Solvable Optimization Problems involving a p-Laplacian Type Operator [⋆]

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Abstract

This paper is concerned with maximization and minimization problems related to a boundary value problem involving the p-Laplacian type operator. These optimization problems are formulated relative to the rearrangement of a fixed function. Under some suitable assumptions, we show that both optimization problems are solvable. Furthermore, we show that the solution of the minimization problem is unique and has some symmetric property if the domain considered is a ball.

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1 Introduction

Let Ω be a smooth bounded domain of $\mathbb{R}^N$ $(N \geq 2)$ and g_0 be a measurable function on Ω, we denote by $\mathscr{R}(g_0)$ the set of all measurable functions g on Ω satisfying

$$\text{meas}\left(\{x \in \Omega : g(x) \geq a\}\right) = \text{meas}\left(\{x \in \Omega : g_0(x) \geq a\}\right), \ \forall a \in \mathbb{R}.$$

Each element of $\mathscr{R}(g_0)$ is called a rearrangement of g_0.

A rearrangement optimization problem is referred to an optimization problem in which the admissible set consists of functions that are rearrangements of a prescribed function. A great deal of attentions have been focussed on rearrangement optimization problems for elliptic boundary value problems in addressing questions such as existence, uniqueness, and symmetry of optimal solutions, see for example [1–17] and the references therein.

Let $1 < p < \infty$, $h(x,t) : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ be a Carathéodory function satisfying suitable growth conditions and $f \in L^q(\Omega)$ with some $1 \leq q < \infty$. Consider the following boundary value problem:

$$(\mathscr{P}) \qquad \begin{cases} \text{div}A(-\nabla u) + h(x,u) = f(x) & \text{in } \ \Omega, \\ u = 0 & \text{on } \ \partial\Omega, \end{cases}$$

where $\text{div}A(-\nabla u)$ is the p-Laplacian type operator which will be defined in the next section. The p-Laplacian type operator $\text{div}A(-\nabla u)$ was introduced in [18] and defined as follows: let $\alpha : \mathbb{R}^N \mapsto [0,\infty)$ be a convex function of class $C^1(\mathbb{R}^N - \{0\})$ satisfying

$$\alpha(t\xi) = t\alpha(\xi) \ for \ t > 0 \ and \ \xi \in \mathbb{R}^N. \tag{1.1}$$

Define $A(0) = 0$ and $A(\xi) = \alpha^{p-1}(\xi)\nabla\alpha(\xi)$ for $\xi \in \mathbb{R}^N - \{0\}$.

Recall that the energy functional $I : W_0^{1,p}(\Omega) \to \mathbb{R}$ corresponding to $(\mathscr{P})$ is given by

$$I(u) = -\frac{1}{p}\int_\Omega A(-\nabla u)\nabla u dx + \int_\Omega H(x,u)dx - \int_\Omega f u dx, \tag{1.2}$$

where $H(x,u) = \int_0^u h(x,t)dt$.

In this paper, we are interested in the following optimization problems:

(Opt_1) Find $f_1 \in \mathscr{R}(f)$ such that $I(u_{f_1}) = \inf_{g\in\mathscr{R}(f)} I(u_g)$,

(Opt_2) Find $f_2 \in \mathscr{R}(f)$ such that $I(u_{f_2}) = \sup_{g\in\mathscr{R}(f)} I(u_g)$,

where u_g denotes the unique solution of $(\mathscr{P})$ with the right-hand side term f replaced by g (Under some conditions, we will prove that $(\mathscr{P})$ has a unique solution in $W_0^{1,p}(\Omega)$, cf. Proposition 3.1 in Section 3).

An important example of the operator $-\text{div}A(-\nabla u)$ is given by $\alpha(\xi) = |\xi|$ in the definition of A, which corresponds to the so-called the p-Laplacian $\Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u)$.

Obviously, (Opt_1) and (Opt_2) are different optimization problems, these problems have been investigated by several authors. In the case of $h = 0$ and $\alpha(\xi) = |\xi|$, after establishing some abstract results, Burton [2] proved that both the problems (Opt_1) and (Opt_2) have solutions for $p = 2$. However, the results obtained in [2] cannot be directly applied to the general case $1 < p < \infty$. So by using a new approach, Cuccu, et al [5] showed that the problem (Opt_1) has a solution for $1 < p < \infty$. But their approach is not efficient for the problem (Opt_2). Marras [13] obtained the solvability of the problem (Opt_2) for $1 < p < \infty$ by using another method. While by replacing f with fu^l $(1 \le l < p)$, Cuccu, et al. [14] obtained a result of uniqueness for a class of p-Laplace equations under non-standard assumptions. In the case of $h \not\equiv 0$ and $\alpha(\xi) = |\xi|$, Qiu et al. [17] have considered a rearrangement optimization problem related to the quasilinear elliptic boundary value problem for $1 < p < \infty$, where under some suitable assumptions, it is shown that both the problems (Opt_1) and (Opt_2) are solvable, which extends the corresponding results in [2,5,13,14].

The purpose of the present paper is to study the optimization problems (Opt_1) and (Opt_2) in the case of $N < p < \infty$, $q = 1$, $\alpha(\xi)$ is a convex function and $h \not\equiv 0$. Firstly, by introducing a truncated function and using the Clarkon inequality, we establish the existence and uniqueness of the solution of the problem $(\mathscr{P})$. Actually, we obtain that the unique solution of the problem $(\mathscr{P})$ is the global minimum point of the energy functional $I(u)$. Moreover, we show that the unique solution is positive if f and h satisfy suitable sign conditions. This is the fundamental part in the studying optimization problems (Opt_1) and (Opt_2). Then we show that the problems (Opt_1) and (Opt_2) are solvable. At last, we show that the unique solution of the problem (Opt_1) is the Schwartz symmetric decreasing rearrangement of f and has some symmetric property if Ω is a ball centered at the origin, which extends the corresponding results in [1,2,5,13,17].

This paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we establish the existence and uniqueness of the solution of the problem $(\mathscr{P})$ in the case of $p > N$. Moreover, we show that the unique solution is positive if f and h satisfy suitable sign conditions. Section 4 is devoted to proving the solvability of problems (Opt_1) and (Opt_2). Furthermore, we show that the solution of the problem (Opt_1) is unique and has some symmetric property if Ω is a ball, and that the unique solution of the problem (Opt_1) is the Schwartz symmetric decreasing rearrangement of f .

