varphi(\cdot) \text{ is } \mathbb{F}\text{-adapted, continuous}, \ \mathbb{E}\left[\sup_{s\in[t,T]}|\varphi(s)|^2\right]<\infty\Bigg\},$$

$$L^2_{\mathbb{F}}(\Omega;L^1(t,T;\mathbb{H})) = \Bigg\{\varphi:[t,T]\times\Omega\to\mathbb{H} \;\Big|\; \varphi(\cdot) \text{ is } \mathbb{F}\text{-progressively measurable}, \ \mathbb{E}\left(\int_t^T |\varphi(s)|ds\right)^2<\infty\Bigg\}.$$

For an $\mathbb{S}^n$-valued function $F(\cdot)$ on $[t,T]$, we use the notation $F(\cdot)\gg 0$ to indicate that $F(\cdot)$ is uniformly positive definite on $[t,T]$, i.e., there exists a constant $\delta>0$ such that

$$F(s)\geq \delta I, \qquad \text{a.e.} \quad s\in[t,T].$$

To formulate our system, we identify the state space of the chain α with a finite set $\mathcal{S}:=\{1,2\ldots,D\}$, where $D\in\mathbb{N}$ and suppose that the chain is homogeneous and irreducible. To specify statistical or probabilistic properties of the chain α, we define the generator $\lambda(t):=[\lambda_{ij}(t)]_{i,j=1,2,\ldots,D}$ of the chain under $\mathbb{P}$. This is also called the rate matrix, or the Q-matrix. Here, for each $i,j=1,2,\ldots,D$, $\lambda_{ij}(t)$ is the constant transition intensity of the chain from state i to state j at time t. Note that $\lambda_{ij}(t)\geq 0$, for $i\neq j$ and $\sum_{j=1}^D\lambda_{ij}(t)=0$, so $\lambda_{ii}(t)\leq 0$. In what follows, for each $i,j=1,2,\ldots,D$ with $i\neq j$, we suppose that $\lambda_{ij}(t)>0$, so $\lambda_{ii}(t)<0$. For each fixed $j=1,2,\cdots,D$, let $N_j(t)$ be the number of jumps into state j up to time t and set

$$\lambda_j(t):=\int_0^t \lambda_{\alpha(s-)\,j} I_{\{\alpha(s-)\neq j\}}ds=\sum_{i=1,i\neq j}^D\int_0^t\lambda_{ij}(s)I_{\{\alpha(s-)=i\}}ds.$$

Following Elliott et al. [8], we have that for each $j=1,2,\cdots,D$,

$$\widetilde{N}_j(t):=N_j(t)-\lambda_j(t) \tag{1}$$

is an $(\mathbb{F},\mathbb{P})$-martingale.

Consider the following controlled Markov regime switching linear stochastic differential equation (SDE, for short) on a finite horizon $[t,T]$:

$$\begin{cases} dX^u(s;t,x,i)=\Big\{A(s,\alpha(s))X^u(s;t,x,i)+B(s,\alpha(s))u(s)+b(s,\alpha(s))\Big\}ds \\ \qquad\qquad +\Big\{C(s,\alpha(s))X^u(s;t,x,i)+D(s,\alpha(s))u(s)+\sigma(s,\alpha(s))\Big\}dW(s), \\ X^u(t;t,x,i)=x, \quad \alpha(t)=i, \qquad s\in[t,T], \end{cases} \tag{2}$$

where $A(\cdot,\cdot), B(\cdot,\cdot), C(\cdot,\cdot), D(\cdot,\cdot)$ are given deterministic matrix-valued functions of proper dimensions, and $b(\cdot,\cdot), \sigma(\cdot,\cdot)$ are vector-valued $\mathbb{F}$-progressively measurable processes. In the above, $X^u(\cdot\,; t, x, i)$, valued in $\mathbb{R}^n$, is the *state process*, and $u(\cdot)$, valued in $\mathbb{R}^m$, is the *control process*. Any $u(\cdot)$ is called an *admissible control* on $[t, T]$, if it belongs to the following Hilbert space:

$$\mathcal{U}[t,T] = \left\{ u : [t,T] \times \Omega \to \mathbb{R}^m \;\middle|\; u(\cdot) \text{ is } \mathbb{F}\text{-progressively measurable, } \mathbb{E}\int_t^T |u(s)|^2 ds < \infty \right\}.$$

For any admissible control $u(\cdot)$, we consider the following general quadratic cost functional:

$$\begin{aligned}
J(t,x,i;u(\cdot)) := \mathbb{E}\Bigg\{ & \Big\langle G(\alpha(T)) X^u(T;t,x,i), X^u(T;t,x,i) \Big\rangle + 2\Big\langle g(\alpha(T)), X^u(T;t,x,i) \Big\rangle \\
& + \Big\langle \bar{G}(\alpha(T)) \mathbb{E}[X^u(T;t,x,i)], \mathbb{E}[X^u(T;t,x,i)] \Big\rangle + 2\Big\langle \bar{g}(\alpha(T)), \mathbb{E}[X^u(T;t,x,i)] \Big\rangle \\
& + \int_t^T \left[\left\langle \begin{pmatrix} Q(s,\alpha(s)) & S(s,\alpha(s))^\top \\ S(s,\alpha(s)) & R(s,\alpha(s)) \end{pmatrix} \begin{pmatrix} X^u(s;t,x,i) \\ u(s) \end{pmatrix}, \begin{pmatrix} X^u(s;t,x,i) \\ u(s) \end{pmatrix} \right\rangle \right] ds \\
& + \int_t^T \left[\left\langle \begin{pmatrix} \bar{Q}(s,\alpha(s)) & \bar{S}(s,\alpha(s))^\top \\ \bar{S}(s,\alpha(s)) & \bar{R}(s,\alpha(s)) \end{pmatrix} \begin{pmatrix} \mathbb{E}[X^u(s;t,x,i)] \\ \mathbb{E}[u(s)] \end{pmatrix}, \begin{pmatrix} \mathbb{E}[X^u(s;t,x,i)] \\ \mathbb{E}[u(s)] \end{pmatrix} \right\rangle \right] ds \\
& + 2\int_t^T \left[\left\langle \begin{pmatrix} q(s,\alpha(s)) \\ \rho(s,\alpha(s)) \end{pmatrix}, \begin{pmatrix} X^u(s;t,x,i) \\ u(s) \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} \bar{q}(s,\alpha(s)) \\ \bar{\rho}(s,\alpha(s)) \end{pmatrix}, \begin{pmatrix} \mathbb{E}[X^u(s;t,x,i)] \\ \mathbb{E}[u(s)] \end{pmatrix} \right\rangle \right] ds \Bigg\},
\end{aligned} \tag{3}$$

where $G(i)$, $\bar{G}(i)$ are symmetric matrices, $Q(\cdot,i)$, $\bar{Q}(\cdot,i)$, $S(\cdot,i)$, $\bar{S}(\cdot,i)$, $R(\cdot,i)$, $\bar{R}(\cdot,i)$, $i = 1,\cdots,D$ are deterministic matrix-valued functions of proper dimensions with $Q(\cdot,i)^\top = Q(\cdot,i)$, $\bar{Q}(\cdot,i)^\top = \bar{Q}(\cdot,i)$, $R(\cdot,i)^\top = R(\cdot,i)$, $\bar{R}(\cdot,i)^\top = \bar{R}(\cdot,i)$; $g(\cdot)$ is allowed to be an $\mathcal{F}_T$-measurable random variable and $\bar{g}(\cdot)$ is a deterministic vector; $q(\cdot,\cdot)$, $\rho(\cdot,\cdot)$ are allowed to be vector-valued $\mathbb{F}$-progressively measurable processes and $\bar{q}(\cdot,\cdot)$, $\bar{\rho}(\cdot,\cdot)$ are vector-valued deterministic functions.

The following standard assumptions will be in force throughout this paper.

