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# Quantum Filtering for Multiple Measurements Driven by Two Single-Photon States

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***Abstract*— In this paper, the problem of quantum filtering with two homodyne detection measurements for a two-level system has been considered. The quantum system is driven by two input light field channels, each of which contains a single photon. A quantum filter based on multiple measurements is designed; both the master equations and stochastic master equations are derived. In addition, numerical simulations for master equations with various pulse shape parameters are also compared.**

**Key Words: Quantum filtering, master equation, single-photon state, homodyne detection**

## I. Introduction

IN quantum optics, the quantum filtering problems have drawn more and more attentions. It is first studied by Belavkin in [1][2] with a framework of continuous measurements. Quantum filter with various Gaussian input fields, such as vacuum state, thermal state and squeezed state, have been considered and investigated in [3]-[6]. For an arbitrary quantum system driven by single-photon states or coherent states, master equations and stochastic master equations have been described in [7], respectively. The conditional dynamic for a cavity driven by a single-photo state has been considered in [8], where both the homodyne detection and photon-counting measurements are simulated. Recently, filters for an arbitrary quantum system driven by continuous-mode multi-photons have been presented in [9]. Non-Markovian embedding technique is adopted in this framework and an example, a two-level system driven by a two-photon state, demonstrates the validity of multi-photon filters. Quantum system driven by single-photon states which is contaminated by vacuum noise has been investigated in [10]. Diffusive plus Poissonian measurements and two diffusive measurements have been discussed, the simulation results also show the improvements of estimation performance with multiple measurements.

In this paper, we study quantum filtering for a two-level system which driven by two single-photon states. Initially, each channel contains one photon and they are independent with each other. Then the two photons interact with the system and excite the atom simultaneously. The filtering equations with two homodyne detection measurements have been derived explicitly and numerical results of master equations for the system are given.

The rest of this paper is organized as follows: Section II gives some basic knowledge about quantum system and quantum filtering. In Section III, quantum filter for multiple measurements is briefly introduced, then we focus on the problem of a two-level system with two homodyne detection driven by two single-photon states. Both the master equations and stochastic master equations are derived explicitly. Some simulation results are presented in Section IV to show the pulse shape parameters' influence on the detection probability. Finally, Section V concludes the paper.

*Notation*. Let $|\eta\rangle$ be initial state of the system and $|0\rangle$ be the vacuum state. We use $X^\dagger = (X^*)^T$ to denote the adjoint operator or complex conjugate transpose. The commutator is defined by $[A, B] = AB - BA$ and superoperator

$$D_L X = L^\dagger X L - \frac{1}{2}(L^\dagger L X + X L^\dagger L).$$

## II. Preliminary

### A. Open Quantum Systems

The system we consider here is an arbitrary quantum system which driven by two single-photon states. The two single-photon state are independent with each other before their interaction with the system. We will use the triple language $(S, L, H)$ [11][13] to describe the dynamic of the system. Here, $S$ is a unitary operator which satisfies $SS^\dagger = S^\dagger S = I$. $L$ denotes the couplings between the system and field, and the initial Hamiltonian of the system is described by a self-adjoint operator $H$.

$a_j$ denotes the annihilation operator of the system while $a_j^*$ is the corresponding creation operator. The canonical commutation relations is satisfied that $[a_j, a_k^*] = \delta_{jk}$. On the other hand, the $j$-th input bosonic field is represented by $b_j(t)$, and $b_j^*(t)$ denotes its adjoint operator. Since the field we discuss is continuous-mode, we have the following property:

$$[b_j(t), b_k^*(r)] = \delta_{jk}\delta(t - r).$$

We assume that the quantum stochastic processes satisfy the following Itō table

| $\times$ | $dt$ | $dB$ | $d\Lambda$ | $dB^\dagger$ |
|---|---|---|---|---|
| $dt$ | 0 | 0 | 0 | 0 |
| $dB$ | 0 | 0 | $dB$ | $dt$ |
| $d\Lambda$ | 0 | 0 | $d\Lambda$ | $dB^\dagger$ |
| $dB^\dagger$ | 0 | 0 | 0 | 0 |

Quantum stochastic process on the tensor product space, i.e., $System \otimes Field$, is characterized by the following

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stochastic differential equation:

$$\begin{aligned} dU(t) =&\Big\{(S-I)d\Lambda(t) + LdB^{\dagger}(t) \\ &- L^{\dagger}SdB(t) - \left(\frac{1}{2}L^{\dagger}L + iH\right)dt\Big\}U(t) \end{aligned} \tag{1}$$

where $U(0) = I$.