2 Preliminaries

We denote by $L^r(\Omega)$ $(1 \le r \le \infty)$ and $W_0^{1,p}(\Omega)$ $(p > 1)$ the usual Sobolev spaces endowed with the norms $\|u\|_{L^r} = (\int_\Omega |u|^r dx)^{1/r}$ if $1 \le r < \infty$, and $\|u\|_\infty =$ ess $\sup_{x\in\Omega}|u(x)|$ if $r = \infty$ and $\|u\| = (\int_\Omega |\nabla u|^p dx)^{1/p}$, respectively. Throughout this paper, C will denote a positive (possibly different) constant. For the p-Laplacian type operator $-\mathrm{div}A(-\nabla u)(p > 1)$, we always assume that $A : \mathbb{R}^N \mapsto \mathbb{R}^N$ satisfies the following conditions: there exist positive constants Γ and γ such that

$$(A(\xi) - A(\eta)) \cdot (\xi - \eta) \ge \gamma(|\xi| + |\eta|)^{p-2}|\xi - \eta|^2 \tag{2.1}$$

$$|A(\xi) - A(\eta)| \le \Gamma(|\xi| + |\eta|)^{p-2}|\xi - \eta| \tag{2.2}$$

for all $\xi, \eta \in R^N$.

Definition 2.1 *By a solution u_f of the problem $(\mathscr{P})$ we mean that $u_f \in W_0^{1,p}(\Omega)$ satisfying*

$$\int_\Omega (-A(-\nabla u_f)\nabla v + h(x, u_f)v - fv)\, dx = 0,\ \forall v \in W_0^{1,p}(\Omega).$$

Definition 2.2 [19, Definition 16.5] *Let $f : \Omega \mapsto [0, \infty)$ be a measurable function. The Schwarz symmetric decreasing rearrangement of f is the function $f^* : B(0, r) \mapsto [0, \infty)$, defined by*

$$f^*(x) = \inf\left\{t \in [0, \infty) : \mu_f(t) \le \omega_N |x|^N\right\}, \forall x \in B(0, r)$$

where ω_N *denotes the volume of the unit ball in* N*-dimensions,* $r := (meas(\Omega)/\omega_N)^{1/N}$ *and* $\mu_f : \mathbb{R} \mapsto [0,\infty)$ *is the distribution function of* f *defined by*

$$\mu_f(t) = meas(\{x \in \Omega : f(x) > t\}).$$

Let I be given in (1.2). It is easy to check that $I \in C^1(W_0^{1,p}(\Omega),\ \mathbb{R})$ and

$$I'(u)v = \int_\Omega (-A(-\nabla u)\nabla v + h(x,u)v - fv)\,dx,\ \forall v \in W_0^{1,p}(\Omega).$$

Therefore, $u \in W_0^{1,p}(\Omega)$ is a weak solution if and only if $I'(u)v = 0$, $\forall v \in W_0^{1,p}(\Omega)$.

It is easy to prove that if $g \in \mathscr{R}(f)$ then $g \in L^q(\Omega)$ and $\|g\|_{L^q} = \|f\|_{L^q}$ (cf. [2, Lemma 2.1]).

The following lemmas will be used through the proofs of our main results.

Lemma 2.1 ([2, Lemma 2.2]) *Assume that* $1 \le r < \infty$ *and given* $f \in L^r(\Omega)$*, denote by* $\overline{\mathscr{R}(f)^{r,w}}$ *the weak closure of* $\mathscr{R}(f)$ *in* $L^r(\Omega)$*, then* $\overline{\mathscr{R}(f)^{r,w}}$ *is convex and weakly compact in* $L^r(\Omega)$*.*

Lemma 2.2 ([2, Lemma 2.9] or [7, Lemma 2.1]) *Let* $f, g : \Omega \mapsto \mathbb{R}$ *be measurable functions and suppose that for each* $t \in \mathbb{R}$*, the level set of* g *at* t*, i.e.,* $\{x \in \Omega : g(x) = t\}$*, has zero measure. Then there exists an increasing (decreasing) function* φ *such that* $\varphi \circ g$ *is a rearrangement of* f *where* $\varphi \circ g$ *denotes a composite function defined by*

$$(\varphi \circ g)(x) = \varphi(g(x)),\ \forall x \in \Omega.$$

Lemma 2.3 ([2, Lemma 2.4] or [7, Lemma 2.2]) *For any* $1 \le r < \infty$ *define* $r' = \frac{r}{r-1}$ *if* $r > 1$ *and* $r' = \infty$ *if* $r = 1$*. Let* $f \in L^r(\Omega)$ *and* $g \in L^{r'}(\Omega)$*. Suppose that there exists an increasing (decreasing) function* $\varphi : \mathbb{R} \mapsto \mathbb{R}$ *such that* $\varphi \circ g \in \mathscr{R}(f)$*. Then* $\varphi \circ g$ *is the unique maximizer (minimizer) of the linear functional* $\int_\Omega hg dx$*, relative to* $h \in \overline{\mathscr{R}(f)^{r,w}}$*.*

Lemma 2.4 ([10, Lemma 2.3]) *Suppose that* $f \in L^r(\Omega)$ *and* $g \in L^{r'}(\Omega)$*, then there exists* $\hat{f} \in \mathscr{R}(f)$ *which maximizes (minimizes) the linear functional* $\int_\Omega hg dx$*, relative to* $h \in \overline{\mathscr{R}(f)^{r,w}}$*.*

Lemma 2.5 ([19, Theorem 16.9]) *Suppose that B is a ball centered at the origin, then*

$$\int_B fg dx \leq \int_B f^* g^* dx,$$

for any non-negative measurable functions f and g, where f^* and g^* are respectively the Schwarz symmetric decreasing rearrangements of f and g, defined in Definition 2.2.

It is well known that $f^* = g^*$ for each $g \in \mathscr{R}(f)$.

Lemma 2.6 ([19, Theorem 16.10]) *Suppose that B is a ball centered at the origin, $u : B \mapsto [0, \infty)$ is a measurable function and $\phi : [0, \infty) \mapsto [0, \infty)$ is a Borel function, then*

$$\int_B \phi \circ u^* dx \leq \int_B \phi \circ u dx.$$

The following result can be deduced from Theorem 1.1 of [20].

Lemma 2.7 *Suppose that B is a ball centered at the origin. If $u \in W_0^{1,p}(B)$ with $1 < p < \infty$ and $u \geq 0$ then $u^{-1}(t, \infty)$ is a translation of $u^{*-1}(t, \infty)$ for every $t \in [0, \text{ess sup}_{x \in B} u(x))$ and*

$$\int_B \alpha^p(-\nabla u) dx \geq \int_B \alpha^p(-\nabla u^*) dx. \tag{2.3}$$

where $\alpha : \mathbb{R}^N \mapsto [0, \infty)$ is a convex function of class $C^1(\mathbb{R}^N - \{0\})$ satisfying (1.1) and there exists a positive constant a_0, such that $\alpha(\xi) = a_0$, for all $\xi \in \mathbb{R}^N$ and $|\xi| = 1$. If the equality holds in (2.3) and the set

$$\left\{ x \in B : \nabla u(x) = 0, 0 < u(x) < \operatorname*{ess\,sup}_{y \in B} u(y) \right\}$$

has zero measure, then $u = u^*$.

