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FREE BOUNDARY PROBLEM FOR AN OPTIMAL INVESTMENT PROBLEM WITH A BORROWING CONSTRAINT

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Abstract. This paper considers an optimal investment problem under CRRA utility with a borrowing constraint. We formulate it into a free boundary problem consisting of a fully nonlinear equation and a linear equation. We prove the existence and uniqueness of the classical solution and present the condition for the existence of the free boundary under a linear constraint on a borrowing rate. Furthermore, we prove that the free boundary is continuous and smooth when the relative risk aversion coefficient is sufficiently small.

1. **Introduction.** Credit decisions depend on supply and demand factors in the real financial market. On the supply side, the lender will decide how much to lend, considering either their potential borrowers' capacity to repay or quantity rationing. On the demand side, the investors' desire to borrowing depends on the price of credit, which is affected by interest rates, inflation, and other macroeconomic conditions. Some institutional investors, tiny private firms, have difficulty in raising money and are impacted by the discriminatory borrowing constraint imposed on them compared with state-owned firms. More collateral is required, and the shortage of credit information exposure leads to the fact that they have less access to sufficient external financing from banks. Individual investors have to face credit ceilings according to the evaluations of their present value of assets, income stream, consumption habits, credit reports, and many other factors. Therefore, investors cannot borrow money as much as they desire at any time. It is meaningful for us to consider a borrowing constraint in the investment problem.

The milestone works on the optimal investment problem in a continuous-time setting are represented by Samuelson [19] and Merton [16, 17], where investors can dynamically adjust their portfolio allocation to the risk-free and risky asset in

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order to maximize a certain linear expected utility over a set of possible terminal payoff over time. Along this line of research, optimal investment problems have been extensively studied under different objectives with constraints. For example, Li, Zhou and Lim [14] consider a continuous-time mean-variance portfolio problem where short-selling stocks are prohibited. Bielecki et al. [2] study a similar problem where the bankruptcy of the wealth process is prohibited. Dai and Yi [4] and Dai, Xu and Zhou [3] develop optimal investment problems with transaction cost under an expected utility model and the mean-variance model, respectively. Li and Xu [13] consider a continuous-time Markowitz's model with bankruptcy prohibition and convex cone portfolio constraints. Guan [8] discusses an investment problem with different interest rates. In this paper, we incorporate the borrowing constraint into our model under the expected utility framework.

The borrowing constraint means the dollar amount allocated in the risky asset cannot exceed an exogenous time-invariant function at any time t, mathematically, that is

$$\pi_t \leq f(X_t),$$

namely, the maximum borrowing rate at time t is $f(X_t) - X_t$. Our control problem is considered under a standard Black-Sholes framework over a finite trading horizon, the associated HJB equation is

$$-V_t - \max_{0\leq\pi\leq f(x)} \Big(\frac{1}{2}\sigma^2\pi^2 V_{xx} + \mu\pi V_x\Big) - rxV_x = 0.$$

When

$$V_x > 0, \quad V_{xx} < 0, \tag{1.1}$$

the optimal strategy follows

$$\pi^* = \operatorname*{argmax}_{0\leq\pi\leq f(x)} \Big(\frac{1}{2}\sigma^2\pi^2 V_{xx} + \mu\pi V_x\Big) = \min\Big\{ -\frac{\mu}{\sigma^2}\frac{V_x}{V_{xx}},\ f(x)\Big\}. \tag{1.2}$$

If $\pi^* = f(x)$, the borrowing rate reaches its upper limit, the corresponding HJB equation is a linear partial differential equation

$$-V_t - \frac{1}{2}\sigma^2 f^2(x) V_{xx} - \mu f(x) V_x - rxV_x = 0.$$

if $\pi^* < f(x)$, the constraint is loose, V satisfies a fully nonlinear equation

$$-V_t + \frac{\mu^2}{2\sigma^2}\frac{V_x^2}{V_{xx}} - rxV_x = 0.$$

The continuous-time optimal investment problem boils down to a free boundary problem. Our model's free boundary is a time-wealth curve, by which the whole domain is divided into two regions based on different types of investment policy.

The topics covered in free boundary problems are diverse, including insurance risk control, option pricing, credit default risk, and portfolio optimization. For example, Asmussen and Taksar [1] study a dividend payout problem of an insurance company under a controlled diffusion model, where dividend strategies depend on the wealth of the insurance company. Yang, Yi and Dai [21] formulate a pricing model of a strike reset option as a free boundary problem where the free boundary corresponds to the optimal reset strategy adopted by the holder of the option. Hu, Liang and Wu [11] propose a free boundary model for pricing a corporate bond with credit rating migration. Guan et al. [9] investigate an optimal stopping problem for an investor whose utility is nonsmooth and nonconcave over a finite time horizon.

Mathematically, the free boundary problem can be distinguished into infinite and finite time horizon problems. In infinite time scenarios, such as perpetual American option pricing, dividend problem with unrestricted dividend rate, where the free boundary degenerates to a point, explicit characterization can be derived by smooth pasting, see [20] for a relatively complete exposition. While in the finite horizon case, technically, it is more complicated than the former. A partial differential equation implicitly specifies the solution to the free boundary problem, and the free boundary is determined by the domain over which the PDE must be solved. Due to its analytical intractability resulting from an additional time variable in the associated HJB equation, we turn to investigate the regularity of the solution to associated PDE and the properties of free boundary using techniques derived from theories in the field of differential equations (See eg., Friedman [5, 6], Gilbarg and Trudinger [7], Guan, Yi and Chen [10], Lieberman[15]).

In our problem, we impose quadratic differentiable and linear growth conditions on $f(x)$. We show that the value function is cubically differentiable in x and quadratically differentiable in t, namely, $V \in C^{3,2}$. In particular, we provide conditions for the existence of the free boundary when $f(x)$ is confined to a linear function. Moreover, we prove that the free boundary is continuous and smooth when the relative risk aversion coefficient is relatively small. In the previous work, the closest one to us is Zariphopoulou [22], where they also incorporate the borrowing constraint in the investment decision. We differ from their work in two aspects. First, we apply PDE techniques to present rigorous proof of a classical solution's existence and uniqueness to the value function. Second, we further investigate the free boundary properties governed by a linear and a fully nonlinear partial differential equation.

The main contributions of this paper are listed as follows. We formulate our problem into a free boundary problem for a fully nonlinear equation and a linear equation and prove that the solution belongs to $C^{3,2}$ space. Moreover, we present the existence and smoothness of the free boundary. The techniques established here are also applicable to other similar problems.

The remainder of this paper is organized as follows. Section 2 formulates a dynamic investment model and provide some properties of the value function. Section 3 gives the HJB equation and presents the main results of the solution listed in Theorem 3.1. In Section 4, we discuss the existence, uniqueness and smoothness of the free boundary and estimate the upper bound when $f(x)$ is a linear function. Numerical examples are shown to visualize the free boundary in Section 5. We prove Theorem 3.1, Lemma A.4 and Theorem 3.2 in Appendix A, B, and C, respectively.

2. **Model formulation.** We consider a financial market with two financial instruments: a bond and a stock. The price of bond S_0 follows an ordinary differential equation:

$$\mathrm{d}S_t^0 = rS_t^0\mathrm{d}t,$$

where r is the risk free interest rate. The price of stock can be described by the classical Black-Sholes dynamics:

$$\mathrm{d}S_t^1 = S_t^1\big((r+\mu)\mathrm{d}t + \sigma\mathrm{d}B_t\big),$$

We assume the investor can trade continuously during the whole period $[0, T]$. By denoting π_t as dollar amount invested in risky asset, the wealth process of the investor becomes

$$\begin{cases} \mathrm{d}X_s &= \big(rX_s + \mu\pi_s\big)\mathrm{d}s + \sigma\pi_s\mathrm{d}B_s, \quad t \le s \le T, \\ X_t &= x, \end{cases} \tag{2.1}$$

where $x > 0$ is the current endowment of the investor, the objective of the investor is to choose the optimal investment strategy π_s to maximize the expected (discounted) utility of her terminal wealth

$$\mathbb{E}_{t,x}\left[e^{-\beta(T-t)}U(X_T)\right],$$

where $\mathbb{E}_{t,x}[\cdot]$ represents the conditional expectation $\mathbb{E}[\cdot|X_t = x]$, β is a constant discount factor, and $U : \mathbb{R}^+ \to \mathbb{R}$ is a concave function.

