A Power Penalty Method For Discrete HJB Equations

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Abstract

We develop a power penalty approach to the discrete Hamilton-Jacobi-Bellman (HJB) equation in $\mathbb{R}^N$ in which the HJB equation is approximated by a nonlinear equation containing a power penalty term. We prove that the solution to this penalized equation converges to that of the HJB equation at an exponential rate with respect to the penalty parameter when the control set is finite and the coefficient matrices are M-matrices. Examples are presented to confirm the theoretical findings and to show the efficiency of the new method.

Keywords. HJB equation; Penalty method; Convergence rate
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1 Introduction

HJB equations are widely used in characterizing many real-world phenomena in engineering, mechanics, economics and finance. Extensive studies of HJB equations have been conducted, see the book [2] and the references therein. As HJB equations have some nonlinear and non-smooth structure, it is generally very difficult to get its analytical solution. Hence, numerical approximation methods are widely used to solve HJB equations, such as Markov chain method [5], valuation iteration method [6], and policy iteration method [1]. However it seems that there is a limited investigation of penalty methods for HJB equations, except that in [11, 12] where the linear penalty method is studied.

Consider the following discrete HJB problem:

Problem 1.1. Find $x \in \mathbb{R}^N$ such that

$$\min_{q\in\mathcal{Q}} \{A(q)\,x - b(q)\} = 0, \tag{1}$$

where $\mathcal{Q} = \{q_1, q_2, \cdots, q_M\} \subset \mathbb{R}^M$ is a finite control set, for every control $q \in \mathcal{Q}$, $A(q) : \mathcal{Q} \to \mathbb{R}^{N\times N}$ and $b(q) : \mathcal{Q} \to \mathbb{R}^N$ refer to an $N \times N$ system matrix and a vector in $\mathbb{R}^N$, respectively defined by $A(q) := (a_{ij}(q))$ and $b(q) := (b_1(q), b_2(q), \cdots, b_N(q))^\top$.

For the sake of concreteness, we introduce the following notations. Denote by $\mathcal{M}$ the set of real-valued $N \times N$ matrices, and let $\mathbb{I}$ be the set of $\{1, \cdots, N\}$. Throughout this paper, for every $x, y \in \mathbb{R}^N$, the notation $y \geq x$ means that $y_i \geq x_i$, $\forall i \in \mathbb{I}$. We also denote by $\min\{x, y\}$ (resp. $\max\{x, y\}$) the vector with components $\min(x_i, y_i)$ (resp. $\max\{x_i, y_i\}$). The definitions extend trivially to other relational operators. With these notations, Equation (1) can also be stated as the following equivalent generalized complementarity form:

$$\underbrace{A(q_1)\,x - b(q_1) \geq 0, \cdots, A(q_M)\,x - b(q_M) \geq 0,}_{M} \quad \prod_{i=1}^{M} [A(q_i)\,x - b(q_i)] = 0,$$

where $\prod_{i=1}^{M}[A(q_i)x - b(q_i)]$ is the Hamard product.

The above complementarity form of the the discrete HJB equation (1) inspires us to propose a power penalty approach to solving Problem 1.1, since the power penalty method has been well developed to approximate standard linear and nonlinear complementarity problems, see [4, 9, 10], etc.

2 Penalty approach

In this section we will present and analyze a power penalty method for Problem 1.1. Consider the following problem:

Problem 2.1. Find $x_\lambda \in \mathbb{R}^N$, such that

$$A(q_1)\,x_\lambda - b(q_1) - \lambda \sum_{i=2}^{M} [b(q_i) - A(q_i)\,x_\lambda]_+^{1/k} = 0, \tag{2}$$

where $\lambda > 1$ is the penalty parameter, $[u]_+ = \max\{u, 0\}$, and for any $y = (y_1, \cdots, y_N)^\top \in \mathbb{R}^N$, $y^\alpha \doteq [y_1^\alpha, \cdots, y_N^\alpha]$.

Problem 2.1 is the penalization corresponding to Problem 1.1, where the penalty term penalizes the violation of all the (control) constraints, except one. The essence is to enforce all the constraints to be satisfied by letting $\lambda \to \infty$. We expect that the solution x_λ of Problem 2.1 converges to that of Problem 1.1. Clearly, the rate of convergence depends on all the parameter in the penalty term. Penalty problems of form (2) for the discrete HJB equation (1) are discussed in [11] when $k = 1$, i.e., the linear penalty method.