3 Existence and Uniqueness for the Solution of Problem ($\mathscr{P}$)

We make the following hypotheses on the function $h(x, t)$:

(h_1) $h(x, t)$ is Carathéodory and is non-decreasing with respect to the second variable for almost all $x \in \Omega$.

(h_2) For each $M > 0$, there exists $\phi_M \in L^1(\Omega)$ such that for all $|t| \le M$

$$|h(x,t)| \le \phi_M(x), \text{ a.e. } x \in \Omega.$$

In this section, we will obtain the existence and uniqueness for the solution of the problem $(\mathscr{P})$.

Proposition 3.1 *Suppose that $N < p < \infty$, $f \in L^1(\Omega)$ and the assumptions (h_1) and (h_2) hold. Then the problem $(\mathscr{P})$ has a unique solution $u_f \in W_0^{1,p}(\Omega)$ and $I(u_f) = \inf_{v \in W_0^{1,p}(\Omega)} I(v)$. Moreover, if in addition $f(x) > 0$ and $h(x,t) \le 0$, $\forall\, t \in \mathbb{R}$ and a.e. $x \in \Omega$, then $u_f > 0$.*

Proof: For each $M > 0$, we introduce the truncated function h_M by

$$h_M(x,t) = \begin{cases} h(x,t), & x \in \Omega,\ |t| \le M, \\ h(x,M), & x \in \Omega,\ t > M, \\ h(x,-M), & x \in \Omega,\ t < -M. \end{cases} \tag{3.1}$$

Let us consider the problem

$$(\mathscr{P}_M) \qquad \begin{cases} \operatorname{div} A(-\nabla u) + h_M(x,u) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We denote the energy functional $E_M : W_0^{1,p}(\Omega) \to \mathbb{R}$ corresponding to the above problem by

$$E_M(u) = \frac{1}{p}\int_\Omega -A(-\nabla u)\nabla u dx + \int_\Omega H_M(x,u)dx - \int_\Omega f u dx,\ u \in W_0^{1,p}(\Omega), \tag{3.2}$$

where $H_M(x,u) = \int_0^u h_M(x,t)dt$.

Firstly, we claim that the problem $(\mathscr{P}_M)$ has a unique solution.

By the assumption (h_2), we can show that for each $M > 0$ the following inequality holds.

$$|H_M(x,t)| \le (|t| + M)\phi_M(x),\ \forall\, t \in \mathbb{R},\ \text{a.e. } x \in \Omega. \tag{3.3}$$

Indeed, if $|t| \le M$, then by (h_2),

$$|H_M(x,t)| = \left|\int_0^t h_M(x,s)ds\right| = \left|\int_0^t h(x,s)ds\right| \le \left|\int_0^t \phi_M(x)ds\right| \le M\phi_M(x).$$

If $t > M$, then we have

$$|H_M(x,t)| \leq \left|\int_0^M h(x,s)ds\right| + \left|\int_M^t h(x,M)ds\right| \leq M\phi_M(x) + (t-M)|h(x,M)| \leq t\phi_M(x).$$

If $t < -M$, then we get

$$\begin{aligned}|H_M(x,t)| &\leq \left|\int_0^{-M} h(x,s)ds\right| + \left|\int_{-M}^t h(x,-M)ds\right| \\ &\leq M\phi_M(x) + (-M-t)|h(x,-M)| \\ &\leq |t|\phi_M(x).\end{aligned}$$

So that in summary (3.3) is valid. This, together with (2.1), (2.2), the Hölder inequality and the Sobolev imbedding theorem, implies that for each $M > 0$

$$\begin{aligned}E_M(u) &= \frac{1}{p}\int_\Omega -A(-\nabla u)\nabla u dx + \int_\Omega H_M(x,u)dx - \int_\Omega f u dx \\ &\geq \frac{\gamma}{p}\|u\|^p - (\|u\|_{L^\infty} + M)\|\phi_M\|_{L^1} - \|f\|_{L^1}\|u\|_{L^\infty} \\ &\geq \frac{\gamma}{p}\|u\|^p - (C\|u\| + M)\|\phi_M\|_{L^1} - C\|f\|_{L^1}\|u\| \to \infty\end{aligned} \tag{3.4}$$

as $\|u\| \to \infty$, which shows that the functional E_M is coercive.

We now prove that the functional E_M is weakly lower semi-continuous (which we will denote by w.l.s.c for short) in $W_0^{1,p}(\Omega)$.

In order to do this, let $v_n \rightharpoonup v$ in $W_0^{1,p}(\Omega)$ as $n \to \infty$. Noting that the embedding $W_0^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega})$ is compact since $p > N$, then $v_n \to v$ in $C(\overline{\Omega})$ as $n \to \infty$. So that for each $\epsilon > 0$ there exists $K > 0$ such that

$$|v_n(x) - v(x)| \leq \epsilon, \quad \forall\ n > K,\ \forall\ x \in \Omega.$$

This, together with (3.3), implies that for all $n > K$ and all $x \in \Omega$

$$|H_M(x,v_n)| \leq (|v_n(x)| + M)\phi_M(x) \leq (|v(x)| + \epsilon + M)\phi_M(x).$$

Since $(|v(x)| + \epsilon + M)\phi_M(x) \in L^1(\Omega)$, we use the dominated convergence theorem to derive that

$$\int_\Omega H_M(x,v_n)dx \to \int_\Omega H_M(x,v)dx \tag{3.5}$$

as $n \to \infty$. Then

$$\liminf_{n\to\infty} E_M(v_n) \geq E_M(v) - \limsup_{n\to\infty} \|f\|_{L^1}\|v_n - v\|_{L^\infty} = E_M(v).$$

That is, the functional E_M is weakly lower semi-continuous. So that the functional E_M has a minimizer $u_M \in W_0^{1,p}(\Omega)$ with

$$E_M(u_M) = \inf_{v \in W_0^{1,p}(\Omega)} E_M(v). \tag{3.6}$$

By assumptions(h_1), (h_2), and using a standard argument (cf. [21, Lemma 2.16]), we can easily show that $E_M \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$, therefore u_M is a solution of the problem $(\mathscr{P}_M)$ satisfying

$$E'_M(u_M)v = \int_\Omega (-A(-\nabla u_M)\nabla v + h(x, u_M)v - fv)\, dx = 0,\ \forall v \in W_0^{1,p}(\Omega). \tag{3.7}$$

Next, we show that u_M is the unique solution of the problem $(\mathscr{P}_M)$.