Denote $L^2_{\mathcal{F}}([0,T];\mathbb{R})$ as the set of all $\mathbb{R}$-valued, $\mathcal{F}_t$- progressively measurable process $g(t,\omega)$ satisfying $\mathbb{E}\int_0^T |g(t,\omega)|^2 \mathrm{d}t < +\infty$. It requires $\pi_s \in L^2_{\mathcal{F}}([t,T];\mathbb{R})$ such that (2.1) admits a unique solution. Meanwhile, no bankruptcy is allowed in this paper that

$$X_s \ge 0, \quad t \le s \le T.$$

In practice, the trading constraint on stock always exists partly due to the existence of leverage (borrowing) constraint, as pointed out in the introduction. Other practical examples, such as Basel Accord imposes a leverage ceiling on banks to prevent them expand the balance sheets. Mutual funds are also faced with leverage monitor by Securities Regulatory Commission. Thus, in many cases, the leverage available need not meet the demand for credit. These scenarios motivate us to consider the investment strategy subject to an upper bound that

$$\pi_s \le f(X_s), \qquad s \in [t,T].$$

Specifically, we assume upper bound is a time invariant function of X_s, on which we impose a quadratic and linear growth condition that

$$f \in C^2([0,+\infty)), \tag{2.2}$$

$$0 \le f(x) \le kx + b, \tag{2.3}$$

$$0 \le f'(x) \le k, \tag{2.4}$$

where k, b are positive constants. Then the admissible investment strategy set can be written as

$$\Pi_t := \{\pi_s \in L^2_{\mathcal{F}}([t,T];\mathbb{R}) \mid \pi_s \le f(X_s),\ X_s \ge 0,\ \forall s \in [t,T]\}.$$

The constant relative risk aversion (CRRA) utility function is commonly employed to measure investors' attitudes towards risk. In this paper, we adopt the CRRA utility function and assume the predetermined discounted factor β is zero for analytical simplicity, i.e., the value function is defined as

$$V(x,t) = \sup_{\pi_s \in \Pi_t} \mathbb{E}_{t,x}\left[\frac{X_T^{1-\gamma}}{1-\gamma}\right] \tag{2.5}$$

for $\gamma > 0$ and $\gamma \ne 1$.

If there is no upper bound restriction for π_t, i.e. $f(x) \equiv +\infty$, the explicit solution of (2.5) can be expressed by

$$\overline{V} := e^{\rho(T-t)}\frac{x^{1-\gamma}}{1-\gamma},$$

where $\rho := \mu^2(1-\gamma)/(2\sigma^2\gamma) + r(1-\gamma)$. In this case, the optimal investment strategy $\overline{\pi}_t$ is proportional to current wealth, namely, $\overline{\pi}_t := \kappa X_t$, where $\kappa := \mu/(\sigma^2\gamma)$.

If we choose admissible strategy $\pi_s \equiv 0$, the particular solution is

$$\underline{V} := e^{\eta(T-t)} \frac{x^{1-\gamma}}{1-\gamma},$$

where $\eta := r(1-\gamma)$. Hence, we get an upper bound and a lower bound on V:

$$\underline{V} \le V \le \overline{V}. \tag{2.6}$$

If $k \ge \kappa$, then $\overline{\pi}_t = \kappa X_t < kX_t + b$, namely the constraint on π_t is not tight, so we have $V = \overline{V}$. Thus, we only need to discuss the case of $k < \kappa$.

Due to the fact that the utility function $U(x) = x^{1-\gamma}/(1-\gamma)$ is increasing and concave, It is not hard to prove that V is also increasing and concave with respect to x. Thus $V_x(\cdot, t)$ is finite almost everywhere for each t. Moreover, according to the concavity property, we get

$$\frac{V(\lambda x, t) - V(x,t)}{(\lambda - 1)x} \le V_x(x,t) \le \frac{V(x,t) - V(x/2,t)}{x/2},$$

for any $\lambda > 1$. Using (2.6), we have

$$\frac{V(x,t) - V(x/2,t)}{x/2} \le \frac{\overline{V}(x,t) - \underline{V}(x/2,t)}{x/2} = \overline{C}x^{-\gamma},$$

where

$$\overline{C} = \frac{2e^{\rho(T-t)} - 2^{\gamma} e^{\eta(T-t)}}{1-\gamma} > 0.$$

On the other hand, using (2.6) again, we get

$$\frac{V(\lambda x, t) - V(x,t)}{(\lambda - 1)x} \ge \frac{\underline{V}(\lambda x, t) - \overline{V}(x,t)}{(\lambda - 1)x} = \underline{C}x^{-\gamma},$$

where

$$\underline{C} = \frac{e^{\eta(T-t)}\lambda^{1-\gamma} - e^{\rho(T-t)}}{(\lambda - 1)(1-\gamma)},$$

It is positive if we choose $\lambda > \exp(\frac{(\rho - \eta)T}{1-\gamma})$. Therefore, we get a growth condition on V_x as

$$\underline{C}x^{-\gamma} \le V_x \le \overline{C}x^{-\gamma}. \tag{2.7}$$

Then we have a boundary condition on $x = 0$ that

$$V_x(0+, t) = +\infty.$$

3. **HJB equation.** Applying dynamic programming method, we obtain the associated HJB equation of problem (2.5) with terminal-boundary condition:

$$\begin{cases} -V_t - \max\limits_{0 \le \pi \le f(x)} \left(\frac{1}{2}\sigma^2 \pi^2 V_{xx} + \mu \pi V_x \right) - rxV_x = 0 \quad \text{in} \quad \Omega := (0, +\infty) \times [0, T], \\ V_x(0+, t) = +\infty, \quad 0 < t < T, \\ V(x, T) = \frac{x^{1-\gamma}}{1-\gamma}, \quad x > 0. \end{cases} \tag{3.1}$$

The following theorem gives that this fully nonlinear problem has a unique classical solution.

Theorem 3.1. *Suppose $f(x)$ satisfies conditions (2.2)-(2.4), then there exists a unique classical solution V to Problem (3.1), and its first order partial derivatives*

$$V_t,\ V_x \in C^{2,1}(\Omega). \tag{3.2}$$

Moreover, there exist the following estimates

$$e^{N(T-t)}\frac{x^{1-\gamma}}{1-\gamma} - C_T \leq V \leq e^{M(T-t)}2^{\gamma}\frac{x^{1-\gamma}}{1-\gamma} + C_T, \tag{3.3}$$

$$e^{N(T-t)}x^{-\gamma} \leq V_x \leq e^{M(T-t)}2^{\gamma}x^{-\gamma}, \tag{3.4}$$

$$V_{xx} < 0, \tag{3.5}$$

$$V_{xt} \leq -NV_x \tag{3.6}$$

where M and N are constants, defined as

$$M := \frac{\mu^2(\gamma+1)}{2\sigma^2\gamma} + (\mu K_f + r) + \frac{1}{2}\gamma(\gamma+1), \quad K_f := \sup_{x\in[0,+\infty)} f'(x),$$

$$N := -\frac{(\mu+\sigma^2 K_f)^2}{2\sigma^2}\frac{\gamma}{\gamma+1} - r\gamma + (\mu k_f + r), \quad k_f := \inf_{x\in[0,+\infty)} f'(x),$$

and $C_T > 0$ only depends on T.

Proof. We put it in Section A. □

Combining the regularity of the solution of Problem (3.1) and stochastic control theory, we can prove the verification theorem.

Theorem 3.2. *If the upper bound function $f(x)$ satisfies condition (2.2)-(2.4), then the solution of problem (3.1) is the value function defined in (2.5).*

Proof. See Appendix C. □

4. **The free boundary for the case $f(x) = kx + b$.** Based on Theorem 3.1, the HJB equation in (3.1) can be rewritten as

$$\begin{cases} -V_t + \frac{\mu^2}{2\sigma^2}\frac{V_x^2}{V_{xx}} - rxV_x = 0, & \text{if } -\frac{\mu}{\sigma^2}\frac{V_x}{V_{xx}} < f(x), \\ -V_t - \frac{1}{2}\sigma^2 f^2(x)V_{xx} - \mu f(x)V_x - rxV_x = 0, & \text{if } -\frac{\mu}{\sigma^2}\frac{V_x}{V_{xx}} \geq f(x). \end{cases} \tag{4.1}$$

It is a free boundary problem consisting of a fully nonlinear equation and a linear equation.

In this section, we consider a special case of the trigger bound $f(x)$ with a linear function that

$$f(x) = kx + b,$$

where $k,\ b > 0$.