(H1) The coefficients of the state equation satisfy the following: for each $i \in \mathcal{S}$,

$$\begin{cases} A(\cdot,i) \in L^1(0,T;\mathbb{R}^{n\times n}),\ B(\cdot,i) \in L^2(0,T;\mathbb{R}^{n\times m}),\ b(\cdot,i) \in L^2_{\mathbb{F}}(\Omega; L^1(0,T;\mathbb{R}^n)), \\ C(\cdot,i) \in L^2(0,T;\mathbb{R}^{n\times n}),\ D(\cdot,i) \in L^\infty(0,T;\mathbb{R}^{n\times m}),\ \sigma(\cdot,i) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n). \end{cases}$$

(H2) The weighting coefficients in the cost functional satisfy the following: for each $i \in \mathcal{S}$,

$$\begin{cases} Q(\cdot,i), \bar{Q}(\cdot,i) \in L^1(0,T;\mathbb{S}^n),\ S(\cdot,i), \bar{S}(\cdot,i) \in L^2(0,T;\mathbb{R}^{m\times n}), \\ g(i) \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n), \quad \bar{g}(i) \in \mathbb{R}^n, \quad R(\cdot,i), \bar{R}(\cdot,i) \in L^\infty(0,T;\mathbb{S}^m), \\ q(\cdot,i) \in L^2_{\mathbb{F}}(\Omega; L^1(0,T;\mathbb{R}^n)), \quad \bar{q}(\cdot,i) \in L^1(0,T;\mathbb{R}^n), \\ \rho(\cdot,i) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m), \quad \bar{\rho}(\cdot,i) \in L^2(0,T;\mathbb{R}^m), \quad G(i), \bar{G}(i) \in \mathbb{S}^n. \end{cases}$$

We can state the mean-field stochastic LQ optimal control problem for the Markov regime switching system as follows.

Problem. (M-MF-SLQ) *For any given initial pair $(t,x,i) \in [0,T) \times \mathbb{R}^n \times \mathcal{S}$, find a $u^*(\cdot) \in \mathcal{U}[t,T]$, such that*

$$J(t,x,i;u^*(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}[t,T]} J(t,x,i;u(\cdot)) := V(t,x,i). \tag{4}$$

Any $u^*(\cdot)\in\mathcal{U}[t,T]$ satisfying (4) is called an *optimal control* of Problem (M-MF-SLQ) for the initial pair (t,x,i), and the corresponding path $X^*(\cdot)\equiv X^{u^*}(\cdot\,;t,x,i)$ is called an *optimal state process*; the pair $(X^*(\cdot),u^*(\cdot))$ is called an *optimal pair*. The function $V(\cdot,\cdot,\cdot)$ is called the *value function* of Problem (M-MF-SLQ). When $b(\cdot,\cdot),\sigma(\cdot,\cdot),g(\cdot),\bar{g}(\cdot),\ q(\cdot,\cdot),\ \bar{q}(\cdot,\cdot),\ \rho(\cdot,\cdot),\ \bar{\rho}(\cdot,\cdot)=0$, we denote the corresponding Problem (M-MF-SLQ) by Problem (M-MF-SLQ)^0. The corresponding cost functional and value function are denoted by $J^0(t,x,i;u(\cdot))$ and $V^0(t,x,i)$, respectively. We now introduce the following definition.

Definition 1.1. (i) Problem (M-MF-SLQ) is said to be *finite at initial pair* $(t,x,i)\in[0,T]\times\mathbb{R}^n\times\mathcal{S}$ if

$$V(t,x,i)>-\infty. \tag{5}$$

Problem (M-MF-SLQ) is said to be *finite at* $t\in[0,T]$ if (5) holds for all $(x,i)\in\mathbb{R}^n\times\mathcal{S}$, and Problem (M-MF-SLQ) is said to be *finite* if (5) holds for all $(t,x,i)\in[0,T]\times\mathbb{R}^n\times\mathcal{S}$.

(ii) An element $u^*(\cdot)\in\mathcal{U}[t,T]$ is called an *open-loop optimal control* of Problem (M-MF-SLQ) for the initial pair $(t,x,i)\in[0,T]\times\mathbb{R}^n\times\mathcal{S}$ if

$$J(t,x,i;u^*(\cdot))\le J(t,x,i;u(\cdot)),\qquad \forall u(\cdot)\in\mathcal{U}[t,T]. \tag{6}$$

If an open-loop optimal control (uniquely) exists for $(t,x,i)\in[0,T]\times\mathbb{R}^n\times\mathcal{S}$, Problem (M-MF-SLQ) is said to be (uniquely) open-loop solvable at $(t,x,i)\in[0,T]\times\mathbb{R}^n\times\mathcal{S}$. Problem (M-MF-SLQ) is said to be (uniquely) open-loop solvable at $t\in[0,T)$ if for the given t, (6) holds for all $(x,i)\in\mathbb{R}^n\times\mathcal{S}$. Problem (M-MF-SLQ) is said to be (uniquely) open-loop solvable (on $[0,T)\times\mathbb{R}^n\times\mathcal{S}$) if it is (uniquely) open-loop solvable at all $(t,x,i)\in[0,T)\times\mathbb{R}^n\times\mathcal{S}$.

To simplify notation of our further analysis, we introduce the following mean-field forward-backward stochastic differential equation (MF-FBSDE for short) on a finite horizon $[t,T]$:

$$\begin{cases} dX^u(s;t,x,i)=\Big\{A(s,\alpha(s))X^u(s;t,x,i)+B(s,\alpha(s))u(s)+b(s,\alpha(s))\Big\}ds\\ \qquad\qquad +\Big\{C(s,\alpha(s))X^u(s;t,x,i)+D(s,\alpha(s))u(s)+\sigma(s,\alpha(s))\Big\}dW(s),\\ dY^u(s;t,x,i)=-\Big\{A(s,\alpha(s))^\top Y^u(s;t,x,i)+C(s,\alpha(s))^\top Z^u(s;t,x,i)\\ \qquad\qquad +\mathbb{E}[\bar{Q}(s,\alpha(s))]\mathbb{E}[X^u(s;t,x,i)]+S(s,\alpha(s))^\top u(s)+\mathbb{E}[\bar{S}(s,\alpha(s))]^\top\mathbb{E}[u(s)]\\ \qquad\qquad +q(s,\alpha(s))+\mathbb{E}[\bar{q}(s,\alpha(s))+Q(s,\alpha(s))X^u(s;t,x,i)]\Big\}ds\\ \qquad\qquad +Z^u(s;t,x,i)dW(s)+\sum_{k=1}^{D}\Gamma^u_k(s;t,x,i)d\widetilde{N}_k(s),\\ X^u(t;t,x,i)=x,\quad \alpha(t)=i,\\ Y^u(T;t,x,i)=G(\alpha(T))X^u(T;t,x,i)+\mathbb{E}[\bar{G}(\alpha(T))]\mathbb{E}[X^u(T;t,x,i)]\\ \qquad\qquad +g(\alpha(T))+\mathbb{E}[\bar{g}(\alpha(T))]. \end{cases} \tag{7}$$

The solution of the above MF-FBSDE is denoted by $(X^u(\cdot\,;t,x,i),\ Y^u(\cdot\,;t,x,i),\ Z^u(\cdot\,;t,x,i),\ \Gamma^u(\cdot\,;t,x,i))$, where $\Gamma^u(\cdot\,;t,x,i):=(\Gamma^u_1(\cdot\,;t,x,i),\cdots,\Gamma^u_D(\cdot\,;t,x,i))$. The following result is concerned with the differentiability of the map $u(\cdot)\mapsto J(t,x,i;u(\cdot))$.

Proposition 1. *Let* (H1)–(H2) *hold and* $(t,i)\in[0,T]\times\mathcal{S}$ *be given. For any* $x\in\mathbb{R}^n$, $\lambda\in\mathbb{R}$ *and* $u(\cdot),v(\cdot)\in\mathcal{U}[t,T]$, *the following holds:*

$$\begin{aligned}&J(t,x,i;u(\cdot)+\lambda v(\cdot))-J(t,x,i;u(\cdot))\\&=\lambda^2J^0(t,0,i;v(\cdot))+2\lambda\mathbb{E}\int_t^T\big\langle\bar{M}(t,i)(x,u)(s),v(s)\big\rangle ds,\end{aligned}\tag{8}$$

where

$$\begin{aligned}\bar{M}(t,i)(x,u)(s):=\ &B(s,\alpha(s))^\top Y^u(s;t,x,i)+D(s,\alpha(s))^\top Z^u(s;t,x,i)\qquad s\in[t,T]\\&+S(s,\alpha(s))X^u(s;t,x,i)+\mathbb{E}[\bar{S}(s,\alpha(s))]\mathbb{E}[X^u(s;t,x,i)]\\&+R(s,\alpha(s))u(s)+\mathbb{E}[\bar{R}(s,\alpha(s))]\mathbb{E}[u(s)]+\rho(s,\alpha(s))+\mathbb{E}[\bar{\rho}(s,\alpha(s))].\end{aligned}\tag{9}$$