Assume that $w \in W_0^{1,p}(\Omega)$ is another solution of the problem $(\mathscr{P}_M)$ and $u_M \neq w$, i.e., there exists a subset $E \subset \Omega$ with positive measure suth that $u_M(x) \neq w(x), \forall x \in E$, then

$$\|u_M - w\| > 0. \tag{3.8}$$

Since $h(x, \cdot)$ is non-decreasing,

$$\int_\Omega (h(x, u_M) - h(x, w))(u_M - w)dx \geq 0. \tag{3.9}$$

From (3.7) and Def. 2.1 we get that for every $v \in W_0^{1,p}(\Omega)$,

$$\int_\Omega (-A(-\nabla u_M)\nabla v + h(x, u_M)v)\, dx = \int_\Omega fvdx, \tag{3.10}$$

$$\int_\Omega (-A(-\nabla w)\nabla v + h(x, w)v)\, dx = \int_\Omega fvdx. \tag{3.11}$$

From (3.10) and (3.11) we obtain

$$\int_\Omega (h(x, u_M) - h(x, w))vdx = \int_\Omega (A(-\nabla u_M) - A(-\nabla w))\nabla vdx,\ \forall v \in W_0^{1,p}(\Omega).$$

Take $v = u_M - w$. Note that $p > N \geq 2$, from the equality above, we have

$$\begin{aligned}
&\int_\Omega (h(x, u_M) - h(x, w))(u_M - w)dx \\
&= \int_\Omega (A(-\nabla u_M) - A(-\nabla w))\nabla(u_M - w)dx \\
&\leq -C \int_\Omega |\nabla(u_M - w)|^p dx = -C\|u_M - w\|^p < 0,
\end{aligned}$$

by the Clarkon inequality (cf. [22, Lemma 4.2]), (2.1) and (3.8). Then the above inequality contradicts (3.9). Therefore we have proved that u_M is the unique solution of the problem $(\mathscr{P}_M)$.

Secondly, we will prove that there exists $\widetilde{M} > 0$ such that the unique solution $u_{\widetilde{M}}$ of the problem $(\mathscr{P}_{\widetilde{M}})$ is also the unique solution of the problem $(\mathscr{P})$. Moreover, $u_{\widetilde{M}} = u_M, \forall\ M \geq \widetilde{M}$.

In fact, since $h(x,\cdot)$ is non-decreasing (cf. (h_1)), it follows that for all $M_1, M_2 > 0$ with $M_1 \leq M_2$,

$$h_{M_1}(x,t) \leq h_{M_2}(x,t) \leq h(x,t), \forall\ t \geq 0,\ \text{a.e. } x \in \Omega,$$

and

$$h(x,t) \leq h_{M_2}(x,t) \leq h_{M_1}(x,t), \forall\ t \leq 0,\ \text{a.e. } x \in \Omega,$$

which, together with (3.1), gives

$$H_{M_2}(x,t) \geq H_{M_1}(x,t),\ \forall\ t \in \mathbb{R},\ \text{a.e. } x \in \Omega. \tag{3.12}$$

By (3.4) and $E_M(u_M) \leq E_M(0) = 0$, there exists $M_0 > 0$ such that

$$\|u_M\| \leq M_0,\ \forall\ M \geq M_0.$$

This together with the Sobolev embedding inequality yields that there exists a positive constant C_0 such that

$$\|u_M\|_{L^\infty} \leq C_0\|u_M\| \leq C_0 M_0,\ \forall M > M_0.$$

Let $\widetilde{M} := \max\{C_0 M_0, M_0\}$, then by (3.1), we have

$$\begin{aligned} h_{\widetilde{M}}(x, u_{\widetilde{M}}) &= h(x, u_{\widetilde{M}}),\ \text{a.e. } x \in \Omega, \\ H_{\widetilde{M}}(x, u_{\widetilde{M}}) &= H(x, u_{\widetilde{M}}),\ \text{a.e. } x \in \Omega. \end{aligned} \tag{3.13}$$

Noting that (3.7), (3.13) and Def. 2.1, we see that $u_{\widetilde{M}}$ is in fact a solution of the problem $(\mathscr{P})$. By using the very same arguments in the above proof, we may show that $u_{\widetilde{M}}$ is the unique solution of the problem $(\mathscr{P})$, which, together with (3.13), implies that $u_M = u_{\widetilde{M}}, \forall\ M \geq \widetilde{M}$.

Thirdly, we shall show that $u_{\widetilde{M}}$ also minimizes the functional I corresponding to the problem $(\mathscr{P})$.

Indeed, similarly as (3.12), we have

$$H_{\widetilde{M}}(x,t) \leq H(x,t),\ \forall\ t \in \mathbb{R},\ \text{a.e. } x \in \Omega. \tag{3.14}$$

Combining (1.2), (3.2), (3.6), (3.13) and (3.14), we get that for each $v \in W_0^{1,p}(\Omega)$,

$$\begin{aligned}
I(u_{\widetilde{M}}) = E_{\widetilde{M}}(u_{\widetilde{M}}) \leq E_{\widetilde{M}}(v) \\
&= \frac{1}{p}\int_\Omega -A(-\nabla v)\nabla v dx + \int_\Omega H_{\widetilde{M}}(x,v)dx - \int_\Omega f v dx \\
&\leq \frac{1}{p}\int_\Omega -A(-\nabla v)\nabla v dx + \int_\Omega H(x,v)dx - \int_\Omega f v dx \\
&= I(v).
\end{aligned}$$

Therefore $u_{\widetilde{M}}$ is a minimizer of the problem $(\mathscr{P})$. Let $u_f = u_{\widetilde{M}}$, then we have proved the first half of this theorem.

To complete the proof, we finally show that $u_f > 0$ if $f(x) > 0$ and $h(x,t) \leq 0$, $\forall\, t \in \mathbb{R}$ and a.e. $x \in \Omega$.

In this case, obviously, $H(x,\cdot)$ is decreasing and so is $H_{\widetilde{M}}(x,\cdot)$. In particular, we have $H_{\widetilde{M}}(x,u_f^+) \leq H_{\widetilde{M}}(x,u_f)$, a.e. $x \in \Omega$. Since $f(x) > 0$, a.e. $x \in \Omega$,

$$\int_\Omega f u_f dx \leq \int_\Omega f u_f^+ dx.$$

It is easy to check that

$$\int_\Omega A(-\nabla u_f^+)(-\nabla u_f^+)dx \leq \int_\Omega A(-\nabla u_f)(-\nabla u_f)dx.$$

Therefore, $E_{\widetilde{M}}(u_f^+) \leq E_{\widetilde{M}}(u_f)$, which shows that u_f^+ is also a minimizer of the functional $E_{\widetilde{M}}$ and then a solution of the problem $(\mathscr{P}_{\widetilde{M}})$. Noting that u_f is the unique solution of $(\mathscr{P}_{\widetilde{M}})$, so $u_f = u_f^+ \geq 0$. Since

$$\text{div} A(-\nabla u_f(x)) = f(x) - h(x,u_f(x)) > 0, \text{ a.e. } x \in \Omega,$$

we have $u_f(x) > 0$, $\forall\, x \in \Omega$ (cf. [23, Theorem 5]). ∎

Remark 3.1 In Proposition 3.1, we obtain that not only the existence of the solution for the problem $(\mathscr{P})$, but also the uniqueness and that the solution is actually the global minimum point of the energy functional $I(u)$ under some suitable conditions. Moreover, we show that the unique solution is positive if f and h satisfy suitable sign conditions.

