The following publication Zhang, G., Petersen, I. R., & Li, J. (2019). Structural characterization of linear quantum systems with application to back-action evading measurement. IEEE Transactions on Automatic Control, 65(7), 3157-3163 is available at https://doi.org/10.1109/TAC.2019.2946234

Structural characterization of linear quantum systems with application to back-action evading measurement

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***Abstract*—The purpose of this paper is to study the structure of quantum linear systems in terms of their Kalman canonical form, which was proposed in a recent paper. Physical realizability conditions for the Kalman canonical form are given, which shed light on various relations among the components of the system matrices for the Kalman canonical form. These relations are used to explore the structure of the spectrum of quantum linear systems. The spectral structure obtained indicates that a quantum linear system is both controllable and observable if it is Hurwitz stable. A new parameterization method for quantum linear systems is proposed. This new parameterization is designed for the Kalman canonical form directly. Consequently, the parameters involved are in a blockwise form in correspondence with the blockwise structure of the Kalman canonical form. This parameter structure can be used to simplify various quantum control design problems. For example, necessary and sufficient conditions for the realization of back-action evading (BAE) measurements are given in terms of these new parameters that specify the Kalman canonical form. Due to their blockwise nature, a small number of parameters are required for realizing BAE measurements. Moreover, it is shown that a refined structure of these physical parameters reveals the noiseless subsystem and invariant subsystems of a given quantum linear system.**

Index Terms— Quantum linear systems; Kalman canonical form; noiseless systems; invariant systems; back-action evading (BAE) measurements.

I. Introduction

The last two decades have witnessed a fast growth in the theoretical investigation and experimental demonstration of quantum control as it is an essential ingredient of quantum information technologies, including quantum communication, quantum computation, quantum cryptography, quantum ultra-precision metrology, and nano-electronics. As with the role played by classical linear systems in classical systems and control theory, quantum linear systems play a significant role in quantum control theory. Quantum linear systems are modeled mathematically by linear quantum stochastic differential equations (linear QSDEs). Although looking like classical stochastic differential equations (SDEs), QSDEs describe the dynamical evolution of quantum-mechanical systems and are fundamentally different from classical SDEs. For example, the system variables of QSDEs are operators, not ordinary random variables as in classical SDEs. Moreover, these operators may not commute with each other. Indeed, this non-commutative nature of variables is fundamental to quantum mechanics as it gives rise to Heisenberg's uncertainty principle. In the field of quantum optics, linear systems are widely used as they are easy to manipulate and, more importantly, they are often good approximations to more general dynamics [7], [35], [40], [31]. Besides their wide applications in quantum optical systems, quantum linear models have found important and successful applications for many other quantum dynamical systems such as opto-mechanical systems [13], [33], [19], circuit quantum electro-dynamical (circuit QED) systems [18], and atomic ensembles [30]. Quantum linear systems driven by Gaussian input fields have been studied extensively, and results like quantum measurement-based feedback control and filtering have been well established [40], [6], [24], [49].

The physical realization of quantum computing calls for a hierarchical quantum network. The bottom level is the one- and two- qubit regime, where a photon interacts with matter (e.g., a trapped ion). Going one level up we enter the regime of quantum logic gates, where typically ten or more qubits operate. One level further up is the fault-tolerant quantum error correction (QEC) architecture regime where hundreds of qubits reside. The final level is the algorithms regime. In order to fulfil a desired quantum computing task, precise control must be exerted at the bottom level. One of the most important objectives of quantum control is to protect the system of interest from the adverse influence of the surrounding environment so that the coherence of the system can be maintained long enough. The ideal situation is to completely isolate the system from the surrounding environment. This is the idea of decoherence-free subsystems (DFSs) and noiseless subsystems, [36], [37], [5], [42], [43], [10], [26]. In the language of linear systems theory, a DFS is not affected by the input field and cannot be observed from the output field either; in other words, it is an uncontrollable and unobservable subsystem in the Kalman canonical form [47]. We are often interested in getting information about a certain quantum quantity, for example, the position or momentum of a nano-mechanical oscillator or the spin of an electron. In quantum information theory, this is often achieved by using a probe field to interact with the system and inferring the required system information from the output field after

This research is supported in part by a Hong Kong Research Grant council (RGC) grant (No. 15206915), the Air Force Office of Scientific Research (AFOSR) and the office of Naval Research Grants (ONRS) under agreement number FA2386-16-1-4065, and the Australian Research Council under grant number DP180101805.

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interaction. Thus, normally the dynamical evolution of the quantum quantity of interest suffers from noise in the probe field. However, if the above information acquisition process can be designed in a clever way such that the quantum quantity of interest is not affected by the noise in the input probe field, but can still be extracted from the output probe field, then such a quantum quantity is called a quantum-nondemolition (QND) variable, [38], [33], [42], [43], [47]. Therefore, a QND variable can be repeatedly measured and is immune to input probe field noise. In the language of linear systems theory, a QND variable is uncontrollable and observable in the Kalman canonical form. A measurement process often involves measurement noise from the surrounding environment. In quantum mechanics, environmental noise can be represented by two conjugate quadrature operators. A fundamental fact in quantum mechanics is that these two noise quadrature operators do not commute. This gives rise to the so-called standard quantum limit (SQL). However, if a measurement process suffers from a noise quadrature (shot noise), but not from the conjugate quadrature noise (measurement back-action noise), then it is called a back-action evading (BAE) measurement, [3], [32, Fig. 2(a)], [41], [43], [50], [25], [44]. As a result, a BAE measurement may be able to beat the SQL, thus enabling extremely high precision measurement. In fact, the idea of BAE measurement originates from the study of gravitational wave detection [13]. In the language of linear systems theory, a BAE measurement is realized if the transfer function from the measurement back-action noise to the measured output is zero.

Controllability and observability are two fundamental notions in modern control theory [1], [48], [15], [4]. Moreover, as discussed above, they are also closely related to DFSs, QND variables and BAE measurements of quantum linear systems. Therefore, it is important to study the controllability and observability of quantum linear systems. Indeed, recently these two fundamental notions have been investigated in the quantum control community. The equivalence between detectability and stabilizability of quantum linear systems has been shown in [39]. In [16], Maalouf and Petersen studied the controllability and observability of passive quantum linear systems. On the basis of these two notions they derived a complex-domain bounded real lemma for passive quantum linear systems [16, Theorem 6.5]. Nurdin [23] proposed model reduction methods for quantum linear systems by means of controllability and observability decompositions; see also [28]. When restricted to the passive case, Yamamoto and Guta showed that controllability and observability are equivalent to each other, and they imply Hurwitz stability [12]. In [10], Gough and Zhang proved that Hurwitz stability, controllability and observability are equivalent to each other for passive quantum linear systems. Moreover, it is shown in [10] that controllability and observability are actually equivalent for general (not necessarily passive) quantum linear systems. The application of controllability and observability in quantum information science has been discussed, for example regarding, DFSs in [36], [37], [5], [42], [43], [10], [26], QND variables in [38], [33], [42], [43], and BAE measurements in [3], [32], [41], [43], [25], [44].

In [47], a Kalman canonical form is proposed for quantum linear systems. More specifically, given a quantum linear system, an orthogonal and blockwise symplectic transformation matrix is constructed which transforms the original system into four subsystems: the controllable and observable (co) subsystem, the controllable and unobservable ($c\bar{o}$) subsystem, the uncontrollable and observable ($\bar{c}o$) subsystem, and the uncontrollable and unobservable ($\bar{c}\bar{o}$) subsystem. The fact that the transformation matrix is required to be symplectic is due to the nature of quantum mechanics; the system variables of the $c\bar{o}$ subsystem are conjugates of those of the $\bar{c}o$ subsystem; this is critical because it means that the commutation relations of the quantum linear system are preserved. In [47], the quantities $\boldsymbol{x}_{co}, \boldsymbol{x}_{\bar{c}\bar{o}}, \boldsymbol{q}_h, \boldsymbol{p}_h$ are used to denote the quadrature operators of the $co, \bar{c}\bar{o}, c\bar{o}, \bar{c}o$ subsystems respectively. The Kaman canonical form has also been derived in Reference [11] by means of an SVD-like factorization.

On the basis of the Kalman canonical form proposed in [47], in this paper we aim to explore deeper the structure of quantum linear systems. As mentioned above, an open quantum system can be described by a set of QSDEs. Due to the nature of quantum-mechanical systems, there are constraints on the coefficients of these QSDEs, which are called physical realizability conditions of quantum systems [14]. In this paper we present physical realizability conditions for quantum linear systems in the Kalman canonical form; see Lemma 3.1. Based on these conditions, we study the relations among the components of the system matrices for the Kalman canonical form; see Theorem 3.1. Interestingly, these relations allow us to expose the nice structure of the spectrum of a quantum linear system in the Kalman canonical form; see Propositions 3.1 and 3.2, Example 3.1 and Theorem 3.2. In particular, it is shown in Theorem 3.2 that if a quantum linear system is Hurwitz stable, then it is both controllable and observable.

A new parameterization method for quantum linear systems is proposed in the paper. We express the system Hamiltonian and the coupling operator explicitly in terms of the partitioned system variables $\boldsymbol{x}_{co}, \boldsymbol{x}_{\bar{c}\bar{o}}, \boldsymbol{q}_h, \boldsymbol{p}_h$. Specifically, let $\bar{\boldsymbol{x}} = [\boldsymbol{q}_h^\top \ \boldsymbol{p}_h^\top \ \boldsymbol{x}_{co}^\top \ \boldsymbol{x}_{\bar{c}\bar{o}}^\top]^\top$. Then the system Hamiltonian is $\boldsymbol{H} = \bar{\boldsymbol{x}}^\top H \bar{\boldsymbol{x}}/2$ where H is a real symmetric matrix, and the coupling operator is $\boldsymbol{L} = \Gamma \bar{\boldsymbol{x}}$ with Γ being a complex matrix. Due to the special structure of the system matrices in the Kalman canonical form, if $\boldsymbol{H}$ and $\boldsymbol{L}$ generate the Kalman canonical form, then the matrices H and Γ should be of specific form. This form is given in Lemma 4.1. Moreover, we also establish the converse: If the matrices H and Γ are of the given specific form, then the resulting QSDEs are formally in the Kalman canonical form; see Lemma 4.3. Finally, as the Kalman canonical form is obtained based on the notions of controllability and observability, we derive further conditions on H and Γ such that the resulting system is indeed the Kalman canonical form; see Theorem 4.1.

BAE measurements are very important in quantum ultra-precision metrology, as they allow one to go beyond the SQL. On the basis of the proposed new parameterization method, necessary and sufficient conditions for the realization of BAE measurements by means of quantum linear systems are given in Theorem 5.2. Moreover, a special case is discussed in Corollary 5.1.

The rest of the paper is organized as follows. The notation commonly used in this paper is summarized in Subsection I-A. Preliminaries are given in Section II, which include quantum linear systems and their Kalman canonical form. The structural properties of the Kalman canonical form are studied in Section III. A new parameterization method is presented in Section IV. Applications to noiseless and invariant subsystems, and the realization of BAE measurements are studied in Section V. Concluding remarks are given in Section VI.

A. Notation

1) x^* denotes the complex conjugate of a complex number x or the adjoint of an operator x. The commutator of two operators X and Y is defined as $[X, Y] \triangleq XY - YX$. If X and Y are two vectors of operators, then their commutator is defined as the matrix of operators $\left[X, Y^\top\right] \triangleq XY^\top - (YX^\top)^\top$.
2) For a matrix $X = [x_{ij}]$ with number or operator entries, $X^\# = [x_{ij}^*]$, $X^\top = [x_{ji}]$ is the transpose, and $X^\dagger = (X^\#)^\top$. Moreover, let $\breve{X} = \left[\begin{smallmatrix} X \\ X^\# \end{smallmatrix}\right]$. $\mathrm{Re}(X)$ and $\mathrm{Im}(X)$ denote the real part and imaginary part of a matrix X, respectively.
3) I_k is the identity matrix and 0_k the zero matrix in $\mathbb{C}^{k\times k}$. δ_{ij} denotes the Kronecker delta. Let $J_k = \mathrm{diag}(I_k, -I_k)$. For a matrix $X \in \mathbb{C}^{2k\times 2r}$, define its $\flat$-adjoint by $X^\flat \triangleq J_r X^\dagger J_k$.
4) Given two matrices U, $V \in \mathbb{C}^{k\times r}$, define $\Delta(U, V) \triangleq [U \;\; V; V^\# \;\; U^\#]$. A matrix with this structure will be called *doubled-up* [9].
5) A matrix $T \in \mathbb{C}^{2k\times 2k}$ is called *Bogoliubov* if it is doubled-up and satisfies $TJ_kT^\dagger = T^\dagger J_k T = J_k \Leftrightarrow TT^\flat = T^\flat T = I_{2k}$.
6) Let $\mathbb{J}_k = \left[\begin{smallmatrix} 0_k & I_k \\ -I_k & 0_k \end{smallmatrix}\right]$. A matrix $S \in \mathbb{C}^{2k\times 2k}$ is called *symplectic*, if it satisfies $S\mathbb{J}_k S^\dagger = S^\dagger \mathbb{J}_k S = \mathbb{J}_k$.

II. Preliminaries

In this section, quantum linear systems are briefly introduced, and their Kalman canonical form, derived in [47], is also presented.

A. Quantum linear systems

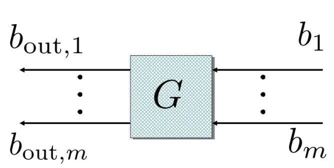

Fig. 1. Open quantum linear system G composed of n harmonic oscillators driven by m input fields.

The open quantum linear system G, as shown in Fig. 1, can be used to model a collection of n quantum harmonic oscillators interacting with m input boson fields. The j-th oscillator, $j = 1, \ldots, n$, may be represented by its annihilation operator $\boldsymbol{a}_j$ and creation operator $\boldsymbol{a}_j^*$ (the adjoint operator of $\boldsymbol{a}_j$). These are operators on an infinite-dimensional Hilbert space and satisfy the *canonical commutation relations (CCRs)* $[\boldsymbol{a}_j(t), \, \boldsymbol{a}_k(t)] = 0$, $[\boldsymbol{a}_j^*(t), \, \boldsymbol{a}_k^*(t)] = 0$, and $[\boldsymbol{a}_j(t), \, \boldsymbol{a}_k^*(t)] = \delta_{jk}$, $\forall j, k = 1, \ldots n, \forall t \in \mathbb{R}^+$. Let $\boldsymbol{a} = [\boldsymbol{a}_1 \; \cdots \; \boldsymbol{a}_n]^\top$. The system Hamiltonian is given by $\boldsymbol{H} = (1/2)\breve{\boldsymbol{a}}^\dagger \Omega \breve{\boldsymbol{a}}$, where $\breve{\boldsymbol{a}} = [\boldsymbol{a}^\top \;\; (\boldsymbol{a}^\#)^\top]^\top$, and $\Omega = \Delta(\Omega_-, \Omega_+) \in \mathbb{C}^{2n\times 2n}$ is a Hermitian matrix with $\Omega_-, \Omega_+ \in \mathbb{C}^{n\times n}$. The coupling of the system to the input fields is described by the operator $\boldsymbol{L} = [C_- \;\; C_+]\breve{\boldsymbol{a}}$, with $C_-, C_+ \in \mathbb{C}^{m\times n}$. The k-th input boson field, $k = 1, \ldots, m$, may be represented in terms of its annihilation operator $\boldsymbol{b}_k(t)$ and creation operator $\boldsymbol{b}_k^*(t)$ (the adjoint operator of $\boldsymbol{b}_k(t)$). These are operators on a symmetric Fock space (a special kind of infinite-dimensional Hilbert space, [27]). The operators $\boldsymbol{b}_k(t)$ and $\boldsymbol{b}_k^*(t)$ satisfy the *singular commutation relations* $[\boldsymbol{b}_j(t), \, \boldsymbol{b}_k(r)] = 0$, $[\boldsymbol{b}_j^*(t), \, \boldsymbol{b}_k^*(r)] = 0$, and $[\boldsymbol{b}_j(t), \, \boldsymbol{b}_k^*(r)] = \delta_{jk}\delta(t-r)$, $\forall j, k = 1, \ldots, m$, $\forall t, r \in \mathbb{R}$. Let $\boldsymbol{b}(t) = [\boldsymbol{b}_1(t) \; \cdots \; \boldsymbol{b}_m(t)]^\top$ and $\breve{\boldsymbol{b}}(t) = [\boldsymbol{b}(t)^\top \;\; (\boldsymbol{b}(t)^\#)^\top]^\top$.