Consequently, the map $u(\cdot)\mapsto J(t,x,i;u(\cdot))$ *is Fréchet differentiable with the Fréchet derivative given by*

$$\mathcal{D}J(t,x,i;u(\cdot))(s)=2\bar{M}(t,i)(x,u)(s),\qquad s\in[t,T],\tag{10}$$

and (8) *can also be written as*

$$\begin{aligned}&J(t,x,i;u(\cdot)+\lambda v(\cdot))-J(t,x,i;u(\cdot))\\&=\lambda^2J^0(t,0,i;v(\cdot))+\lambda\mathbb{E}\int_t^T\big\langle\mathcal{D}J(t,x,i;u(\cdot))(s),v(s)\big\rangle ds.\end{aligned}\tag{11}$$

Proof. By the linearity of the state equation, $X^{u+\lambda v}(\cdot;t,x,i)=X^u(\cdot;t,x,i)+\lambda X_0^v(\cdot;t,0,i)$. Hence,

$$\begin{aligned}&J(t,x,i;u(\cdot)+\lambda v(\cdot))-J(t,x,i;u(\cdot))\\&=\lambda\mathbb{E}\bigg\{\Big\langle G(\alpha(T))[2X^u(T;t,x,i)+\lambda X_0^v(T;t,0,i)],X_0^v(T;t,0,i)\Big\rangle\\&\quad+\Big\langle\bar{G}(\alpha(T))\Big(2\mathbb{E}[X^u(T;t,x,i)]+\lambda\mathbb{E}[X_0^v(T;t,0,i)]\Big),\mathbb{E}[X_0^v(T;t,0,i)]\Big\rangle\\&\quad+2\Big\langle g(\alpha(T)),X_0^v(T;t,0,i)\Big\rangle+2\Big\langle\bar{g}(\alpha(T)),\mathbb{E}[X_0^v(T;t,0,i)]\Big\rangle\\&\quad+\int_t^T\bigg[\bigg\langle\begin{pmatrix}Q&S^\top\\S&R\end{pmatrix}\begin{pmatrix}2X^u(s;t,x,i)+\lambda X_0^v(s;t,0,i)\\2u(s)+\lambda v(s)\end{pmatrix},\begin{pmatrix}X_0^v(s;t,0,i)\\v(s)\end{pmatrix}\bigg\rangle\bigg]ds\\&\quad+\int_t^T\bigg[\bigg\langle\begin{pmatrix}\bar{Q}&\bar{S}^\top\\\bar{S}&\bar{R}\end{pmatrix}\begin{pmatrix}2\mathbb{E}[X^u(s;t,x,i)]+\lambda\mathbb{E}[X_0^v(s;t,0,i)]\\2\mathbb{E}[u(s)]+\lambda\mathbb{E}[v(s)]\end{pmatrix},\begin{pmatrix}\mathbb{E}[X_0^v(s;t,0,i)]\\\mathbb{E}[v(s)]\end{pmatrix}\bigg\rangle\bigg]ds\\&\quad+2\int_t^T\bigg[\bigg\langle\begin{pmatrix}q(s,\alpha(s))\\\rho(s,\alpha(s))\end{pmatrix},\begin{pmatrix}X_0^v(s;t,0,i)\\v(s)\end{pmatrix}\bigg\rangle+\bigg\langle\begin{pmatrix}\bar{q}(s,\alpha(s))\\\bar{\rho}(s,\alpha(s))\end{pmatrix},\begin{pmatrix}\mathbb{E}[X_0^v(s;t,0,i)]\\\mathbb{E}[v(s)]\end{pmatrix}\bigg\rangle\bigg]ds\bigg\},\\&=2\lambda\mathbb{E}\bigg\{\Big\langle G(\alpha(T))X^u(T;t,x,i)+\mathbb{E}[\bar{G}(\alpha(T))]\mathbb{E}[X^u(T;t,x,i)]\\&\qquad+g(\alpha(T))+\mathbb{E}[\bar{g}(\alpha(T))],X_0^v(T;t,0,i)\Big\rangle\\&\quad+\int_t^T\bigg[\Big\langle Q(s,\alpha(s))X^u(s;t,x,i)+\mathbb{E}[\bar{Q}(s,\alpha(s))]\mathbb{E}[X^u(s;t,x,i)]+S(s,\alpha(s))^\top u(s)\\&\qquad+\mathbb{E}[\bar{S}(s,\alpha(s))]^\top\mathbb{E}[u(s)]+q(s,\alpha(s))+\mathbb{E}[\bar{q}(s,\alpha(s))],X_0^v(s;t,0,i)\Big\rangle\\&\qquad+\Big\langle S(s,\alpha(s))X^u(s;t,x,i)+\mathbb{E}[\bar{S}(s,\alpha(s))]\mathbb{E}[X^u(s;t,x,i)]+R(s,\alpha(s))^\top u(s)\\&\qquad+\mathbb{E}[\bar{R}(s,\alpha(s))]^\top\mathbb{E}[u(s)]+\rho(s,\alpha(s))+\mathbb{E}[\bar{\rho}(s,\alpha(s))],v(s)\Big\rangle\bigg]ds\bigg\}+\lambda^2J^0(t,0,i;v(\cdot)).\end{aligned}$$

Now applying Itô's formula to $s \mapsto \langle Y^u(s;t,x,i), X_0^v(s;t,0,i) \rangle$, we have

$$
\begin{aligned}
&\mathbb{E}\Big\langle G(\alpha(T))X^u(T;t,x,i) + \mathbb{E}[\bar{G}(\alpha(T))]\mathbb{E}[X^u(T;t,x,i)] + g(\alpha(T)) + \mathbb{E}[\bar{g}(\alpha(T))], X_0^v(T;t,0,i)\Big\rangle \\
&= \mathbb{E}\int_t^T \Big\{ -\Big\langle A(s,\alpha(s))^\top Y^u(s;t,x,i) + C(s,\alpha(s))^\top Z^u(s;t,x,i) + Q(s,\alpha(s))X^u(s;t,x,i) \\
&\quad + q(s,\alpha(s)) + \mathbb{E}[\bar{Q}(s,\alpha(s))]\mathbb{E}[X^u(s;t,x,i)] + S(s,\alpha(s))^\top u(s) + \mathbb{E}[\bar{S}(s,\alpha(s))]^\top \mathbb{E}[u(s)] \\
&\quad + \mathbb{E}[\bar{q}(s,\alpha(s))], X_0^v(s;t,0,i)\Big\rangle + \Big\langle A(s,\alpha(s))X_0^v(s;t,0,i) + B(s,\alpha(s))v(s), Y^u(s;t,x,i)\Big\rangle \\
&\quad + \Big\langle C(s,\alpha(s))X_0^v(s;t,0,i) + D(s,\alpha(s))v(s), Z^u(s;t,x,i)\Big\rangle\Big\}ds, \\
&= \mathbb{E}\int_t^T \Big\{\Big\langle B(s,\alpha(s))^\top Y^u(s;t,x,i) + D(s,\alpha(s))^\top Z^u(s;t,x,i), v(s)\Big\rangle \\
&\quad - \Big\langle Q(s,\alpha(s))X^u(s;t,x,i) + q(s,\alpha(s)) + \mathbb{E}[\bar{Q}(s,\alpha(s))]\mathbb{E}[X^u(s;t,x,i)] + S(s,\alpha(s))^\top u(s) \\
&\quad + \mathbb{E}[\bar{S}(s,\alpha(s))]^\top \mathbb{E}[u(s)] + \mathbb{E}[\bar{q}(s,\alpha(s))], X_0^v(s;t,0,i)\Big\rangle\Big\}ds.
\end{aligned}
$$

Combining the above equalities, we obtain (8). ☐

From the above, we have the following result, which gives a characterization for the optimal controls of Problem (M-MF-SLQ).