Define $j_t(X) = U^{\dagger}(t)(X_{system} \otimes I_{field})U(t)$ in the Heisenberg picture, the temporal evolution of $j_t(X)$ can be derived by the Itō table and (1):

$$\begin{aligned} dj_t(X) =& j_t(\mathcal{L}_G X)dt + j_t([L^{\dagger}, X]S)dB(t) \\ &+ j_t(S^{\dagger}[X, L])dB^{\dagger}(t) + j_t(S^{\dagger}XS - X)d\Lambda(t). \end{aligned}$$

### B. Quantum Filtering

Homodyne detection and photon-counting measurements are commonly used in quantum filtering [9]. For homodyne detection, the measurement can be given in the quadrature form

$$Y(t) = U^{\dagger}(t)(I_{system} \otimes (B(t) + B^{\dagger}(t)))U(t),$$

where $B(t)$ is the integrated annihilation operator and defined as $B(t) = \int_0^t b(r)dr$. The measurement should satisfy the commutation relation

$$[Y(r), Y(t)] = 0, \quad 0 \le r \le t,$$

and the non-demolition property

$$[X(t), Y(s)] = 0, \quad 0 \le s \le t.$$

Quantum conditional expectation is defined by

$$\pi_t(X) = \mathbb{E}[j_t(X)|\mathcal{Y}_t],$$

where $\mathcal{Y}_t$ is characterized by $Y(r) : 0 \le r \le t$. Based on the past information $\mathcal{Y}_t$, we want to minimize the estimation $\mathbb{E}[(\pi_t(X) - j_t(X))^2]$ for system observable $j_t(X)$.

## III. Quantum Filter with Multi-measurements

### A. System Depiction

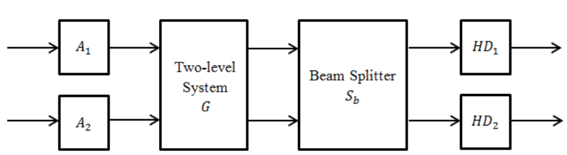


Fig. 1. Quantum system depiction

In Fig. 1, $A_1$, $A_2$ and $G$ are two-level systems with the following $(S, L, H)$ language:

$$A_1 = (I, L_1, 0), \quad A_2 = (I, L_2, 0), \quad G = (I_2, L, 0),$$

where $L_1 = \lambda_1(t)\sigma_{-1}$, $L_2 = \lambda_2(t)\sigma_{-2}$ and $L = \begin{bmatrix} \sqrt{\kappa_1}\sigma_- \\ \sqrt{\kappa_2}\sigma_- \end{bmatrix}$. Here, $\sigma_{-1}$, $\sigma_{-2}$ and $\sigma_-$ are lowering operators and $\lambda_1(t)$, $\lambda_2(t)$ are given by

$$\lambda_1(t) = \frac{\xi_1(t)}{\sqrt{w_1(t)}}, \quad \lambda_2(t) = \frac{\xi_2(t)}{\sqrt{w_2(t)}},$$

where $w_1(t) = \int_t^{\infty} |\xi_1(s)|^2 ds$, $w_2(t) = \int_t^{\infty} |\xi_2(s)|^2 ds$ and $\xi_1(t)$, $\xi_2(t)$ are the input pulse shapes in the first and second channels respectively.

By adding a beam splitter $S_b = (S_b, 0, 0)$ with parameter

$$S_b = \begin{bmatrix} \sqrt{1-r^2} & ir \\ ir & \sqrt{1-r^2} \end{bmatrix}, 0 \leqslant r \leqslant 1$$

and the concatenation and series products [11], we can derive the whole system

$$(A_1 \boxplus A_2) \triangleright G \triangleright S_b = (S_t, L_t, H_t), \tag{2}$$

where

$$\begin{aligned} S_t &= \begin{bmatrix} \sqrt{1-r^2} & ir \\ ir & \sqrt{1-r^2} \end{bmatrix}, \\ L_t &= \begin{bmatrix} \sqrt{1-r^2}(L_1 + \sqrt{\kappa_1}\sigma_-) + ir(L_2 + \sqrt{\kappa_2}\sigma_-) \\ ir(L_1 + \sqrt{\kappa_1}\sigma_-) + \sqrt{1-r^2}(L_2 + \sqrt{\kappa_2}\sigma_-) \end{bmatrix}, \\ H_t &= \frac{\sqrt{\kappa_1}\sigma_+ L_1 + \sqrt{\kappa_2}\sigma_+ L_2 - \sqrt{\kappa_1}L_1^{\dagger}\sigma_- - \sqrt{\kappa_2}L_2^{\dagger}\sigma_-}{2i}. \end{aligned} \tag{3}$$