The dynamics of the open quantum linear system in Fig. 1 is described by the following quantum stochastic differential equations (QSDEs), ([9, Eq. (26)], [46, Eqs. (14)-(15)])

$$\begin{aligned} \dot{\breve{\boldsymbol{a}}}(t) &= \mathcal{A}\breve{\boldsymbol{a}}(t) + \mathcal{B}\breve{\boldsymbol{b}}(t), \\ \breve{\boldsymbol{b}}_{\mathrm{out}}(t) &= \mathcal{C}\breve{\boldsymbol{a}}(t) + \breve{\boldsymbol{b}}(t), \quad t \geq 0, \end{aligned} \tag{1}$$

where the system matrices are given by

$$\mathcal{C} = \Delta(C_-, \; C_+), \; \mathcal{B} = -\mathcal{C}^\flat, \text{ and } \mathcal{A} = -\imath J_n \Omega - \frac{1}{2}\mathcal{C}^\flat \mathcal{C}.$$

These system matrices satisfy

$$\mathcal{A} + \mathcal{A}^\flat + \mathcal{B}\mathcal{B}^\flat = 0, \tag{2a}$$

$$\mathcal{B} = -\mathcal{C}^\flat. \tag{2b}$$

Moreover, from Eq. (2a) one can get

$$\Omega = \frac{\imath}{2}(J_n\mathcal{A} - \mathcal{A}^\dagger J_n). \tag{3}$$

The dynamical evolution of an open quantum system is often given by means of stochastic differential equations (SDEs). However, not every SDE describes a valid *quantum-mechanical* system. Those SDEs, which indeed describe valid quantum-mechanical dynamical systems, are often called QSDEs. Quantum mechanics imposes constraints on the SDEs. As far as linear dynamics is concerned, these constraints are expressed as Eqs. (2a)-(2b). On one hand, given the coupling operator $\boldsymbol{L}$ and the system Hamiltonian $\boldsymbol{H}$, a quantum linear system is generated whose system matrices $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ satisfy Eqs. (2a)-(2b); on the other hand, given matrices $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ that satisfy Eqs. (2a)-(2b), the coupling operator $\boldsymbol{L}$ is determined by the matrix $\mathcal{C}$, and the system Hamiltonian $\boldsymbol{H}$ is determined by $\mathcal{A}$ as can be seen in Eq. (3). In the quantum control literature, Eqs. (2a)-(2b) are called the *physical realizability conditions* for a quantum linear system; see, e.g., [14], [20], [21], [22], [29], [23], [46] for more details.

B. Kalman canonical form of quantum linear systems

The Kalman decomposition of quantum linear systems has been proposed in [47]. In this subsection, some results from [47] are summarized for completeness.

A unitary and blockwise Bogoliubov coordinate transformation matrix T is defined in [47, Eq. (47)], which is

$$T \triangleq \left[\begin{array}{cc|cc|cc} Z_3 & 0 & Z_1 & 0 & Z_2 & 0 \\ 0 & Z_3^{\#} & 0 & Z_1^{\#} & 0 & Z_2^{\#} \end{array}\right], \tag{4}$$

where $Z_1 \in \mathbb{C}^{n\times n_1}$, $Z_2 \in \mathbb{C}^{n\times n_2}$, and $Z_3 \in \mathbb{C}^{n\times n_3}$ ($n_1, n_2, n_3 \geq 0$ and $n_1+n_2+n_3 = n$). T is called blockwise Bogoliubov as it satisfies

$$T^\dagger J_n T = \begin{bmatrix} J_{n_3} & 0 & 0 \\ 0 & J_{n_1} & 0 \\ 0 & 0 & J_{n_2} \end{bmatrix}. \tag{5}$$

The following result is proved in [47].

Lemma 2.1: [47, Theorem 4.2] The unitary and blockwise Bogoliubov coordinate transformation

$$\begin{bmatrix} \breve{\boldsymbol{a}}_h \\ \breve{\boldsymbol{a}}_{co} \\ \breve{\boldsymbol{a}}_{\bar{c}\bar{o}} \end{bmatrix} \triangleq T^\dagger \breve{\boldsymbol{a}}$$

transforms the quantum linear system (1) into the form

$$\begin{aligned} \begin{bmatrix} \dot{\breve{\boldsymbol{a}}}_h(t) \\ \dot{\breve{\boldsymbol{a}}}_{co}(t) \\ \dot{\breve{\boldsymbol{a}}}_{\bar{c}\bar{o}}(t) \end{bmatrix} &= \bar{\mathcal{A}} \begin{bmatrix} \breve{\boldsymbol{a}}_h(t) \\ \breve{\boldsymbol{a}}_{co}(t) \\ \breve{\boldsymbol{a}}_{\bar{c}\bar{o}}(t) \end{bmatrix} + \bar{\mathcal{B}} \breve{\boldsymbol{b}}(t), \\ \breve{\boldsymbol{b}}_{\rm out}(t) &= \bar{\mathcal{C}} \begin{bmatrix} \breve{\boldsymbol{a}}_h(t) \\ \breve{\boldsymbol{a}}_{co}(t) \\ \breve{\boldsymbol{a}}_{\bar{c}\bar{o}}(t) \end{bmatrix} + \breve{\boldsymbol{b}}(t), \end{aligned}$$

where

$$\begin{aligned} &\bar{\mathcal{A}} \triangleq T^\dagger \mathcal{A} T = \begin{bmatrix} \mathcal{A}_h & \mathcal{A}_{12} & \mathcal{A}_{13} \\ \mathcal{A}_{21} & \mathcal{A}_{co} & 0 \\ \mathcal{A}_{31} & 0 & \mathcal{A}_{\bar{c}\bar{o}} \end{bmatrix}, \\ &\bar{\mathcal{B}} \triangleq T^\dagger \mathcal{B} = \begin{bmatrix} \mathcal{B}_h \\ \mathcal{B}_{co} \\ 0 \end{bmatrix}, \quad \bar{\mathcal{C}} \triangleq \mathcal{C} T = \begin{bmatrix} \mathcal{C}_h & \mathcal{C}_{co} & 0 \end{bmatrix}. \end{aligned} \tag{6}$$

$\breve{\boldsymbol{a}}_h$, $\breve{\boldsymbol{a}}_{co}$, and $\breve{\boldsymbol{a}}_{\bar{c}\bar{o}}$ are vectors of system variables, of dimensions $2n_3$, $2n_1$, and $2n_2$ respectively.

Next, we look at the Kalman canonical form in the real quadrature operator representation. Define the matrix $\Pi \in \mathbb{C}^{2n_3\times 2n_3}$ by

$$\Pi \triangleq \begin{bmatrix} I_{n_a} & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_{n_b} \\ 0 & 0 & I_{n_a} & 0 \\ 0 & I_{n_b} & 0 & 0 \end{bmatrix},$$

where $0 \leq n_a, n_b \leq n_3$, and $n_a + n_b = n_3$. Let

$$\tilde{V}_{n_3} = \Pi V_{n_3}, \tag{7}$$

where

$$V_k \triangleq \frac{1}{\sqrt{2}} \begin{bmatrix} I_k & I_k \\ -\imath I_k & \imath I_k \end{bmatrix}, \quad k \in \mathbb{Z}^+ \tag{8}$$

is a unitary matrix. Define two more matrices

$$\tilde{V}_n \triangleq \begin{bmatrix} \tilde{V}_{n_3} & & \mathbf{0} \\ & V_{n_1} & \\ \mathbf{0} & & V_{n_2} \end{bmatrix}$$

and

$$\hat{T} \triangleq T \tilde{V}_n^\dagger. \tag{9}$$

The following result is proved in [47], which presents the Kalman canonical form of the quantum linear system (1).

Lemma 2.2: The transformations

$$\begin{bmatrix} \boldsymbol{q}_h \\ \boldsymbol{p}_h \\ \boldsymbol{x}_{co} \\ \boldsymbol{x}_{\bar{c}\bar{o}} \end{bmatrix} \triangleq \hat{T}^\dagger \breve{\boldsymbol{a}}, \tag{10a}$$

$$\begin{bmatrix} \boldsymbol{q}_{\rm in} \\ \boldsymbol{p}_{\rm in} \end{bmatrix} \equiv \boldsymbol{u} \triangleq V_m \breve{\boldsymbol{b}}, \quad \begin{bmatrix} \boldsymbol{q}_{\rm out} \\ \boldsymbol{p}_{\rm out} \end{bmatrix} \equiv \boldsymbol{y} \triangleq V_m \breve{\boldsymbol{b}}_{\rm out} \tag{10b}$$

convert the system (1) into the following real quadrature form, as given in [47, Theorem 4.4],

$$\begin{aligned} \left[\begin{array}{c} \dot{\boldsymbol{q}}_h(t) \\ \dot{\boldsymbol{p}}_h(t) \\ \hline \dot{\boldsymbol{x}}_{co}(t) \\ \hline \dot{\boldsymbol{x}}_{\bar{c}\bar{o}}(t) \end{array}\right] &= \bar{A} \left[\begin{array}{c} \boldsymbol{q}_h(t) \\ \boldsymbol{p}_h(t) \\ \hline \boldsymbol{x}_{co}(t) \\ \hline \boldsymbol{x}_{\bar{c}\bar{o}}(t) \end{array}\right] + \bar{B}\boldsymbol{u}(t), \\ \boldsymbol{y}(t) &= \bar{C} \left[\begin{array}{c} \boldsymbol{q}_h(t) \\ \boldsymbol{p}_h(t) \\ \hline \boldsymbol{x}_{co}(t) \\ \hline \boldsymbol{x}_{\bar{c}\bar{o}}(t) \end{array}\right] + \boldsymbol{u}(t), \end{aligned} \tag{11}$$

where the real matrices $\bar{A}, \bar{B}, \bar{C}$ are of the form

$$\bar{A} \triangleq \left[\begin{array}{cc|c|c} A_h^{11} & A_h^{12} & A_{12} & A_{13} \\ 0 & A_h^{22} & 0 & 0 \\ \hline 0 & A_{21} & A_{co} & 0 \\ \hline 0 & A_{31} & 0 & A_{\bar{c}\bar{o}} \end{array}\right], \tag{12a}$$

$$\bar{B} \triangleq \left[\begin{array}{c} B_h \\ 0 \\ \hline B_{co} \\ \hline 0 \end{array}\right], \tag{12b}$$

$$\bar{C} \triangleq \left[\begin{array}{cc|c|c} 0 & C_h & C_{co} & 0 \end{array}\right]. \tag{12c}$$

After a re-arrangement, the system (11) becomes

$$\begin{aligned} \begin{bmatrix} \dot{\boldsymbol{q}}_h(t) \\ \dot{\boldsymbol{x}}_{co}(t) \\ \dot{\boldsymbol{x}}_{\bar{c}\bar{o}}(t) \\ \dot{\boldsymbol{p}}_h(t) \end{bmatrix} &= \begin{bmatrix} A_h^{11} & A_{12} & A_{13} & A_h^{12} \\ 0 & A_{co} & 0 & A_{21} \\ 0 & 0 & A_{\bar{c}\bar{o}} & A_{31} \\ 0 & 0 & 0 & A_h^{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_h(t) \\ \boldsymbol{x}_{co}(t) \\ \boldsymbol{x}_{\bar{c}\bar{o}}(t) \\ \boldsymbol{p}_h(t) \end{bmatrix} \\ &\quad + \begin{bmatrix} B_h \\ B_{co} \\ 0 \\ 0 \end{bmatrix} \boldsymbol{u}(t), \\ \boldsymbol{y}(t) &= [0 \; C_{co} \; 0 \; C_h] \begin{bmatrix} \boldsymbol{q}_h(t) \\ \boldsymbol{x}_{co}(t) \\ \boldsymbol{x}_{\bar{c}\bar{o}}(t) \\ \boldsymbol{p}_h(t) \end{bmatrix} + \boldsymbol{u}(t). \end{aligned} \tag{13}$$

The corresponding system block diagram is shown in Fig. 2.

III. Structure of the Kalman canonical form

In this section, we present the real quadrature counterpart of the physical realizability conditions (2a)-(2b); namely, the physical realizability conditions for the Kalman canonical

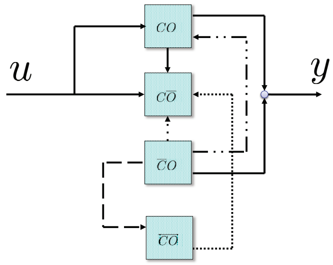

Fig. 2. The Kalman canonical form of a quantum linear system; see [47, Fig. 2].

form (11). Relations among the various components of the system matrices $\bar{A}, \bar{B}, \bar{C}$ in Eqs. (12a)-(12c) are derived. These properties allow us to investigate the spectral structure of the Kalman canonical form (11).

A. Physical realizability conditions

The following result gives the real quadrature counterpart of the physical realizability conditions (2a)-(2b); see also [14].

Lemma 3.1: For the Kalman canonical form (11), the following conditions hold:

$$\begin{aligned} \bar{A}\bar{\mathbb{J}}_n + \bar{\mathbb{J}}_n \bar{A}^\top + \bar{B}\mathbb{J}_m \bar{B}^\top &= 0, && (14a)\\ \bar{B} &= \bar{\mathbb{J}}_n \bar{C}^\top \mathbb{J}_m, && (14b)\end{aligned}$$

where

$$\bar{\mathbb{J}}_n \triangleq \begin{bmatrix} \mathbb{J}_{n_3} & 0 & 0\\ 0 & \mathbb{J}_{n_1} & 0 \\ 0 & 0 & \mathbb{J}_{n_2}\end{bmatrix}. \quad (15)$$

Proof. According to Eqs. (5) and (9), we have

$$\begin{aligned} \hat{T}^\dagger J_n \hat{T} = \tilde{V}_n T^\dagger J_n T \tilde{V}_n^\dagger &= \tilde{V}_n \begin{bmatrix} J_{n_3} & 0 & 0\\ 0 & J_{n_1} & 0 \\ 0 & 0 & J_{n_2}\end{bmatrix} \tilde{V}_n^\dagger \\ &= \begin{bmatrix} \imath\mathbb{J}_{n_3} & 0 & 0\\ 0 & \imath\mathbb{J}_{n_1} & 0 \\ 0 & 0 & \imath\mathbb{J}_{n_2}\end{bmatrix} = \imath\bar{\mathbb{J}}_n, \quad (16)\end{aligned}$$

with $\bar{\mathbb{J}}_n$ as given in Eq. (15). Pre- and post-multiplying both sides of Eq. (2a) by $\hat{T}^\dagger$ and $\hat{T}$ respectively gives

$$\begin{aligned} & \hat{T}^\dagger \mathcal{A}\hat{T} + \hat{T}^\dagger \mathcal{A}^\flat \hat{T} + \hat{T}^\dagger \mathcal{B}\mathcal{B}^\flat \hat{T} \\ = \; & \hat{T}^\dagger \mathcal{A}\hat{T} + \hat{T}^\dagger J_n \hat{T}\hat{T}^\dagger \mathcal{A}^\dagger \hat{T}\hat{T}^\dagger J_n \hat{T} \\ & + \hat{T}^\dagger \mathcal{B} V_m^\dagger V_m J_m V_m^\dagger V_m \mathcal{B}^\dagger \hat{T}\hat{T}^\dagger J_n \hat{T} \\ = \; & \bar{A} + (\imath\bar{\mathbb{J}}_n)\bar{A}^\top(\imath\bar{\mathbb{J}}_n) + \bar{B}(\imath\mathbb{J}_m)\bar{B}^\top(\imath\bar{\mathbb{J}}_n) \\ = \; & 0,\end{aligned}$$

where Eq. (16) has been used to derive the second step. Thus,

$$\bar{A} - \bar{\mathbb{J}}_n \bar{A}^\top \bar{\mathbb{J}}_n - \bar{B}\mathbb{J}_m \bar{B}^\top \bar{\mathbb{J}}_n = 0. \quad (17)$$

Post-multiplying by $\bar{\mathbb{J}}_n$ on both sides of Eq. (17) yields Eq. (14a). On the other hand, from the coordinate transformations (10a)-(10b), we know that $\bar{B} = \hat{T}^\dagger \mathcal{B} V_m^\dagger$. Then by Eqs. (2b) and (16), we have

$$\begin{aligned} \bar{B} &= \hat{T}^\dagger \mathcal{B} V_m^\dagger = -\hat{T}^\dagger \mathcal{C}^\flat V_m^\dagger = -\hat{T}^\dagger J_n \mathcal{C}^\dagger J_m V_m^\dagger \\ &= -\hat{T}^\dagger J_n \hat{T}\hat{T}^\dagger \mathcal{C}^\dagger V_m^\dagger V_m J_m V_m^\dagger \\ &= \bar{\mathbb{J}}_n \bar{C}^\top \mathbb{J}_m,\end{aligned}$$

which is (14b). □

Remark 3.1: From the proof of Lemma 3.1, it can be seen that Eqs. (14a)-(14b) can be derived directly from Eq. (2a)-(2b) via Eqs. (10a)-(10b), irrespective of the specific forms of the system matrices $\bar{A}, \bar{B}, \bar{C}$ in Eqs. (12a)-(12c) for the Kalman canonical form (11). Nevertheless, the transformations (10a)-(10b) are constructed in a careful way in [47] to generate the Kalman canonical form of quantum linear systems.