Define the following two regions

$$\mathcal{S} := \{(x,t)|\pi^* < kx + b\}, \quad \mathcal{R} := \{(x,t)|\pi^* = kx + b\},$$

where

$$\pi^* := \min\Big\{ -\frac{\mu}{\sigma^2}\frac{V_x}{V_{xx}},\ kx + b\Big\}, \tag{4.2}$$

is the optimal investment on risky asset.

4.1. **The condition on the existence of free boundary.** Before we discuss the existence of these two regions, we present two lemmas.

Lemma 4.1. *Let*

$$\theta := \frac{(\mu+\sigma^2 k)^2}{\mu^2}\frac{\gamma}{\gamma+1} + \frac{2\sigma^2 r\gamma}{\mu^2} - \frac{2\sigma^2 k}{\mu} - 1,$$

we have the estimation

$$\partial_x\Big(\frac{V_x}{V_{xx}}\Big) \le \theta \quad \textit{in} \quad \mathcal{S}. \tag{4.3}$$

Proof. Differentiating the first equation of (4.1) w.r.t. x we have

$$-V_{xt} - \frac{\mu^2}{2\sigma^2}\Big(\frac{V_x}{V_{xx}}\Big)^2 V_{xxx} - rxV_{xx} + (\frac{\mu^2}{\sigma^2} - r)V_x = 0 \quad \text{in} \quad \mathcal{S}.$$

Dividing by V_x and noting that

$$V_{xxx} = -\frac{(V_{xx})^2}{V_x}\Big[\partial_x\Big(\frac{V_x}{V_{xx}}\Big) - 1\Big]$$

we obtain

$$-\frac{V_{xt}}{V_x} + \frac{\mu^2}{2\sigma^2}\partial_x\Big(\frac{V_x}{V_{xx}}\Big) - \frac{\mu^2}{2\sigma^2} - rx\frac{V_{xx}}{V_x} + (\frac{\mu^2}{\sigma^2} - r) = 0 \quad \text{in} \quad \mathcal{S}.$$

Using (3.5) and (3.6), namely,

$$V_{xx} < 0, \quad V_{xt} \le NV_x,$$

we get

$$\partial_x\Big(\frac{V_x}{V_{xx}}\Big) \le \frac{2\sigma^2}{\mu^2}(r - N) - 1 = \theta \quad \text{in} \quad \mathcal{S}.$$

□

Lemma 4.2. *For any* $t \in [0,T]$,

$$\lim_{x\to 0} \pi^*(x,t) = 0. \tag{4.4}$$

Proof. We first prove that for any $t \in [0,T]$,

$$\liminf_{x\to 0} \pi^*(x,t) = 0. \tag{4.5}$$

If it is not true, there exist $t_0 \in [0,T]$ and $\delta > 0$, such that

$$\pi^*(x,t_0) \ge \delta, \quad x \in (0,\delta),$$

which implies

$$-\frac{\mu}{\sigma^2}\frac{V_x}{V_{xx}}(x,t_0) \ge \delta, \quad x \in (0,\delta).$$

So

$$\ln(V_x(\delta,t_0)) - \ln(V_x(x,t_0)) = \int_x^\delta \frac{V_{xx}}{V_x}(y,t_0)\mathrm{d}y \ge -\frac{\mu}{\sigma^2\delta}(\delta - x), \quad x \in (0,\delta).$$

Using the first inequality in (3.4), we derive

$$\ln(V_x(\delta,t_0)) - N(T-t) + \gamma\ln x \ge -\frac{\mu}{\sigma^2\delta}(\delta - x), \quad x \in (0,\delta).$$

Taking $x \to 0+$, we get a contradiction that $-\infty \ge \mu/\sigma^2$. Therefore, (4.5) holds.

Furthermore, by (4.3) we have

$$\lim_{x\to 0}\pi^*(x,t)=\liminf_{x\to 0}\pi^*(x,t)=0.$$

□

Intuitively, investors tend to fully invest in the riskless asset for value maintenance when they are on the brink of bankruptcy. Our lemma proves that the optimal investment strategy in risky assets is zero when the wealth state is close to zero, which is consistent with the practice.

The following proposition gives the condition of the existence of $\mathcal{R}$.

Proposition 4.3. *If $k\ge\kappa=\mu/(\sigma^2\gamma)$, we have*

$$\pi^*(x,t)<kx+b,\quad \forall x>0,\ 0\le t\le T,$$

namely $\mathcal{R}=\emptyset$. Otherwise, if $k<\kappa=\mu/(\sigma^2\gamma)$, then for any $t\in[0,T]$, we have

$$\{x>0|\pi^*(x,t)=kx+b\}\neq\emptyset. \tag{4.6}$$

Proof. If $k\ge\kappa=\mu/(\sigma^2\gamma)$, according to the discussion in Section 2, $\pi^*(x,t)=\kappa x<kx+b$.

In the case that $k<\kappa$, if (4.6) fails, there exists a $t_0\in[0,T]$, such that $\pi^*(x,t_0)<kx+b$ for all $x>0$, i.e.,

$$-\frac{\mu}{\sigma^2}\frac{V_x}{V_{xx}}(x,t_0)<kx+b,\quad x>0.$$

Note that

$$\begin{aligned}\ln(V_x(x,t_0))-\ln(V_x(1,t_0))&=\int_1^x\frac{V_{xx}}{V_x}(y,t_0)\mathrm{d}y\\&<-\frac{\mu}{\sigma^2}\int_1^x\frac{1}{ky+b}\mathrm{d}y=-\frac{\mu}{\sigma^2k}\Big(\ln(kx+b)-\ln(k+b)\Big).\end{aligned}$$

Using the first inequality in (3.4), we have

$$N(T-t)-\gamma\ln x-\ln(V_x(1,t_0))<-\frac{\mu}{\sigma^2k}\Big(\ln(kx+b)-\ln(k+b)\Big).$$

Dividing by $\ln x$ and taking $x\to+\infty$ we get $-\gamma\le-\mu/(\sigma^2k)$, which is a contradiction. □

Now, we define a function

$$g(t):=\inf\{x>0|\pi^*(x,t)=kx+b\}.$$

Due to (4.4) and (4.6), we obtain $0<g(t)<+\infty$ when $k<\kappa:=\mu/(\sigma^2\gamma)$. We know from the definition of $g(t)$ that $\{(x,t)|x<g(t)\}\subset\mathcal{S}$. But up to now, whether $\pi^*(x,t)=kx+b$ when $x\ge g(t)$ is still unknown. In other words, we could not ascertain $g(t)$ is the unique free boundary line. In the next section, we will prove $g(t)$ is the unique and smooth free boundary when γ is small enough.

4.2. **The smoothness of free boundary for small γ.**

Theorem 4.4. *If γ is small enough such that*

$$q := -\frac{(\mu+\sigma^2 k)^2}{\sigma^2\mu}\frac{\gamma}{\gamma+1} - \frac{2r\gamma}{\mu} + \frac{\mu}{\sigma^2} + k > 0,$$

(which implies $k < \kappa$), the free boundary line $g(t)$ is unique, i.e.,

$$\mathcal{S} = \{(x,t)|x < g(t)\}, \quad \mathcal{R} = \{(x,t)|x \geq g(t)\}. \tag{4.7}$$

Moreover, we have $g \in C^1([0,T])$ and

$$0 < g(t) \leq \frac{b}{q},$$

$$g(T) = \frac{b}{\frac{\mu}{\sigma^2\gamma} - k} > 0. \tag{4.8}$$

Proof. Define

$$I(x,t) := -\frac{\mu}{\sigma^2}\frac{V_x}{V_{xx}} - (kx+b). \tag{4.9}$$

Note that (3.2) and $V_{xx}(g(t),t) \neq 0$ imply $I \in C^{1,1}$ in a neighborhood of the line $x = g(t)$. Using (4.3) we get

$$\begin{aligned} I_x &\geq -\frac{\mu}{\sigma^2}\theta - k \\ &\geq -\frac{\mu}{\sigma^2}\Big(\frac{(\mu+\sigma^2 k)^2}{\mu^2}\frac{\gamma}{\gamma+1} + \frac{2\sigma^2 r\gamma}{\mu^2} - \frac{2\sigma^2 k}{\mu} - 1\Big) - k \\ &= q > 0 \quad \text{in} \quad \mathcal{S}, \end{aligned}$$

which implies $\{(x,t)|x \geq g(t)\} \subset \mathcal{R}$, and will imply (4.7).