Theorem 1.2. *Let* (H1)–(H2) *hold and* $(t,x,i) \in [0,T)\times\mathbb{R}^n\times\mathcal{S}$ *be given. Let* $u(\cdot) \in \mathcal{U}[t,T]$ *and* $(X^u(\cdot;t,x,i),\ Y^u(\cdot;t,x,i),\ Z^u(\cdot;t,x,i),\ \Gamma^u(\cdot;t,x,i))$ *be the adapted solution to* (7)*. Then* $u(\cdot)$ *is an optimal control of Problem (M-MF-SLQ) for the initial pair* (t,x,i) *if and only if*

$$J^0(t,0,i;u(\cdot)) \geq 0, \qquad \forall u(\cdot) \in \mathcal{U}[t,T], \tag{12}$$

and the following stationary condition holds:

$$
\begin{aligned}
\mathcal{D}J(t,x,i;u(\cdot))(s) = 2\Big\{ &B(s,\alpha(s))^\top Y^u(s;t,x,i) + D(s,\alpha(s))^\top Z^u(s;t,x,i) \\
&+ S(s,\alpha(s))X^u(s;t,x,i) + \mathbb{E}[\bar{S}(s,\alpha(s))]\mathbb{E}[X^u(s;t,x,i)] \\
&+ R(s,\alpha(s))u(s) + \mathbb{E}[\bar{R}(s,\alpha(s))]\mathbb{E}[u(s)] \\
&+ \rho(s,\alpha(s)) + \mathbb{E}[\bar{\rho}(s,\alpha(s))]\Big\} = 0, \ {}^1\text{a.e.} \ \ {}^2\text{a.s.}
\end{aligned} \tag{13}
$$

Proof. By (8), it is clear that $u(\cdot)$ is an optimal control of Problem (M-MF-SLQ) for the initial pair (t,x,i) if and only if

$$
\begin{aligned}
&\lambda^2 J^0(t,0,i;v(\cdot)) + \lambda\mathbb{E}\int_t^T \big\langle \mathcal{D}J(t,x,i;u(\cdot))(s), v(s)\big\rangle ds \\
&= J(t,x,i;u(\cdot)+\lambda v(\cdot)) - J(t,x,i;u(\cdot)) \geq 0, \qquad \forall\lambda\in\mathbb{R}, \forall v(\cdot)\in\mathcal{U}[t,T],
\end{aligned}
$$

which is equivalent to (12) and the following:

$$\mathbb{E}\int_t^T \big\langle \mathcal{D}J(t,x,i;u(\cdot))(s), v(s)\big\rangle ds = 0, \qquad \forall v(\cdot)\in\mathcal{U}[t,T].$$

Note that the above equality holds for all $v(\cdot) \in \mathcal{U}[t,T]$ if and only if $\mathcal{D}J(t,x,i;u(\cdot)) = 0$. The result therefore follows. ☐

[1]a.e. means that (13) holds almost everywhere $s \in [t,T]$.
[2]a.s. means that (13) holds almost surely $\omega \in \Omega$.

Remark 1. Note that if $u(\cdot)$ happens to be an open-loop optimal control of Problem (M-MF-SLQ), then the *stationary condition* (13) holds, which brings a coupling into the FBSDE (7). We call (7), together with the stationary condition (13), the *optimality system* for the open-loop optimal control of Problem (M-MF-SLQ).

2. Finiteness and open-loop solvability of problem (M-MF-SLQ).

We begin with a representation of the cost functional. By the argument in [27], we can define bounded linear operators $\mathcal{L}_t : \mathcal{U}[t,T] \to L^2_{\mathbb{F}}(t,T;\mathbb{R}^n)$ and $\widehat{\mathcal{L}}_t : \mathcal{U}[t,T] \to L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$ by $u(\cdot) \mapsto X_0^u(\cdot;t,0,i)$ and $u(\cdot) \mapsto X_0^u(T;t,0,i)$, respectively, *via* the SDE corresponding to the $X_0^u(\cdot;t,0,i)$. Then

$$
\begin{aligned}
J^0(t,0,i;u(\cdot)) &= \mathbb{E}\bigg\{ \Big\langle G(\alpha(T))X_0^u(T;t,0,i), X_0^u(T;t,0,i) \Big\rangle \\
&\quad + \Big\langle \bar{G}(\alpha(T))\mathbb{E}[X_0^u(T;t,0,i)], \mathbb{E}[X_0^u(T;t,0,i)] \Big\rangle \\
&\quad + \int_t^T \left[\left\langle \begin{pmatrix} Q & S^\top \\ S & R \end{pmatrix} \begin{pmatrix} X_0^u(s;t,0,i) \\ u(s) \end{pmatrix}, \begin{pmatrix} X_0^u(s;t,0,i) \\ u(s) \end{pmatrix} \right\rangle \right] ds \\
&\quad + \int_t^T \left[\left\langle \begin{pmatrix} \bar{Q} & \bar{S}^\top \\ \bar{S} & \bar{R} \end{pmatrix} \begin{pmatrix} \mathbb{E}[X_0^u(s;t,0,i)] \\ \mathbb{E}[u(s)] \end{pmatrix}, \begin{pmatrix} \mathbb{E}[X_0^u(s;t,0,i)] \\ \mathbb{E}[u(s)] \end{pmatrix} \right\rangle \right] ds \bigg\} \\
&= \langle G(\alpha(T))\widehat{\mathcal{L}}_t u, \widehat{\mathcal{L}}_t u \rangle + \langle \bar{G}(\alpha(T))\mathbb{E}[\widehat{\mathcal{L}}_t u], \mathbb{E}[\widehat{\mathcal{L}}_t u] \rangle + \langle Q(\cdot,\alpha(\cdot))\mathcal{L}_t u(\cdot), \mathcal{L}_t u(\cdot) \rangle \\
&\quad + 2\langle S(\cdot,\alpha(\cdot))\mathcal{L}_t u(\cdot), u(\cdot) \rangle + \langle R(\cdot,\alpha(\cdot))u(\cdot), u(\cdot) \rangle + \langle \bar{Q}(\cdot,\alpha(\cdot))\mathbb{E}[\mathcal{L}_t u(\cdot)], \mathbb{E}[\mathcal{L}_t u(\cdot)] \rangle \\
&\quad + 2\langle \bar{S}(\cdot,\alpha(\cdot))\mathbb{E}[\mathcal{L}_t u(\cdot)], \mathbb{E}[u(\cdot)] \rangle + \langle \bar{R}(\cdot,\alpha(\cdot))\mathbb{E}[u(\cdot)], \mathbb{E}[u(\cdot)] \rangle \\
&= \Big\langle \Big[\widehat{\mathcal{L}}_t^* \Big(G(\alpha(T)) + \mathbb{E}^*\bar{G}(\alpha(T))\mathbb{E} \Big) \widehat{\mathcal{L}}_t + \mathcal{L}_t^* \Big(Q(\cdot,\alpha(\cdot)) + \mathbb{E}^*\bar{Q}(\cdot,\alpha(\cdot))\mathbb{E} \Big) \mathcal{L}_t \\
&\quad + \Big(R(\cdot,\alpha(\cdot)) + \mathbb{E}^*\bar{R}(\cdot,\alpha(\cdot))\mathbb{E} \Big) + \Big(S(\cdot,\alpha(\cdot)) + \mathbb{E}^*\bar{S}(\cdot,\alpha(\cdot))\mathbb{E} \Big) \mathcal{L}_t \\
&\quad + \mathcal{L}_t^* \Big(S(\cdot,\alpha(\cdot))^\top + \mathbb{E}^*\bar{S}(\cdot,\alpha(\cdot))^\top \mathbb{E} \Big) \Big] u(\cdot), u(\cdot) \Big\rangle.
\end{aligned}
$$

Denote

$$
\begin{aligned}
\mathcal{M}_t := \quad & \widehat{\mathcal{L}}_t^* \Big(G(\alpha(T)) + \mathbb{E}^*\bar{G}(\alpha(T))\mathbb{E} \Big) \widehat{\mathcal{L}}_t + \mathcal{L}_t^* \Big(Q(\cdot,\alpha(\cdot)) + \mathbb{E}^*\bar{Q}(\cdot,\alpha(\cdot))\mathbb{E} \Big) \mathcal{L}_t \\
& + \Big(R(\cdot,\alpha(\cdot)) + \mathbb{E}^*\bar{R}(\cdot,\alpha(\cdot))\mathbb{E} \Big) + \Big(S(\cdot,\alpha(\cdot)) + \mathbb{E}^*\bar{S}(\cdot,\alpha(\cdot))\mathbb{E} \Big) \mathcal{L}_t \\
& + \mathcal{L}_t^* \Big(S(\cdot,\alpha(\cdot))^\top + \mathbb{E}^*\bar{S}(\cdot,\alpha(\cdot))^\top \mathbb{E} \Big),
\end{aligned}
\tag{14}
$$

which is a bounded self-adjoint linear operator on $\mathcal{U}[t,T]$. Then by Proposition 1, the cost functional $J(t,x,i;u(\cdot))$ can be written as

$$
\begin{aligned}
J(t,x,i;u(\cdot)) = \langle \mathcal{M}_t u(\cdot), u(\cdot) \rangle + \langle \mathcal{D}J(t,x,i;0), u(\cdot) \rangle + J(t,x,i;0), \\
\forall (t,x,i) \in [0,T] \times \mathbb{R}^n \times \mathcal{S}, \quad \forall u(\cdot) \in \mathcal{U}[t,T].
\end{aligned}
\tag{15}
$$

Remark 2. It is important to point out that the general inner product notation $\langle \cdot,\cdot \rangle$ is also used to denote the inner product under different Hilbert spaces. However, it will not cause any trouble to understand the main idea of this paper. $\mathcal{L}_t^*$ is the adjoint operator of $\mathcal{L}_t$ and $\mathbb{E}^*$ is the adjoint operator of $\mathbb{E}$ such that $\langle \mathcal{L}_t[a], b \rangle = \langle a, \mathcal{L}_t^*[b] \rangle$ and $\langle \mathbb{E}[a], b \rangle = \langle a, \mathbb{E}^*[b] \rangle$, $\forall a,b \in \mathbb{H}$, where $\mathbb{H}$ represents a general Hilbert space.