Substituting Eqs. (12a)-(12c) into Eqs. (14a)-(14b) we can obtain the following result, which shows the relations among the components of the system matrices $\bar{A}, \bar{B}, \bar{C}$ for the Kalman canonical form (11).

Theorem 3.1: For the Kalman canonical form (11), the following conditions hold:

$$\begin{aligned} A_h^{22\top} &= -A_h^{11}, && (18a)\\ -A_h^{12} + A_h^{12\top} + B_h \mathbb{J}_m B_h^\top &= 0, && (18b)\\ -A_{12} + A_{21}^\top \mathbb{J}_{n_1} + B_h \mathbb{J}_m B_{co}^\top \mathbb{J}_{n_1} &= 0, && (18c)\\ A_{31}^\top \mathbb{J}_{n_2} &= A_{13}, && (18d)\\ \mathbb{J}_{n_1} A_{co} + A_{co}^\top \mathbb{J}_{n_1} - \mathbb{J}_{n_1} B_{co}\mathbb{J}_m B_{co}^\top \mathbb{J}_{n_1} &= 0, && (18e)\\ \mathbb{J}_{n_2} A_{\bar{c}\bar{o}} + A_{\bar{c}\bar{o}}^\top \mathbb{J}_{n_2} &= 0, && (18f)\end{aligned}$$

and

$$\begin{aligned} B_h &= C_h^\top \mathbb{J}_m, && (18g)\\ B_{co} &= \mathbb{J}_{n_1} C_{co}^\top \mathbb{J}_m. && (18h)\end{aligned}$$

Proof. Pre- and post- multiplying by $\bar{\mathbb{J}}_n$ on both sides of Eq. (14a) yields

$$\bar{\mathbb{J}}_n \bar{A} + \bar{A}^\top \bar{\mathbb{J}}_n - \bar{\mathbb{J}}_n \bar{B}\mathbb{J}_m \bar{B}^\top \bar{\mathbb{J}}_n = 0. \quad (19)$$

Substituting

$$\bar{\mathbb{J}}_n \bar{A} = \begin{bmatrix} 0 & A_h^{22} & 0 & 0\\ -A_h^{11} & -A_h^{12} & -A_{12} & -A_{13}\\ 0 & \mathbb{J}_{n_1}A_{21} & \mathbb{J}_{n_1}A_{co} & 0\\ 0 & \mathbb{J}_{n_2}A_{31} & 0 & \mathbb{J}_{n_2}A_{\bar{c}\bar{o}}\end{bmatrix},$$

$$\bar{A}^\top \bar{\mathbb{J}}_n = -(\bar{\mathbb{J}}_n \bar{A})^\top,$$

and

$$\begin{aligned} & \bar{\mathbb{J}}_n \bar{B}\mathbb{J}_m \bar{B}^\top \bar{\mathbb{J}}_n \\ = & \begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & -B_h\mathbb{J}_m B_h^\top & -B_h \mathbb{J}_m B_{co}^\top \mathbb{J}_{n_1} & 0\\ 0 & \mathbb{J}_{n_1}B_{co}\mathbb{J}_m B_h^\top & \mathbb{J}_{n_1}B_{co}\mathbb{J}_m B_{co}^\top \mathbb{J}_{n_1} & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\end{aligned}$$

into Eq. (19), we have

$$\begin{aligned} & \bar{\mathbb{J}}_n \bar{A} + \bar{A}^\top \bar{\mathbb{J}}_n - \bar{\mathbb{J}}_n \bar{B}\mathbb{J}_m \bar{B}^\top \bar{\mathbb{J}}_n \\ = & \begin{bmatrix} 0 & A_h^{22} + A_h^{11\top} & 0 & 0\\ -A_h^{11} - A_h^{22\top} & -A_h^{12} + A_h^{12\top} + B_h\mathbb{J}_m B_h^\top & -A_{12} + A_{21}^\top \mathbb{J}_{n_1} + B_h \mathbb{J}_m B_{co}^\top \mathbb{J}_{n_1} & -A_{13} + A_{31}^\top \mathbb{J}_{n_2}\\ 0 & \mathbb{J}_{n_1}A_{21} + A_{12}^\top - \mathbb{J}_{n_1}B_{co}\mathbb{J}_m B_h^\top & \mathbb{J}_{n_1}A_{co} + A_{co}^\top \mathbb{J}_{n_1} - \mathbb{J}_{n_1}B_{co}\mathbb{J}_m B_{co}^\top \mathbb{J}_{n_1} & 0\\ 0 & \mathbb{J}_{n_2}A_{31} + A_{13}^\top & 0 & \mathbb{J}_{n_2}A_{\bar{c}\bar{o}} + A_{\bar{c}\bar{o}}^\top \mathbb{J}_{n_2}\end{bmatrix} \\ = & \; 0,\end{aligned}$$

which is equivalent to Eqs. (18a)-(18f). Moreover, Eqs. (18g)-(18h) are a direct consequence of Eq. (14b) and the form of the matrices $\bar{B}$ in Eq. (12b) and $\bar{C}$ in Eq. (12c). □

B. Structure of the spectrum

The specific relations among the components of the system matrices, established in Theorem 3.1 in the previous subsection, can be used to study the structure of the poles of quantum linear systems, which is the focus of this subsection. We denote the set of eigenvalues of a matrix A by $\sigma(A)$.

Let us first look at the $\bar{c}\bar{o}$ subsystem by ignoring the other modes, which is

$$\dot{\boldsymbol{x}}_{\bar{c}\bar{o}}(t) = A_{\bar{c}\bar{o}}\boldsymbol{x}_{\bar{c}\bar{o}}(t). \tag{20}$$

The following result shows that the poles of this subsystem are symmetric about both the real and imaginary axes.

Proposition 3.1: If $\lambda \in \sigma(A_{\bar{c}\bar{o}})$, then $-\lambda, \lambda^*, -\lambda^* \in \sigma(A_{\bar{c}\bar{o}})$.

Proof. Let λ and μ be an eigenvalue and eigenvector of the matrix $A_{\bar{c}\bar{o}}$; i.e., $A_{\bar{c}\bar{o}}\mu = \lambda\mu$. By Eq. (18f), $-\lambda\mathbb{J}_{n_2}\mu = -\mathbb{J}_{n_2}A_{\bar{c}\bar{o}}\mu = A_{\bar{c}\bar{o}}^\top\mathbb{J}_{n_2}\mu$. In other words, $-\lambda$ is an eigenvalue of $A_{\bar{c}\bar{o}}^\top$ with the corresponding eigenvector $\mathbb{J}_{n_2}\mu$. As $\sigma(A_{\bar{c}\bar{o}}) = \sigma(A_{\bar{c}\bar{o}}^\top)$, we have $-\lambda \in \sigma(A_{\bar{c}\bar{o}})$. Therefore, if $\lambda \in \sigma(A_{\bar{c}\bar{o}})$, then $-\lambda, \lambda^*, -\lambda^* \in \sigma(A_{\bar{c}\bar{o}})$. □

In the quantum linear system (11), if we ignore the $\bar{c}\bar{o}$ and co subsystems, we obtain the following subsystem

$$\begin{aligned}\begin{bmatrix} \dot{\boldsymbol{q}}_h(t) \\ \dot{\boldsymbol{p}}_h(t) \end{bmatrix} &= \begin{bmatrix} A_h^{11} & A_h^{12} \\ 0 & A_h^{22} \end{bmatrix}\begin{bmatrix} \boldsymbol{q}_h(t) \\ \boldsymbol{p}_h(t) \end{bmatrix} \\ &\quad + \begin{bmatrix} B_h \\ 0 \end{bmatrix}\boldsymbol{u}(t), \\ \boldsymbol{y}(t) &= \begin{bmatrix} 0 & C_h \end{bmatrix}\begin{bmatrix} \boldsymbol{q}_h(t) \\ \boldsymbol{p}_h(t) \end{bmatrix} + \boldsymbol{u}(t).\end{aligned} \tag{21}$$

In this paper, the system (21) is called the "h" subsystem.

Proposition 3.2: For the "h" subsystem (21), we have

1) The set of the poles is given by

$$\sigma\left(A_h^{11}\right) \cup \sigma\left(-A_h^{11}\right). \tag{22}$$

2) If λ is a pole of this subsystem, then so are $-\lambda, \lambda^*, -\lambda^*$.

Proof. Eq. (22) in Item 1) is an immediate consequence of Eq. (18a), while Item 2) follows Item 1). □

If $C_+ = 0$ and $\Omega_+ = 0$, the resulting quantum linear system (1) is said to be passive, [45], [46], [12], [43], [10], [47]. In the passive case, the existence of purely imaginary poles is equivalent to the existence of the $\bar{c}\bar{o}$ subsystem. This has been proved in [47, Theorem 3.2]. In the general case, by Proposition 3.2, the poles of the "h" subsystem are symmetric about the real and imaginary axes. However, this spectral property does not guarantee the existence of an "h" subsystem. This is shown by the following counter-example.

Example 3.1: Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 2 & -1 \end{bmatrix}.$$

It is easy to see that the two poles of the system (A, B, C) are -1 and 1, which are symmetric about the real and imaginary axes. Moreover, this system is physically realizable as it satisfies Eqs. (14a)-(14b). However, this system is both controllable and observable. Thus it is not an "h" subsystem.

Finally, we look at the the co subsystem by ignoring the other modes, which is

$$\begin{aligned}\dot{\boldsymbol{x}}_{co}(t) &= A_{co}\boldsymbol{x}_{co}(t) + B_{co}\boldsymbol{u}(t), \\ \boldsymbol{y}(t) &= C_{co}\boldsymbol{x}_{co}(t) + \boldsymbol{u}(t).\end{aligned} \tag{23}$$

In general, the poles of a co subsystem are not symmetric about the real and imaginary axes. For example, let $n = m = 1$. $\Omega_- = C_+ = 0$, and $C_- = \Omega_+ = 1$. It is easy to see that the resulting quantum linear system is both controllable and observable; in other words, it is a co system. However, the poles of this system are $1/2$ and $-3/2$, which are not symmetric about the real and imaginary axes.

The spectral structure of a quantum linear system established above implies the following result.

Theorem 3.2: If a quantum linear system is Hurwitz stable, then it is both controllable and observable.

Proof. Assume a given quantum linear system is Hurwitz stable; i.e., all its poles are on the open left-half plane. Without loss of generality, suppose the system is in the Kalman canonical form. According to Eq. (13), the poles of the system are those of the $\bar{c}\bar{o}$, co, and "h" subsystems. However, by Propositions 3.1 and 3.2, there are no the $\bar{c}\bar{o}$ and "h" subsystems. In other words, in this case, the only subsystem in the Kalman canonical form is the co subsystem. This means that the quantum linear system is both controllable and observable. □

Theorem 3.2 tells us that Hurwitz stability implies controllability and observability; in general the converse is not true, as shown by Example 3.1 above. However, for the passive case, Theorem 3.2 can be strengthened to the following interesting result.

Corollary 3.1: For a passive quantum linear system, the properties of Hurwitz stability, controllability and observability are all equivalent.

In fact, Corollary 3.1 has already been proved in [10, Lemma 2]. Nevertheless, by Theorem 3.2, Hurwitz stability implies controllability and observability. On the other hand, it has been shown in [12, Lemma 3.1] that controllability and observability imply Hurwitz stability.

We end this section with a final remark.

Remark 3.2: A $2d \times 2d$ real matrix N is said to be a Hamiltonian matrix if the matrix $\mathbb{J}_d N$ is symmetric; see, e.g., [2, Fact 3.19.1]. If a Hamiltonian matrix has λ as an eigenvalue, then $-\lambda, \lambda^*, -\lambda^*$ are also its eigenvalues. Later in Lemma 4.3 and Remark 4.4 we will show that both the matrices $A_{\bar{c}\bar{o}}$ and $\begin{bmatrix} A_h^{11} & 0 \\ 0 & A_h^{22} \end{bmatrix}$ are Hamiltonian matrices, while in general the matrix A_{co} is not.

IV. A NEW PARAMETERIZATION METHOD FOR QUANTUM LINEAR SYSTEMS

In this section, we propose a new parameterization method for quantum linear systems. The proposed parameterization is given in the real quadrature operator representation, and will generate a quantum linear system that is naturally in the Kalman canonical form (11).

A. Basic set-up

Let $\boldsymbol{q}_h, \boldsymbol{p}_h$, $\boldsymbol{x}_{co}$, and $\boldsymbol{x}_{\bar{c}\bar{o}}$ be quadrature operators of a quantum linear system. Here, $\boldsymbol{x}_{co}, \boldsymbol{x}_{\bar{c}\bar{o}}, \boldsymbol{q}_h, \boldsymbol{p}_h$ are vectors of operators of dimensions $2n_1, 2n_2, n_3$, and n_3 respectively, with $n_1, n_2, n_3 \geq 0$. Moreover, these operators are assumed to satisfy the following CCRs:

$$\left[\begin{bmatrix} \boldsymbol{q}_h \\ \boldsymbol{p}_h \end{bmatrix}, \begin{bmatrix} \boldsymbol{q}_h \\ \boldsymbol{p}_h \end{bmatrix}^\top\right] = \imath \mathbb{J}_{n_3}, \tag{24a}$$

$$\left[\boldsymbol{x}_{co}, \boldsymbol{x}_{co}^\top\right] = \imath \mathbb{J}_{n_1}, \tag{24b}$$

$$\left[\boldsymbol{x}_{\bar{c}\bar{o}}, \boldsymbol{x}_{\bar{c}\bar{o}}^\top\right] = \imath \mathbb{J}_{n_2}, \tag{24c}$$

and

$$\begin{aligned} & [\boldsymbol{x}_{co},\ \boldsymbol{x}_{\bar{c}\bar{o}}^\top] = [\boldsymbol{x}_{co},\ \boldsymbol{q}_h^\top] = [\boldsymbol{x}_{co},\ \boldsymbol{p}_h^\top] \\ = \ & [\boldsymbol{x}_{\bar{c}\bar{o}},\ \boldsymbol{q}_h^\top] = [\boldsymbol{x}_{\bar{c}\bar{o}},\ \boldsymbol{p}_h^\top] = 0. \end{aligned}$$

Let $n = n_1 + n_2 + n_3$. That is, the system consists of n quantum harmonic oscillators. Define a vector of operators of dimension $2n$,

$$\bar{\boldsymbol{x}} \triangleq \begin{bmatrix} \boldsymbol{q}_h \\ \boldsymbol{p}_h \\ \boldsymbol{x}_{co} \\ \boldsymbol{x}_{\bar{c}\bar{o}} \end{bmatrix}. \tag{25}$$

Also, let the system Hamiltonian be

$$\boldsymbol{H} = \frac{1}{2}\bar{\boldsymbol{x}}^\top H \bar{\boldsymbol{x}}, \tag{26}$$

where the real matrix $H \in \mathbb{R}^{2n\times 2n}$ is symmetric. Let the coupling operator be

$$\boldsymbol{L} = \Gamma \bar{\boldsymbol{x}}, \tag{27}$$

where $\Gamma \in \mathbb{C}^{m\times 2n}$. Thus, the system has m input channels. We aim to find conditions on the matrices H and Γ such that the resulting QSDEs generated by the system Hamiltonian $\boldsymbol{H}$ in Eq. (26) and the coupling operator $\boldsymbol{L}$ in Eq. (27) are in the Kalman canonical form (11).

B. The necessary condition

In this subsection, we suppose that the parameterization (25)-(27) indeed yields the Kalman canonical form (11), and then find the forms of the parameters involved.

By means of Lemma 3.1 and Theorem 3.1 given in the previous section, we can derive the following result, which presents a necessary condition for the system Hamiltonian $\boldsymbol{H}$ in Eq. (26) and the coupling operator $\boldsymbol{L}$ in Eq. (27) to yield QSDEs in the Kalman canonical form (11).