Since $I(g(t),t) = 0$ and $I(0+,t) = -b$ (by Lemma 4.2), combining with $I_x \geq q > 0$ when $x < g(t)$, we have

$$b = I(g(t),t) - I(0+,t) \geq qg(t),$$

which implies

$$g(t) \leq \frac{b}{q}.$$

Now, we prove the continuity of $g(t)$. Suppose it is not true, there exists a $t_0 \in [0,T]$ such that $g(t)$ is discontinuous at t_0, i.e.,

$$x_1 := \liminf_{t\to t_0} g(t) < x_2 := \limsup_{t\to t_0} g(t), \tag{4.10}$$

by the continuity of I defined by (4.9), we have $I(x,t_0) = 0$, $\forall x \in [x_1,x_2]$, thus $I_x(x,t_0) = 0$, $\forall x \in [x_1,x_2]$. However, since $I_x \geq q$ in $\mathcal{S}$ and the continuity of I_x, we have $I_x(x_1,t_0) \geq q > 0$. This contradiction implies (4.10) is impossible. Consequently, $g(t) \in C([0,T])$.

Now, we prove $g(t) \in C^1([0,T])$. Note that

$$I(g(t),t) = 0, \quad t \in [0,T].$$

It follows from implicit differentiation that

$$I_x(g(t),t)g'(t) + I_t(g(t),t) = 0, \quad t \in [0,T].$$

Note that $I_x \geq q$ in $\mathcal{S}$ and the continuity of I_x imply $I_x(g(t),t) > 0$. Then we can derive

$$g'(t) = \frac{I_t(g(t),t)}{I_x(g(t),t)} \in C([0,T]),$$

Thus $g(t) \in C^1([0,T])$.

Finally, we ascertain $g(T)$. The terminal condition $V(x,T) = x^{1-\gamma}/(1-\gamma)$ leads to

$$-\frac{\mu}{\sigma^2}\frac{V_x}{V_{xx}}(x,T) = \frac{\mu}{\sigma^2}\frac{1}{\gamma}x.$$

Thus $g(T)$ is the root of the equation

$$\frac{\mu}{\sigma^2}\frac{1}{\gamma}x = kx + b.$$

Then we have (4.8). The proof is complete. □

5. **A numerical example.** We provide numerical results to the free boundary and with the following parameters unless otherwise specified:

$$\gamma = 0.5, \quad \mu = 0.4, \quad \sigma^2 = 0.35, \quad r = 0.05, \quad T = 3, \quad k = 1.2, \quad b = 1.5.$$

which satisfy the conditions in Theorem 4.4. We approximate the problem in a bounded domain $[x_{\min}, x_{\max}] \times [0,T]$ with boundary conditions

$$V\Big|_{x=x_{\min}} = 0, \quad V_x\Big|_{x=x_{\max}} = 0.$$

and further choose the following parameters:

$$\Delta x = 0.01, \quad \Delta t = 0.001, \quad x_{\min} = 0.01, \quad x_{max} = 50,$$

where Δt is the time step size and Δx is the value size. We change the direction of time by taking $\tau = T - t$ as remaining maturity. Figure 1 depicts the free boundary $g(\tau)$ for different parameters k and b. We can see that the free boundary is a smooth curve, which is consistent with the result of Theorem 4.5. Besides, with different coefficients in $f(x)$, the corresponding free boundaries all exhibit decreasing trends as time approaches maturity.

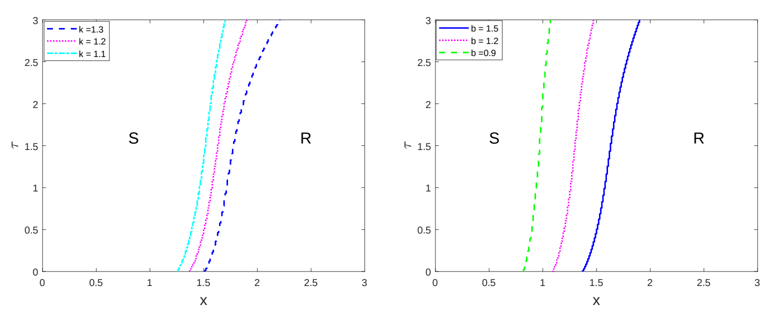

FIGURE 1. Free boundaries with various k and b

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Appendix A. The proof of Theorem 3.1.

A.1. **The equation on $W := V_x$.** Now, we derive the equation on $W := V_x$ and discuss its regularity. The advantage is that it can transform the fully non-linear equation of V into a quasilinear equation of W.

From (4.2) we see that the optimal π^* is a function of V_x/V_{xx} and x, we define this function by

$$A(z,x) := \min\Big\{-\frac{\mu}{\sigma^2}z, f(x)\Big\}, \quad z<0,\ x>0.$$

For convenience of discussion, set $A(z,x) = f(x)$ if $z \geq 0$ such that

$$A(z,x) = \begin{cases} -\frac{\mu}{\sigma^2}z, & \text{if} \quad 0 < -\frac{\mu}{\sigma^2}z < f(x), \\ f(x), & \text{others}. \end{cases}$$

Then we can write

$$\pi^* = A\big(\frac{V_x}{V_{xx}}, x\big).$$

Without precluding the case $V_{xx} = 0$, we further define

$$\begin{aligned} A(\pm\infty, x) &:= \lim_{z\to\pm\infty} A(z,x) = f(x), \\ A_z(\pm\infty, x) &:= \lim_{z\to\pm\infty} A_z(z,x) = 0, \\ A_x(\pm\infty, x) &:= \lim_{z\to\pm\infty} A_x(z,x) = f'(x). \end{aligned}$$

Furthermore, if we regard π^* as a function of (V_x, V_{xx}, x), which is denoted by

$$G(u,v,x) := A\big(\frac{u}{v}, x\big).$$

We found that it is Lipschitz continuous in $[\varepsilon, +\infty) \times (-\infty, +\infty) \times [0, L]$ for any fixed $\varepsilon,\ L > 0$ since

$$\begin{aligned} |G_u(u,v,x)| &= \Big|A_z\big(\frac{u}{v}, x\big)\frac{1}{v}\Big| = -\frac{\mu}{\sigma^2}\frac{u}{v}\frac{1}{u} \leq \frac{f(L)}{\varepsilon}, \\ |G_v(u,v,x)| &= \Big| - A_z\big(\frac{u}{v}, x\big)\frac{u}{v^2}\Big| = \frac{\sigma^2}{\mu}\big(-\frac{\mu}{\sigma^2}\frac{u}{v}\big)^2\frac{1}{u} \leq \frac{\sigma^2}{\mu}\frac{f^2(L)}{\varepsilon} \end{aligned}$$

if $0 < -\frac{\mu}{\sigma^2}\frac{u}{v} < f(x)$ and $G(u,v,x) = f(x) \in C^1(\mathbb{R}^+)$ if $-\frac{\mu}{\sigma^2}\frac{u}{v} \geq f(x)$ or $-\frac{\mu}{\sigma^2}\frac{u}{v} \leq 0$.

Now, (3.1) can be rewritten as the following terminal-boundary value problem on fully nonlinear equation.

$$\begin{cases} -V_t - \mathcal{L}V = 0 \quad \text{in} \quad \Omega, \\ V_x(0+, t) = +\infty, \quad 0 < t < T, \\ V(x,T) = \frac{x^{1-\gamma}}{1-\gamma}, \quad x > 0, \end{cases} \tag{A.1}$$

where the operator $\mathcal{L}$ is defined by

$$\mathcal{L}V := \frac{1}{2}\sigma^2 A^2\Big(\frac{V_x}{V_{xx}}, x\Big)V_{xx} + \mu A\Big(\frac{V_x}{V_{xx}}, x\Big)V_x + rxV_x.$$

Note that if

$$0 < -\frac{\mu}{\sigma^2}\frac{V_x}{V_{xx}} < f(x),$$

then

$$\partial_x(\mathcal{L}V) = \frac{\mu^2}{2\sigma^2}\Big(\frac{V_x}{V_{xx}}\Big)^2 V_{xxx} - \frac{\mu^2}{\sigma^2}V_x + rxV_{xx} + rV_x,$$

otherwise,

$$\partial_x(\mathcal{L}V) = \frac{1}{2}\sigma^2 f(x)^2 V_{xxx} + (\mu + \sigma^2 f'(x))f(x)V_{xx} + rxV_{xx} + (r + \mu f'(x))V_x,$$

they can be merged into

$$\begin{aligned}\partial_x(\mathcal{L}V) =& \frac{1}{2}\sigma^2 A^2\Big(\frac{V_x}{V_{xx}}, x\Big)V_{xxx} + \Big(\mu + \sigma^2 f'(x)\Big)A\Big(\frac{V_x}{V_{xx}}, x\Big)V_{xx} \\ &+ rxV_{xx} + \Big(r + \mu f'(x)\Big)V_x,\end{aligned}$$

which is a quasilinear operator on V_x, we denote it by $\mathcal{T}$, i.e.,

$$\begin{aligned}\mathcal{T}W :=& \frac{1}{2}\sigma^2 A^2\Big(\frac{W}{W_x}, x\Big)W_{xx} + \Big(\mu + \sigma^2 f'(x)\Big)A\Big(\frac{W}{W_x}, x\Big)W_x \\ &+ rxW_x + \Big(r + \mu f'(x)\Big)W,\end{aligned} \tag{A.2}$$

thus, $W = V_x$ satisfies the following terminal-boundary value problem

$$\begin{cases} -W_t - \mathcal{T}W = 0 & \text{in} \quad \Omega, \\ W(0+, t) = +\infty, & 0 < t < T, \\ W(x, T) = x^{-\gamma}, & x > 0. \end{cases} \tag{A.3}$$