Now let us introduce the following conditions:

(H3) The following holds:

$$
J^0(t,0,i;u(\cdot)) \geq 0, \qquad \forall u(\cdot) \in \mathcal{U}[t,T].
$$

(H4) There exists a constant $\delta > 0$ such that

$$J^0(t,0,i;u(\cdot)) \geq \delta \mathbb{E}\int_t^T |u(s)|^2 ds, \qquad \forall u(\cdot) \in \mathcal{U}[t,T].$$

From (15), we see that the map $u(\cdot) \mapsto J(t,x,i;u(\cdot))$ is convex if and only if

$$\mathcal{M}_t \geq 0,$$

which is equivalent to (H3), when we let $u = 0$, $v = u$ and $\lambda = 1$ in (11). And $u(\cdot) \mapsto J(t,x,i;u(\cdot))$ is uniformly convex if and only if

$$\mathcal{M}_t \geq \delta I, \qquad \text{for some } \delta > 0,$$

which is equivalent to (H4), when we let $u = 0$, $v = u$ and $\lambda = 1$ in (11). The following result tells us that (H3) is necessary for the finiteness (and open-loop solvability) of Problem (M-MF-SLQ) at t, and (H4) is sufficient for the open-loop solvability of Problem (M-MF-SLQ) at t.

Proposition 2. *Let* (H1)–(H2) *hold and* $t \in [0,T)$ *be given. We have the following:*

(i) *If Problem (M-MF-SLQ) is finite at* t*, then* (H3) *must hold.*

(ii) *Suppose* (H4) *holds. Then Problem (M-MF-SLQ) is uniquely open-loop solvable at* t*, and the unique optimal control for the initial pair* (t,x,i) *is given by*

$$u^*(\cdot) = -\frac{1}{2}\mathcal{M}_t^{-1}\mathcal{D}J(t,x,i;0)(\cdot). \tag{16}$$

Moreover,

$$V(t,x,i) = J(t,x,i;0) - \frac{1}{4}\left|\mathcal{M}_t^{-\frac{1}{2}}\mathcal{D}J(t,x,i;0)\right|^2. \tag{17}$$

Proof.

(i) We prove the result by contradiction. Suppose that $J^0(t,0,i;u(\cdot)) < 0$ for some $u(\cdot) \in \mathcal{U}[t,T]$. By Proposition 1, we have

$$\begin{aligned} &J(t,x,i;\lambda u(\cdot)) \\ &= J(t,x,i;0) + \lambda^2 J^0(t,0,i;u(\cdot)) + \lambda \mathbb{E}\int_t^T \langle \mathcal{D}J(t,x,i;0)(s), u(s)\rangle\, ds, \\ &\qquad \forall \lambda \in \mathbb{R}. \end{aligned}$$

Letting $\lambda \to \infty$, we obtain that

$$V(t,x,i) \leq \lim_{\lambda\to\infty} J(t,x,i;\lambda u(\cdot)) = -\infty,$$

which is a contradiction.

(ii) Suppose (H4) holds. Then the operator $\mathcal{M}_t$ is invertible, and

$$\begin{aligned} J(t,x,i;u(\cdot)) &= \left|\mathcal{M}_t^{\frac{1}{2}}u + \frac{1}{2}\mathcal{M}_t^{-\frac{1}{2}}\mathcal{D}J(t,x,i;0)\right|^2 + J(t,x,i;0) - \frac{1}{4}\left|\mathcal{M}_t^{-\frac{1}{2}}\mathcal{D}J(t,x,i;0)\right|^2, \\ &\geq J(t,x,i;0) - \frac{1}{4}\left|\mathcal{M}_t^{-\frac{1}{2}}\mathcal{D}J(t,x,i;0)\right|^2, \qquad \forall x \in \mathbb{R}^n, \forall u(\cdot) \in \mathcal{U}[t,T]. \end{aligned}$$

Note that the equality in the above holds if and only if

$$u = -\frac{1}{2}\mathcal{M}_t^{-1}\mathcal{D}J(t,x,i;0).$$

The result therefore follows.

□

Due to the necessity of (H3) for the finiteness of Problem (M-MF-SLQ), we will assume (H3) holds for the rest of this paper. Now for any $\varepsilon > 0$, we consider state equation (2) and the following cost functional:

$$\begin{aligned} J_\varepsilon(t,x,i;u(\cdot)) &:= J(t,x,i;u(\cdot)) + \varepsilon\mathbb{E}\int_t^T |u(s)|^2 ds \\ &= \langle (\mathcal{M}_t + \varepsilon I)u, u\rangle + \langle \mathcal{D}J(t,x,i;0), u\rangle + J(t,x,i;0). \end{aligned} \tag{18}$$

Denote the corresponding optimal control problem and value function by Problem $(\text{M-MF-SLQ})_\varepsilon$ and $V_\varepsilon(\cdot,\cdot,\cdot)$, respectively. By Proposition 2, part (ii), for any $x \in \mathbb{R}^n$, Problem $(\text{M-MF-SLQ})_\varepsilon$ admits a unique optimal control

$$u^*_\varepsilon(\cdot) = -\frac{1}{2}(\mathcal{M}_t + \varepsilon I)^{-1}\mathcal{D}J(t,x,i;0)(\cdot), \tag{19}$$

and the value function is given by

$$V_\varepsilon(t,x,i) = J(t,x,i;0) - \frac{1}{4}\left|(\mathcal{M}_t + \varepsilon I)^{-\frac{1}{2}}\mathcal{D}J(t,x,i;0)\right|^2. \tag{20}$$

Before we give the main result of this section, we first present the following lemma.

Lemma 2.1. *Let $\mathcal{H}$ be a Hilbert space with norm $|\cdot|$ and $\theta, \theta_n \in \mathcal{H}$, $n = 1, 2, \cdots$.*

(i) *If $\theta_n \to \theta$ weakly, then $|\theta| \le \varliminf_{n\to\infty} |\theta_n|$.*

(ii) *$\theta_n \to \theta$ strongly if and only if*

$$|\theta_n| \to |\theta| \qquad \text{and} \qquad \theta_n \to \theta \quad \text{weakly.}$$

Now we are ready to state the main result of this section.

Theorem 2.2. *Let (H1)–(H3) hold and the initial pair $(t,x,i) \in [0,T) \times \mathbb{R}^n \times \mathcal{S}$ is given. We have the following:*

(i) *$\lim_{\varepsilon\to\infty} V_\varepsilon(t,x,i) = V(t,x,i)$. In particular, Problem (M-MF-SLQ) is finite at (t,x,i) if and only if $\{V_\varepsilon(t,x,i)\}_{\varepsilon>0}$ is bounded from below.*

(ii) *The sequence $\{u^*_\varepsilon(\cdot)\}_{\varepsilon>0}$ defined by (19) is a minimizing sequence of $u(\cdot) \mapsto J(t,x,i;u(\cdot))$, i.e.*

$$\lim_{\varepsilon\to\infty} J(t,s,i;u^*_\varepsilon(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}[t,T]} J(t,x,i;u(\cdot)) = V(t,x,i). \tag{21}$$

(iii) *The following statements are equivalent:*
   (a) *Problem (M-MF-SLQ) is open-loop solvable at (t,x,i);*
   (b) *The sequence $\{u^*_\varepsilon(\cdot)\}_{\varepsilon>0}$ is bounded in $\mathcal{U}[t,T]$;*
   (c) *The sequence $\{u^*_\varepsilon(\cdot)\}_{\varepsilon>0}$ admits a weakly convergent subsequence;*
   (d) *The sequence $\{u^*_\varepsilon(\cdot)\}_{\varepsilon>0}$ admits a strongly convergent subsequence;*
   In this case, the weak (strong) limit of any weakly (strongly) convergent subsequence of $\{u^*_\varepsilon(\cdot)\}_{\varepsilon>0}$ is an optimal control of Problem (M-MF-SLQ) at (t,x,i).

Proof.