Lemma 4.1: If the system Hamiltonian $\boldsymbol{H}$ in Eq. (26) and the coupling operator $\boldsymbol{L}$ in Eq. (27) lead to QSDEs in the Kalman canonical form (11), then the real symmetric matrix H must be of the form

$$H = \left[\begin{array}{cc|c|c} 0 & H_h^{12} & 0 & 0 \\ H_h^{12^\top} & H_h^{22} & H_{12} & H_{13} \\ \hline 0 & H_{12}^\top & H_{co} & 0 \\ \hline 0 & H_{13}^\top & 0 & H_{\bar{c}\bar{o}} \end{array}\right], \tag{28}$$

where

$$H_h^{12} = -A_h^{22}, \tag{29a}$$

$$H_h^{22} = A_h^{12} - B_h \mathbb{J}_m B_h^\top/2, \tag{29b}$$

$$H_{12} = A_{12} - B_h \mathbb{J}_m B_{co}^\top \mathbb{J}_{n_1}/2, \tag{29c}$$

$$H_{13} = A_{13}, \tag{29d}$$

$$H_{co} = -\mathbb{J}_{n_1} A_{co} + \mathbb{J}_{n_1} B_{co} \mathbb{J}_m B_{co}^\top \mathbb{J}_{n_1}/2, \tag{29e}$$

$$H_{\bar{c}\bar{o}} = -\mathbb{J}_{n_2} A_{\bar{c}\bar{o}}, \tag{29f}$$

and the complex matrix Γ must satisfy

$$\begin{bmatrix} \Gamma \\ \Gamma^\# \end{bmatrix} = \left[\begin{array}{cc|c|c} 0 & \Gamma_h & \Gamma_{co} & 0 \end{array}\right], \tag{30}$$

where

$$\begin{aligned} \Gamma_h &= V_m^\dagger C_h, \\ \Gamma_{co} &= V_m^\dagger C_{co}. \end{aligned} \tag{31}$$

Proof. Suppose that the system Hamiltonian $\boldsymbol{H}$ in Eq. (26) and the coupling operator $\boldsymbol{L}$ in Eq. (27) indeed lead to QSDEs in the Kalman canonical form (11). However, in the annihilation-creation operator representation, $\boldsymbol{H}$ and $\boldsymbol{L}$ also lead to the QSDEs (1). As shown in Subsection II-B, the QSDEs (1) and the QSDEs (11) are related by the coordinate transformations in Eqs. (10a)-(10b). Thus, by Eqs. (25) and (10a), we have

$$\boldsymbol{H} = \frac{1}{2}\bar{\boldsymbol{x}}^\dagger H \bar{\boldsymbol{x}} = \frac{1}{2}\breve{\boldsymbol{a}}^\dagger \hat{T} H \hat{T}^\dagger \breve{\boldsymbol{a}}.$$

Therefore,

$$H = \hat{T}^\dagger \Omega \hat{T}. \tag{32}$$

On the other hand, from

$$\begin{bmatrix} \boldsymbol{L} \\ \boldsymbol{L}^\# \end{bmatrix} = \begin{bmatrix} \Gamma \\ \Gamma^\# \end{bmatrix} \bar{\boldsymbol{x}} = \begin{bmatrix} \Gamma \\ \Gamma^\# \end{bmatrix} \hat{T}^\dagger \breve{\boldsymbol{a}} = \mathcal{C}\breve{\boldsymbol{a}},$$

we have

$$\begin{bmatrix} \Gamma \\ \Gamma^\# \end{bmatrix} = \mathcal{C}\hat{T} = \mathcal{C} T \tilde{V}_n^\dagger. \tag{33}$$

Substituting Eqs. (3) and (16) into Eq. (32), together with Theorem 3.1, we have

$$\begin{aligned} H &= \frac{\imath}{2}\hat{T}^\dagger \left(J_n \mathcal{A} - \mathcal{A}^\dagger J_n\right)\hat{T} \\ &= \frac{\imath}{2}\left(\hat{T}^\dagger J_n \hat{T}\hat{T}^\dagger \mathcal{A}\hat{T} - \hat{T}^\dagger \mathcal{A}^\dagger \hat{T}\hat{T}^\dagger J_n \hat{T}\right) \\ &= \frac{1}{2}\left(\bar{A}^\top \bar{\mathbb{J}}_n - \bar{\mathbb{J}}_n \bar{A}\right) \\ &= \begin{bmatrix} 0 & -A_h^{22} & 0 & 0 \\ -A_h^{22^\top} & A_h^{12} - B_h \mathbb{J}_m B_h^\top/2 & A_{12} - B_h \mathbb{J}_m B_{co}^\top \mathbb{J}_{n_1}/2 & A_{13} \\ 0 & A_{12}^\top - \mathbb{J}_{n_1} B_{co} \mathbb{J}_m B_h^\top/2 & -\mathbb{J}_{n_1} A_{co} + \mathbb{J}_{n_1} B_{co} \mathbb{J}_m B_{co}^\top \mathbb{J}_{n_1}/2 & 0 \\ 0 & A_{13}^\top & 0 & -\mathbb{J}_{n_2} A_{\bar{c}\bar{o}} \end{bmatrix}, \end{aligned}$$

which yields Eqs. (29a)-(29f). Moreover, by Eqs. (12c) and (33), we get

$$\begin{aligned}\begin{bmatrix}\Gamma\\ \Gamma^{\#}\end{bmatrix} &= \mathcal{C}T\tilde{V}_n^{\dagger} = \bar{\mathcal{C}}\tilde{V}_n^{\dagger}\\ &= V_m^{\dagger}\left[\begin{array}{cc|c|c} 0 & C_h & C_{co} & 0\end{array}\right]\tilde{V}_n\tilde{V}_n^{\dagger}\\ &= \left[\begin{array}{cc|c|c} 0 & V_m^{\dagger}C_h & V_m^{\dagger}C_{co} & 0\end{array}\right],\end{aligned}$$

which yields Eq. (31). □

Remark 4.1: By Eqs. (27) and (30), the matrices Γ_{co} and Γ_h are of the form

$$\Gamma_{co} = \begin{bmatrix}\Gamma_{co,q} & \Gamma_{co,p}\\ \Gamma_{co,q}^{\#} & \Gamma_{co,p}^{\#}\end{bmatrix} \in \mathbb{C}^{2m\times 2n_1}, \tag{34a}$$

and

$$\Gamma_h = \begin{bmatrix}\Gamma_{h,p}\\ \Gamma_{h,p}^{\#}\end{bmatrix} \in \mathbb{C}^{2m\times n_3}, \tag{34b}$$

respectively. Using Eq. (31), it suffices to establish the following result to confirm Eqs. (34a)-(34b) .

Lemma 4.2: The system matrices C_{co} and C_h in Eq. (6) satisfy respectively

$$\begin{aligned}&V_m^{\dagger}C_{co} \qquad\qquad (35a)\\ &= \frac{1}{\sqrt{2}}\begin{bmatrix} C_-Z_1 + C_+Z_1^{\#} & \imath(C_-Z_1 - C_+Z_1^{\#})\\ (C_-Z_1 + C_+Z_1^{\#})^{\#} & (\imath(C_-Z_1 - C_+Z_1^{\#}))^{\#}\end{bmatrix},\end{aligned}$$

and

$$V_m^{\dagger}C_h = \sqrt{2}\begin{bmatrix}\imath C_-Z_3\Theta^{\#}\\ (\imath C_-Z_3\Theta^{\#})^{\#}\end{bmatrix}, \tag{35b}$$

where

$$\Theta \triangleq \begin{bmatrix} I_{n_a} & 0\\ 0 & \imath I_{n_b}\end{bmatrix}.$$

Proof. By Eqs. (4) and (6),

$$\begin{aligned}\mathcal{C}_h &= \Delta(C_-Z_3,\ C_+Z_3^{\#}), & (36a)\\ \mathcal{C}_{co} &= \Delta(C_-Z_1,\ C_+Z_1^{\#}). & (36b)\end{aligned}$$

Using Eqs. (6) and (9), we get

$$\begin{aligned}&\left[\begin{array}{cc|c|c} 0 & C_h & C_{co} & 0\end{array}\right]\\ &= \bar{C} = V_m\mathcal{C}\hat{T} = V_m\mathcal{C}T\tilde{V}_n^{\dagger} = V_m\left[\begin{array}{c|c|c}\mathcal{C}_h & \mathcal{C}_{co} & 0\end{array}\right]\tilde{V}_n^{\dagger}\\ &= V_m\left[\begin{array}{c|c|c}\mathcal{C}_h & \mathcal{C}_{co} & 0\end{array}\right]\begin{bmatrix}\tilde{V}_{n_3}^{\dagger} & & \mathbf{0}\\ & V_{n_1}^{\dagger} & \\ \mathbf{0} & & V_{n_2}^{\dagger}\end{bmatrix}.\end{aligned}$$

Therefore,

$$\begin{aligned}\begin{bmatrix} 0 & V_m^{\dagger}C_h\end{bmatrix} &= \mathcal{C}_h\tilde{V}_{n_3}^{\dagger}, & (37a)\\ V_m^{\dagger}C_{co} &= \mathcal{C}_{co}V_{n_1}^{\dagger}. & (37b)\end{aligned}$$

Substituting Eq. (36b) into Eq. (37b) yields

$$\begin{aligned}&V_m^{\dagger}C_{co}\\ &= \frac{1}{\sqrt{2}}\begin{bmatrix} C_-Z_1 + C_+Z_1^{\#} & \imath(C_-Z_1 - C_+Z_1^{\#})\\ (C_-Z_1 + C_+Z_1^{\#})^{\#} & (\imath(C_-Z_1 - C_+Z_1^{\#}))^{\#}\end{bmatrix},\end{aligned}$$

which gives Eq. (35a). On the other hand, the matrix $\tilde{V}_{n_3}^{\dagger}$ defined in (7) can be re-written as

$$\tilde{V}_{n_3} = \Pi V_{n_3} = \frac{1}{\sqrt{2}}\begin{bmatrix}\Theta & \Theta^{\dagger}\\ -\imath\Theta & \imath\Theta^{\dagger}\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}\Theta & \Theta^{\#}\\ -\imath\Theta & \imath\Theta^{\#}\end{bmatrix}.$$

Substitution of Eq. (36a) into Eq. (37a) gives

$$\begin{aligned}&\begin{bmatrix} 0 & V_m^{\dagger}C_h\end{bmatrix} \qquad\qquad (38)\\ =&\frac{1}{\sqrt{2}}\begin{bmatrix} C_-Z_3\Theta^{\#} + C_+Z_3^{\#}\Theta & \imath(C_-Z_3\Theta^{\#} - C_+Z_3^{\#}\Theta)\\ C_+^{\#}Z_3\Theta^{\#} + C_-^{\#}Z_3^{\#}\Theta & -\imath(C_-^{\#}Z_3^{\#}\Theta - C_+^{\#}Z\Theta^{\#})\end{bmatrix}.\end{aligned}$$

Therefore,

$$\begin{aligned}C_-Z_3\Theta^{\#} + C_+Z_3^{\#}\Theta &= 0,\\ C_-Z_3\Theta^{\#} - C_+Z_3^{\#}\Theta &= 2C_-Z_3\Theta^{\#}. \qquad (39)\end{aligned}$$

Substituting Eq. (39) into Eq. (38) gives Eq. (35b). □

C. The sufficient condition

We have shown in Lemma 4.1 that if the system Hamiltonian $\boldsymbol{H}$ in Eq. (26) and the coupling operator $\boldsymbol{L}$ in Eq. (27) generate QSDEs in the Kalman canonical form (11), then the real symmetric matrix H has the form (28) and the matrix Γ satisfies (30). In this subsection, we establish the converse result.

Lemma 4.3: If the real symmetric matrix H for the system Hamiltonian (26) is of the form (28) and the complex matrix Γ for the coupling operator (27) satisfies Eq. (30), then the QSDEs generated are of the form (11), with the matrices $\bar{A}$, $\bar{B}$ and $\bar{C}$ in Eqs. (12a)-(12c) given by

$$\begin{aligned}A_h^{11} &= H_h^{12^{\top}}, & (40a)\\ A_h^{12} &= H_h^{22} - \imath\Gamma_h^{\dagger}J_m\Gamma_h/2, & (40b)\\ A_h^{22} &= -H_h^{12}, & (40c)\\ A_{12} &= H_{12} - \imath\Gamma_h^{\dagger}J_m\Gamma_{co}/2, & (40d)\\ A_{13} &= H_{13}, & (40e)\\ A_{co} &= \mathbb{J}_{n_1}H_{co} - \imath\mathbb{J}_{n_1}\Gamma_{co}^{\dagger}J_m\Gamma_{co}/2, & (40f)\\ A_{\bar{c}\bar{o}} &= \mathbb{J}_{n_2}H_{\bar{c}\bar{o}}, & (40g)\\ A_{21} &= \mathbb{J}_{n_1}H_{12}^{\top} - \imath\mathbb{J}_{n_1}\Gamma_{co}^{\dagger}J_m\Gamma_h/2, & (40h)\\ A_{31} &= \mathbb{J}_{n_2}H_{13}^{\top}, & (40i)\\ B_h &= \Gamma_h^{\dagger}V_m^{\dagger}\mathbb{J}_m, & (40j)\\ B_{co} &= \mathbb{J}_{n_1}\Gamma_{co}^{\dagger}V_m^{\dagger}\mathbb{J}_m, & (40k)\\ C_h &= V_m\Gamma_h, & (40l)\\ C_{co} &= V_m\Gamma_{co}. & (40m)\end{aligned}$$

Proof. As there are m input fields, we write the coupling operator $\boldsymbol{L}$ as $\boldsymbol{L} = [\boldsymbol{L}_1\ \cdots\ \boldsymbol{L}_m]^{\top}$. Given the system Hamiltonian $\boldsymbol{H}$ and coupling operator $\boldsymbol{L}$, the temporal evolution of a system variable $\boldsymbol{X}$ is given by, ([7], [14], [9]),

$$\begin{aligned}&d\boldsymbol{X}(t)\\ =&-\imath[\boldsymbol{X}(t),\boldsymbol{H}(t)]dt + \frac{1}{2}\sum_{j=1}^{m}\boldsymbol{L}_j(t)^*[\boldsymbol{X}(t),\ \boldsymbol{L}_j(t)]dt\\ &+\frac{1}{2}\sum_{j=1}^{m}[\boldsymbol{L}_j(t)^*,\boldsymbol{X}(t)]\boldsymbol{L}_j(t)dt\\ &+\sum_{j=1}^{m}d\boldsymbol{B}_j(t)^*[\boldsymbol{X}(t),\ \boldsymbol{L}_j(t)] + \sum_{j=1}^{m}[\boldsymbol{L}_j(t)^*,\ \boldsymbol{X}(t)]d\boldsymbol{B}_j(t),\end{aligned}$$

where $\boldsymbol{B}_j(t) \equiv \int_0^t \boldsymbol{b}_j(\tau)d\tau$ are quantum Wiener processes $(j = 1,\ldots,m)$. When $\boldsymbol{X}$ is one of the elements of $\bar{\boldsymbol{x}}$ and $\boldsymbol{L}$

is of the form (27), then the above equation can be re-written in a more compact form as

$$\begin{aligned}&d\boldsymbol{X}(t)\\=&-\imath[\boldsymbol{X}(t),\boldsymbol{H}(t)]dt\\&-\frac{1}{2}\boldsymbol{L}(t)^{\dagger}[\boldsymbol{L}(t),\boldsymbol{X}(t)]dt+\frac{1}{2}\boldsymbol{L}(t)^{\top}[\boldsymbol{L}(t)^{\#},\boldsymbol{X}(t)]dt\\&-d\boldsymbol{B}(t)^{\dagger}[\boldsymbol{L}(t),\boldsymbol{X}(t)]+d\boldsymbol{B}(t)^{\top}[\boldsymbol{L}(t)^{\#},\boldsymbol{X}(t)]\\=&-\imath[\boldsymbol{X}(t),\boldsymbol{H}(t)]dt-\frac{1}{2}\breve{\boldsymbol{L}}(t)^{\dagger}J_m[\breve{\boldsymbol{L}}(t),\boldsymbol{X}(t)]dt\\&-d\breve{\boldsymbol{B}}(t)^{\dagger}J_m[\breve{\boldsymbol{L}}(t),\boldsymbol{X}(t)].\end{aligned}\tag{41}$$

Informally, Eq. (41) can be re-written as

$$\begin{aligned}\dot{\boldsymbol{X}}(t)&=-\imath[\boldsymbol{X}(t),\boldsymbol{H}(t)]-\frac{1}{2}\breve{\boldsymbol{L}}(t)^{\dagger}J_m[\breve{\boldsymbol{L}}(t),\ \boldsymbol{X}(t)]\\&\quad-\breve{\boldsymbol{b}}(t)^{\dagger}J_m[\breve{\boldsymbol{L}}(t),\ \boldsymbol{X}(t)].\end{aligned}\tag{42}$$