By Itō calculus, the evolution of output fields

$$dB_{\text{out}}(t) = S_t dB_t + L_t dt,$$

and the general measurement equation

$$dY(t) = F^* dB^*_{\text{out}}(t) + F dB_{\text{out}}(t) + G\text{diag}(d\Lambda_{\text{out}}(t)),$$

with $F = I$, $G = 0$, which means that both channels are with homodyne detection measurements, we can get the measurement stochastic equations

$$\begin{aligned} dY_{1,t} =& \sqrt{1-r^2}\Big[dB_1(t) + dB_1^{\dagger}(t) + (L_1 + \sqrt{\kappa_1}\sigma_-)dt \\ &+ (L_1^{\dagger} + \sqrt{\kappa_1}\sigma_+)dt\Big] + ir\Big[dB_2(t) - dB_2^{\dagger}(t) \\ &+ (L_2 + \sqrt{\kappa_2}\sigma_-)dt - (L_2^{\dagger} + \sqrt{\kappa_2}\sigma_+)dt\Big], \\ dY_{2,t} =& ir\Big[dB_1(t) - dB_1^{\dagger}(t) + (L_1 + \sqrt{\kappa_1}\sigma_-)dt \\ &- (L_1^{\dagger} + \sqrt{\kappa_1}\sigma_+)dt\Big] + \sqrt{1-r^2}\Big[dB_2(t) + dB_2^{\dagger}(t) \\ &+ (L_2 + \sqrt{\kappa_2}\sigma_-)dt + (L_2^{\dagger} + \sqrt{\kappa_2}\sigma_+)dt\Big]. \end{aligned} \tag{4}$$

Then the expectation and correlation of the measurements can be derived as

$$\begin{aligned} \tilde{\pi}_t(dY_{1,t}) =& \sqrt{1-r^2}\tilde{\pi}_t(L_1 + L_1^{\dagger} + \sqrt{\kappa_1}\sigma_- + \sqrt{\kappa_1}\sigma_+)dt \\ &+ ir\tilde{\pi}_t(L_2 - L_2^{\dagger} + \sqrt{\kappa_2}\sigma_- - \sqrt{\kappa_2}\sigma_+)dt, \\ \tilde{\pi}_t(dY_{2,t}) =& \sqrt{1-r^2}\tilde{\pi}_t(L_2 + L_2^{\dagger} + \sqrt{\kappa_2}\sigma_- + \sqrt{\kappa_2}\sigma_+)dt \\ &+ ir\tilde{\pi}_t(L_1 - L_1^{\dagger} + \sqrt{\kappa_1}\sigma_- - \sqrt{\kappa_1}\sigma_+)dt, \end{aligned} \tag{5}$$

and

$$\begin{aligned} \tilde{\pi}_t(dY_{1,t}dY_{1,t}) &= \tilde{\pi}_t(dY_{2,t}dY_{2,t}) = dt, \\ \tilde{\pi}_t(dY_{1,t}dY_{2,t}) &= \tilde{\pi}_t(dY_{2,t}dY_{1,t}) = 0. \end{aligned} \tag{6}$$

The Lindblad superoperator for the whole system may be expressed in the form

$$\begin{aligned}&\mathcal{L}_{L_t}(A_1\otimes A_2\otimes X)=\mathcal{D}_{L_1}A_1\otimes A_2\otimes X\\&+A_1\otimes\mathcal{D}_{L_2}A_2\otimes X+(\kappa_1+\kappa_2)A_1\otimes A_2\otimes\mathcal{D}_{\sigma_-}X\\&+\sqrt{\kappa_1}L_1^\dagger A_1\otimes A_2\otimes[X,\sigma_-]+\sqrt{\kappa_1}A_1L_1\otimes A_2\otimes[\sigma_+,X]\\&+\sqrt{\kappa_2}A_1\otimes L_2^\dagger A_2\otimes[X,\sigma_-]+\sqrt{\kappa_2}A_1\otimes A_2L_2\otimes[\sigma_+,X],\end{aligned}$$

for any ancilla operator $A_1$, $A_2$ and system operator $X$.

According to Theorem 3.2 [14], we can derive the following result directly.