4 Existence of Solutions of Problems (Opt_1) and (Opt_2)

We first consider the problem (Opt_1).

Theorem 4.1 *Suppose that $N < p < \infty$, $f \in L^1(\Omega)$ and the assumptions (h_1) and (h_2) hold. Then there exists $f_1 \in \mathscr{R}(f)$ which solves the problem (Opt_1).*

Proof: We first show that $K := \inf_{g\in\mathscr{R}(f)} I(u_g)$ is finite. Similar to the proof of Proposition 3.1, we know that there exists a constant $\widetilde{M} > 0$ such that $\forall\ g \in \mathscr{R}(f)$ and $\forall\ M \geq \widetilde{M}$, the unique solution u_g of the problem

$$(\mathscr{P}_{M,g}) \qquad \begin{cases} \mathrm{div}A(-\nabla u) + h_M(x,u) = g(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

is the unique solution of the problem

$$(\mathscr{P}_{g}) \qquad \begin{cases} \mathrm{div}A(-\nabla u) + h(x,u) = g(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

Moreover, $\|u_g\|_{L^\infty} \leq \widetilde{M}$, where $h_M(x,u)$ is defined by (3.1). Similarly as in the proof of (3.4), we have

$$I(u_g) = E_{\widetilde{M}}(u_g) \geq \frac{\gamma}{p}\|u_g\|^p - (C\|u_g\| + \widetilde{M})\|\phi_M\|_{L^1} - C\|g\|_{L_1}\|u_g\|, \tag{4.1}$$

which implies that K is finite since $\|g\|_{L_1} = \|f\|_{L_1}$ and $p > N$. Now we choose a sequence $\{g_i\} \subset \mathscr{R}(f)$ such that $I(u_i) \to K$ as $i \to \infty$, where $u_i := u_{g_i}$ for each $i \in \mathbb{N}$. It follows from (4.1) that $\{u_i\}$ is bounded in $W_0^{1,p}(\Omega)$. Going if necessary to a subsequence, $\{u_i\}$ weakly converges to $u \in W_0^{1,p}(\Omega)$ and strongly converges to u in $C(\overline{\Omega})$ since $p > N$. Also, the boundedness of $\{g_i\}$ in $L^1(\Omega)$ (since $\|g_i\|_{L_1} \equiv \|f\|_{L_1}$) implies, going if necessary to a subsequence, that $\{g_i\}$ converges weakly to some $\bar{g} \in \overline{\mathscr{R}(f)}^{1,w}$, the weak closure of $\mathscr{R}(f)$ in $L^1(\Omega)$. Therefore,

$$\left|\int_\Omega (g_i u_i - \bar{g}u)dx\right| \leq \|g_i\|_{L^1}\|u_i - u\|_\infty + \left|\int_\Omega (g_i - \bar{g})udx\right| \to 0 \tag{4.2}$$

as $i \to \infty$. Similarly as in the proof of (3.5), we also have

$$\int_\Omega H_{\widetilde{M}}(x,u_i)dx \to \int_\Omega H_{\widetilde{M}}(x,u)dx. \tag{4.3}$$

By (4.2), (4.3) and the weak lower semi-continuity of the norm in the $W_0^{1,p}(\Omega)$, we obtain that

$$K = \lim_{i\to\infty} I(u_i) \geq \frac{1}{p}\int_\Omega -A(-\nabla u)\nabla u dx + \int_\Omega (H(x,u) - \bar{g}u)dx. \tag{4.4}$$

From Lemma 2.4 we infer the existence of $\hat{f} \in \mathscr{R}(f)$ which maximizes the linear functional $\int_\Omega hudx$, relative to $h \in \overline{\mathscr{R}(f)}^{1,w}$. As a consequence,

$$\int_\Omega \bar{g}udx \leq \int_\Omega \hat{f}udx.$$

Combining with (4.4), we get

$$K \geq \frac{1}{p}\int_\Omega -A(-\nabla u)\nabla u dx + \int_\Omega (H(x,u) - \hat{f}u)dx. \tag{4.5}$$

By Proposition 3.1,

$$\begin{aligned} I(\hat{u}) &= \inf_{v\in W_0^{1,p}(\Omega)} \int_\Omega \left(\frac{1}{p}A(-\nabla v)(-\nabla v) + H(x,v) - \hat{f}v\right) dx \\ &\leq \frac{1}{p}\int_\Omega -A(-\nabla u)\nabla u dx + \int_\Omega (H(x,u) - \hat{f}u)dx, \end{aligned} \tag{4.6}$$

where $\hat{u} = u_{\hat{f}}$.

It follows from (4.5) and (4.6) that $I(\hat{u}) \leq K$.

On the other hand, recall that $K = \inf_{g\in\mathscr{R}(f)} I(u_g)$ and $\hat{f} \in \mathscr{R}(f)$, we must have $K \leq I(\hat{u})$. So that $K = I(\hat{u})$. We complete the proof by letting $f_1 = \hat{f}$. ∎

We now consider the problem (Opt_2). Our results for the problem (Opt_2) are the following.

Theorem 4.2 *Suppose that $N < p < \infty$, $f \in L^1(\Omega)$ and the assumptions (h_1) and (h_2) hold. Moreover, if $f(x) > 0$ and $h(x,t) \leq 0$, $\forall\, t \in \mathbb{R}$ and a.e. $x \in \Omega$, then there exists $f_2 \in \mathscr{R}(f)$ which solves the problem (Opt_2), i.e.,*

$$I(u_{f_2}) = \sup_{g\in\mathscr{R}(f)} I(u_g).$$

By using Proposition 3.1, under assumptions of Theorem 4.1 , we can define the functional $\Phi_1 : L^1(\Omega) \mapsto \mathbb{R}$ by $\Phi_1(g) = I(u_g)$.

Before proving Theorem 4.2, we shall show the following lemmas.

Lemma 4.1 *Under the assumptions of Theorem 4.2 , we have*

(I) *The functional* $\Phi_1|_{\overline{\mathscr{R}(f)^{1,w}}}$ *is weakly continuous.*

(II) *The functional* $\Phi_1|_{\overline{\mathscr{R}(f)^{1,w}}}$ *is strictly concave.*

(III) *The functional* Φ_1 *is Gâteaux differentiable at each* $g \in \overline{\mathscr{R}(f)^{1,w}}$ *with derivative* $-u_g$.