It should be noted that Eq. (42) should be understood as (41). Using the coordinate transformation (10b), Eq. (42) becomes

$$\begin{aligned}\dot{\boldsymbol{X}}(t)&=-\imath[\boldsymbol{X}(t),\boldsymbol{H}(t)]-\frac{1}{2}\breve{\boldsymbol{L}}(t)^{\dagger}J_m[\breve{\boldsymbol{L}}(t),\ \boldsymbol{X}(t)]\\&\quad-\boldsymbol{u}(t)^{\top}V_mJ_m[\breve{\boldsymbol{L}}(t),\ \boldsymbol{X}(t)].\end{aligned}\tag{43}$$

Substituting the elements of $\bar{\boldsymbol{x}}$ into Eq. (43) and transposing both sides of the resulting equation, we have

$$\begin{aligned}\dot{\bar{\boldsymbol{x}}}(t)^{\top}&=-\imath[\bar{\boldsymbol{x}}(t)^{\top},\ \boldsymbol{H}(t)]-\frac{1}{2}\breve{\boldsymbol{L}}(t)^{\dagger}J_m[\breve{\boldsymbol{L}}(t),\ \bar{\boldsymbol{x}}(t)^{\top}]\\&\quad-\boldsymbol{u}(t)^{\top}V_mJ_m[\breve{\boldsymbol{L}}(t),\ \bar{\boldsymbol{x}}(t)^{\top}].\end{aligned}\tag{44}$$

After system-field interaction, the output fields

$$\breve{\boldsymbol{b}}_{\rm out}(t)=\breve{\boldsymbol{L}}(t)+\breve{\boldsymbol{b}}(t),$$

are generated, which, by the coordinate transformation Eq. (10b), in the real quadrature operator representation are

$$\boldsymbol{y}(t)=V_m\breve{\boldsymbol{L}}(t)+\boldsymbol{u}(t).\tag{45}$$

Given the matrix H in Eq. (28), the Hamiltonian $\boldsymbol{H}$ in Eq. (26) can be re-written as

$$\begin{aligned}\boldsymbol{H}=&\frac{1}{2}\boldsymbol{q}_h^{\top}H_h^{12}\boldsymbol{p}_h+\frac{1}{2}\boldsymbol{p}_h^{\top}H_h^{12^{\top}}\boldsymbol{q}_h+\frac{1}{2}\boldsymbol{p}_h^{\top}H_h^{22}\boldsymbol{p}_h\\&+\frac{1}{2}\boldsymbol{p}_h^{\top}H_{12}\boldsymbol{x}_{co}+\frac{1}{2}\boldsymbol{x}_{co}^{\top}H_{12}^{\top}\boldsymbol{p}_h\\&+\frac{1}{2}\boldsymbol{x}_{co}^{\top}H_{co}\boldsymbol{x}_{co}+\frac{1}{2}\boldsymbol{x}_{\bar{c}\bar{o}}^{\top}H_{\bar{c}\bar{o}}\boldsymbol{x}_{\bar{c}\bar{o}}\\&+\frac{1}{2}\boldsymbol{p}_h^{\top}H_{13}\boldsymbol{x}_{\bar{c}\bar{o}}+\frac{1}{2}\boldsymbol{x}_{\bar{c}\bar{o}}^{\top}H_{13}^{\top}\boldsymbol{p}_h.\end{aligned}\tag{46}$$

After standard, although tedious calculation, one can obtain

$$\begin{aligned}[\boldsymbol{q}_h,\boldsymbol{H}]&=\imath(H_h^{12^{\top}}\boldsymbol{q}_h+H_h^{22}\boldsymbol{p}_h+H_{12}\boldsymbol{x}_{co}+H_{13}\boldsymbol{x}_{\bar{c}\bar{o}}),\\ [\boldsymbol{p}_h,\boldsymbol{H}]&=-\imath H_h^{12}\boldsymbol{p}_h,\\ [\boldsymbol{x}_{co},\boldsymbol{H}]&=\imath\left(\mathbb{J}_{n_1}H_{12}^{\top}\boldsymbol{p}_h+\mathbb{J}_{n_1}H_{co}\boldsymbol{x}_{co}\right),\\ [\boldsymbol{x}_{\bar{c}\bar{o}},\boldsymbol{H}]&=\imath\mathbb{J}_{n_2}H_{13}^{\top}\boldsymbol{p}_h+\imath\mathbb{J}_{n_2}H_{\bar{c}\bar{o}}\boldsymbol{x}_{\bar{c}\bar{o}}.\end{aligned}$$

The above equations can be written in a more compact form

$$-\imath[\bar{\boldsymbol{x}},\boldsymbol{H}]=\begin{bmatrix}H_h^{12^{\top}}&H_h^{22}&H_{12}&H_{13}\\0&-H_h^{12}&0&0\\0&\mathbb{J}_{n_1}H_{12}^{\top}&\mathbb{J}_{n_1}H_{co}&0\\0&\mathbb{J}_{n_2}H_{13}^{\top}&0&\mathbb{J}_{n_2}H_{\bar{c}\bar{o}}\end{bmatrix}\bar{\boldsymbol{x}}.\tag{47}$$

Similarly, for the complex matrix Γ satisfying (30), we get

$$\begin{aligned}&[\boldsymbol{q}_h,\breve{\boldsymbol{L}}^{\dagger}]=\imath\Gamma_h^{\dagger},\quad[\boldsymbol{p}_h,\breve{\boldsymbol{L}}^{\dagger}]=0,\\&[\boldsymbol{x}_{co},\breve{\boldsymbol{L}}^{\dagger}]=\imath\mathbb{J}_{n_1}\Gamma_{co}^{\dagger},\quad[\boldsymbol{x}_{\bar{c}\bar{o}},\breve{\boldsymbol{L}}^{\dagger}]=0.\end{aligned}$$

Consequently,

$$[\bar{\boldsymbol{x}},\breve{\boldsymbol{L}}^{\dagger}]=\imath\begin{bmatrix}\Gamma_h^{\dagger}\\0\\\mathbb{J}_{n_1}\Gamma_{co}^{\dagger}\\0\end{bmatrix}.\tag{48}$$

Eq. (48) can be re-written as

$$[\breve{\boldsymbol{L}}(t),\bar{\boldsymbol{x}}(t)^{\top}]=[\breve{\boldsymbol{L}}(t),\bar{\boldsymbol{x}}(t)^{\dagger}]=-\imath\left[\Gamma_h\ \ 0\ \ -\Gamma_{co}\mathbb{J}_{n_1}\ \ 0\right].\tag{49}$$

By Eqs. (44), (47) and (49), we get

$$\begin{aligned}\dot{\bar{\boldsymbol{x}}}(t)=&(\dot{\bar{\boldsymbol{x}}}(t)^{\top})^{\dagger}\\=&\begin{bmatrix}H_h^{12^{\top}}&H_h^{22}&H_{12}&H_{13}\\0&-H_h^{12}&0&0\\0&\mathbb{J}_{n_1}H_{12}^{\top}&\mathbb{J}_{n_1}H_{co}&0\\0&\mathbb{J}_{n_2}H_{13}^{\top}&0&\mathbb{J}_{n_2}H_{\bar{c}\bar{o}}\end{bmatrix}\bar{\boldsymbol{x}}(t)\\&-\frac{\imath}{2}\begin{bmatrix}0&\Gamma_h^{\dagger}J_m\Gamma_h&\Gamma_h^{\dagger}J_m\Gamma_{co}&0\\0&0&0&0\\0&\mathbb{J}_{n_1}\Gamma_{co}^{\dagger}J_m\Gamma_h&\mathbb{J}_{n_1}\Gamma_{co}^{\dagger}J_m\Gamma_{co}&0\\0&0&0&0\end{bmatrix}\bar{\boldsymbol{x}}(t)\\&+\begin{bmatrix}\Gamma_h^{\dagger}V_m^{\dagger}\mathbb{J}_m\\0\\\mathbb{J}_{n_1}\Gamma_{co}^{\dagger}V_m^{\dagger}\mathbb{J}_m\\0\end{bmatrix}\boldsymbol{u}(t),\end{aligned}$$

from which Eqs. (40a)-(40k) follow. Finally, by Eq. (30),

$$V_m\breve{\boldsymbol{L}}=V_m(\Gamma_h\boldsymbol{p}_h+\Gamma_{co}\boldsymbol{x}_{co}).\tag{50}$$

Substituting Eq. (50) into Eq. (45) yields Eqs. (40l)-(40m). □

Remark 4.2: By the form of Γ_{co} and Γ_h in Eqs. (34a)-(34b) respectively, it can be readily shown that the matrices A_h^{12}, A_{12}, A_{co}, A_{21}, B_h, and B_{co} in Lemma 4.3 are all real matrices.

Remark 4.3: Notice that the system Hamiltonian $\boldsymbol{H}$ in Eq. (26) can be written as that in Eq. (46). On the other hand, by Eqs. (30), (34a), and (34b), the coupling operator operator $\boldsymbol{L}$ in Eq. (27) can be re-written as

$$\boldsymbol{L}=\Gamma_{h,p}\boldsymbol{p}_h+[\Gamma_{co,q}\ \Gamma_{co,p}]\boldsymbol{x}_{co}.\tag{51}$$

Remark 4.4: By Eq. (40g), $\mathbb{J}_{n_2}A_{\bar{c}\bar{o}}=-H_{\bar{c}\bar{o}}$ is symmetric. Thus, $A_{\bar{c}\bar{o}}$ is a Hamiltonian matrix. Similarly, by Eqs. (40a) and (40c), the matrix

$$\mathbb{J}_{n_3}\begin{bmatrix}A_h^{11}&0\\0&A_h^{22}\end{bmatrix}=-\begin{bmatrix}0&H_h^{12}\\H_h^{12^{\top}}&0\end{bmatrix}$$

is symmetric. Therefore, the matrix $\begin{bmatrix}A_h^{11}&0\\0&A_h^{22}\end{bmatrix}$ is also a Hamiltonian matrix. On the other hand, if the matrix A_{co} in Eq. (40f) is a Hamiltonian matrix, then $\Gamma_{co}^{\dagger}J_m\Gamma_{co}=\Gamma_{co}^{\top}J_m\Gamma_{co}^{\#}$ must hold. However, by Eq. (34a) it can be readily shown that $\Gamma_{co}^{\dagger}J_m\Gamma_{co}+\Gamma_{co}^{\top}J_m\Gamma_{co}^{\#}=0$. Therefore, in general the matrix A_{co} is not a Hamiltonian matrix. This remark, together with Remark 3.2, describes the spectral structure of the co, $\bar{c}\bar{o}$, and "h" subsystems.

D. Controllability and observability

Given the real symmetric matrix H in Eq. (28) and complex matrix Γ satisfying Eq. (30), Lemma 4.3 provides a way for constructing the system matrices $\bar{A}$, $\bar{B}$ and $\bar{C}$ in the Kalman canonical form. However, to guarantee that the QSDEs (11) are indeed the quantum Kalman canonical form, certain controllability and observability conditions have to be satisfied. This problem is investigated in this subsection.

We first prove the following two results.

Lemma 4.4: For the system (11), the following statements are equivalent.

(i) (A_h^{11}, B_h) is controllable;
(ii) (A_h^{22}, C_h) is observable;
(iii) (H_h^{12}, Γ_h) is observable.

Proof. This result is a consequence of Theorem 3.1 and Lemma 4.3. Notice $(A_h^{11}, B_h) = (-A_h^{22^\top}, \Gamma_h^\dagger V_m^\dagger \mathbb{J}_m) = (H_h^{12^\top}, \Gamma_h^\dagger V_m^\dagger \mathbb{J}_m)$. As a result, (A_h^{11}, B_h) is controllable $\Longleftrightarrow$ $(H_h^{12^\top}, \Gamma_h^\dagger V_m^\dagger \mathbb{J}_m)$ is controllable $\Longleftrightarrow$ $(H_h^{12}, \mathbb{J}_m V_m \Gamma_h)$ is observable $\Longleftrightarrow$ $((H_h^{12}, \Gamma_h)$ is observable. This establishes the equivalence between (i) and (iii). On the other hand, because $(A_h^{22}, C_h) = (-H_h^{12}, V_m\Gamma_h)$, (A_h^{22}, C_h) is observable $\Longleftrightarrow$ $(-H_h^{12}, V_m\Gamma_h)$ is observable $\Longleftrightarrow$ (H_h^{12}, Γ_h) is observable. This establishes the equivalence between (ii) and (iii). □

Lemma 4.5: For the system (11), the following statements are equivalent.

(i) (A_{co}, B_{co}) is controllable;
(ii) (A_{co}, C_{co}) is observable;
(iii) $(\mathbb{J}_{n_1} H_{co}, \Gamma_{co})$ is observable.

Proof. This result can be proved in a similar way as in the proof of Lemma 4.4. Notice that $(A_{co}, B_{co}) = (\mathbb{J}_{n_1} H_{co} - \imath \mathbb{J}_{n_1} \Gamma_{co}^\dagger J_m \Gamma_{co}/2, \mathbb{J}_{n_1} \Gamma_{co}^\dagger V_m^\dagger \mathbb{J}_m)$. Hence, (A_{co}, B_{co}) is controllable $\Longleftrightarrow$ $(\mathbb{J}_{n_1} H_{co} - \imath \mathbb{J}_{n_1} \Gamma_{co}^\dagger J_m \Gamma_{co}/2, \mathbb{J}_{n_1} \Gamma_{co}^\dagger V_m^\dagger \mathbb{J}_m)$ is controllable $\Longleftrightarrow$ $(\mathbb{J}_{n_1} H_{co}, \mathbb{J}_{n_1} \Gamma_{co}^\dagger)$ is controllable. The last statement follows since if $x^\dagger(\mathbb{J}_{n_1} H_{co} - \imath \mathbb{J}_{n_1} \Gamma_{co}^\dagger J_m \Gamma_{co}/2) = \lambda x^\dagger$ and $x^\dagger \mathbb{J}_{n_1} \Gamma_{co}^\dagger V_m^\dagger \mathbb{J}_m = 0$, then $x^\dagger \mathbb{J}_{n_1} \Gamma_{co}^\dagger = 0$. As a result, $x^\dagger \mathbb{J}_{n_1} H_{co} = \lambda x^\dagger$. On the other hand, if $x^\dagger \mathbb{J}_{n_1} H_{co} = \lambda x^\dagger$ and $x^\dagger \mathbb{J}_{n_1} \Gamma_{co}^\dagger = 0$, then $x^\dagger \mathbb{J}_{n_1} \Gamma_{co}^\dagger V_m^\dagger \mathbb{J}_m = 0$ and $x^\dagger(\mathbb{J}_{n_1} H_{co} - \imath \mathbb{J}_{n_1} \Gamma_{co}^\dagger J_m \Gamma_{co}/2) = x^\dagger \mathbb{J}_{n_1} H_{co} = \lambda x^\dagger$. Now $(\mathbb{J}_{n_1} H_{co}, \mathbb{J}_{n_1} \Gamma_{co}^\dagger)$ is controllable $\Longleftrightarrow$ $(H_{co}\mathbb{J}_{n_1}, \Gamma_{co}^\dagger)$ is controllable $\Longleftrightarrow$ $(\mathbb{J}_{n_1} H_{co}, \Gamma_{co})$ is observable. This establishes the equivalence between (i) and (iii). On the other hand, notice that $(A_{co}, C_{co}) = (\mathbb{J}_{n_1} H_{co} - \imath \mathbb{J}_{n_1} \Gamma_{co}^\dagger J_m \Gamma_{co}/2, V_m \Gamma_{co})$. Hence, (A_{co}, C_{co}) is observable $\Longleftrightarrow$ $(\mathbb{J}_{n_1} H_{co} - \imath \mathbb{J}_{n_1} \Gamma_{co}^\dagger J_m \Gamma_{co}/2, V_m \Gamma_{co})$ is observable $\Longleftrightarrow$ $(\mathbb{J}_{n_1} H_{co}, V_m \Gamma_{co})$ is observable. The last statement holds since if $(\mathbb{J}_{n_1} H_{co} - i \mathbb{J}_{n_1} \Gamma_{co}^\dagger J_m \Gamma_{co}/2)x = \lambda x$, and $V_m \Gamma_{co} x = 0$, then $\Gamma_{co} x = 0$ and $\mathbb{J}_{n_1} H_{co} x = \lambda x$. On the other hand, if $\Gamma_{co} x = 0$ and $\mathbb{J}_{n_1} H_{co} x = \lambda x$, then $V_m \Gamma_{co} x = 0$ and $(\mathbb{J}_{n_1} H_{co} - \imath \mathbb{J}_{n_1} \Gamma_{co}^\dagger J_m \Gamma_{co}/2)x = \mathbb{J}_{n_1} H_{co} x = \lambda x$. Now $(\mathbb{J}_{n_1} H_{co}, V_m \Gamma_{co})$ is observable $\Longleftrightarrow$ $(\mathbb{J}_{n_1} H_{co}, \Gamma_{co})$ is observable. This establishes the equivalence between (ii) and (iii). □

Combining Lemma 4.1 and Lemmas 4.3-4.5, we obtain the main result of this subsection.