*Theorem 1:* Let $\{Y_{i,t}, i=1,2,\dots,N\}$ be a set of $N$ compatible measurement outputs for a quantum system $G$. With vacuum initial state, the corresponding joint measurement quantum filter is given by

$$\begin{aligned}d\tilde{\pi}_t(A_1\otimes A_2\otimes X)=&\tilde{\pi}_t(\mathcal{L}_{L_t}(A_1\otimes A_2\otimes X))dt\\&+\beta_t^T\left[dY_t-\tilde{\pi}_t(dY_t)\right],\end{aligned}\tag{7}$$

where $dW_{i,t}=dY_{i,t}-\tilde{\pi}_t(dY_{i,t})$ is a martingale process for each measurement output and $\beta$ is the corresponding gain given by

$$\begin{aligned}\beta_t^T=&\frac{1}{\tilde{\pi}_t(dY_tdY_t^T)}\Big\{\tilde{\pi}_t(A_1\otimes A_2\otimes XdY_t^T)\\&-\tilde{\pi}_t(A_1\otimes A_2\otimes X)\tilde{\pi}_t(dY_t^T)\\&+\tilde{\pi}_t\left([L_t^\dagger,A_1\otimes A_2\otimes X]S_tdB_tdY_t^T\right)\Big\}.\end{aligned}\tag{8}$$

In Fig. 1, the corresponding gain $\beta=\left[\begin{array}{cc}\beta_1 & \beta_2\end{array}\right]$ can be calculated by (5) and (6),

$$\begin{aligned}\beta_1=&\sqrt{1-r^2}\tilde{\pi}_t\Big[A_1L_1\otimes A_2\otimes X+L_1^\dagger A_1\otimes A_2\otimes X\\&+\sqrt{\kappa_1}A_1\otimes A_2\otimes X\sigma_-+\sqrt{\kappa_1}A_1\otimes A_2\otimes\sigma_+X\Big]\\&+ir\tilde{\pi}_t\Big[A_1\otimes A_2L_2\otimes X-A_1\otimes L_2^\dagger A_2\otimes X\\&+\sqrt{\kappa_2}A_1\otimes A_2\otimes X\sigma_--\sqrt{\kappa_2}A_1\otimes A_2\otimes\sigma_+X\Big]\\&-\tilde{\pi}_t(A_1\otimes A_2\otimes X)\left[\sqrt{1-r^2}k_{11}(t)+irk_{12}(t)\right],\\\beta_2=&ir\tilde{\pi}_t\Big[A_1L_1\otimes A_2\otimes X-L_1^\dagger A_1\otimes A_2\otimes X\\&+\sqrt{\kappa_1}A_1\otimes A_2\otimes X\sigma_--\sqrt{\kappa_1}A_1\otimes A_2\otimes\sigma_+X\Big]\\&+\sqrt{1-r^2}\tilde{\pi}_t\Big[A_1\otimes A_2L_2\otimes X+A_1\otimes L_2^\dagger A_2\otimes X\\&+\sqrt{\kappa_2}A_1\otimes A_2\otimes X\sigma_-+\sqrt{\kappa_2}A_1\otimes A_2\otimes\sigma_+X\Big]\\&-\tilde{\pi}_t(A_1\otimes A_2\otimes X)\left[irk_{21}(t)+\sqrt{1-r^2}k_{22}(t)\right],\end{aligned}\tag{9}$$

where

$$\begin{aligned}k_{11}(t)&=\tilde{\pi}_t(L_1+L_1^\dagger+\sqrt{\kappa_1}\sigma_-+\sqrt{\kappa_1}\sigma_+)dt,\\k_{12}(t)&=\tilde{\pi}_t(L_2-L_2^\dagger+\sqrt{\kappa_2}\sigma_--\sqrt{\kappa_2}\sigma_+)dt,\\k_{21}(t)&=\tilde{\pi}_t(L_1-L_1^\dagger+\sqrt{\kappa_1}\sigma_--\sqrt{\kappa_1}\sigma_+)dt,\\k_{22}(t)&=\tilde{\pi}_t(L_2+L_2^\dagger+\sqrt{\kappa_2}\sigma_-+\sqrt{\kappa_2}\sigma_+)dt.\end{aligned}$$

### *B. Master Equations*

In what follows, write

$$\pi_t^{jk;mn}(x)=\mathrm{Tr}\left\{(\rho^{jk;mn}(t))^\dagger x\right\},\quad j,k,m,n=0,1,$$

and set

$$\mathcal{D}^\star_{\sigma_-}\rho=\sigma_-\rho\sigma_+-\frac{1}{2}\sigma_+\sigma_-\rho-\frac{1}{2}\rho\sigma_+\sigma_-,$$

we only present master equations in Schrodinger picture due to page limitation. The master equations is given by (10) with initial conditions:

$$\rho^{11;11}(0)=\rho^{00;11}(0)=\rho^{11;00}(0)=\rho^{00;00}(0)=|\eta\rangle\langle\eta|.$$