Proof:

(I) Let $\{g_n\} \subset \overline{\mathscr{R}(f)^{1,w}}$ be such that $g_n \rightharpoonup g$ in $L^1(\Omega)$ as $n \to \infty$. By Proposition 3.1, we may respectively denote by u_n and u_g the unique solutions to the problems $(\mathscr{P}_{g_n})$ and $(\mathscr{P}_g)$. Moreover,

$$I(u_g) = \inf_{v \in W_0^{1,p}(\Omega)} \int_\Omega \left(\frac{1}{p} A(-\nabla v)(-\nabla v) + H(x, v) - gv \right) dx$$

and

$$I(u_{g_n}) = \inf_{v \in W_0^{1,p}(\Omega)} \int_\Omega \left(\frac{1}{p} A(-\nabla v)(-\nabla v) + H(x, v) - g_n v \right) dx.$$

We claim that

$$\lim_{n \to \infty} \Phi_1(g_n) = \Phi_1(g). \tag{4.7}$$

Indeed, we have

$$\begin{aligned} &\Phi_1(g) + \int_\Omega (g - g_n) u_g dx \\ &= \int_\Omega \left(\frac{1}{p} A(-\nabla u_g)(-\nabla u_g) + H(x, u_g) - g_n u_g \right) dx \\ &\geq \Phi_1(g_n) \\ &= \int_\Omega \left(\frac{1}{p} A(-\nabla u_n)(-\nabla u_n) + H(x, u_n) - g u_n \right) dx + \int_\Omega (g - g_n) u_n dx \\ &\geq \Phi_1(g) + \int_\Omega (g - g_n) u_n dx. \end{aligned} \tag{4.8}$$

For any $v \in L^\infty(\Omega)$, since $g_n \rightharpoonup g$ in $L^1(\Omega)$ as $n \to \infty$,

$$\lim_{n \to \infty} \int_\Omega (g_n - g) v dx = 0. \tag{4.9}$$

In particular,

$$\lim_{n \to \infty} \int_\Omega (g_n - g) u_g dx = 0. \tag{4.10}$$

From (4.8) and (4.10), to prove the claim, we only need to show that

$$\lim_{n\to\infty}\int_\Omega (g_n - g)u_n dx = 0. \tag{4.11}$$

In fact, by (4.1) and the fact that $I(u_n) \le I(0) = 0$, we get

$$0 \ge I(u_n) \ge \frac{\gamma}{p}\|u_n\|^p - C\|u_n\|,$$

which implies that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Since $\{g_n\} \subset \overline{\mathscr{R}(f)^{1,w}}$ and $g \in \overline{\mathscr{R}(f)^{1,w}}$, we clearly have $\|g\|_{L^1} \le \|f\|_{L^1}$ and $\|g_n\|_{L^1} \le \|f\|_{L^1}$ for all $n \in \mathbb{N}$. Hence,

$$\left|\int_\Omega (g_n - g)u_n dx\right| \le C\|g_n - g\|_{L^1}\|u_n\| \le C.$$

Now we can choose a subsequence $\{u_{n_j}\}$ such that

$$\lim_{j\to\infty}\left|\int_\Omega (g_{n_j} - g)u_{n_j} dx\right| = \limsup_{n\to\infty}\left|\int_\Omega (g_n - g)u_n dx\right|.$$

Noting that $\{u_{n_j}\}$ is also bounded in $W_0^{1,p}(\Omega)$, going if necessary to a subsequence, we may assume that $u_{n_j} \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and $u_{n_j} \to u$ in $L^\infty(\Omega)$ as $j \to \infty$.

By (4.9) and the Hölder inequality, we obtain that

$$\begin{aligned}\left|\int_\Omega (g_{n_j} - g)u_{n_j} dx\right| &\le \left|\int_\Omega (g_{n_j} - g)(u_{n_j} - u) dx\right| + \left|\int_\Omega (g_{n_j} - g)u dx\right| \\ &\le \|g_{n_j} - g\|_{L^1}\|u_{n_j} - u\|_{L^\infty} + \left|\int_\Omega (g_{n_j} - g)u dx\right| \to 0\end{aligned}$$

as $j \to \infty$. So that

$$0 \le \liminf_{n\to\infty}\left|\int_\Omega (g_n - g)u_n dx\right| \le \limsup_{n\to\infty}\left|\int_\Omega (g_n - g)u_n dx\right| \le 0,$$

which implies (4.11), and then the claim (4.7) is valid.

(II) Let $g, h \in \overline{\mathscr{R}(f)^{1,w}}$ and $v \in W_0^{1,p}(\Omega)$, then for all $t \in (0,1)$, we have

$$\begin{aligned}&\int_\Omega \left(\frac{1}{p}A(-\nabla v)(-\nabla v) + H(x,v) - (tg + (1-t)h)v\right) dx \\ &= t\int_\Omega \left(\frac{1}{p}A(-\nabla v)(-\nabla v) + H(x,v) - gv\right) dx \\ &\quad + (1-t)\int_\Omega \left(\frac{1}{p}A(-\nabla v)(-\nabla v) + H(x,v) - hv\right) dx.\end{aligned}$$

By taking the infimum relative to $v \in W_0^{1,p}(\Omega)$ in both sides of the above equality, we get

$$\Phi_1(tg + (1-t)h) \geq t\Phi_1(g) + (1-t)\Phi_1(h),$$

that is, the concavity of Φ_1 has been proved. Now, suppose that equality holds in the above inequality for some $t \in (0,1)$. Then, denote by u_t the solution of the problem $(\mathscr{P})$ corresponding to $tg + (1-t)h$, we have

$$\begin{aligned}
& t\int_\Omega \left(\frac{1}{p}A(-\nabla u_t)(-\nabla u_t) + H(x,u_t) - gu_t\right) dx \\
& \quad + (1-t)\int_\Omega \left(\frac{1}{p}A(-\nabla u_t)(-\nabla u_t) + H(x,u_t) - hu_t\right) dx \\
= {} & t\int_\Omega \left(\frac{1}{p}A(-\nabla u_g)(-\nabla u_g) + H(x,u_g) - gu_g\right) dx \\
& \quad + (1-t)\int_\Omega \left(\frac{1}{p}A(-\nabla u_h)(-\nabla u_h) + H(x,u_h) - hu_h\right) dx.
\end{aligned}$$

It follows that

$$\begin{aligned}
& \int_\Omega \left(\frac{1}{p}A(-\nabla u_t)(-\nabla u_t) + H(x,u_t) - gu_t\right) dx \\
= {} & \int_\Omega \left(\frac{1}{p}A(-\nabla u_g)(-\nabla u_g) + H(x,u_g) - gu_g\right) dx, \\
& \int_\Omega \left(\frac{1}{p}A(-\nabla u_t)(-\nabla u_t) + H(x,u_t) - hu_t\right) dx \\
= {} & \int_\Omega \left(\frac{1}{p}A(-\nabla u_h)(-\nabla u_h) + H(x,u_h) - hu_h\right) dx.
\end{aligned}$$

By the uniqueness of the minimizer of the functional I, we must have $u_t = u_g = u_h$. Moreover, since

$$\begin{aligned}
\mathrm{div}A(-\nabla u_g(x)) + h(x,u_g(x)) = g(x), \quad & \text{a.e. in } \Omega, \\
\mathrm{div}A(-\nabla u_h(x)) + h(x,u_h(x)) = h(x), \quad & \text{a.e. in } \Omega,
\end{aligned}$$

if $u_g = u_h$, we must have $g(x) = h(x)$ a.e. in Ω, and the strict concavity is proved.