Linear convergence rate are also obtained therein. In the next section we will establish an exponential convergence rate $\mathcal{O}(\lambda^k)$ for the power penalty approach (2) for solving the discrete HJB equation (1), under some mild assumptions on the system matrix $A(q)$ and the vector $b(q)$.

Before further discussions, we first make the following assumptions on $A(q)$:

(A) The matrix $A(q)$ is a strictly diagonally dominant M-matrix for every $q \in \mathcal{Q}$.

Remark 2.1. It is easily shown that, under the assumption (A), Problem 1.1 has a unique solution (c.f., [12]). Moreover, for any $A(q) \in \mathcal{M}, q \in \mathcal{Q}$, both $A(q)$ and $A^{-1}(q)$ can be bounded, since there are only finitely many composition that can be assumed. Similarly, $b(q)$ can be bounded as well. In the same way, we can infer that $\|A(q)\| \leq C$ and $\|A^{-1}(q)\| \leq C$ with C a constant.

Remark 2.2. It is worth noting that the above assumption is normally guaranteed by a proper discretization method such as the upwind finite difference/finite element or a fitted finite volume method for 2nd order elliptic partial differential equations (see, for example, [13]).

In the rest of our discussion, we assume the assumption (A) is fulfilled.

3 Convergence analysis

We first investigate the convergence property of the power penalty method.

3.1 Monotonic convergence property

We first shown that the solution to Problem 2.1 is bounded in the following lemma.

Lemma 3.1. *Let x_λ be the solution to (2) for any λ. Then, x_λ is bounded for any $\lambda > 1$, i.e., there exists a positive constant C, independent of λ and k, such that*

$$\|x_\lambda\|_\infty \leq C. \tag{3}$$

Proof. Rearranging (2), we get

$$A(q_1)\, x_\lambda = b(q_1) + \lambda \sum_{i=2}^{M} \left[b(q_i) - A(q_i)\, x_\lambda\right]_+^{1/k}, \tag{4}$$

which implies that

$$A(q_1)\, x_\lambda \geq b(q_1).$$

Note that $A(q_1)$ is a strictly diagonally dominant M-matrix. Hence, $A^{-1}(q_1) > 0$. Thus, we immediately get that

$$x_\lambda \geq A^{-1}(q_1)\, b(q_1). \tag{5}$$

Meanwhile, it follows from (4) that for every $i \in \mathbb{I}$, either

$$\left(A(q_1)\, x_\lambda\right)_i = \left(b(q_1)\right)_i,$$

or, $\exists q_{j(i)} \in \mathcal{Q}$ associated with i, s.t.

$$\left(A\left(q_{j(i)}\right) x_\lambda\right)_i \leq \left(b\left(q_{j(i)}\right)\right)_i.$$

Now, introducing a matrix, denoting $A^* \in \mathcal{M}$, to be the matrix having the ith row as that of $(A(q_{j(i)}))_i$, $i \in \mathbb{I}$ and introducing b^* correspondingly, we get

$$A^* x_\lambda \leq b^*.$$

From the construction of A^*, it follows that the new matrix A^* is also a strictly diagonally dominant M-matrix. Thus, the above inequality gives

$$x_\lambda \leq \left(A^*\right)^{-1} b^*. \tag{6}$$

Combining (5) and (6) and using the fact $A(q)$, $A^{-1}(q)$ and $b(q)$ can be bounded (see Remark 2.1), we complete the proof. □

We next show that the solution of the penalized Problem 2.1 is always less than the that of the discrete HJB problem 1.1, component-wisely.