*Remark 1:* It can be easily verified that

$$\rho^{jk;mn}(t)=(\rho^{kj;nm}(t))^\dagger,\quad j,k,m,n=0,1.$$

### *C. Quantum Filtering Equations*

If we define

$$\pi_t^{jk;mn}(X)=\frac{\tilde{\pi}_t(Q_1^{jk}\otimes Q_2^{mn}\otimes X)}{w_1^{jk}(t)w_2^{mn}(t)},\quad j,k=0,1,$$

where $Q_1^{jk}$, $Q_2^{mn}$, $w_1^{jk}$, $w_2^{mn}$ are given by

$$\begin{aligned}w_i^{jk}&=\left[\begin{array}{cc}w_i^{00} & w_i^{01}\\ w_i^{10} & w_i^{11}\end{array}\right]=\left[\begin{array}{cc}w_i(t) & \sqrt{w_i(t)}\\ \sqrt{w_i(t)} & 1\end{array}\right],\\Q_i^{jk}&=\left[\begin{array}{cc}Q_i^{00} & Q_i^{01}\\ Q_i^{10} & Q_i^{11}\end{array}\right]=\left[\begin{array}{cc}\sigma_{+i}\sigma_{-i} & \sigma_{+i}\\ \sigma_{-i} & I\end{array}\right],\quad i=1,2.\end{aligned}$$

Due to page limitation, the following theorem only presents the quantum filter in Heisenberg picture for the two-level system $G$, and we only give the explicit form of $d\pi_t^{11;11}(X)$, the other 15 stochastic differential equations can be derived similarly.

*Theorem 2:* Let $\{Y_{i,t}, i=1,2\}$ be the two homodyne detection measurements for a quantum system $G$. With single-photon input states $|1_{\xi_1}\rangle$ and $|1_{\xi_2}\rangle$ in each channel, the quantum filter for the conditional expectation in the Heisenberg picture is given by (11). Here,

$$\begin{aligned}k_{11}(t)=&\xi_1(t)\pi_t^{10;11}(I)+\xi_1^*(t)\pi_t^{01;11}(I)\\&+\sqrt{\kappa_1}\pi_t^{11;11}(\sigma_-+\sigma_+),\\k_{12}(t)=&\xi_2(t)\pi_t^{11;10}(I)-\xi_2^*(t)\pi_t^{11;01}(I)\\&+\sqrt{\kappa_2}\pi_t^{11;11}(\sigma_--\sigma_+),\\k_{21}(t)=&\xi_1(t)\pi_t^{10;11}(I)-\xi_1^*(t)\pi_t^{01;11}(I)\\&+\sqrt{\kappa_1}\pi_t^{11;11}(\sigma_--\sigma_+),\\k_{22}(t)=&\xi_2(t)\pi_t^{11;10}(I)+\xi_2^*(t)\pi_t^{11;01}(I)\\&+\sqrt{\kappa_2}\pi_t^{11;11}(\sigma_-+\sigma_+),\end{aligned}$$

the innovation processes $W_1(t)$ and $W_2(t)$ are given by

$$\begin{aligned}dW_1(t)&=dY_{1,t}-\left[\sqrt{1-r^2}k_{11}(t)+irk_{12}(t)\right]dt,\\dW_2(t)&=dY_{2,t}-\left[irk_{21}(t)+\sqrt{1-r^2}k_{22}(t)\right]dt,\end{aligned}$$

respectively. We have $\pi_t^{jk;mn}(X)=(\pi_t^{kj;nm}(X^\dagger))^\dagger$, the initial conditions are

$$\pi_t^{jk;mn}(X)=\begin{cases}\langle\eta|X|\eta\rangle, & j=k\text{ and }m=n,\\ 0, & \text{otherwise.}\end{cases}$$