A.2. **Approximation method.** If we regard equations in (A.1) and (A.3) as linear equations, coefficients of the second order term will not have positive lower bounds, i.e. (A.1) and (A.3) will not satisfy the parabolic condition. Therefore, we define

$$\mathcal{L}_\varepsilon V := \mathcal{L}V + \frac{\varepsilon^2}{2}V_{xx}$$

and

$$\mathcal{T}_\varepsilon W := \mathcal{T}W + \frac{\varepsilon^2}{2}W_{xx}.$$

Denote

$$\mathcal{Q}_\varepsilon := (\varepsilon, \frac{1}{\varepsilon}) \times [0, T].$$

Consider the following approximation problem of (A.3) in bounded domain

$$\begin{cases} -W^\varepsilon_t - \mathcal{T}_\varepsilon W^\varepsilon = 0 & \text{in} \quad \mathcal{Q}_\varepsilon, \\ W^\varepsilon(\varepsilon, t) = e^{N^+(T-t)}\varepsilon^{-\gamma}, & 0 < t < T, \\ (\varepsilon\gamma W^\varepsilon + W^\varepsilon_x)(\frac{1}{\varepsilon}, t) = 0, & 0 < t < T, \\ W^\varepsilon(x, T) = x^{-\gamma}, & \varepsilon < x < \frac{1}{\varepsilon}, \end{cases} \tag{A.4}$$

where $N^+ = \max\{N, 0\}$. We will begin with problem (A.4) to prove the existence and the properties of solution to the origin problem (A.1).

To the approximation problem (A.4), we have

Lemma A.1. *If the upper bound function $f(x)$ satisfies conditions (2.2)-(2.4), there exists a unique solution $W^\varepsilon \in C^{2,1}(\mathcal{Q}_\varepsilon) \cap C(\overline{\mathcal{Q}_\varepsilon})$ of problem (A.4). Moreover, it satisfies*

$$e^{N(T-t)}x^{-\gamma} \le W^\varepsilon \le e^{M(T-t)}2^\gamma(x + \varepsilon)^{-\gamma}, \tag{A.5}$$

where M, N are positive constants defined in Theorem 3.1.

Proof. Using the Leray-Schauder fixed point theorem (see [5] or [7]) and embedding theorem (see [5]) we can prove problem (A.4) has at least a solution belongs to the space

$$\left\{W^\varepsilon \in C^{1+\alpha,\frac{1+\alpha}{2}}\big(\mathcal{Q}_\varepsilon\big) \;\middle|\; 0 \le W^\varepsilon \le e^{TN^+}\varepsilon^{-\gamma}\right\}$$

for some $0 < \alpha < 1$. Moreover, Schauder estimation (see [12]) implies this solution $W^\varepsilon \in C^{2+\alpha,1+\frac{\alpha}{2}}\big(\mathcal{Q}_\varepsilon\big)$.

Now we give the proof to (A.5). Denote

$$\phi(x,t) := e^{N(T-t)}x^{-\gamma}.$$

Note that

$$\phi > 0, \quad \phi_x < 0, \quad \phi_{xx} > 0.$$

So we have

$$\begin{aligned}
-\phi_t - \mathcal{T}_\varepsilon\phi =& -\phi_t - \frac{1}{2}\sigma^2 A^2\Big(\frac{\phi}{\phi_x},x\Big)\phi_{xx} - \frac{\varepsilon^2}{2}\phi_{xx}\\
& - \Big(\mu + \sigma^2 f'(x)\Big)A\Big(\frac{\phi}{\phi_x},x\Big)\phi_x - rx\phi_x - \Big(r + \mu f'(x)\Big)\phi\\
\le& -\phi_t + \max_{a\in\mathbb{R}}\Big(-\frac{1}{2}\sigma^2 a^2\phi_{xx} - \Big(\mu + \sigma^2 f'(x)\Big)a\phi_x\Big)\\
& - rx\phi_x - \Big(r + \mu f'(x)\Big)\phi\\
\le& -\phi_t + \frac{\Big(\mu + \sigma^2 f'(x)\Big)^2\phi_x^2}{2\sigma^2\phi_{xx}} - rx\phi_x - \Big(r + \mu f'(x)\Big)\phi\\
\le& e^{N(T-t)}x^{-\gamma}\Big(N + \frac{\Big(\mu + \sigma^2 K_f\Big)^2}{2\sigma^2}\frac{\gamma}{\gamma+1} + r\gamma - \Big(r + \mu k_f\Big)\Big)\\
=& 0.
\end{aligned}$$

Since $\phi(\varepsilon,t) = e^{N(T-t)}\varepsilon^{-\gamma} \le e^{N^+(T-t)}\varepsilon^{-\gamma} = W^\varepsilon(\varepsilon,t)$, $(\varepsilon\gamma\phi + \phi_x)(\frac{1}{\varepsilon},t) = 0 = (\varepsilon\gamma W^\varepsilon + W^\varepsilon_x)(\frac{1}{\varepsilon},t)$ and $\phi(x,T) = x^{-\gamma} = W^\varepsilon(x,T)$, we can obtain the first inequality in (A.5) by using the comparison principle to the quasilinear equation (see [6] or [18]).

Similarly, denote

$$\Phi(x,t) := e^{M(T-t)}2^\gamma(x+\varepsilon)^{-\gamma}.$$

Note that

$$\Phi > 0, \quad \Phi_x < 0, \quad \Phi_{xx} > 0, \quad A\Big(\frac{\Phi}{\Phi_x}\Big) \le \frac{\mu}{\sigma^2}\Big|\frac{\Phi}{\Phi_x}\Big|.$$

So we have

$$\begin{aligned}
-\Phi_t-\mathcal{T}_\varepsilon\Phi=&-\Phi_t-\frac{1}{2}\sigma^2A^2\Big(\frac{\Phi}{\Phi_x},x\Big)\Phi_{xx}-\frac{\varepsilon^2}{2}\Phi_{xx}-\Big(\mu+\sigma^2f'(x)\Big)A\Big(\frac{\Phi}{\Phi_x},x\Big)\Phi_x\\
&-rx\Phi_x-\Big(r+\mu f'(x)\Big)\Phi\\
\geq&-\Phi_t-\frac{\mu^2}{2\sigma^2}\Big(\frac{\Phi}{\Phi_x}\Big)^2\Phi_{xx}-\frac{\varepsilon^2}{2}\Phi_{xx}-\Big(r+\mu f'(x)\Big)\Phi\\
\geq& e^{M(T-t)}2^\gamma\Big[(x+\varepsilon)^{-\gamma}\Big(M-\frac{\mu^2(\gamma+1)}{2\sigma^2\gamma}-(r+\mu f'(x))\Big)\\
&-\frac{\varepsilon^2}{2}\gamma(\gamma+1)(x+\varepsilon)^{-\gamma-2}\Big]\\
\geq& e^{M(T-t)}2^\gamma(x+\varepsilon)^{-\gamma}\Big(M-\frac{\mu^2(\gamma+1)}{2\sigma^2\gamma}-(r+\mu K_f)-\frac{1}{2}\gamma(\gamma+1)\Big)\\
=&0.
\end{aligned}$$

Due to the fact that $M\geq N^+$ and $2^\gamma(x+\varepsilon)^{-\gamma}\geq x^{-\gamma}$, $\forall x\geq\varepsilon$, we get $\Phi(\varepsilon,t)\geq W^\varepsilon(\varepsilon,t)$ and $\Phi(x,T)\geq W^\varepsilon(x,T)$. Moreover, $(\varepsilon\gamma\Phi+\Phi_x)(\frac{1}{\varepsilon},t)=e^{M(T-t)}2^\gamma\gamma\varepsilon^2(\frac{1}{\varepsilon}+\varepsilon)^{-\gamma-1}\geq 0$. According to the comparison principle to quasilinear equation, the second inequality in (A.5) holds. $\square$

Proposition A.2. *For $\varepsilon>0$, we have*

$$W_x^\varepsilon\leq 0. \tag{A.6}$$

Proof. It is easy to prove that the function $h(x,t):=e^{N^+(T-t)}\varepsilon^{-\gamma}$ is a super-solution to problem (A.4), together with $W^\varepsilon(\varepsilon,t)=h(\varepsilon,t)$, we have $W_x^\varepsilon(\varepsilon,t)\leq h_x(\varepsilon,t)=0$. The right boundary condition in (A.4) yields $W_x^\varepsilon(\frac{1}{\varepsilon},t)=-\varepsilon\gamma W^\varepsilon(\frac{1}{\varepsilon},t)\leq 0$. Combining with terminal condition $W_x^\varepsilon(x,T)=-\gamma x^{-\gamma-1}<0$, (A.6) holds on the parabolic boundary of $\mathcal{Q}_\varepsilon$.