Theorem 4.1: Suppose that the real matrix H in Eq. (28) and complex matrix Γ in Eq. (30) satisfy the following conditions:

(i) $H_h^{22} = H_h^{22^\top}$, $H_{\rm co} = H_{\rm co}^\top$, and $H_{\bar{\rm c}\bar{\rm o}} = H_{\bar{\rm c}\bar{\rm o}}^\top$;
(ii) (H_h^{12}, Γ_h) is observable;
(iii) $(\mathbb{J}_{n_1} H_{co}, \Gamma_{co})$ is observable.

Then the resulting QSDEs (11) are in the Kalman canonical form. In other words, $\boldsymbol{x}_{co}$ is both controllable and observable, $\boldsymbol{x}_{\bar{c}\bar{o}}$ is neither controllable nor observable, $\boldsymbol{q}_h$ is controllable and unobservable, and $\boldsymbol{p}_h$ is uncontrollable and observable. Conversely, if the system Hamiltonian $\boldsymbol{H}$ in Eq. (26) and the coupling operator $\boldsymbol{L}$ in Eq. (27) generate the QSDEs (11) in the Kalman canonical form, then the conditions (28), (30), and (i)-(iii) above must be satisfied.

Remark 4.5: In the Kalman canonical form (11), $\boldsymbol{q}_h$ is controllable but unobservable, while $\boldsymbol{p}_h$ is observable but uncontrollable. Therefore, $\boldsymbol{p}_h$ is a vector of QND variables, [38], [33], [42], [43], [47]. Moreover, as shown in Lemma 4.4, the observability of (A_h^{22}, C_h) is equivalent to the controllability of (A_h^{11}, B_h) and both of them are equivalent to the observability of (H_h^{12}, Γ_h). In fact, according to Lemma 4.3, the matrix pair (H_h^{12}, Γ_h) determines the "h" subsystem whose quadratures $\boldsymbol{p}_h$ are QND variables. Therefore, the existence of an observable pair (H_h^{12}, Γ_h) generates QND variables for the whole quantum linear system.

V. Applications

In this section, we apply the theory proposed in the previous sections to some problems arising in quantum information science, in particular noiseless subsystems and invariant subsystems, and the realization of BAE measurements.

A. Noiseless subsystems and invariant subsystems

It follows from the form of the matrix H in (28) that the "h" subsystem (21), in general, interacts with the "co" subsystem (23) and "$\bar{c}\bar{o}$" subsystem (20) via the sub-matrices H_{12} and H_{13}, respectively. As far as the Kalman canonical form is concerned, the sub-matrices H_{12} and H_{13} are free parameters. In this subsection, we explore the structures of these and other matrices to further reveal the relationships among "h", "co" and "$\bar{c}\bar{o}$" subsystems. More specifically, we reveal noiseless and invariant subsystems of a quantum linear system. To this end, we first introduce the following concept for quantum linear systems.

Definition 5.1: If a quantum linear system G can be written in a concatenation form[1] $G = G_1 \boxplus G_2$, then we say that G_1 and G_2 are *invariant* subsystems of G. Moreover, an invariant subsystem is called a *noiseless* subsystem if it is isolated from the environment.

Linear as well as finite-level noiseless systems and invariant systems have been studied in, e.g., [34], [42], [26], [47].

In terms of the (H, Γ) representation in Eqs. (28) and (30), we are in a position to construct the noiseless subsystem and

[1]Given two open quantum systems $G_1 \triangleq (\boldsymbol{S}_1, \boldsymbol{L}_1, \boldsymbol{H}_1)$ and $G_2 \triangleq (\boldsymbol{S}_2, \boldsymbol{L}_2, \boldsymbol{H}_2)$, their concatenation product is defined to be $G_1 \boxplus G_2 \triangleq \left(\begin{bmatrix} \boldsymbol{S}_1 & 0 \\ 0 & \boldsymbol{S}_2 \end{bmatrix}, \begin{bmatrix} \boldsymbol{L}_1 \\ \boldsymbol{L}_2 \end{bmatrix}, \boldsymbol{H}_1 + \boldsymbol{H}_2 \right)$. See [8] for more details.

the invariant subsystems arising in the quantum linear system (11). To begin with, let us consider the noiseless subsystem.

Lemma 5.1: The "$\bar{c}\bar{o}$" subsystem of the quantum linear system (11) has a noiseless subsystem $G_{\bar{c}\bar{o}}$ if there exists an orthogonal and blockwise symplectic matrix $\mathcal{P}_{\bar{c}\bar{o}}$ such that

$$\mathcal{P}_{\bar{c}\bar{o}}\boldsymbol{x}_{\bar{c}\bar{o}} = \begin{bmatrix} \boldsymbol{x}_{\bar{c}\bar{o}1} \\ \boldsymbol{x}_{\bar{c}\bar{o}2} \end{bmatrix}, \ \mathcal{P}_{\bar{c}\bar{o}}H_{\bar{c}\bar{o}}\mathcal{P}_{\bar{c}\bar{o}}^\top = \begin{bmatrix} H_{\bar{c}\bar{o}1} & 0 \\ 0 & H_{\bar{c}\bar{o}2} \end{bmatrix}, \tag{52a}$$

$$H_{13}\mathcal{P}_{\bar{c}\bar{o}}^\top = \begin{bmatrix} H_{131} & 0 \end{bmatrix}, \tag{52b}$$

where the system variables $\boldsymbol{x}_{\bar{c}\bar{o}1}$ and $\boldsymbol{x}_{\bar{c}\bar{o}2}$ satisfy the following CCRs

(A1) $$\left[\begin{bmatrix} \boldsymbol{x}_{\bar{c}\bar{o}1} \\ \boldsymbol{x}_{\bar{c}\bar{o}2} \end{bmatrix}, \begin{bmatrix} \boldsymbol{x}_{\bar{c}\bar{o}1} \\ \boldsymbol{x}_{\bar{c}\bar{o}2} \end{bmatrix}^\top \right] = \imath \begin{bmatrix} \mathbb{J}_{n_2-n_4} & 0 \\ 0 & \mathbb{J}_{n_4} \end{bmatrix}$$

with $n_4 > 0$. In this case, the noiseless subsystem $G_{\bar{c}\bar{o}}$ is given by

$$\dot{\boldsymbol{x}}_{\bar{c}\bar{o}2}(t) = \mathbb{J}_{n_4}H_{\bar{c}\bar{o}2}\boldsymbol{x}_{\bar{c}\bar{o}2}(t). \tag{53}$$

Remark 5.1: According to Eq. (24c), the system variable $\boldsymbol{x}_{\bar{c}\bar{o}}$ satisfies the CCRs. In order to construct a noiseless subsystem which is itself a quantum-mechanical system, the entries of $\boldsymbol{x}_{\bar{c}\bar{o}}$ need to be combined in an appropriate way, as has been done by the first equation in (52a). Moreover, condition (A1) in Lemma 5.1 gives the CCRs for the physical quantities $\boldsymbol{x}_{\bar{c}\bar{o}1}$ and $\boldsymbol{x}_{\bar{c}\bar{o}2}$. Finally, it can be readily seen from Eqs. (52a)-(52b) that the noiseless subsystem is indeed the one in Eq. (53).

Compared with noiseless subsystems, general invariant subsystems are more complicated as the interaction between the quantum subsystem and the fields also needs to be considered. To make this clearer, we will study these invariant subsystems contained in the quantum linear system (11) in the sequel.

Lemma 5.2: The "co" subsystem of the quantum linear system (11) has an invariant subsystem G_{co} if there exists an orthogonal and blockwise symplectic matrix $\mathcal{P}_{co}$ such that

$$\mathcal{P}_{co}\boldsymbol{x}_{co} = \begin{bmatrix} \boldsymbol{x}_{co1} \\ \boldsymbol{x}_{co2} \end{bmatrix}, \ \Gamma_{co}\mathcal{P}_{co}^\top = \begin{bmatrix} \Gamma_{co1} & \Gamma_{co2} \end{bmatrix}, \quad \mathcal{P}_{co}H_{co}\mathcal{P}_{co}^\top = \begin{bmatrix} H_{co1} & 0 \\ 0 & H_{co2} \end{bmatrix}, \tag{54a}$$

$$H_{12}\mathcal{P}_{co}^\top = \begin{bmatrix} H_{121} & 0 \end{bmatrix}, \tag{54b}$$

where the system variables $\boldsymbol{x}_{co1}$ and $\boldsymbol{x}_{co2}$ satisfy the condition

(B1) $$\left[\begin{bmatrix} \boldsymbol{x}_{co1} \\ \boldsymbol{x}_{co2} \end{bmatrix}, \begin{bmatrix} \boldsymbol{x}_{co1} \\ \boldsymbol{x}_{co2} \end{bmatrix}^\top \right] = \imath \begin{bmatrix} \mathbb{J}_{n_1-n_5} & 0 \\ 0 & \mathbb{J}_{n_5} \end{bmatrix}$$

with $n_5 > 0$;

and the constant matrices Γ_{co1} and Γ_{co2} satisfy the condition

(B2) each row of $\begin{bmatrix} \Gamma_h & \Gamma_{co1} \mid \Gamma_{co2} \end{bmatrix}$ is in the form of either

$$\begin{bmatrix} \Gamma_h^i & \Gamma_{co1}^i \mid 0 \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} 0 & 0 \mid \Gamma_{co2}^i \end{bmatrix},$$

where Γ_h^i, Γ_{co1}^i and Γ_{co2}^i, $i = 1, \cdots, 2m$, are the ith rows of the matrices Γ_h, Γ_{co1} and Γ_{co2}.

Denote the set of indices of nonzero rows of Γ_{co2} by $\mathbb{I}_{co} = \{i_1, \cdots, i_{2m_1}\}$, where $m_1 \leq m$, and $1 \leq i_1 < \cdots < i_{2m_1} \leq 2m$. Define

$$\hat{\Gamma}_{co2} \triangleq \begin{bmatrix} \Gamma_{co2}^{i_1} \\ \vdots \\ \Gamma_{co2}^{i_{2m_1}} \end{bmatrix}, \ \boldsymbol{y}_{co}(t) \triangleq \begin{bmatrix} \boldsymbol{y}_{i_1}(t) \\ \vdots \\ \boldsymbol{y}_{i_{2m_1}}(t) \end{bmatrix},$$
$$\boldsymbol{u}_{co}(t) \triangleq \begin{bmatrix} \boldsymbol{u}_{i_1}(t) & \cdots & \boldsymbol{u}_{i_{2m_1}}(t) \end{bmatrix},$$

where $\boldsymbol{u}_i(t)$ and $\boldsymbol{y}_i(t)$ are the ith column and ith row of $\boldsymbol{u}(t)$ and $\boldsymbol{y}(t)$, respectively. In this case, the invariant controllable and observable subsystem G_{co} is given by

$$\begin{aligned} \dot{\boldsymbol{x}}_{co2}(t) &= (\mathbb{J}_{n_5}H_{co2} - \frac{\imath}{2}\mathbb{J}_{n_5}\hat{\Gamma}_{co2}^\dagger J_{m_1}\hat{\Gamma}_{co2})\boldsymbol{x}_{co2}(t) \\ &\quad + \mathbb{J}_{n_5}\hat{\Gamma}_{co2}^\dagger V_{m_1}^\dagger \mathbb{J}_{m_1}\boldsymbol{u}_{co}(t), \\ \boldsymbol{y}_{co}(t) &= V_{m_1}\hat{\Gamma}_{co2}\boldsymbol{x}_{co2}(t) + \boldsymbol{u}_{co}(t). \end{aligned} \tag{55}$$

Proof. Define $\boldsymbol{H}_{co1} \triangleq \frac{1}{2}\boldsymbol{x}_{co1}^T H_{co1}\boldsymbol{x}_{co1}$, $\boldsymbol{H}_{co2} \triangleq \frac{1}{2}\boldsymbol{x}_{co2}^T H_{co2}\boldsymbol{x}_{co2}$, $\boldsymbol{H}_{coh} \triangleq \frac{1}{2}\boldsymbol{p}_h^T H_{121}\boldsymbol{x}_{co1} + \frac{1}{2}\boldsymbol{x}_{co1}^T H_{121}^T\boldsymbol{p}_h$. Since H_{co} and H_{12} are in the form (54a)-(54b), the system Hamiltonian $\boldsymbol{H}_{co}$ for the "co" subsystem can be rewritten as follows

$$\boldsymbol{H}_{co} = \boldsymbol{H}_{co1} + \boldsymbol{H}_{co2} + \boldsymbol{H}_{coh}. \tag{56}$$

On the other hand, it is worth mentioning that $\boldsymbol{L}$ is a column vector whose elements represent the coupling of each field with the quantum system. This means that swapping the elements in $\boldsymbol{L}$ does not change the coupling relationship between the fields and the quantum system. Since condition (B2) holds, we can re-arrange the elements in $\boldsymbol{L}$ to transform it into the following form

$$\begin{bmatrix} \boldsymbol{L}_{co1} \\ \boldsymbol{L}_{co2} \end{bmatrix}. \tag{57}$$

where $\begin{bmatrix} \boldsymbol{L}_{co2} \\ \boldsymbol{L}_{co2}^\# \end{bmatrix} = \hat{\Gamma}_{co2}\boldsymbol{x}_{co2}$. By Definition 5.1, Eqs. (56) and (57) imply that the subsystem G_{co} in (55) is an invariant subsystem. In terms of the form (54a), it follows from Lemma 4.5 that G_{co} is both controllable and observable. □

In a similar way, we can derive the following result for the "h" subsystem, whose proof is omitted.

Lemma 5.3: The "h" subsystem of the quantum linear system (11) has an invariant subsystem G_h if there exists an orthogonal and blockwise symplectic matrix $\mathcal{P}_h$ such that

$$\mathcal{P}_h \begin{bmatrix} \boldsymbol{q}_h \\ \boldsymbol{p}_h \end{bmatrix} = \begin{bmatrix} \boldsymbol{q}_{h1} \\ \boldsymbol{p}_{h1} \\ \hline \boldsymbol{q}_{h2} \\ \boldsymbol{p}_{h2} \end{bmatrix}, \ \Gamma_h\mathcal{P}_h^\top = \begin{bmatrix} \Gamma_{h1} & \Gamma_{h2} \end{bmatrix},$$

$$\mathcal{P}_h \begin{bmatrix} 0 & H_h^{12} \\ H_h^{12^T} & H_h^{22} \end{bmatrix} \mathcal{P}_h^\top = \begin{bmatrix} 0 & H_{h1}^{12} & 0 & 0 \\ H_{h1}^{12^\top} & H_{h1}^{22} & 0 & 0 \\ 0 & 0 & 0 & H_{h2}^{12} \\ 0 & 0 & H_{h2}^{12^\top} & H_{h2}^{22} \end{bmatrix}, \tag{58a}$$

$$\mathcal{P}_h \begin{bmatrix} 0 \\ H_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ H_{12}^1 \\ 0 \\ 0 \end{bmatrix}, \ \mathcal{P}_h \begin{bmatrix} 0 \\ H_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ H_{13}^1 \\ 0 \\ 0 \end{bmatrix}, \tag{58b}$$

where the system variables $\boldsymbol{q}_{h1}$, $\boldsymbol{p}_{h1}$, $\boldsymbol{q}_{h2}$, and $\boldsymbol{p}_{h2}$ satisfy the condition

(C1)

$$\left[\begin{bmatrix}\boldsymbol{q}_{h1}\\ \boldsymbol{p}_{h1}\\ \boldsymbol{q}_{h2}\\ \boldsymbol{p}_{h2}\end{bmatrix},\begin{bmatrix}\boldsymbol{q}_{h1}\\ \boldsymbol{p}_{h1}\\ \boldsymbol{q}_{h2}\\ \boldsymbol{p}_{h2}\end{bmatrix}^\top\right]=\imath\begin{bmatrix}\mathbb{J}_{n_3-n_6} & 0\\ 0 & \mathbb{J}_{n_6}\end{bmatrix}$$

with $n_6>0$;

and constant matrices Γ_{h1} and Γ_{h2} satisfy the condition

(C2) each row of $\left[\begin{array}{cc|c}\Gamma_{h1} & \Gamma_{co} & \Gamma_{h2}\end{array}\right]$ is in the form

$$\left[\begin{array}{cc|c}\Gamma_{h1}^i & \Gamma_{co}^i & 0\end{array}\right],\quad\text{or}\quad\left[\begin{array}{cc|c}0 & 0 & \Gamma_{h2}^i\end{array}\right],$$

where Γ_{h1}^i, Γ_{h2}^i and Γ_{co}^i, $i=1,\cdots,2m$, are the ith rows of the matrices Γ_{h1}, Γ_{h2} and Γ_{co}.