(III) Let $\{t_n\}$ be a sequence of positive numbers such that $t_n \to 0$ as $n \to \infty$. Let $g \in \overline{\mathscr{R}(f)^{1,w}}$, $h \in L^1(\Omega)$ and $g_n = g + t_n h$, the corresponding solution of the problem $(\mathscr{P}_{g_n})$ is denoted by u_n. Then, by (4.8), we find

$$\Phi_1(g) - \int_\Omega t_n h u_n dx \leq \Phi_1(g + t_n h) \leq \Phi_1(g) - \int_\Omega t_n h u_g dx.$$

So that

$$-\int_\Omega hu_n dx \le \frac{\Phi_1(g+t_nh)-\Phi_1(g)}{t_n} \le -\int_\Omega hu_g dx.$$

We claim that

$$\lim_{n\to\infty}\int_\Omega hu_n dx = \int_\Omega hu_g dx. \tag{4.12}$$

In fact, similarly as in the proof of the part (I), there exist a subsequence $\{u_{n_j}\}$ and $u \in W_0^{1,p}(\Omega)$ such that

$$\limsup_{n\to\infty}\left|\int_\Omega h(u_n-u_g)dx\right| = \lim_{j\to\infty}\left|\int_\Omega h(u_{n_j}-u_g)dx\right|.$$

and $u_{n_j} \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ as $j\to\infty$. We only need to show that $u = u_g$.

Similarly as (4.4), we have

$$\begin{aligned} I(u_g) = \Phi_1(g) &= \lim_{j\to\infty}\Phi_1(g_{n_j}) \\ &\ge \frac{1}{p}\int_\Omega A(-\nabla u)(-\nabla u)dx + \int_\Omega (H(x,u)-gu)dx \\ &= I(u) \ge I(u_g). \end{aligned}$$

By the uniqueness of the minimizer of the functional I, we must have $u = u_g$ so that (4.12) is valid.

Therefore,

$$\lim_{n\to\infty}\frac{\Phi_1(g+t_nh)-\Phi_1(g)}{t_n} = -\int_\Omega hu_g dx.$$

Since the sequence $\{t_n\}$ is arbitrary, it follows that

$$\lim_{t\to 0^+}\frac{\Phi_1(g+th)-\Phi_1(g)}{t} = -\int_\Omega hu_g dx.$$

In the same way we can show that

$$\lim_{t\to 0^-}\frac{\Phi_1(g+th)-\Phi_1(g)}{t} = -\int_\Omega hu_g dx.$$

Thus we have proved that Φ_1 is Gâteaux differentiable at g with derivative $-u_g$. ∎

Lemma 4.2 *Under the assumptions of Theorem 4.2, there exists a unique* $\widetilde{f} \in \overline{\mathscr{R}(f)^{1,w}}$ *which maximizes* $\Phi_1|_{\overline{\mathscr{R}(f)^{1,w}}}$. *Moreover,*

$$\int_\Omega \widetilde{u}\widetilde{f}dx \le \int_\Omega \widetilde{u}hdx,\ \forall h \in \overline{\mathscr{R}(f)^{1,w}}, \tag{4.13}$$

where $\widetilde{u} = u_{\widetilde{f}}$.

Proof: By Lemma 2.1 and the weak continuity of $\Phi_1|_{\overline{\mathscr{R}(f)^{1,w}}}$, we know that a maximizer $\widetilde{f}$ exists in $\overline{\mathscr{R}(f)^{1,w}}$. It follows from Lemma 4.1 that $\Phi_1|_{\overline{\mathscr{R}(f)^{1,w}}}$ is strictly concave, and so the maximizer $\widetilde{f}$ is unique. For each $h \in \overline{\mathscr{R}(f)^{1,w}}$ and $t \in (0,1)$, we define $f_t = \widetilde{f} + t(h - \widetilde{f})$, then $f_t \in \overline{\mathscr{R}(f)^{1,w}}$ since $\overline{\mathscr{R}(f)^{1,w}}$ is convex (cf. Lemma 2.1). Noting that Φ_1 is Gâteaux differentiable at $\widetilde{f}$ with derivative $-\widetilde{u}$ (cf. Lemma 4.1), we have

$$\Phi_1(f_t) = \Phi_1(\widetilde{f}) - t\int_\Omega \widetilde{u}(h - \widetilde{f})dx + o(t).$$

Since $\Phi_1(\widetilde{f}) \geq \Phi_1(f_t)$, we find

$$\Phi_1(\widetilde{f}) \geq \Phi_1(\widetilde{f}) - t\int_\Omega \widetilde{u}(h - \widetilde{f})dx + o(t).$$

It follows that

$$0 \geq -\int_\Omega \widetilde{u}(h - \widetilde{f})dx + \frac{o(t)}{t}.$$

letting $t \to 0$ in the above inequality, we see that

$$\int_\Omega \widetilde{u}\widetilde{f}dx \leq \int_\Omega \widetilde{u}hdx.$$

We finish the proof by noting that h is chosen arbitrarily in $\overline{\mathscr{R}(f)^{1,w}}$. ∎

Proof of Theorem 4.2: Let $\widetilde{f}$ be as in Lemma 4.2. Since $\widetilde{u}$ satisfies

$$\mathrm{div}A(-\nabla\widetilde{u}(x)) = \widetilde{f}(x) - h(x, \widetilde{u}(x)) > 0, \text{ a.e. } x \in \Omega,$$

it follows that each level set of $\widetilde{u}$ has zero measure (cf. [24, Lemma 7.7]). By Lemma 2.2, there exists a decreasing function φ such that $\varphi \circ \widetilde{u}$ is a rearrangement of f, i.e., $\varphi \circ \widetilde{u} \in \mathscr{R}(f)$. Hence, we can apply Lemma 2.3 to deduce that $\varphi \circ \widetilde{u}$ is the unique minimizer of the linear functional $\int_\Omega h\widetilde{u}dx$, relative to $h \in \overline{\mathscr{R}(f)^{1,w}}$. This and (4.13) obviously imply $\widetilde{f} = \varphi \circ \widetilde{u} \in \mathscr{R}(f)$. We complete the proof by choosing $f_2 = \widetilde{f}$. ∎

By Theorem 4.1, we see that the problem (Opt_1) is solvable if h and f satisfy some suitable conditions. If, in addition, the domain Ω in the problem $(\mathscr{P})$ has some symmetric property, then the solution of (Opt_1) is unique.