$$
\begin{aligned}
\dot{\rho}^{11;11}(t) =&(k_1+k_2)\mathcal{D}^{\star}_{\sigma_-}\rho^{11;11}(t)+\sqrt{k_1}\xi_1(t)[\rho^{01;11}(t),\sigma_+]+\sqrt{k_1}\xi_1^*(t)[\sigma_-,\rho^{10;11}(t)]\\
&+\sqrt{k_2}\xi_2(t)[\rho^{11;01}(t),\sigma_+]+\sqrt{k_2}\xi_2^*(t)[\sigma_-,\rho^{11;10}(t)],\\
\dot{\rho}^{10;11}(t) =&(k_1+k_2)\mathcal{D}^{\star}_{\sigma_-}\rho^{10;11}(t)+\sqrt{k_1}\xi_1(t)[\rho^{00;11}(t),\sigma_+]+\sqrt{k_2}\xi_2(t)[\rho^{10;01}(t),\sigma_+]+\sqrt{k_2}\xi_2^*(t)[\sigma_-,\rho^{10;10}(t)],\\
\dot{\rho}^{01;11}(t) =&(k_1+k_2)\mathcal{D}^{\star}_{\sigma_-}\rho^{01;11}(t)+\sqrt{k_1}\xi_1^*(t)[\sigma_-,\rho^{00;11}(t)]+\sqrt{k_2}\xi_2(t)[\rho^{01;01}(t),\sigma_+]+\sqrt{k_2}\xi_2^*(t)[\sigma_-,\rho^{01;10}(t)],\\
\dot{\rho}^{00;11}(t) =&(k_1+k_2)\mathcal{D}^{\star}_{\sigma_-}\rho^{00;11}(t)+\sqrt{k_2}\xi_2(t)[\rho^{00;01}(t),\sigma_+]+\sqrt{k_2}\xi_2^*(t)[\sigma_-,\rho^{00;10}(t)];\\
\dot{\rho}^{11;10}(t) =&(k_1+k_2)\mathcal{D}^{\star}_{\sigma_-}\rho^{11;10}(t)+\sqrt{k_1}\xi_1(t)[\rho^{01;10}(t),\sigma_+]+\sqrt{k_1}\xi_1^*(t)[\sigma_-,\rho^{10;10}(t)]+\sqrt{k_2}\xi_2(t)[\rho^{11;00}(t),\sigma_+],\\
\dot{\rho}^{10;10}(t) =&(k_1+k_2)\mathcal{D}^{\star}_{\sigma_-}\rho^{10;10}(t)+\sqrt{k_1}\xi_1(t)[\rho^{00;10}(t),\sigma_+]+\sqrt{k_2}\xi_2(t)[\rho^{10;00}(t),\sigma_+],\\
\dot{\rho}^{01;10}(t) =&(k_1+k_2)\mathcal{D}^{\star}_{\sigma_-}\rho^{01;10}(t)+\sqrt{k_1}\xi_1^*(t)[\sigma_-,\rho^{00;10}(t)]+\sqrt{k_2}\xi_2(t)[\rho^{01;00}(t),\sigma_+],\\
\dot{\rho}^{00;10}(t) =&(k_1+k_2)\mathcal{D}^{\star}_{\sigma_-}\rho^{00;10}(t)+\sqrt{k_2}\xi_2(t)[\rho^{00;00}(t),\sigma_+];\\
\dot{\rho}^{11;01}(t) =&(k_1+k_2)\mathcal{D}^{\star}_{\sigma_-}\rho^{11;01}(t)+\sqrt{k_1}\xi_1(t)[\rho^{01;01}(t),\sigma_+]+\sqrt{k_1}\xi_1^*(t)[\sigma_-,\rho^{10;01}(t)]+\sqrt{k_2}\xi*_2(t)[\sigma_-,\rho^{11;00}(t)],\\
\dot{\rho}^{10;01}(t) =&(k_1+k_2)\mathcal{D}^{\star}_{\sigma_-}\rho^{10;01}(t)+\sqrt{k_1}\xi_1(t)[\rho^{00;01}(t),\sigma_+]+\sqrt{k_2}\xi_2^*(t)[\sigma_-,\rho^{10;00}(t)],\\
\dot{\rho}^{01;01}(t) =&(k_1+k_2)\mathcal{D}^{\star}_{\sigma_-}\rho^{01;01}(t)+\sqrt{k_1}\xi_1^*(t)[\sigma_-,\rho^{00;01}(t)]+\sqrt{k_2}\xi_2^*(t)[\sigma_-,\rho^{01;00}(t)],\\
\dot{\rho}^{00;01}(t) =&(k_1+k_2)\mathcal{D}^{\star}_{\sigma_-}\rho^{00;01}(t)+\sqrt{k_2}\xi_2^*(t)[\sigma_-,\rho^{00;00}(t)];\\
\dot{\rho}^{11;00}(t) =&(k_1+k_2)\mathcal{D}^{\star}_{\sigma_-}\rho^{11;00}(t)+\sqrt{k_1}\xi_1(t)[\rho^{01;00}(t),\sigma_+]+\sqrt{k_1}\xi_1^*(t)[\sigma_-,\rho^{10;00}(t)],\\
\dot{\rho}^{10;00}(t) =&(k_1+k_2)\mathcal{D}^{\star}_{\sigma_-}\rho^{10;00}(t)+\sqrt{k_1}\xi_1(t)[\rho^{00;00}(t),\sigma_+],\\
\dot{\rho}^{01;00}(t) =&(k_1+k_2)\mathcal{D}^{\star}_{\sigma_-}\rho^{01;00}(t)+\sqrt{k_1}\xi_1^*(t)[\sigma_-,\rho^{00;00}(t)],\\
\dot{\rho}^{00;00}(t) =&(k_1+k_2)\mathcal{D}^{\star}_{\sigma_-}\rho^{00;00}(t),
\end{aligned}
\tag{10}
$$