Lemma 3.2. *Let $\lambda > 1$ and $k > 0$. Assume that x_λ and x^* are the solutions of the penalized Problem 2.1 and that of the discrete HJB Problem 1.1, respectively. Then*

$$x_\lambda \leq x^*.$$

Proof. Since x_λ is the solution of the penalized Problem 2.1, we have

$$A(q_1)\, x_\lambda - b(q_1) - \lambda \sum_{i=2}^{M} \left[b(q_i) - A(q_i)\, x_\lambda\right]_+^{1/k} = 0. \tag{7}$$

Define two disjoint nonempty index subsets J_1 and J_2 of $\mathbb{I}$ as follows

$$J_1 = \left\{ j \,\middle|\, \left(\sum_{i=2}^{M} \left[b(q_i) - A(q_i)\, x_\lambda\right]_+^{1/k} \right)_j = 0 \right\}, \tag{8}$$

$$J_2 = \left\{ j \,\middle|\, \left(\sum_{i=2}^{M} \left[b(q_i) - A(q_i)\, x_\lambda\right]_+^{1/k} \right)_j > 0 \right\}. \tag{9}$$

We still distinguish the following two cases.

- For $j \in J_1$, it follows from (8) that $\left(A\left(q_i\right) x_\lambda - b\left(q_i\right)\right)_j \geq 0,$ for all $q_i \in Q$. Hence,

  $$\left(A\left(q_1\right) x_\lambda - b\left(q_1\right)\right)_j = 0.$$

  Moreover, from the fact x^* is the solution to the discrete HJB Problem 1.1 it follows that

  $$\left(A\left(q_1\right) x^* - b\left(q_1\right)\right)_j \geq \left(A\left(q^*\right) x^* - b\left(q^*\right)\right)_j = \min_{q \in \mathcal{Q}} \left(A\left(q\right) x - b\left(q\right)\right)_j = 0,$$

  with $q^* = \arg\min_{q \in \mathcal{Q}} \left\{A\left(q\right) x - b\left(q\right)\right\}$. Thus, combining the above two equations, we obtain

  $$\left(A\left(q_1\right)\left(x^* - x_\lambda\right)\right)_j \geq 0, \quad j \in J_1.$$

- For $j \in J_2$, as we have shown, there exists at least one control $q_{i(j)} \in Q$ associated with j, such that

  $$\left(b\left(q_{i(j)}\right) - A\left(q_{i(j)}\right) x_\lambda\right)_j > 0,$$

  which is equivalent to

  $$\left(A\left(q_{i(j)}\right) x_\lambda - b\left(q_{i(j)}\right)\right)_j < 0.$$

  As the first case, it also holds that

  $$\left(A\left(q_{i(j)}\right) x^* - b\left(q_{i(j)}\right)\right)_j \geq \min_{q \in \mathcal{Q}} \left(A\left(q\right) x - b\left(q\right)\right)_j = 0.$$

  Thus, combining the above two equations, we obtain

  $$\left(A\left(q_{i(j)}\right)\left(x^* - x_\lambda\right)\right)_j > 0, \quad j \in J_2.$$

Now, we again introduce a matrix, still denoting $A^* \in \mathcal{M}$, to be the matrix having the jth row as that of $(A(q_1))_j$, $j \in J_1$ and of $(A(q_{i(j)}))_j$, $j \in J_2$. Therefore, we have

$$A^*\left(x^* - x_\lambda\right) \geq 0.$$

Hence, providing that A^* is an M-matrix, we have that on the whole index set $\mathbb{I}$

$$x^* \geq x_\lambda.$$

□

Lemma 3.3. *Let $\lambda_2 > \lambda_1 > 1$, and x_{λ_1} and x_{λ_2} be the solutions of Problem 2.1 with $\lambda = \lambda_1, \lambda_2$, respectively. Then*

$$x_{\lambda_1} < x_{\lambda_2}.$$

Proof. From the fact that x_{λ_1} and x_{λ_2} be the solutions of Problem 2.1 with $\lambda = \lambda_1, \lambda_2$, respectively, and $\lambda_2 > \lambda_1 > 1$, it follows that

$$\begin{aligned}
&A(q_1)\, x_{\lambda_1} - b(q_1) - \lambda_1 \sum_{i=2}^{M} [b(q_i) - A(q_i)\, x_{\lambda_1}]_+^{1/k} = 0 \\
=&A(q_1)\, x_{\lambda_2} - b(q_1) - \lambda_2 \sum_{i=2}^{M} [b(q_i) - A(q_i)\, x_{\lambda_2}]_+^{1/k} = 0 \\
\leq&A(q_1)\, x_{\lambda_2} - b(q_1) - \lambda_1 \sum_{i=2}^{M} [b(q_i) - A(q_i)\, x_{\lambda_2}]_+^{1/k}.
\end{aligned}$$