Denote the set of indices of nonzero rows of Γ_{h2} by $\mathbb{I}_h=\{i_1,\cdots,i_{2m_2}\}$, where $m_2\le m$ and $1\le i_1<\cdots<i_{2m_2}\le 2m$. Define

$$\hat{\Gamma}_{h2}\triangleq\begin{bmatrix}\Gamma_{h2}^{i_1}\\ \vdots\\ \Gamma_{h2}^{i_{2m_2}}\end{bmatrix},\quad \boldsymbol{y}_h(t)\triangleq\begin{bmatrix}\boldsymbol{y}_{i_1}(t)\\ \vdots\\ \boldsymbol{y}_{i_{2m_2}}(t)\end{bmatrix},$$
$$\boldsymbol{u}_h(t)\triangleq\begin{bmatrix}\boldsymbol{u}_{i_1}(t) & \cdots & \boldsymbol{u}_{i_{2m_2}}(t)\end{bmatrix},$$

where $\boldsymbol{u}_i(t)$ and $\boldsymbol{y}_i(t)$ are the ith column and ith row of $\boldsymbol{u}(t)$ and $\boldsymbol{y}(t)$, respectively. Then the invariant subsystem G_h is given by

$$\begin{aligned}\begin{bmatrix}\dot{\boldsymbol{q}}_{h2}(t)\\ \dot{\boldsymbol{p}}_{h2}(t)\end{bmatrix}&=\begin{bmatrix}H_{h2}^{12\top} & H_{h2}^{22}-\frac{\imath}{2}\hat{\Gamma}_{h2}^\dagger J_{m_2}\hat{\Gamma}_{h2}\\ 0 & -H_{h2}^{12}\end{bmatrix}\begin{bmatrix}\boldsymbol{q}_{h2}(t)\\ \boldsymbol{p}_{h2}(t)\end{bmatrix}\\&\quad+\begin{bmatrix}\hat{\Gamma}_{h2}^\dagger V_{m_2}^\dagger\mathbb{J}_{m_2}\\ 0\end{bmatrix}\boldsymbol{u}_h(t),\\ \boldsymbol{y}_h(t)&=V_{m_2}\hat{\Gamma}_{h2}\boldsymbol{p}_{h2}(t)+\boldsymbol{u}_h(t).\end{aligned}\tag{59}$$

Let $m_3=m-m_1-m_2$. Denote the set of indices of nonzero rows of $\left[\begin{array}{cc}\Gamma_{h1} & \Gamma_{co1}\end{array}\right]$ by $\mathbb{I}_m=\{i_1,\cdots,i_{2m_3}\}$, $1\le i_1<\cdots<i_{2m_3}\le 2m$. Define

$$\left[\hat{\Gamma}_{h1}\ \hat{\Gamma}_{co1}\right]\triangleq\begin{bmatrix}\Gamma_{h1}^{i_1} & \Gamma_{co1}^{i_1}\\ \vdots & \vdots\\ \Gamma_{h1}^{i_{2m_3}} & \Gamma_{co1}^{i_{2m_3}}\end{bmatrix},\quad \boldsymbol{y}_m(t)\triangleq\begin{bmatrix}\boldsymbol{y}_{i_1}(t)\\ \vdots\\ \boldsymbol{y}_{i_{2m_3}}(t)\end{bmatrix},$$
$$\boldsymbol{u}_m(t)\triangleq\left[\boldsymbol{u}_{i_1}(t)\ \cdots\ \boldsymbol{u}_{i_{2m_3}}(t)\right],$$

where Γ_{h1}^i, Γ_{co1}^i, $i=1,\cdots,2m$, are the ith row of matrices Γ_{h1}, Γ_{co1}, and $\boldsymbol{u}_i(t)$, $\boldsymbol{y}_i(t)$ are the ith column and ith row of $\boldsymbol{u}(t)$ and $\boldsymbol{y}(t)$, respectively.

Based on Lemmas 5.1-5.3, we are now in a position to propose the following result.

Theorem 5.1: The quantum linear system (11) is in the following concatenation form

$$G=G_{\bar{c}\bar{o}}\boxplus G_{co}\boxplus G_h\boxplus G_m,\tag{60}$$

where $G_{\bar{c}\bar{o}}$ is the noiseless subsystem given in Lemma 5.1, G_{co} is the invariant controllable and observable subsystem given in Lemma 5.2, G_h is the invariant subsystem given in Lemma 5.3, and G_m is an invariant subsystem of the form

$$\begin{aligned}\begin{bmatrix}\dot{\boldsymbol{q}}_{h1}(t)\\ \dot{\boldsymbol{p}}_{h1}(t)\\ \hline \dot{\boldsymbol{x}}_{co1}(t)\\ \hline \dot{\boldsymbol{x}}_{\bar{c}\bar{o}1}(t)\end{bmatrix}&=\vec{A}\begin{bmatrix}\boldsymbol{q}_{h1}(t)\\ \boldsymbol{p}_{h1}(t)\\ \hline \boldsymbol{x}_{co1}(t)\\ \hline \boldsymbol{x}_{\bar{c}\bar{o}1}(t)\end{bmatrix}+\vec{B}\boldsymbol{u}_m(t),\\ \boldsymbol{y}_m(t)&=\vec{C}\begin{bmatrix}\boldsymbol{q}_{h1}(t)\\ \boldsymbol{p}_{h1}(t)\\ \hline \boldsymbol{x}_{co1}(t)\\ \hline \boldsymbol{x}_{\bar{c}\bar{o}1}(t)\end{bmatrix}+\boldsymbol{u}_m(t),\end{aligned}\tag{61}$$

where

$$\vec{A}\triangleq\left[\begin{array}{cc|c|c}A_{h1}^{11} & A_{h1}^{12} & A_{m12} & A_{m13}\\ 0 & A_{h1}^{22} & 0 & 0\\ \hline 0 & A_{m21} & A_{co1} & 0\\ \hline 0 & A_{m31} & 0 & A_{\bar{c}\bar{o}1}\end{array}\right],\quad \vec{B}\triangleq\left[\begin{array}{c}B_{h1}\\ 0\\ \hline B_{co1}\\ \hline 0\end{array}\right],$$
$$\vec{C}\triangleq\left[\begin{array}{cc|c|c}0 & C_{h1} & C_{co1} & 0\end{array}\right],$$

with $A_{h1}^{11}=H_{h1}^{12\top}$, $A_{h1}^{12}=H_{h1}^{22}-\frac{\imath}{2}\hat{\Gamma}_{h1}^\dagger J_{m_3}\hat{\Gamma}_{h1}$, $A_{h1}^{22}=-H_{h1}^{12}$, $A_{m12}=H_{m12}-\frac{\imath}{2}\hat{\Gamma}_{h1}^\dagger J_{m_3}\hat{\Gamma}_{co1}$, $A_{m13}=H_{m13}$, $A_{co1}=\mathbb{J}_{n_1-n_5}H_{co1}-\frac{\imath}{2}\mathbb{J}_{n_1-n_5}\hat{\Gamma}_{co1}^\dagger J_{m_3}\hat{\Gamma}_{co1}$, $A_{\bar{c}\bar{o}1}=\mathbb{J}_{n_2-n_4}H_{\bar{c}\bar{o}1}$, $A_{m21}=\mathbb{J}_{n_1-n_5}H_{m12}^\top-\frac{\imath}{2}\mathbb{J}_{n_1-n_5}\hat{\Gamma}_{co1}^\dagger J_{m_3}\hat{\Gamma}_{h1}$, $A_{m31}=\mathbb{J}_{n_2-n_4}H_{m13}^\top$, $B_{h1}=\hat{\Gamma}_{h1}^\dagger V_{m_3}\mathbb{J}_{m_3}$, $B_{co1}=\mathbb{J}_{n_1-n_5}\hat{\Gamma}_{co1}^\dagger V_{m_3}\mathbb{J}_{m_3}$, $C_{h1}=V_{m_3}\hat{\Gamma}_{h1}$, and $C_{co1}=V_{m_3}\hat{\Gamma}_{co1}$, provided that

(i) there exist orthogonal and blockwise symplectic matrices $\mathcal{P}_{\bar{c}\bar{o}}$, $\mathcal{P}_{co}$ and $\mathcal{P}_h$ satisfying (52a), (54a), (58a), and

$$\mathcal{P}_h\begin{bmatrix}0\\ H_{12}\end{bmatrix}\mathcal{P}_{co}^\top=\begin{bmatrix}0 & 0\\ H_{m12} & 0\\ 0 & 0\\ 0 & 0\end{bmatrix},$$
$$\mathcal{P}_h\begin{bmatrix}0\\ H_{13}\end{bmatrix}\mathcal{P}_{\bar{c}\bar{o}}^\top=\begin{bmatrix}0 & 0\\ H_{m13} & 0\\ 0 & 0\\ 0 & 0\end{bmatrix};$$

(ii) (A1), (B1)-(B2) and (C1)-(C2) hold;

(iii) each row of $\left[\begin{array}{cc|c|c}\Gamma_{h1} & \Gamma_{co1} & \Gamma_{h2} & \Gamma_{co2}\end{array}\right]$ is in one of the following forms

$$\left[\begin{array}{cc|c|c}\Gamma_{h1} & \Gamma_{co1} & 0 & 0\end{array}\right],$$

or

$$\left[\begin{array}{cc|c|c}0 & 0 & \Gamma_{h2} & 0\end{array}\right],$$

or

$$\left[\begin{array}{cc|c|c}0 & 0 & 0 & \Gamma_{co2}\end{array}\right],$$

where Γ_{h1}^i, Γ_{h2}^i, Γ_{co1}^i and Γ_{co2}^i, $i=1,\cdots,2m$, are the ith row of matrices Γ_{h1}, Γ_{h2}, Γ_{co1} and Γ_{co2}.

A block diagram for a quantum linear system in the form (60) is shown in Fig. 3.

Remark 5.2: We have the following observations on Theorem 5.1.

(i) The noiseless subsystem $G_{\bar{c}\bar{o}}$ is a subsystem of the $\bar{c}\bar{o}$ subsystem, as can be seen from Eq. (53); similarly,

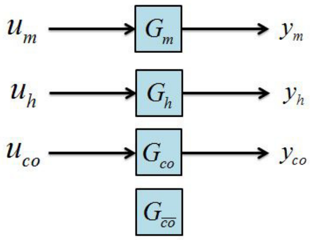

Fig. 3. Block diagram for a quantum linear system in the form $G = G_{\bar{c}\bar{o}} \boxplus G_{co} \boxplus G_h \boxplus G_m$, as given in Theorem 5.1.

the invariant subsystem G_{co} is a subsystem of the co subsystem, as can be seen from Eq. (55); and the invariant subsystem G_h is a subsystem of the "h" subsystem, as can be seen from Eq. (59);

(ii) The invariant subsystem G_m is a mixture of "h", co, and $\bar{c}\bar{o}$ subsystems, as can be seen from Eq. (61).

(iii) Comparison of Figs. 2 and 3 tells us that the system (60) in Theorem 5.1 involves partitioning the system input and output, while the original Kalman canonical form (11) does not.

(iv) It is worthwhile to notice that the system decomposition in Fig. 3 is very general. In some cases, some of the subsystems in Fig. 3 may not exist; this can be easily seen from the conditions in Lemmas 5.1-5.3 and Theorem 5.1. On the other hand, there might be more than one invariant co, $\bar{c}\bar{o}$, or "h" subsystems. Indeed, the system in Example 5.2 can be decomposed into two invariant co subsystems, each of which is a harmonic oscillator driven by a single input field.

It is interesting to see that the subsystem G_m is in the Kalman canonical form (11), while $G_{\bar{c}\bar{o}}$, G_h, and G_{co} are in the form of (20), (21), and (23) respectively. Therefore, the quantum linear system (11) is decomposed into four subsystems which are decoupled from each other, and one of which is a smaller Kalman canonical form. This means that the Kalman canonical form (11), in general, may not reveal the noiseless subsystem and the invariant subsystems of a given quantum linear system. In Theorem 5.1, a refined decomposition of the matrices H and Γ shows that a quantum linear system can be expressed in the form (60) by an appropriate coordinate transformation. Finally, the following should be noted. In the Heisenberg picture of quantum mechanics, a quantum linear system G may be put into a concatenation form, as shown in Fig. 3, where four possible subsystems are decoupled form each other. However, the initial state of the whole system G may still be a state superposed among all these subsystems.

B. BAE measurements

In this subsection, we consider the realization of BAE measurements. We present necessary and sufficient conditions for the quantum linear system (11) to realize a BAE measurement. These necessary and sufficient conditions are given explicitly in terms of the physical parameters H and Γ.

It is mentioned in Remark 4.5 that QND variables are related to the "h" subsystem in the Kalman canonical form (11). In contrast, as BAE measurements are an input-output property, they are determined completely by the co subsystem. For the quantum linear system (11), the transfer function from $\boldsymbol{u}$ to $\boldsymbol{y}$ is

$$\Xi_{\boldsymbol{u}\to\boldsymbol{y}}(s) = C_{co}(sI - A_{co})^{-1}B_{co} + I.$$

Partition the matrices B_{co} and C_{co} as

$$B_{co} = [B_{co,q}\ B_{co,p}], \quad C_{co} = \begin{bmatrix} C_{co,q} \\ C_{co,p} \end{bmatrix},$$

respectively. Then the transfer function from $\boldsymbol{p}_{\rm in}$ to $\boldsymbol{q}_{\rm out}$ is

$$\Xi_{\boldsymbol{p}_{\rm in}\to\boldsymbol{q}_{\rm out}}(s) = C_{co,q}(sI - A_{co})^{-1}B_{co,p}. \tag{63}$$

Similarly, the transfer function from $\boldsymbol{q}_{\rm in}$ to $\boldsymbol{p}_{\rm out}$ is

$$\Xi_{\boldsymbol{q}_{\rm in}\to\boldsymbol{p}_{\rm out}}(s) = C_{co,p}(sI - A_{co})^{-1}B_{co,q}. \tag{64}$$

The following is the main result of this subsection, which presents necessary and sufficient conditions for the realization of BAE measurements by the quantum linear system (11).

Theorem 5.2:

(i) The quantum linear system (11) realizes the BAE measurements of $\boldsymbol{q}_{\rm out}$ with respect to $\boldsymbol{p}_{\rm in}$; i.e.,

$$\Xi_{\boldsymbol{p}_{\rm in}\to\boldsymbol{q}_{\rm out}}(s) \equiv 0 \tag{65}$$

if and only if

$$\begin{bmatrix} \mathrm{Re}\,(\Gamma_{co,q}) & \mathrm{Re}\,(\Gamma_{co,p}) \end{bmatrix} (sI - \mathbb{J}_{n_1}H_{co})^{-1} \times \begin{bmatrix} \mathrm{Re}(\Gamma_{co,p}^\top) \\ -\mathrm{Re}(\Gamma_{co,q}^\top) \end{bmatrix} \equiv 0; \tag{66}$$

(ii) The quantum linear system (11) realizes the BAE measurements of $\boldsymbol{p}_{\rm out}$ with respect to $\boldsymbol{q}_{\rm in}$; i.e.,

$$\Xi_{\boldsymbol{q}_{\rm in}\to\boldsymbol{p}_{\rm out}}(s) \equiv 0 \tag{67}$$

if and only if

$$\begin{bmatrix} \mathrm{Im}\,(\Gamma_{co,q}) & \mathrm{Im}\,(\Gamma_{co,p}) \end{bmatrix} (sI - \mathbb{J}_{n_1}H_{co})^{-1} \times \begin{bmatrix} \mathrm{Im}(\Gamma_{co,p}^\top) \\ -\mathrm{Im}(\Gamma_{co,q}^\top) \end{bmatrix} \equiv 0. \tag{68}$$

Proof. From Eqs. (34a) and (40m), we have

$$\begin{aligned} C_{co,q} &= \sqrt{2}\begin{bmatrix} \mathrm{Re}\,(\Gamma_{co,q}) & \mathrm{Re}\,(\Gamma_{co,p}) \end{bmatrix}, \\ C_{co,p} &= \sqrt{2}\begin{bmatrix} \mathrm{Im}\,(\Gamma_{co,q}) & \mathrm{Im}\,(\Gamma_{co,p}) \end{bmatrix}. \end{aligned} \tag{69}$$

By Eq. (18h), the following can be obtained

$$B_{co,q} = -\mathbb{J}_{n_1}C_{co,p}^\top, \quad B_{co,p} = \mathbb{J}_{n_1}C_{co,q}^\top. \tag{70}$$

Moreover, by Eq. (29e), we have

$$\begin{aligned} A_{co} &= \mathbb{J}_{n_1}H_{co} + B_{co}\mathbb{J}_m B_{co}^\top \mathbb{J}_{n_1}/2 \\ &= \mathbb{J}_{n_1}H_{co} - \mathbb{J}_{n_1}C_{co,p}^\top C_{co,q}/2 + \mathbb{J}_{n_1}C_{co,q}^\top C_{co,p}/2. \end{aligned} \tag{71}$$

(i) According to Eq. (63), Eq. (65) is equivalent to

$$C_{co,q}A_{co}^k B_{co,p} = 0, \quad k = 0, 1, \cdots. \tag{72}$$

Moreover, by Eqs. (69) and (70), Eq. (66) is equivalent to

$$C_{co,q}(\mathbb{J}_{n_1}H_{co})^k B_{co,p} = 0, \quad k = 0, 1, \cdots. \tag{73}$$

Thus, it suffices to establish the equivalence between Eqs. (72) and (73).