(i) For any $\varepsilon_2 > \varepsilon_1 > 0$, we have

$$J_{\varepsilon_2}(t,x,i;u(\cdot)) \ge J_{\varepsilon_1}(t,x,i;u(\cdot)) \ge J(t,x,i;u(\cdot)), \qquad \forall u(\cdot) \in \mathcal{U}[t,T],$$

which implies that

$$V_{\varepsilon_2}(t,x,i) \ge V_{\varepsilon_1}(t,x,i) \ge V(t,x,i), \qquad \forall \varepsilon_2 > \varepsilon_1 > 0.$$

So, the limit $\lim_{\varepsilon\to0} V_\varepsilon(t,x,i)$ exists and

$$\bar{V}(t,x,i) \equiv \lim_{\varepsilon\to0} V_\varepsilon(t,x,i) \ge V(t,x,i). \tag{22}$$

On the other hand, for any $K,\delta>0$, we can find a $u^\delta(\cdot)\in\mathcal{U}[t,T]$, such that

$$\begin{aligned} V_\varepsilon(t,x,i) &\le J(t,x,i;u^\delta(\cdot))+\varepsilon\mathbb{E}\int_t^T|u^\delta(s)|^2ds\\ &\le \max\{V(t,x,i),-K\}+\delta+\varepsilon\mathbb{E}\int_t^T|u^\delta(s)|^2ds. \end{aligned}$$

Letting $\varepsilon\to0$, we obtain

$$\bar V(t,x,i)\le\max\{V(t,x,i),-K\}+\delta.\qquad \forall K,\delta>0,$$

from which we see

$$\bar V(t,x,i)\le V(t,x,i). \tag{23}$$

Combining (22)–(23), we get the desired result.

(ii) If $V(t,x,i)>-\infty$, then by (i), we have

$$\begin{aligned} &\varepsilon\mathbb{E}\int_t^T|u_\varepsilon^*(s)|^2ds\\ &=J_\varepsilon(t,x,i;u_\varepsilon^*(\cdot))-J(t,x,i;u_\varepsilon^*(\cdot))=V_\varepsilon(t,x,i)-J(t,x,i;u_\varepsilon^*(\cdot))\\ &\le V_\varepsilon(t,x,i)-V(t,x,i)\to0\qquad\text{as}\qquad\varepsilon\to0. \end{aligned}$$

Therefore,

$$\lim_{\varepsilon\to0}J(t,x,i;u_\varepsilon^*(\cdot))=\lim_{\varepsilon\to0}\Big[V_\varepsilon(t,x,i)-\varepsilon\mathbb{E}\int_t^T|u_\varepsilon^*(s)|^2ds\Big]=V(t,x,i).$$

If $V(t,x,i)=-\infty$, then by (i), we have

$$J(t,x,i;u_\varepsilon^*(\cdot))\le J_\varepsilon(t,x,i;u_\varepsilon^*(\cdot))=V_\varepsilon(t,x,i)\to-\infty\qquad\text{as}\qquad\varepsilon\to0,$$

and (21) still holds.

(iii) (b) $\Rightarrow$ (c) and (d) $\Rightarrow$ (c) are obvious. We next prove (c) $\Rightarrow$ (a). Let $\{u_{\varepsilon_k}^*(\cdot)\}_{k\ge1}$ be a weakly convergent subsequence of $\{u_\varepsilon^*(\cdot)\}_{\varepsilon>0}$ with weak limit $u^*(\cdot)$. Then $\{u_{\varepsilon_k}^*(\cdot)\}_{k\ge1}$ is bounded in $\mathcal{U}[t,T]$. For any $u(\cdot)\in\mathcal{U}[t,T]$, we have

$$J(t,x,i;u_{\varepsilon_k}^*(s))+\varepsilon_k\mathbb{E}\int_t^T|u_{\varepsilon_k}^*(s)|^2ds=V_{\varepsilon_k}(t,x,i)\le J(t,x,i;u(\cdot))+\varepsilon_k\mathbb{E}\int_t^T|u(s)|^2ds. \tag{24}$$

Note that $u(\cdot)\mapsto J(t,x,i;u(\cdot))$ is sequentially weakly lower semi-continuous. Letting $k\to\infty$ in (24), we get

$$J(t,x,i;u^*(\cdot))\le\varliminf_{k\to\infty}J(t,x,i;u_{\varepsilon_k}^*(s))\le J(t,x,i;u(\cdot)),\qquad\forall u(\cdot)\in\mathcal{U}[t,T].$$

Therefore, $u^*(\cdot)$ is an optimal control of Problem (M-MF-SLQ) at (t,x,i). Now it remains to show (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (d). Suppose $v^*(\cdot)$ is an optimal control of Problem (M-MF-SLQ) at (t,x,i). Then for any $\varepsilon>0$, we have

$$\begin{cases} V_\varepsilon(t,x,i)=J_\varepsilon(t,x,i;u_\varepsilon^*(\cdot))\ge V(t,x,i)+\varepsilon\mathbb{E}\displaystyle\int_t^T|u_\varepsilon^*(s)|^2ds,\\ V_\varepsilon(t,x,i)\le J_\varepsilon(t,x,i;v^*(\cdot))=V(t,x,i)+\varepsilon\mathbb{E}\displaystyle\int_t^T|v^*(s)|^2ds, \end{cases}$$

from which we see that

$$\mathbb{E}\int_t^T|u_\varepsilon^*(s)|^2ds\le\frac{V_\varepsilon(t,x,i)-V(t,x,i)}{\varepsilon}\le\mathbb{E}\int_t^T|v^*(s)|^2ds,\qquad\forall\varepsilon>0. \tag{25}$$

Therefore, $\{u_\varepsilon^*(\cdot)\}_{\varepsilon>0}$ is bounded in Hilbert space $\mathcal{U}[t,T]$ and hence admits a weakly convergent subsequence $\{u_{\varepsilon_k}^*(\cdot)\}_{k\ge1}$. Let $u^*(\cdot)$ be the weak limit of $\{u_{\varepsilon_k}^*(\cdot)\}_{k\ge1}$. By the proof of (c) $\Rightarrow$ (a), we see that $u^*(\cdot)$ is also an optimal control of Problem (M-MF-SLQ) at (t, x, i). Replacing $v^*(\cdot)$ with $u^*(\cdot)$ in (25), we have

$$\mathbb{E}\int_t^T |u_\varepsilon^*(s)|^2 ds \le \mathbb{E}\int_t^T |u^*(s)|^2 ds, \qquad \forall \varepsilon > 0. \tag{26}$$

Also, by Lemma 2.1, part (i),

$$\mathbb{E}\int_t^T |u^*(s)|^2 ds \le \varliminf_{k\to\infty} \mathbb{E}\int_t^T |u_{\varepsilon_k}^*(s)|^2 ds. \tag{27}$$

Combining (26)–(27), we have

$$\mathbb{E}\int_t^T |u^*(s)|^2 ds = \lim_{k\to\infty} \mathbb{E}\int_t^T |u_{\varepsilon_k}^*(s)|^2 ds.$$

Then it follows from Lemma 2.1, part (ii), that $\{u_{\varepsilon_k}^*(\cdot)\}_{k\ge 1}$ converges to $u^*(\cdot)$ strongly. □

3. **Some examples.** In this section, we give some examples to demonstrate our results more clearly. As a start, our model can degenerate to the usual mean-field problem and regime switching problem. Meanwhile, the classical mean-variance portfolio selection problem can be regarded as an application of our proposed problem.