This implies that

$$A(q_1)(x_{\lambda_1} - x_{\lambda_2}) \leq \lambda_1 \left(\sum_{i=2}^{M} [b(q_i) - A(q_i)\, x_{\lambda_1}]_+^{1/k} - \sum_{i=2}^{M} [b(q_i) - A(q_i)\, x_{\lambda_2}]_+^{1/k} \right). \tag{10}$$

Defining two disjoint nonempty index subsets J_1 and J_2 of $\mathbb{I}$ as follows

$$J_1 = \left\{ j \,\middle|\, \left(\sum_{i=2}^{M} [b(q_i) - A(q_i)\, x_{\lambda_1}]_+^{1/k} \right)_j \leq \left(\sum_{i=2}^{M} [b(q_i) - A(q_i)\, x_{\lambda_2}]_+^{1/k} \right)_j \right\}, \tag{11}$$

$$J_2 = \left\{ j \,\middle|\, \left(\sum_{i=2}^{M} [b(q_i) - A(q_i)\, x_{\lambda_1}]_+^{1/k} \right)_j > \left(\sum_{i=2}^{M} [b(q_i) - A(q_i)\, x_{\lambda_2}]_+^{1/k} \right)_j \right\}. \tag{12}$$

On one hand, it follows from (10) and (11) that

$$(A(q_1)(\bar{x}_\lambda - x_\lambda))_j \leq 0, \quad \forall j \in J_1.$$

On the other hand, (12) implies for every $j \in J_2$ there exists at least one control $q_{i(j)} \in Q$ associated with j, such that

$$\left(\left[b\left(q_{i(j)}\right) - A\left(q_{i(j)}\right) x_{\lambda_1} \right]_+^{1/k} \right)_j > \left(\left[b\left(q_{i(j)}\right) - A\left(q_{i(j)}\right) x_{\lambda} \right]_+^{1/k} \right)_j,$$

which, by virtue of the monotonicity of the operator $[\cdot]_+^{1/k}$, further implies that

$$\left(A\left(q_{i(j)}\right) x_{\lambda_1} \right)_j \leq \left(A\left(q_{i(j)}\right) x_{\lambda_2} \right)_j, \quad \forall j \in J_2,$$

i.e.

$$\left(A\left(q_{i(j)}\right)(x_{\lambda_1} - x_{\lambda_2}) \right)_j \leq 0, \quad \forall j \in J_2.$$

Now, as we did in the proof of Lemma 3.2, we introduce a matrix, still denoting $A^* \in \mathcal{M}$ to be the matrix having the jth row as that of of $(A(q_1))_j$, $j \in J_1$ and of $(A(q_{i(j)}))_j$, $j \in J_2$. Hence, we have

$$A^*(x_{\lambda_1} - x_{\lambda_2}) \leq 0,$$

which implies on the whole index set $\mathbb{I}$

$$x_{\lambda_1} \leq x_{\lambda_2},$$

since A^* is also a strictly diagonally dominant M-matrix. □

With the above lemmas, we now establish the following monotonic convergence result for the $l_{1/k}$ penalty method.

Theorem 3.1. *Let $\{\lambda_m\}, m = 1, 2, \cdots,$ be a monotonically increasing sequence tending to positive infinity as $m \to \infty$. Assume that x_{λ_m} is the solution of Problem 2.1 with $\lambda = \lambda_m$, and x^* is the solution of Problem 1.1. Then the sequence $\{x_{\lambda_m}\}$ is monotonically increasing and convergent to x^*.*

Proof. It follows from Lemmas 3.3 and 3.2 that

$$x_{\lambda_1} \leq x_{\lambda_2} \leq \cdots \leq x_{\lambda_i} \leq \cdots \leq x^*.$$

This implies that there exists some $x^\star$ such that

$$\lim_{m\to\infty} x_{\lambda_m} = x^\star.$$

Since x_{λ_m} is the solution of Problem 2.1 with $\lambda = \lambda_m$, there must hold

$$A(q_1)\, x_{\lambda_m} - b(q_1) = \lambda_m \sum_{i=2}^{M} [b(q_i) - A(q_i)\, x_{\lambda_m}]_+^{1/k} \geq 0. \tag{13}$$