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$$
\begin{aligned}
d\pi_t^{11;11}(X) =&\Big\{(\kappa_1+\kappa_2)\pi_t^{11;11}(\mathcal{D}_{\sigma_-}X)+\sqrt{\kappa_1}\xi_1^*(t)\pi_t^{01;11}([X,\sigma_-])+\sqrt{\kappa_1}\xi_1(t)\pi_t^{10;11}([\sigma_+,X])\\
&+\sqrt{\kappa_2}\xi_2^*(t)\pi_t^{11;01}([X,\sigma_-])+\sqrt{\kappa_2}\xi_2(t)\pi_t^{11;10}([\sigma_+,X])\Big\}dt\\
&+\Big\{\sqrt{1-r^2}\Big[\xi_1(t)\pi_t^{10;11}(X)+\xi_1^*(t)\pi_t^{01;11}(X)+\sqrt{\kappa_1}\pi_t^{11;11}(X\sigma_-)+\sqrt{\kappa_1}\pi_t^{11;11}(\sigma_+X)\Big]\\
&+ir\Big[\xi_2(t)\pi_t^{11;10}(X)-\xi_2^*(t)\pi_t^{11;01}(X)+\sqrt{\kappa_2}\pi_t^{11;11}(X\sigma_-)-\sqrt{\kappa_2}\pi_t^{11;11}(\sigma_+X)\Big]\\
&-\pi_t^{11;11}(X)\Big[\sqrt{1-r^2}k_{11}(t)+irk_{12}(t)\Big]\Big\}dW_1(t)\\
&+\Big\{ir\Big[\xi_1(t)\pi_t^{10;11}(X)-\xi_1^*(t)\pi_t^{01;11}(X)+\sqrt{\kappa_1}\pi_t^{11;11}(X\sigma_-)-\sqrt{\kappa_1}\pi_t^{11;11}(\sigma_+X)\Big]\\
&+\sqrt{1-r^2}\Big[\xi_2(t)\pi_t^{11;10}(X)+\xi_2^*(t)\pi_t^{11;01}(X)+\sqrt{\kappa_2}\pi_t^{11;11}(X\sigma_-)+\sqrt{\kappa_2}\pi_t^{11;11}(\sigma_+X)\Big]\\
&-\pi_t^{11;11}(X)\Big[irk_{21}(t)+\sqrt{1-r^2}k_{22}(t)\Big]\Big\}dW_2(t)
\end{aligned}
\tag{11}
$$

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## IV. Simulation Results

Assume the atom is initially in the ground state, i.e., $|\eta\rangle=|g\rangle$. Annihilation operator is $\sigma_-=|g\rangle\langle e|$, and creation operator is $\sigma_+=|e\rangle\langle g|$. The pulse shapes of the two photons are given by

$$\xi_i(t)=\left(\frac{\Omega_i^2}{2\pi}\right)^{\frac{1}{4}}\exp\left\{-\frac{\Omega_i^2}{4}(t-t_i)^2\right\},\quad i=1,2.$$

The exciting probability is defined as

$$P_e(t)=\mathrm{Tr}\{\rho^{11;11}(t)|e\rangle\langle e|\}=\langle e|\rho^{11;11}(t)|e\rangle,$$

where $\rho^{11;11}(t)$ is the solution to (10).