Taking derivative to the equation in (A.4) with respect to x, we obtain the following equation on W_x^ε in divergence form as follows

$$\begin{aligned}
&-\partial_tW_x^\varepsilon-\partial_x\Big[\Big(\frac{\sigma^2}{2}A^2+\frac{\varepsilon^2}{2}\Big)\partial_xW_x^\varepsilon\Big]-\Big[\Big(\mu+\sigma^2f'(x)\Big)A+rx\Big]\partial_xW_x^\varepsilon\\
&-\Big(\mu+\sigma^2f'(x)\Big)\Big[\Big(1-\frac{W^\varepsilon W_{xx}^\varepsilon}{(W_x^\varepsilon)^2}\Big)A_z+A_x\Big]W_x^\varepsilon-\Big[\sigma^2f'(x)'A+\mu f'(x)+2r\Big]W_x^\varepsilon=0.
\end{aligned}$$

where we slightly abuse the notation for simplicity that

$$A=A\Big(\frac{W^\varepsilon}{W_x^\varepsilon},x\Big),\quad A_x=A_x\Big(\frac{W^\varepsilon}{W_x^\varepsilon},x\Big),\quad A_z=A_z\Big(\frac{W^\varepsilon}{W_x^\varepsilon},x\Big).$$

After adjustment, we get

$$\begin{aligned}
&-\partial_tW_x^\varepsilon-\partial_x\Big[\Big(\frac{\sigma^2}{2}A^2+\frac{\varepsilon^2}{2}\Big)\partial_xW_x^\varepsilon\Big]\\
&\qquad-\Big[\Big(\mu+\sigma^2f'(x)\Big)A-\Big(\mu+\sigma^2f'(x)\Big)A_z\frac{W^\varepsilon}{W_x^\varepsilon}+rx\Big]\partial_xW_x^\varepsilon\\
&\quad-\Big[\Big(\mu+\sigma^2f'(x)\Big)\Big(A_z+A_x\Big)+\sigma^2f'(x)'A+\mu f'(x)+2r\Big]W_x^\varepsilon=0.
\end{aligned}\tag{A.7}$$

Note that

$$A\Big(\frac{W^\varepsilon}{W_x^\varepsilon},x\Big),\quad A_z\Big(\frac{W^\varepsilon}{W_x^\varepsilon},x\Big)\frac{W^\varepsilon}{W_x^\varepsilon},\quad A_x\Big(\frac{W^\varepsilon}{W_x^\varepsilon},x\Big)$$

are bounded in $[\varepsilon, \varepsilon^{-1}] \times [0, T]$. Using the maximum principle of divergence form (see [6] or [18]), we obtain $W^\varepsilon_x \leq 0$. □

Proposition A.3. *For any $\varepsilon > 0$, we have*

$$W^\varepsilon_t \leq -NW^\varepsilon. \tag{A.8}$$

Proof. Denote $w(x,t) := e^{-N(T-t)}W^\varepsilon(x,t)$ and $\overline{w}(x,t) = w(x, t-h)$. Then both w and $\overline{w}$ satisfy the same following equation

$$-w_t - \mathcal{T}_\varepsilon w + Nw = 0.$$

The first inequality in (A.5) yields $w \geq x^\gamma$, which implies

$$w(x,T) = x^\gamma \leq w(x, T-h) = \overline{w}(x,T).$$

Since $w(\varepsilon, t) = e^{(N^+-N)(T-t)}\varepsilon^{-\gamma}$ and $(\varepsilon\gamma w + w)(\frac{1}{\varepsilon}, t) = 0$ are decreasing in t, by comparison principle, we get $\overline{w} \geq w$ in $\mathcal{Q}_\varepsilon$, i.e. $w := e^{-N(T-t)}W^\varepsilon(x,t)$ is decreasing in t, which implies the desired result (A.8). □

The following lemma shows that the equation in (A.4) satisfies the uniform parabolic condition interior.

Lemma A.4. *For any $d > a > 0$, there exists a $\delta > 0$, which is independent of ε (but depends on a, d), such that*

$$A\Big(\frac{W^\varepsilon}{W^\varepsilon_x}, x\Big) \geq \delta, \quad (x,t) \in [a,d] \times [0,T].$$

Proof. Since the proof is somewhat technical, we put it in Appendix B. □

Now, suppose W^ε is the solution of (A.4), and define

$$V^\varepsilon(x,t) = \int_1^x W^\varepsilon(y,t)\mathrm{d}y + \int_t^T h_\varepsilon(t)\mathrm{d}t + \frac{1}{1-\gamma}, \tag{A.9}$$

where

$$h_\varepsilon(t) := \frac{1}{2}\sigma^2 A^2\Big(\frac{W^\varepsilon}{W^\varepsilon_x}, x\Big)W^\varepsilon_x + \frac{\varepsilon^2}{2}W^\varepsilon_x + \mu A\Big(\frac{W^\varepsilon}{W^\varepsilon_x}, x\Big)W^\varepsilon + rxW^\varepsilon\bigg|_{(1,t)}.$$

Then $V^\varepsilon_x = W^\varepsilon$. So we have

$$\partial_x(-V^\varepsilon_t - \mathcal{L}_\varepsilon V^\varepsilon) = -W^\varepsilon_t - \mathcal{T}_\varepsilon W^\varepsilon = 0.$$

Moreover, note that

$$(-V^\varepsilon_t - \mathcal{L}_\varepsilon V^\varepsilon)(1,t) = 0,$$

we derive

$$(-V^\varepsilon_t - \mathcal{L}_\varepsilon V^\varepsilon)(x,t) = (-V^\varepsilon_t - \mathcal{L}_\varepsilon V^\varepsilon)(1,t) + \int_1^x \partial_x(-V^\varepsilon_t - \mathcal{L}_\varepsilon V^\varepsilon)(y,t)\mathrm{d}y = 0.$$

Therefore, V^ε satisfies the following equation.

$$\begin{cases} -V^\varepsilon_t - \mathcal{L}_\varepsilon V^\varepsilon = 0 & \text{in} \quad Q_\varepsilon, \\ V^\varepsilon(x,T) = \frac{x^{1-\gamma}}{1-\gamma}, & \varepsilon < x < 1/\varepsilon. \end{cases} \tag{A.10}$$

Lemma A.5. *There exists a $0 < \alpha < 1$ such that, for any $[a,d] \subset (0, +\infty)$,*

$$|V^\varepsilon|_{C^{3+\alpha, \frac{3+\alpha}{2}}([a,d]\times[0,T])} \leq C, \tag{A.11}$$

where C is independent of ε.