3.1. **Example 1.** In this example, we take off the regime-switching in the system (2) and cost functional (3). Then our state equation is

$$\begin{cases} dX^u(s;t,x) = \Big\{A(s)X^u(s;t,x) + B(s)u(s) + b(s)\Big\}ds \\ \qquad\qquad + \Big\{C(s)X^u(s;t,x) + D(s)u(s) + \sigma(s)\Big\}dW(s), \\ X^u(t;t,x) = x, \qquad s \in [t,T], \end{cases} \tag{28}$$

and cost functional is

$$\begin{aligned} J(t,x;u(\cdot)) :=& \mathbb{E}\Bigg\{ \Big\langle GX^u(T;t,x), X^u(T;t,x)\Big\rangle + 2\Big\langle g, X^u(T;t,x)\Big\rangle \\ &+ \Big\langle \bar{G}\mathbb{E}[X^u(T;t,x)], \mathbb{E}[X^u(T;t,x)]\Big\rangle + 2\Big\langle \bar{g}, \mathbb{E}[X^u(T;t,x)]\Big\rangle \\ &+ \int_t^T \left[\left\langle \begin{pmatrix} Q(s) & S(s)^\top \\ S(s) & R(s) \end{pmatrix} \begin{pmatrix} X^u(s;t,x) \\ u(s) \end{pmatrix}, \begin{pmatrix} X^u(s;t,x) \\ u(s) \end{pmatrix} \right\rangle\right] ds \\ &+ \int_t^T \left[\left\langle \begin{pmatrix} \bar{Q}(s) & \bar{S}(s)^\top \\ \bar{S}(s) & \bar{R}(s) \end{pmatrix} \begin{pmatrix} \mathbb{E}[X^u(s;t,x)] \\ \mathbb{E}[u(s)] \end{pmatrix}, \begin{pmatrix} \mathbb{E}[X^u(s;t,x)] \\ \mathbb{E}[u(s)] \end{pmatrix} \right\rangle\right] ds \\ &+ 2\int_t^T \Bigg[\left\langle \begin{pmatrix} q(s) \\ \rho(s) \end{pmatrix}, \begin{pmatrix} X^u(s;t,x) \\ u(s) \end{pmatrix} \right\rangle \\ &\qquad + \left\langle \begin{pmatrix} \bar{q}(s) \\ \bar{\rho}(s) \end{pmatrix}, \begin{pmatrix} \mathbb{E}[X^u(s;t,x)] \\ \mathbb{E}[u(s)] \end{pmatrix} \right\rangle \Bigg] ds \Bigg\}. \end{aligned} \tag{29}$$

This problem has been well studied by Sun [21]. By the results of this paper, we have Proposition 2 which is consistent with Proposition 3.1 in [21]. And for such mean-field SLQ problem (MF-SLQ, for short), we are able to introduce the following Riccati equations

$$\begin{cases} \dot{P} + PA + A^\top P + C^\top PC + Q - (PB + C^\top PD + S^\top) \\ \quad \times (R + D^\top PD)^\dagger (B^\top P + D^\top PC + S) = 0, \qquad \text{a.e. } s \in [t, T], \\ P(T) = G, \end{cases} \tag{30}$$

and

$$\begin{cases} \dot{\Pi} + \Pi A + A^\top \Pi + C^\top \Pi C + Q + \bar{Q} + C^\top PC - (\Pi B + C^\top PD + S^\top + \bar{S}^\top) \\ \quad \times (R + \bar{R} + D^\top PD)^\dagger (B^\top \Pi + D^\top PC + S + \bar{S}) = 0, \qquad \text{a.e. } s \in [t, T], \\ \Pi(T) = G + \bar{G}. \end{cases} \tag{31}$$

Then we can deduce the feedback form of the open-loop solution of such mean-field SLQ problems like [21]. By our results, this problem is open-loop solvability and has feedback form when its cost functional is uniformly convex and Riccati equations have a strongly regular solution. The open-loop solution of such Problem $(\text{MF-SLQ})^0$ has a feedback form:

$$u^* = \Theta(X^* - \mathbb{E}[X^*]) + \bar{\Theta}\mathbb{E}[X^*],$$

where

$$\begin{cases} \Theta = -(R + D^\top PD)^{-1}(B^\top P + D^\top PC + S), \\ \bar{\Theta} = -(R + \bar{R} + D^\top PD)^{-1}(B^\top \Pi + D^\top PC + S + \bar{S}). \end{cases}$$

For instance, we consider the following Problem $(\text{MF-SLQ})^0$ with one-dimensional state equation

$$\begin{cases} dX(s) = X(s)ds + u(s)dW(s), \qquad s \in [0, T], \\ X(0) = 1, \end{cases}$$

and cost functional

$$J(0, 1; u(\cdot)) = \mathbb{E}\Bigg\{ 2|X(T)|^2 - 2|\mathbb{E}[X(T)]|^2 + \int_0^T \Big(-4|X(s)|^2 - |u(s)|^2 + 4|\mathbb{E}[X(s)]|^2 + |\mathbb{E}[u(s)]|^2 \Big) ds \Bigg\}.$$

In this example,

$$\begin{cases} A = 1, \quad B = 0, \quad C = 0, \quad D = 1, \quad G = 2, \quad \bar{G} = -2, \\ Q = -4, \quad \bar{Q} = 4, \quad S = \bar{S} = 0, \quad R = -1, \quad \bar{R} = 1. \end{cases}$$

Obviously, the related Riccati equations (30), (31) are

$$\begin{cases} \dot{P} + 2P - 4 = 0, \qquad s \in [0, T], \\ P(T) = 2, \end{cases} \quad \text{and} \quad \begin{cases} \dot{\Pi} + 2\Pi = 0, \qquad s \in [0, T], \\ \Pi(T) = 0. \end{cases}$$

It is easy to see that $P \equiv 2$ and $\Pi \equiv 0$ is the unique solution of Riccati equation, respectively. So we get $u^* = 0$, which implies $V^0(0, 1) = 0$. Clearly, it is really optimal control because $V^0(0, 1) \geq 0$ and $u^* = 0$ obtains the minimum.

3.2. **Example 2.** In this example, we take off the mean-field terms in the system (2) and cost functional (3). Then our state equation is

$$\begin{cases} dX^u(s;t,x,i) = \Big\{A(s,\alpha(s))X^u(s;t,x,i) + B(s,\alpha(s))u(s) + b(s,\alpha(s))\Big\}ds \\ \qquad\qquad\qquad + \Big\{C(s,\alpha(s))X^u(s;t,x,i) + D(s,\alpha(s))u(s) + \sigma(s,\alpha(s))\Big\}dW(s), \\ X^u(t;t,x,i) = x, \quad \alpha(t) = i, \qquad s \in [t,T], \end{cases} \tag{32}$$

and cost functional is

$$\begin{aligned} J(t,x,i;u(\cdot)) :=& \mathbb{E}\Bigg\{ \Big\langle G(\alpha(T))X^u(T;t,x,i), X^u(T;t,x,i)\Big\rangle + 2\Big\langle g(\alpha(T)), X^u(T;t,x,i)\Big\rangle \\ &+ \int_t^T \left[\left\langle \begin{pmatrix} Q & S^\top \\ S & R \end{pmatrix}\begin{pmatrix} X^u(s;t,x,i) \\ u(s) \end{pmatrix}, \begin{pmatrix} X^u(s;t,x,i) \\ u(s) \end{pmatrix}\right\rangle\right] ds \\ &+ 2\int_t^T \left\langle \begin{pmatrix} q(s,\alpha(s)) \\ \rho(s,\alpha(s)) \end{pmatrix}, \begin{pmatrix} X^u(s;t,x,i) \\ u(s) \end{pmatrix}\right\rangle ds \Bigg\}. \end{aligned} \tag{33}$$

This problem has been studied by Zhang [31]. By the results of this paper, we have Theorem 1.2 which is consistent with Theorem 4.1 in [31]. And for such Markov SLQ problem (M-SLQ, for short), we are also able to introduce the following Riccati equation

$$\begin{cases} \dot P(s,\alpha(s)) + P(s,\alpha(s))A(s,\alpha(s)) + A(s,\alpha(s))^\top P(s,\alpha(s)) + Q(s,\alpha(s)) \\ \quad + C(s,\alpha(s))^\top P(s,\alpha(s))C(s,\alpha(s)) - \hat S(s,\alpha(s))^\top \hat R(s,\alpha(s))^\dagger \hat S(s,\alpha(s)) \\ \quad + \displaystyle\sum_{k=1}^{D} \lambda_{\alpha(s-)k}(s)P(s,k) = 0, \qquad\qquad \text{a.e. } s \in [t,T], \\ P(T,\alpha(T)) = G(\alpha(T)), \end{cases} \tag{34}$$

where

$$\begin{cases} \hat S(s,\alpha(s)) := B(s,\alpha(s))^\top P(s,\alpha(s)) + D(s,\alpha(s))^\top P(s,\alpha(s))C(s,\alpha(s)) + S(s,\alpha(s)), \\ \hat R(s,\alpha(s)) := R(s,\alpha(s)) + D(s,\alpha(s))^\top P(s,\alpha(s))D(s,\alpha(s)). \end{cases}$$

Then we can deduce the feedback form of the open-loop solution of such Markov SLQ problems like [31]. By our results, this problem is open-loop solvability and has feedback form when its cost functional is uniformly convex and Riccati equations have a strongly regular solution. The open-loop solution of such Problem (M-SLQ)^0 has a feedback form:

$$u^*(s) = \Theta(s,\alpha(s))X^*(s),$$

where

$$\Theta(s,\alpha(s)) = -\hat R(s,\alpha(s))^{-1}\hat S(s,\alpha(s)).$$

For an instance, consider the following Problem (M-SLQ)^0 with one-dimensional state equation

$$\begin{cases} dX(s) = A(\alpha(s))X(s)ds + u(s)dW(s), \qquad s \in [0,T], \\ X(0) = 1, \quad \alpha(0) = 1, \end{cases}$$

and cost functional

$$J(0,1;u(\cdot)) = \mathbb{E}\bigg\{G(\alpha(T))|X(T)|^2 + \int_0^T \Big(-4|X(s)|^2 - |u(s)|^2\Big)ds\bigg\}.$$