Firstly, we show that Eq. (73) implies Eq. (72). Suppose Eq. (73) holds. Then

$$C_{co,q}B_{co,p}=0. \tag{74}$$

Assume that

$$C_{co,q}A_{co}^{l}B_{co,p}=0,\quad \forall l\leq k-1. \tag{75}$$

By Eqs. (70) and (71), direct matrix manipulations yield

$$\begin{aligned} & C_{co,q}A_{co}^{k}B_{co,p} \\ =\ & C_{co,q}\mathbb{J}_{n_1}H_{co}A_{co}^{k-1}B_{co,p} \\ & -C_{co,q}\mathbb{J}_{n_1}C_{co,p}^{\top}\boxed{C_{co,q}A_{co}^{k-1}B_{co,p}}/2 \\ & +\boxed{C_{co,q}B_{co,p}}C_{co,p}A_{co}^{k-1}B_{co,p}/2 \\ =\ & C_{co,q}(\mathbb{J}_{n_1}H_{co})A_{co}^{k-1}B_{co,p} \\ =\ & \cdots \\ =\ & C_{co,q}(\mathbb{J}_{n_1}H_{co})^{k}B_{co,p} \\ =\ & 0, \end{aligned}$$

where the two terms in the boxes above are both equal to zero due to Eqs. (74) and (75). Therefore, by mathematical induction, Eq. (72) holds.

Secondly, assume that Eq. (72) holds. Clearly. Eq. (74) holds. Assume that

$$C_{co,q}(\mathbb{J}_{n_1}H_{co})^{l}B_{co,p}=0,\quad \forall l\leq k-1. \tag{76}$$

By Eqs. (70) and (71),

$$\begin{aligned} & C_{co,q}(\mathbb{J}_{n_1}H_{co})^{k}B_{co,p} \\ =\ & C_{co,q}A_{co}(\mathbb{J}_{n_1}H_{co})^{k-1}B_{co,p} \\ & +C_{co,q}\mathbb{J}_{n_1}C_{co,p}^{\top}\boxed{C_{co,q}(\mathbb{J}_{n_1}H_{co})^{k-1}B_{co,p}}/2 \\ & -\boxed{C_{co,q}B_{co,p}}C_{co,p}(\mathbb{J}_{n_1}H_{co})^{k-1}B_{co,p}/2 \\ =\ & C_{co,q}A_{co}(\mathbb{J}_{n_1}H_{co})^{k-1}B_{co,p} \\ =\ & \cdots \\ =\ & C_{co,q}A_{co}^{k}B_{co,p} \\ =\ & 0, \end{aligned}$$

where the two terms in the boxes above are both equal to zero due to Eqs. (74) and (76). Thus, by mathematical induction, Eq. (73) holds. Thus the equivalence between Eqs. (72) and (73) has been established.

(ii) The proof follows in a similar way as that of (i), and thus is omitted. □

The following corollary presents a special case of Theorem 5.2.

Corollary 5.1: Let a quantum linear system be parametrized by the Hamiltonian $\boldsymbol{H}=\frac{1}{2}\breve{\boldsymbol{x}}^{\top}H\breve{\boldsymbol{x}}$ and the coupling operator $\boldsymbol{L}=\Gamma\breve{\boldsymbol{x}}$, where $\breve{\boldsymbol{x}}$ satisfies the CCRs $[\breve{\boldsymbol{x}},\ \breve{\boldsymbol{x}}^{\top}]=\imath\mathbb{J}_n$,

$$H=\begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}, \tag{77}$$

and

$$\Gamma=[\Gamma_q\quad \Gamma_p]$$

with $\Gamma_q,\Gamma_p\in\mathbb{C}^{m\times n}$.

(i) The system is controllable and observable and $\Xi_{\boldsymbol{p}_{\rm in}\to\boldsymbol{q}_{\rm out}}(s)\equiv 0$ if and only if

$$\mathrm{Re}\,(\Gamma_q)\perp\mathrm{Re}\,(\Gamma_p), \tag{78}$$

and

$$\mathrm{rank}\left(\begin{bmatrix} \breve{\Gamma} \\ \breve{\Gamma}J_n \end{bmatrix}\right)=2n, \tag{79}$$

where $\breve{\Gamma}\triangleq\begin{bmatrix} \Gamma \\ \Gamma^{\#} \end{bmatrix}$;

(ii) The system is controllable and observable and $\Xi_{\boldsymbol{q}_{\rm in}\to\boldsymbol{p}_{\rm out}}(s)\equiv 0$ if and only if

$$\mathrm{Im}\,(\Gamma_q)\perp\mathrm{Im}\,(\Gamma_p), \tag{80}$$

and (79) hold.

Proof. Let H be as in Eq. (77). Then $\mathbb{J}_nH=J_n$ and

$$\begin{aligned} & \mathrm{rank}\left(\begin{bmatrix} \breve{\Gamma} \\ \breve{\Gamma}\mathbb{J}_nH \\ \vdots \\ \breve{\Gamma}(\mathbb{J}_nH)^{2n-1} \end{bmatrix}\right) \qquad (81) \\ =\ & \mathrm{rank}\left(\begin{bmatrix} \breve{\Gamma} \\ \breve{\Gamma}J_n \\ \vdots \\ \breve{\Gamma}(J_n)^{2n-1} \end{bmatrix}\right)=\mathrm{rank}\left(\begin{bmatrix} \breve{\Gamma} \\ \breve{\Gamma}J_n \end{bmatrix}\right). \end{aligned}$$

Therefore, the observability of $(\mathbb{J}_nH,\breve{\Gamma})$ is equivalent to Eq. (79). In a similar way, by Lemma 4.5, the controllability of the system is also equivalent to Eq. (79).

(i) By Eq. (77), it can be seen that Eq. (66) in Theorem 5.2 is equivalent to

$$\begin{aligned} & \mathrm{Re}(\Gamma)(\mathbb{J}_nH)^k\mathbb{J}_n\mathrm{Re}(\Gamma^{\top})=\mathrm{Re}(\Gamma)\begin{bmatrix} I_{n_1} & 0 \\ 0 & -I_{n_1} \end{bmatrix}^k\mathbb{J}_n\mathrm{Re}(\Gamma^{\top}) \\ & \qquad =0,\quad k=0,1,\ldots. \end{aligned} \tag{82}$$

However, Eq. (82) is equivalent to

$$\mathrm{Re}\,(\Gamma_q)\,\mathrm{Re}\,(\Gamma_p^{\top})-\mathrm{Re}\,(\Gamma_p)\,\mathrm{Re}\,(\Gamma_q^{\top})=0,$$

and

$$\mathrm{Re}\,(\Gamma_q)\,\mathrm{Re}\,(\Gamma_p^{\top})+\mathrm{Re}\,(\Gamma_p)\,\mathrm{Re}\,(\Gamma_q^{\top})=0,$$

which are equivalent to Eq. (78).

(ii) In a similar way, it can be shown that Eq. (68) in Theorem 5.2 is equivalent to

$$\mathrm{Im}\,(\Gamma_q)\,\mathrm{Im}(\Gamma_p^{\top})-\mathrm{Im}\,(\Gamma_p)\,\mathrm{Im}(\Gamma_q^{\top})=0,$$

and

$$\mathrm{Im}\,(\Gamma_q)\,\mathrm{Im}\,(\Gamma_p^{\top})+\mathrm{Im}\,(\Gamma_p)\,\mathrm{Im}\,(\Gamma_q^{\top})=0,$$

which are equivalent to Eq. (80). □

Example 5.1: This example is used to illustrate Corollary 5.1. Let $n=m=1$. Choose

$$H=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\quad \Gamma_q=\imath,\quad \Gamma_p=-\imath.$$

Clearly, H is of the form (77), and Eq. (78) is satisfied. Moreover, Eq. (79) holds, but Eq. (80) does not. In fact, with the above system parameters, it is easy to see that this controllable and observable system is described by the QSDEs:

$$\begin{bmatrix} \dot{\boldsymbol{q}}(t) \\ \dot{\boldsymbol{p}}(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \boldsymbol{q}(t) \\ \boldsymbol{p}(t) \end{bmatrix} + \sqrt{2} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_{\rm in}(t) \\ \boldsymbol{p}_{\rm in}(t) \end{bmatrix},$$

$$\begin{bmatrix} \boldsymbol{q}_{\rm out}(t) \\ \boldsymbol{p}_{\rm out}(t) \end{bmatrix} = \sqrt{2} \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \boldsymbol{q}(t) \\ \boldsymbol{p}(t) \end{bmatrix} + \begin{bmatrix} \boldsymbol{q}_{\rm in}(t) \\ \boldsymbol{p}_{\rm in}(t) \end{bmatrix}.$$

It can be verified that

$$\Xi_{\boldsymbol{p}_{\rm in} \to \boldsymbol{q}_{\rm out}}(s) \equiv 0, \quad \Xi_{\boldsymbol{q}_{\rm in} \to \boldsymbol{p}_{\rm out}}(s) = \frac{2}{s-1} - \frac{2}{s+1} \neq 0.$$

Finally, for this system, the Hamiltonian $\boldsymbol{H}$ and the coupling operator $\boldsymbol{L}$ are respectively

$$\boldsymbol{H} = \frac{\boldsymbol{qp} + \boldsymbol{pq}}{2} = \frac{\boldsymbol{a}_{co}^2 - (\boldsymbol{a}_{co}^*)^2}{2\imath},$$

$$\boldsymbol{L} = \imath(\boldsymbol{p} - \boldsymbol{q}) = \frac{1+\imath}{\imath\sqrt{2}} \boldsymbol{a}_{co} - \left(\frac{1+\imath}{\imath\sqrt{2}}\right)^* \boldsymbol{a}_{co}^*.$$

This system can be physically realized by means of quantum optical devices; see, e.g., [14], [20], [23], [46].

Example 5.2: This example, taken from [43], is used to illustrate Theorem 5.2 as well a Theorem 5.1. This example considers the Michelson's interferometer which is one of the simplest devices for gravitational wave detection, see [43, Fig. 3(c)]. The interferometer contains two identical mechanical oscillators with position quadratures $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, and momentum quadratures $\boldsymbol{p}_1$, $\boldsymbol{p}_2$, respectively. The resonant frequency and mass of the mechanical oscillators are denoted by ω_m and m, respectively. The input coherent light field (the probe field $\hat{W}_1$ in [43, Fig. 3(c)]) and the input vacuum light field ($\hat{W}_2$ in [43, Fig. 3(c)]) are described by their respective position quadratures $\boldsymbol{q}_{{\rm in},1}$, $\boldsymbol{q}_{{\rm in},2}$, and momentum quadratures $\boldsymbol{p}_{{\rm in},1}$, $\boldsymbol{p}_{{\rm in},2}$. Let λ be the coupling strength between the probe field and the mechanical oscillators. It is assumed that the mechanical oscillators are subjected to forces F and $-F$. Then the dynamics of the system, given in [43, Eq. (19)], is described by the following QSDEs.

$$\begin{bmatrix} \dot{\boldsymbol{q}}_1 \\ \dot{\boldsymbol{q}}_2 \\ \dot{\boldsymbol{p}}_1 \\ \dot{\boldsymbol{p}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1/m & 0 \\ 0 & 0 & 0 & 1/m \\ -m\omega_m^2 & 0 & 0 & 0 \\ 0 & -m\omega_m^2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_1 \\ \boldsymbol{q}_2 \\ \boldsymbol{p}_1 \\ \boldsymbol{p}_2 \end{bmatrix} + \sqrt{\lambda} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_{{\rm in},1} \\ \boldsymbol{q}_{{\rm in},2} \\ \boldsymbol{p}_{{\rm in},1} \\ \boldsymbol{p}_{{\rm in},2} \end{bmatrix},$$

$$\begin{bmatrix} \boldsymbol{q}_{{\rm out},1} \\ \boldsymbol{q}_{{\rm out},2} \\ \boldsymbol{p}_{{\rm out},1} \\ \boldsymbol{p}_{{\rm out},2} \end{bmatrix} = \sqrt{\lambda} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_1 \\ \boldsymbol{q}_2 \\ \boldsymbol{p}_1 \\ \boldsymbol{p}_2 \end{bmatrix} + \begin{bmatrix} \boldsymbol{q}_{{\rm in},1} \\ \boldsymbol{q}_{{\rm in},2} \\ \boldsymbol{p}_{{\rm in},1} \\ \boldsymbol{p}_{{\rm in},2} \end{bmatrix}. \tag{83}$$

Note that the system (83) is both controllable and observable. It follows from Lemma 4.1 that

$$H = \begin{bmatrix} m\omega_m^2 & 0 & 0 & 0 \\ 0 & m\omega_m^2 & 0 & 0 \\ 0 & 0 & 1/m & 0 \\ 0 & 0 & 0 & 1/m \end{bmatrix}, \quad \Gamma = \sqrt{\frac{\lambda}{2}} \begin{bmatrix} \imath & \imath & 0 & 0 \\ \imath & -\imath & 0 & 0 \end{bmatrix}. \tag{84}$$

It is easy to check that H and Γ in Eq. (84) satisfy the condition (66), but not the condition (68). Indeed, denote $\boldsymbol{q}_{\rm in} = \begin{bmatrix} \boldsymbol{q}_{{\rm in},1} \\ \boldsymbol{q}_{{\rm in},2} \end{bmatrix}$, $\boldsymbol{p}_{\rm in} = \begin{bmatrix} \boldsymbol{p}_{{\rm in},1} \\ \boldsymbol{p}_{{\rm in},2} \end{bmatrix}$, $\boldsymbol{q}_{\rm out} = \begin{bmatrix} \boldsymbol{q}_{{\rm out},1} \\ \boldsymbol{q}_{{\rm out},2} \end{bmatrix}$, and $\boldsymbol{p}_{\rm out} = \begin{bmatrix} \boldsymbol{p}_{{\rm out},1} \\ \boldsymbol{p}_{{\rm out},2} \end{bmatrix}$. It turns out that

$$\Xi_{\boldsymbol{p}_{\rm in} \to \boldsymbol{q}_{\rm out}}(s) \equiv 0, \quad \Xi_{\boldsymbol{q}_{\rm in} \to \boldsymbol{p}_{\rm out}}(s) = \frac{2}{m(s^2 + \omega_m^2)} I_2 \neq 0.$$

Therefore the QSDEs (83) only realizes the BAE measurements of $\boldsymbol{q}_{\rm out}$ with respect to $\boldsymbol{p}_{\rm in}$. Finally, by Lemma 5.2, it can be easily verified that the orthogonal and blockwise symplectic matrix

$$\mathcal{P} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

transforms the system (83) to two controllable and observable subsystems which are decoupled from each other.

Remark 5.3: It is worthwhile to notice that the properties of system (2) in [32] and the system in Fig. 3(b) in [43] can also be checked by using Theorem 5.2.

VI. Conclusion

In this paper, we have investigated the structure of quantum linear systems. Physical realizability conditions have been demonstrated, which reveal the relations among the system matrices of a quantum linear system in its Kalman canonical form. These relations have been used to study the spectral structure of quantum linear systems, based on which it has been shown that a quantum linear system is both controllable and observable if it is Hurwitz stable. A new parameterization method has been proposed which generates the Kalman canonical form directly. Further decomposition of these physical parameters reveals the noiseless and invariant subsystems of a given quantum linear system. Necessary and sufficient conditions for the realizations of quantum back-action evading (BAE) measurements have also been explicitly proposed in terms of physical parameters. The *system analysis* results presented in this paper may be useful for quantum control engineering.

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