Proof. Note that W^ε is uniformly bounded in any bounded region $[a,d]\times[0,T]\subset\Omega$. Since the coefficients of the second derivative of (A.4) have uniform positive upper and lower bounds which are independent of ε in $[a,d]\times[0,T]$, i.e., (A.4) satisfies the uniform parabolic condition in $[a,d]\times[0,T]$. Taking $C^{\alpha,\frac{\alpha}{2}}$ interior estimate (see [6] or [15]), we obtain

$$|W^\varepsilon|_{C^{\alpha,\frac{\alpha}{2}}([a,d]\times[0,T])}\le C, \tag{A.12}$$

where C is independent of ε. Using $C^{\alpha,\frac{\alpha}{2}}$ interior estimate to (A.7) yields

$$|W^\varepsilon_x|_{C^{\alpha,\frac{\alpha}{2}}([a,d]\times[0,T])}\le C. \tag{A.13}$$

Therefore, according to the definition in (A.9), V^ε is uniformly bounded in $[a,d]\times[0,T]$. Using $C^{\alpha,\frac{\alpha}{2}}$ interior estimate to (A.10), we derive

$$|V^\varepsilon|_{C^{\alpha,\frac{\alpha}{2}}([a,d]\times[0,T])}\le C.$$

According to (A.12), (A.13) and the equation in (A.10), we have

$$|V^\varepsilon_t|_{C^{\alpha,\frac{\alpha}{2}}([a,d]\times[0,T])}\le C.$$

Hence, $|V^\varepsilon|_{C^{2+\alpha,1+\frac{\alpha}{2}}([a,d]\times[0,T])}$ is uniformly bounded. Furthermore, taking Schauder interior estimate to (A.4) (see [5] or [15]), we have

$$|V^\varepsilon_x|_{C^{2+\alpha,1+\frac{\alpha}{2}}([a,d]\times[0,T])}\le C,$$

which implies the desired result (A.11). □

A.3. **Existence and uniqueness of solution to the original problem.** By Lemma A.5, problem (A.10) has at least one solution $V^\varepsilon\in C^{3+\alpha,\frac{3+\alpha}{2}}([\varepsilon,\frac{1}{\varepsilon}]\times[0,T])$, such that for any region $\mathcal{Q}=[a,d]\times[0,T]\subset\Omega$, there exists a subsequence, which is denoted by V^ε, satisfying $V^\varepsilon\to V$ in $C^{3,\frac{3}{2}}(\mathcal{Q})$. Therefore, V satisfies the equation and the terminal condition of (A.1).

Taking derivative for the PDE in (A.1) with respect to t, we obtain the following equation

$$-\partial_t V_t-\frac{1}{2}\sigma^2A^2\Big(\frac{V_x}{V_{xx}},x\Big)\partial_{xx}V_t-\mu A\Big(\frac{V_x}{V_{xx}},x\Big)\partial_x V_t-rx\partial_x V_t=0.$$

Since $V\in C^{3+\alpha,\frac{3+\alpha}{2}}(\Omega)$ and $A(\frac{V_x}{V_{xx}},x)$ belongs to $C^{\alpha,\frac{\alpha}{2}}$ with positive upper and lower bounds in any bounded region contained in Ω, we obtain

$$V_t\in C^{2,1}(\Omega)$$

using Schauder interior estimate.

Following from (A.5) and (A.8) we have (3.4) and (3.6). (A.6) implies (3.5). Also, we derive (3.3) from (A.9).

Finally, we prove its uniqueness. Suppose that $V_1,\ V_2\in C^{2,1}(\Omega)$ are two solutions to problem (3.1) with the growth condition:

$$|V_i|\le C(x^{1-\gamma}+1),\quad i=1,2, \tag{A.14}$$

for a large constant $C>0$.

Define a barrier function

$$\Phi^L:=4e^{\beta(T-t)}C\frac{x^2+1}{L}\quad \text{in } [0,L]\times[0,T],$$

where $\beta > 0$ is undetermined. Note that

$$\begin{aligned} & -\partial_t \Phi^L - \sup_{0\le\pi\le f(x)} \Big(\frac{1}{2}\sigma^2\pi^2\partial_{xx}\Phi^L + \mu\pi\partial_x\Phi^L\Big) - rx\partial_x\Phi^L \\ & \qquad = \frac{4e^{\beta(T-t)}C}{L}\Big(\beta(x^2+1) - \sigma^2 f^2(x) - 2\mu f(x)x - 2rx\Big). \end{aligned}$$

We choose β large enough to get

$$-\partial_t \Phi^L - \sup_{0\le\pi\le f(x)} \Big(\frac{1}{2}\sigma^2\pi^2\partial_{xx}\Phi^L + \mu\pi\partial_x\Phi^L\Big) - rx\partial_x\Phi^L \ge 0.$$

Introducing $V_2^\varepsilon(x,t) := V_2(x+\varepsilon,t)$, we derive

$$\begin{aligned} & -\partial_t V_2^\varepsilon - \sup_{0\le\pi\le f(x)} \Big(\frac{1}{2}\sigma^2\pi^2\partial_{xx}V_2^\varepsilon + \mu\pi\partial_x V_2^\varepsilon\Big) - rx\partial_x V_2^\varepsilon \\ \ge & -\partial_t V_2^\varepsilon - \sup_{0\le\pi\le f(x+\varepsilon)} \Big(\frac{1}{2}\sigma^2\pi^2\partial_{xx}V_2^\varepsilon + \mu\pi\partial_x V_2^\varepsilon\Big) - r(x+\varepsilon)\partial_x V_2^\varepsilon \\ = & 0. \end{aligned}$$

Together with the equation

$$-\partial_t V_1 - \sup_{0\le\pi\le f(x)} \Big(\frac{1}{2}\sigma^2\pi^2\partial_{xx}V_1 + \mu\pi\partial_x V_1\Big) - rx\partial_x V_1 = 0,$$

we obtain

$$\begin{aligned} & -\partial_t\Big(V_1 - V_2^\varepsilon - \Phi^L\Big) - \sup_{0\le\pi\le f(x)} \Big[\frac{1}{2}\sigma^2\pi^2\partial_{xx}\Big(V_1 - V_2^\varepsilon - \Phi^L\Big) \\ & \qquad + \mu\pi\partial_x\Big(V_1 - V_2^\varepsilon - \Phi^L\Big)\Big] - rx\partial_x\Big(V_1 - V_2^\varepsilon - \Phi^L\Big) \le 0. \end{aligned}$$

Moreover,

$$(V_1 - V_2^\varepsilon - \Phi^L)(x,T) = \frac{x^{1-\gamma}}{1-\gamma} - \frac{(x+\varepsilon)^{1-\gamma}}{1-\gamma} - \Phi^L(x,T) \le 0,$$

and owing to (A.14) and $\partial_x V_1(0+,t) = +\infty$, we have $(V_1 - V_2^\varepsilon - \Phi^L)(L,t) \le 0$ and $\partial_x(V_1 - V_2^\varepsilon - \Phi^L)(0+,t) = +\infty \ge 0$, respectively. Applying the maximum principle, we get $V_1 - V_2^\varepsilon - \Phi^L \le 0$ in $[0,L]\times[0,T]$. For the fixed point $(x,t)\in\Omega$, we choose L satisfying $x < L$ to get $(V_1 - V_2^\varepsilon - \Phi^L)(x,t) \le 0$. Taking $L \to +\infty$ and $\varepsilon \to 0$, we have $V_1 \le V_2$.

Appendix B. **The prove of Lemma A.4.** By Lemma A.1, we know that W^ε has a uniform positive lower bound in $[a,d]\times[0,T]$. Hence, we only need to prove the following result.

Lemma B.1. *For any $a > d > 0$, there exists a $C > 0$ which is independent of ε, such that*

$$W_x^\varepsilon \ge -C \quad in \quad [a,d]\times[0,T]. \tag{B.15}$$

Proof. Define

$$\mathcal{S}_\varepsilon = \Big\{(x,t)\in\Omega \;\Big|\; -\frac{\mu}{\sigma^2}\frac{W^\varepsilon}{W_x^\varepsilon} < f(x) \quad \text{and} \quad W_x^\varepsilon < 0\Big\},$$
$$\mathcal{R}_\varepsilon = \Big\{(x,t)\in\Omega \;\Big|\; -\frac{\mu}{\sigma^2}\frac{W^\varepsilon}{W_x^\varepsilon} \ge f(x) \quad \text{or} \quad W_x^\varepsilon \ge 0\Big\}.$$

It is obvious that (B.15) holds in $\mathcal{R}_\varepsilon \cap ([a,d]\times[0,T])$. Now, we focus on $\mathcal{S}_\varepsilon \cap ([a,d]\times[0,T])$. By the PDE of (A.4), we have

$$-W_t^\varepsilon - \frac{\mu^2}{2\sigma^2}\Big(\frac{W^\varepsilon}{W_x^\varepsilon}\Big)^2 W_{xx}^\varepsilon - \frac{\varepsilon^2}{2}W_{xx}^\varepsilon - rxW_x^\varepsilon + (\frac{\mu^2}{\sigma^2}-r)W^\varepsilon = 0 \quad \text{in} \quad \mathcal{S}_\varepsilon.$$

Note that

$$\partial_x\Big(\frac{W^\varepsilon}{W_x^\varepsilon}\Big) = 1 - \frac{W^\varepsilon}{(W_x^\varepsilon)^2}W_{xx}^\varepsilon \quad \text{and} \quad W_{xx}^\varepsilon = -\frac{(W_x^\varepsilon)^2}{W^\varepsilon}\Big[\partial_x\Big(\frac{W^\varepsilon}{W_x^\varepsilon}\Big)-1\Big].$$