In Fig. 2, the master equations for the two-level system are simulated. Particularly, the red line denotes the two photons have same peak arrival time $t_1=t_2=3$ and same ratio for their own decay rate $\Omega_1=5.84\kappa_1$, $\Omega_2=5.84\kappa_2$, and we can get the maximum value of exciting probability $P_e=0.44$. When we let $\kappa_2=0$, it means that the system has only one input channel. Then the problem reduces to quantum filtering for a two-level system driven by a single-photon state, which has been considered in [7]. Meanwhile, let $\Omega_1=1.46\kappa_1$, the maximum value of exciting probability $P_e=0.8$ (black line) is consistent with the simulation result in [7]. Moreover, we also considered the case of two pulse shapes

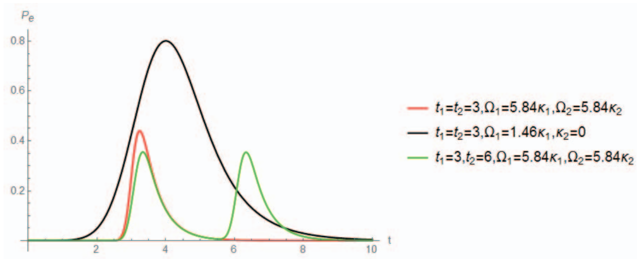


Fig. 2. (Color online) The exciting probability for the two-level system

with different peak arrival times, see the green line. In this case, the maximum exciting probability $P_e = 0.36$ can be attained two times.

## V. Conclusions

In this paper, we have derived the master equations and filtering equations for a two-level system driven by two single-photon states. Particularly, two homodyne detection measurements are applied and the influence of photon pulse shape parameters on the detection probability have shown with numerical simulation results. By simulation, it seems that the maximum of exciting probability can be achieved with same peak arrival time and ratio for bandwidth, i.e., $\Omega = 5.84\kappa$.

## References


[1] V.P. Belavkin, "Nondemolition measurements, nonlinear filtering and dynamic programming of quantum stochastic processes," *Modeling and Control of Systems*, pp. 245-265, Springer, 1989.

[2] V.P. Belavkin, "Quantum filtering of Markov signals with white quantum noise," *Quantum Communications and Measurement*, pp. 381-391, Springer, 1995.

[3] R. Dum, A.S. Parkins, P. Zoller and C.W. Gardiner, "Monte Carlo simulation of master equations in quantum optics for vacuum, thermal, and squeezed reservoirs," *Physical Review A*, vol. 46, no. 7, pp. 4382, 1992.

[4] C.W. Gardiner and P. Zoller, *Quantum Noise*, Springer, 2004.

[5] H.I. Nurdin, "Quantum filtering for multiple input multiple output systems driven by arbitrary zero-Mean jointly Gaussian input fields," *Russian Journal of Mathematical Physics*, vol. 21, no. 3, pp. 386-398, 2014.

[6] H.M. Wiseman and G.J. Milburn, *Quantum Measurement and Control*, Cambridge University Press, 2009.

[7] J.E. Gough, M.R. James, H.I. Nurdin and J. Combes, "Quantum filtering for systems driven by fields in single-photon states or superposition of coherent states," *Physical Review A*, vol. 86, no. 4, pp. 043819, 2012.

[8] A.R.R. Carvalho, M.R. Hush and M.R. James, "Cavity driven by a single photon: Conditional dynamics and nonlinear phase shift," *Physical Review A*, vol. 86, no. 2, pp. 023806, 2012.

[9] H. Song, G. Zhang and Z. Xi, "Continuous-mode multi-photon filtering," *arXiv preprint: 1307.7367*, 2013.

[10] Z. Dong, G. Zhang and H. Amini, "Quantum filtering for multiple measurements driven by fields in single-photon states," *arXiv preprint: 1511.08335*, accepted, 2015.

[11] J.E. Gough and M.R. James, "The series product and its application to quantum feedforward and feedback networks," *IEEE Transactions on Automatic Control*, vol. 54, pp. 2530-2544, 2009.

[12] G. Zhang and M. R. James, Direct and indirect couplings in coherent feedback control of linear quantum systems, IEEE Trans. Automat. Contr., vol. 56, no. 7, pp. 1535-1550, 2011.

[13] G. Zhang and M.R. James, "Quantum feedback networks and control: a brief survey," *Chinese Science Bulletin*, vol. 57, no. 18, pp. 2200-2214, 2012.

[14] M.F. Emzir, M.J. Woolley and I.R. Petersen, "Quantum filtering for multiple diffusive and Poissonian measurements," *Journal of Physics A: Mathematical and Theoretical*, vol. 48, no. 38, pp. 385302-385319, 2015.