Theorem 4.3 *Suppose that $N < p < \infty$, Ω is a ball centered at the origin, $f \in L^1(\Omega)$ and $f(x) > 0$, the assumptions (h_1) and (h_2) hold, and $h(x,t) = h(t) \leq 0$,*

$\forall\ t \in \mathbb{R}$, *a.e.* $x \in \Omega$. *Assume that* $\alpha : R^N \mapsto [0,\infty)$ *is a convex function of class* $C^1(R^N - \{0\})$ *satisfying* (1.1) *and there exists a positive constant* a_0*, such that* $\alpha(\xi) = a_0$*, for all* $\xi \in R^N$ *and* $|\xi| = 1$*. Then the problem* (Opt_1) *has a unique solution* f_1 *and* $f_1 = f^*$*, where* f^* *is the Schwarz symmetric decreasing rearrangement of* f (cf. Def. 2.2).

Proof: By Theorem 4.1, the problem (Opt_1) has a solution f_1. We denote by $u_1 := u_{f_1}$, the unique solution of the problem $(\mathscr{P}_{f_1})$. Since

$$\mathrm{div}A(-\nabla u_1(x)) = f_1(x) - h(u_1(x)) > 0, \text{ a.e. } x \in \Omega,$$

which implies that every level set of u_1 has zero measure (cf. [24, Lemma 7.7]). By Lemmas 2.2 and 2.3, there exists an increasing function φ such that $\varphi \circ u_1 \in \mathscr{R}(f)$ is the unique maximizer of the functional $\int_\Omega hu_1$, relative to $h \in \overline{\mathscr{R}(f)^{1,w}}$.

Firstly, we claim that f_1 is also a maximizer of the functional $\int_\Omega hu_1$, relative to $h \in \overline{\mathscr{R}(f)^{1,w}}$.

In fact, we notice that for each $g \in \mathscr{R}(f)$,

$$\begin{aligned}
&\frac{1}{p}\int_\Omega A(-\nabla u_1)(-\nabla u_1)dx + \int_\Omega (H(u_1) - f_1 u_1)dx \\
&= I(u_1) \le I(u_g) \\
&= \frac{1}{p}\int_\Omega A(-\nabla u_g)(-\nabla u_g)dx + \int_\Omega (H(u_g) - g u_g)dx \\
&\le \frac{1}{p}\int_\Omega A(-\nabla u_1)(-\nabla u_1)dx + \int_\Omega (H(u_1) - g u_1)dx,
\end{aligned}$$

which implies that

$$\int_\Omega f_1 u_1 dx \ge \int_\Omega g u_1 dx, \ \forall g \in \mathscr{R}(f). \tag{4.14}$$

If $g \in \overline{\mathscr{R}(f)^{1,w}}$ then we may choose a sequence $\{g_n\} \subset \mathscr{R}(f)$ such that $\{g_n\}$ converge weakly to g in $L^1(\Omega)$. By (4.14), we get

$$\int_\Omega f_1 u_1 dx \ge \int_\Omega g_n u_1 dx \to \int_\Omega g u_1 dx$$

as $n \to \infty$. So that

$$\int_\Omega f_1 u_1 dx \ge \int_\Omega g u_1 dx, \forall g \in \overline{\mathscr{R}(f)^{1,w}}$$

and our claim is valid, so that $f_1 = \varphi \circ u_1 \in \mathscr{R}(f)$ by the uniqueness of the maximizer.

Secondly, we claim that

$$\int_\Omega \alpha^p(-\nabla u_1^*)dx = \int_\Omega \alpha^p(-\nabla u_1)dx. \tag{4.15}$$

Indeed, since $\int_\Omega A(-\nabla u)(-\nabla u)dx = \int_\Omega \alpha^p(-\nabla u)dx, \forall u \in W_0^{1,p}(\Omega)$,

$$\begin{aligned}
&\frac{1}{p}\int_\Omega \alpha^p(-\nabla u_1)dx + \int_\Omega (H(u_1) - f_1u_1)dx \\
&= \frac{1}{p}\int_\Omega A(-\nabla u_1)(-\nabla u_1)dx + \int_\Omega (H(u_1) - f_1u_1)dx \\
&\le \frac{1}{p}\int_\Omega A(-\nabla u_{f^*})(-\nabla u_{f^*})dx + \int_\Omega (H(u_{f^*}) - f^*u_{f^*})dx \\
&= \frac{1}{p}\int_\Omega \alpha^p(-\nabla u_{f^*})dx + \int_\Omega (H(u_{f^*}) - f^*u_{f^*})dx \\
&\le \frac{1}{p}\int_\Omega \alpha^p(-\nabla u_1^*)dx + \int_\Omega (H(u_1^*) - f^*u_1^*)dx.
\end{aligned}$$

Therefore, from Lemma 2.5 and Lemma 2.6 that

$$\frac{1}{p}\int_\Omega (\alpha^p(-\nabla u_1^*) - \alpha^p(-\nabla u_1))dx \ge \int_\Omega (H(u_1) - H(u_1^*) + f^*u_1^* - f_1u_1)dx \ge 0,$$

which, together with (2.3), implies that (4.15) holds.

Finally, we claim that

$$\text{meas}\left(\left\{x \in \Omega : \nabla u_1 = 0,\ 0 < u_1(x) < \operatorname*{ess\,sup}_{y\in\Omega} u_1(y)\right\}\right) = 0. \tag{4.16}$$

In fact, for each $x_0 \in \Omega$ such that $0 < u_1(x_0) < \text{ess sup}_{x\in\Omega} u_1(x)$, we set $S = \{x \in \Omega : u_1(x) \ge u_1(x_0)\}$, which is then a closed ball by Lemma 2.7. If we define $u(x) = u_1(x) - u_1(x_0)$, then we have $\text{div}A(-\nabla u(x)) = \text{div}A(-\nabla u_1(x)) > 0$, a.e. $x \in \Omega$. By the strong maximum principle (cf. [23, Theorem 5]), we deduce that $u(x) > 0$ in the interior $\mathring{S}$ of S. So that $u_1(x) > u_1(x_0)$ for all $x \in \mathring{S}$. Hence x_0 must be a boundary point of S. By the Hopf boundary lemma, we derive $\frac{\partial u}{\partial \nu}(x_0) = \frac{\partial u_1}{\partial \nu}(x_0) \neq 0$, where ν stands for the outward unit normal to ∂S at x_0. This means that

$$\left\{x \in \Omega : \nabla u_1 = 0,\ 0 < u_1(x) < \operatorname*{ess\,sup}_{y\in\Omega} u_1(y)\right\} = \emptyset,$$

so that (4.16) is ture.

Now, by using Lemma 2.7 and noting (4.15) and (4.16), we see that $u_1 = u_1^*$. Hence $f_1 = \varphi\circ u_1^*$ is a spherically symmetric decreasing function. It follows that f_1 coincides its Schwarz rearrangement, i.e., $f_1 = f_1^*$. Recall that $g^* = f^*$, $\forall g \in \mathscr{R}(f)$, we then derive that $f_1 = f^*$ since $f_1 \in \mathscr{R}(f)$. ∎

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