Then we obtain

$$\begin{aligned}-\frac{W_t^\varepsilon}{W^\varepsilon} + \frac{1}{2}\Big[\frac{\mu^2}{\sigma^2}+\varepsilon^2\Big(\frac{W_x^\varepsilon}{W^\varepsilon}\Big)^2\Big]\partial_x\Big(\frac{W^\varepsilon}{W_x^\varepsilon}\Big)&\\ -\frac{1}{2}\Big[\frac{\mu^2}{\sigma^2}+\varepsilon^2\Big(\frac{W_x^\varepsilon}{W^\varepsilon}\Big)^2\Big] - rx\frac{W_x^\varepsilon}{W^\varepsilon} + (\frac{\mu^2}{\sigma^2}-r) &= 0 \quad \text{in} \quad \mathcal{S}_\varepsilon.\end{aligned}$$

Using (A.6) and (A.8), we get

$$\partial_x\Big(\frac{W^\varepsilon}{W_x^\varepsilon}\Big) \le \frac{2(N-\frac{\mu^2}{\sigma^2}+r) + \Big[\frac{\mu^2}{\sigma^2}+\varepsilon^2\Big(\frac{W_x^\varepsilon}{W^\varepsilon}\Big)^2\Big]}{\frac{\mu^2}{\sigma^2}+\varepsilon^2\Big(\frac{W_x^\varepsilon}{W^\varepsilon}\Big)^2} \quad \text{in} \quad \mathcal{S}_\varepsilon. \tag{B.16}$$

Let

$$\lambda := \max\Big\{1, \frac{2(N+r)-\frac{\mu^2}{\sigma^2}}{\frac{\mu^2}{\sigma^2}}\Big\},$$

then

$$\partial_x\Big(\frac{W^\varepsilon}{W_x^\varepsilon}\Big) \le \lambda \quad \text{in} \quad \mathcal{S}_\varepsilon.$$

Thus,

$$\begin{aligned}\partial_x\Big(\frac{(W^\varepsilon)^{-\lambda}}{W_x^\varepsilon}\Big) &= \partial_x\Big((W^\varepsilon)^{-(\lambda+1)}\frac{W^\varepsilon}{W_x^\varepsilon}\Big)\\ &= (W^\varepsilon)^{-(\lambda+1)}\Big[\partial_x\Big(\frac{W^\varepsilon}{W_x^\varepsilon}\Big)-(\lambda+1)\Big]\\ &\le -(W^\varepsilon)^{-(\lambda+1)} \quad \text{in} \quad \mathcal{S}_\varepsilon.\end{aligned}$$

According to estimation (A.5), there exist two constants $C_2 > C_1 > 0$ independent of ε such that

$$C_1 \le W^\varepsilon \le C_2, \quad (x,t)\in[\frac{a}{2},d]\times[0,T].$$

Hence, we obtain

$$\partial_x\Big(\frac{(W^\varepsilon)^{-\lambda}}{W_x^\varepsilon}\Big) \le -C_2^{-(\lambda+1)} \quad \text{in} \quad \mathcal{S}_\varepsilon \cap [\frac{a}{2},d]\times[0,T]. \tag{B.17}$$

For any $(x,t)\in\mathcal{S}_\varepsilon\cap([a,d]\times[0,T])$, let $y=\sup\{z\in(a/2,x)|(z,t)\in\mathcal{R}_\varepsilon\}$, then we obtain $\{(z,t)|y<z<x\}\subset\mathcal{S}_\varepsilon$. If $y=a/2$, i.e., $\{(z,t)|a/2<z<x\}\subset\mathcal{S}_\varepsilon$, by (B.17), we get

$$\begin{aligned}\Big(\frac{(W^\varepsilon)^{-\lambda}}{W_x^\varepsilon}\Big)(x,t) &\le \Big(\frac{(W^\varepsilon)^{-\lambda}}{W_x^\varepsilon}\Big)(\frac{a}{2},t) - \Big(x-\frac{a}{2}\Big)C_2^{-(\lambda+1)}\\ &\le -\Big(x-\frac{a}{2}\Big)C_2^{-(\lambda+1)} \le -\frac{a}{2}C_2^{-(\lambda+1)}.\end{aligned}$$

Therefore,

$$W_x^\varepsilon(x,t) \geq -\frac{2}{a} C_2^{(\lambda+1)} (W^\varepsilon)^{-\lambda}(x,t) \geq -\frac{2}{a} C_2^{(\lambda+1)} C_1^{-\lambda},$$

which implies (B.15).

Otherwise, if $y > a/2$, by (B.17), we obtain

$$\begin{aligned}\Big(\frac{(W^\varepsilon)^{-\lambda}}{W_x^\varepsilon}\Big)(x,t) \leq \Big(\frac{(W^\varepsilon)^{-\lambda}}{W_x^\varepsilon}\Big)(y,t) = \Big(\frac{W^\varepsilon}{W_x^\varepsilon}\frac{1}{(W^\varepsilon)^{\lambda+1}}\Big)(y,t) \\ = -\frac{\sigma^2}{\mu} f(y) \frac{1}{(W^\varepsilon)^{\lambda+1}(y,t)} \leq -\frac{\sigma^2}{\mu} f(\frac{a}{2}) \frac{1}{C_2^{\lambda+1}},\end{aligned}$$

which also implies (B.15). □

Appendix C. **The proof of Theorem 3.2.** Suppose V is the solution of (3.1). For fixed $x > 0$ and $t < T$ and any admissible π_s, suppose X_s is the strong solution of (2.1), we choose $\tau_n = \inf\{s \geq t | V_x(X_s) \geq n\}$ such that $Y_s := \int_t^{s\wedge\tau_n} V_x(X_u)\sigma\pi_u \mathrm{d}W_u$ is a martingale and $\tau_n \to +\infty$ when $n \to +\infty$. By Itô formula and the HJB equation on V,

$$\begin{aligned}&\mathbb{E}_{t,x}[V(X_{T\wedge\tau_n}, T\wedge\tau_n)] - V(x,t) \\ &\quad = \mathbb{E}_{t,x}\Big[\int_t^{T\wedge\tau_n} \Big(V_t + \frac{1}{2}\sigma^2\pi_s^2 V_{xx} + \mu\pi_s V_x - rxV_x\Big)(X_s,s)\mathrm{d}s\Big] \leq 0.\end{aligned}$$

Using the estimate (3.3) we have

$$|V(X_{T\wedge\tau_n}, T\wedge\tau_n)| \leq C(1 + \sup_{s\in[t,T]} |X_s|^{1-\gamma}),$$

by the standard SDE theorem, the right hand side is integrable, so we can apply the dominated convergence theorem and the terminal condition to obtain

$$V(x,t) \geq \mathbb{E}_{t,x}[\lim_{n\to\infty} V(X_{T\wedge\tau_n}, T\wedge\tau_n)] = \mathbb{E}_{t,x}[U(X_T)].$$

Therefore, we have

$$V(x,t) \geq \sup_{\pi_s\in\Pi_t} \mathbb{E}_{t,x}[U(X_T)].$$

On the other hand, suppose X_s^* is the solution of the following SDE

$$\begin{cases} \mathrm{d}X_s^* &= \big(rX_s^* + \mu\pi^*(X_s^*,s)\big)\mathrm{d}s + \sigma\pi^*(X_s^*,s)\mathrm{d}B_s, \quad t \leq s \leq T, \\ X_t^* &= x. \end{cases} \tag{C.18}$$

Applying Itô formula to $V(X_s^*, s)$ for $s \in [t,T]$(after an eventual localization for removing the stochastic integral term in the expectation), we get

$$\begin{aligned}V(x,t) =&\mathbb{E}_{t,x}[V(X_T^*,T)] - \mathbb{E}_{t,x}\Big[\int_t^T \Big(V_t + \frac{1}{2}\sigma^2\pi_s^{*2}V_{xx} + \mu\pi_s^* V_x - rxV_x\Big)(X_s^*,s)\mathrm{d}s\Big] \\ =&\mathbb{E}_{t,x}[V(X_T^*,T)] \\ =&U(X_T^*),\end{aligned}$$

where π^* is defined in (4.2). This shows that $V(x,t) = \sup_{\pi_s\in\Pi_t} \mathbb{E}_{t,x}[U(X_T)]$ and $\pi_t^* := \pi^*(X_t^*,t)$ is the optimal control.

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