In this example,

$$\begin{cases} A(1) = 1, \quad A(2) = 2, \quad B = C = 0, \quad D = 1, \quad Q = -4, \\ G(1) = 2, \quad G(2) = 1, \quad S = \bar{S} = 0, \quad R = -1, \quad \mathcal{S} = 1, 2. \end{cases}$$

Obviously, the related Riccati equation (34) is (noticing $\sum_{k=1}^{D} \lambda_{ik} = 0$, $\forall i \in \mathcal{S}$)

$$\begin{cases} \dot{P}(s,1) + 2P(s,1) - 4 = 0, \qquad s \in [0,T], \\ P(T,1) = 2, \end{cases}$$

and

$$\begin{cases} \dot{P}(s,2) + 4P(s,2) - 4 = 0, \qquad s \in [0,T], \\ P(T,2) = 1. \end{cases}$$

It is easy to see that

$$P(s,i) = \begin{cases} 2, \quad i = 1, \\ 1, \quad i = 2, \end{cases}$$

is the unique solution of the Riccati equation. So we get $u^* = 0$, which implies $V^0(0,1) = P(0,1) = 2$.

3.3. **Example 3: Mean-Variance portfolio selection.** For financial applications, we can regard the mean-variance portfolio selection problem as a special case of the M-MF-SLQ problem. We will give a specific case to show this and we will apply the settings in [33]. For simplicity, we consider a market in which two assets are traded continuously. One of the assets is a bank account whose price $P_0(t)$ is subject to the following stochastic ordinary differential equation:

$$\begin{cases} dP_0(t) = r(t)P_0(t)dt, \qquad t \in [0,T], \\ P_0(0) = p_0 > 0, \end{cases} \tag{35}$$

where $r(t) \geq 0$ is given as the interest rate process and is independent with the different market modes. The other asset is stock whose price process $P(t)$ satisfies the following system of stochastic differential equation:

$$\begin{cases} dP(t) = P(t)\Big[b_p(t,\alpha(t))dt + \sigma_p(t,\alpha(t))dW(t)\Big], \qquad t \in [0,T], \\ P(0) = p > 0, \end{cases} \tag{36}$$

where, for each $i = 1, 2, \ldots, D$, $b_p(t,i)$ is the appreciation rate process and $\sigma_p(t,i)$ is the volatility or the dispersion rate of the stock corresponding to $\alpha(t) = i$. Setting $B_p(t,i) := b_p(t,i) - r(t)$, we can write the wealth equation as

$$\begin{cases} dX(t) = [r(t)X(t) + B_p(t,\alpha(t))u(t)]dt + \sigma_p(t,\alpha(t))u(t)dW(t), \\ X(0) = x_0, \quad \alpha(0) = i_0. \end{cases} \tag{37}$$

It is clear that we can get (37) by setting the coefficients $A(s,\alpha(s)) = r(s)$, $B(s,\alpha(s)) = B_p(s,\alpha(s))$, $D(s,\alpha(s)) = \sigma_p(s,\alpha(s))$, $C(s,\alpha(s)) = b(s,\alpha(s)) = \sigma(s,\alpha(s)) = 0$, and initial pair $(t,x,i) = (0,x_0,i_0)$ in (2). Our objective is to find an admissible portfolio $u(\cdot)$ to minimize the variance of the terminal wealth and, meanwhile, to maximize the expectation of the terminal wealth. The problem can be stated as follows, which is an equivalent statement.

Problem. (MV) *For given initial wealth x_0, initial market mode i_0 and trade-off λ, find a $u^*(\cdot) \in \mathcal{U}[t,T]$ minimizing*

$$J_{MV}(x_0, i_0; u(\cdot)) := \mathbb{E}\Big[(X(T) - \mathbb{E}[X(T)])^2\Big] - \lambda \mathbb{E}[X(T)]. \tag{38}$$

Comparing (3) and (38), we can set $G(\alpha(T)) = 1$, $\bar{G}(\alpha(T)) = -1$, $\bar{g}(\alpha(T)) = -\frac{\lambda}{2}$, and all of the rest coefficients are zero. Then the Problem (MV) becomes a special case of Problem (M-MF-SLQ). By the Proposition 2, we know the optimal portfolio selection is given by (16), i.e.

$$u^*(\cdot) = -\frac{1}{2}\mathcal{M}_t^{-1}\mathcal{D}J_{MV}(x_0, i_0; 0)(\cdot).$$

By (13), we have

$$\mathcal{D}J_{MV}(x_0, i_0; 0)(\cdot) = 2\Big[B(s, \alpha(s))Y^0(s; 0, x_0, i_0) + D(s, \alpha(s))Z^0(s; 0, x_0, i_0)\Big].$$

And it is easy to compute out the solution to the FBSDE (7) when $u(\cdot) = 0$ with initial pair $(t, x, i) = (0, x_0, i_0)$. So we have

$$\begin{cases} X^0(s; 0, x_0, i_0) = x_0 e^{r(s)s}, \\ Y^0(s; 0, x_0, i_0) = -\dfrac{\lambda}{2} e^{r(s)(s-T)}, \\ Z^0(s; 0, x_0, i_0) = \Gamma_k^0(s; 0, x_0, i_0) = 0. \end{cases} \tag{39}$$

Now, it remains to calculate the representation of the functional $\mathcal{M}_t^{-1}$. It is difficult to deduce the explicit form of $\mathcal{M}_t^{-1}$ but we have already get

$$\mathcal{M}_t u^*(s) = \frac{\lambda}{2}(b_p(s, \alpha(s)) - r(s))e^{r(s)(s-T)}.$$

By the definition of $\mathcal{M}_t$ in (14), we have

$$\mathcal{M}_t = \widehat{\mathcal{L}}_t^*\Big(1 - \mathbb{E}^*\mathbb{E}\Big)\widehat{\mathcal{L}}_t,$$

i.e.

$$\begin{aligned} \langle\, \mathcal{M}_t u^*(s), u^*(s)\,\rangle \quad &= \langle\, \widehat{\mathcal{L}}_t u^*(s), \widehat{\mathcal{L}}_t u^*(s)\,\rangle - \langle\, \mathbb{E}[\widehat{\mathcal{L}}_t u^*(s)], \mathbb{E}[\widehat{\mathcal{L}}_t u^*(s)]\,\rangle, \\ &= \langle\, X^{u^*}(T; 0, 0, i_0), X^{u^*}(T; 0, 0, i_0)\,\rangle \\ &\quad - \langle\, \mathbb{E}[X^{u^*}(T; 0, 0, i_0)], \mathbb{E}[X^{u^*}(T; 0, 0, i_0)]\,\rangle\,. \end{aligned} \tag{40}$$

Due to the linearity of the state equation, we can figure out that

$$X^{u^*}(T; 0, 0, i_0) = e^{r(T)T}\Big[\int_0^T e^{-r(t)t}B_p(t, \alpha(t))u^*(t)dt + \int_0^T e^{-r(t)t}\sigma_p(t, \alpha(t))u^*(t)dW(t)\Big]. \tag{41}$$

Replacing $X^{u^*}(T; 0, 0, i_0)$ in (40) by (41), we can get the optimal portfolio selection $u^*(\cdot)$.

4. **Conclusion.** In this paper, we have studied the mean-field stochastic linear quadratic optimal control problem of Markov regime switching system. We have derived the characteristics of the solution. In particular, based on the cost function defined by (14), and in Proposition 2 we proved that the convexity of the cost functional (H3) is necessary for the finiteness of the Problem (M-MF-SLQ), whereas uniform convexity of the cost functional (H4) is sufficient for the open-loop solvability of the problem. Finally, in Theorem 2.2 by considering a family of uniformly

convex cost functionals, a characterization of the finiteness of the problem is derived and a minimizing sequence, whose convergence is equivalent to the open-loop solvability of the problem, is constructed.

There are a few possible extension to our work. For instance, we can introduce some Riccati equations to decouple the MF-FBSDE (7) and investigate the relationship between the solvability of the Riccati equations and the convexity of the cost functionals. Besides, we can further investigate the closed-loop solvability of the Problem (M-MF-SLQ) and the solvability of the related Riccati equations.

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