Letting $m \to \infty$ in (13), we get

$$A(q_1)\, x^\star - b(q_1) \geq 0.$$

Furthermore, reforming (13) gives

$$\sum_{i=2}^{M} [b(q_i) - A(q_i)\, x_{\lambda_m}]_+^{1/k} = \frac{A(q_1)\, x_{\lambda_m} - b(q_1)}{\lambda_m}. \tag{14}$$

Thus, letting $m \to \infty$ in (14), we get

$$\sum_{i=1}^{M} [b(q_i) - A(q_i)\, x^\star]_+^{1/k} = 0,$$

since $A(q_1)$, $b(q_1)$ and x_{λ_m} are bounded. This implies that $b(q) - A(q)\, x^\star \leq 0$, $\forall q \in \mathcal{Q}$. Hence,

$$A(q)\, x^\star - b(q) \geq 0, \quad \forall q \in \mathcal{Q}.$$

Specifically,

$$\min_{q \in \mathcal{Q}} \{A(q)x^\star - b(q)\} \geq 0. \tag{15}$$

In what follows we will prove that $\min_{q\in\mathcal{Q}} \{A(q)x^\star - b(q)\} \leq 0$. In doing so, we distinguish two disjoint nonempty index subsets J_1 and J_2 of $\mathbb{I}$ as we did in (8) and (9).

On one hand, as seen in the proof of Lemma 3.2, for any given $j \in J_{1,}$ we have

$$(A(q_i)x_{\lambda_m} - b(q_i))_j \geq 0, \forall q_i \in Q \quad \text{ and } \ (A(q_1)x_{\lambda_m} - b(q_1))_j = 0,$$

which gives

$$\min_{q \in \mathcal{Q}} (A(q)x_{\lambda_m} - b(q))_j = 0, \quad j \in J_1. \tag{16}$$

On the other hand, for any given $j \in J_2$, based on (9), we can deduce that there exists at least one control $q_{i(j)} \in Q$ associated with j, such that

$$(b(q_i) - A(q_i)x_{\lambda_m})_j = \max_{q \in \mathcal{Q}} (b(q) - A(q)x_{\lambda_m})_j > 0.$$

which means

$$\min_{q \in \mathcal{Q}} (A(q)x_{\lambda_m} - b(q))_j < 0, \quad j \in J_2. \tag{17}$$

Summarizing (16) and (17) we deduce that on the whole index set $\mathbb{I}$

$$\min_{q \in \mathcal{Q}} \{A(q)x_{\lambda_m} - b(q)\} \leq 0. \tag{18}$$

Letting $m \to \infty$ in (18), we get

$$\min_{q \in \mathcal{Q}} \{A(q)x^\star - b(q)\} \leq 0. \tag{19}$$

In view of (15) and (19), we eventually have

$$\min_{q \in \mathcal{Q}} \{A(q)x^\star - b(q)\} = 0.$$

This shows that $x^\star$ solves the discrete HJB Problem 1.1. Since the discrete HJB Problem 1.1 has a unique solution (see, Remark 2.1), we obtain

$$\lim_{m\to\infty} x_{\lambda_m} = x^\star = x^*.$$

□

Remark 3.1. Clearly, Theorem 3.1 gives a constructive proof of the existence of a solution to Problem 1.1.

3.2 Exponential convergence rate

We first give an error estimation of the solution to Problem 2.1.

Theorem 3.2. *Assume that x_λ is the solution of Problem 2.1 for every $\lambda > 1$. There exists a constant $C > 0$, independent of λ, such that*

$$\left\|\min_{q\in\mathcal{Q}} \{A(q)\, x_\lambda - b(q)\}\right\|_\infty \le \frac{C}{\lambda^k}.$$

Proof. It follows from Lemma 3.1 and Remark 2.1 that both x_λ and $A(q_1)$ are bounded, which implies

$$\lambda \sum_{i=2}^{M} [b(q_i) - A(q_i)\, x_\lambda]_+^{1/k} = A(q_1)\, x_\lambda - b(q_1) \le C.$$

Hence, for any $q \in \mathcal{Q}$

$$[b(q) - A(q)\, x_\lambda]_+ \le \frac{C}{\lambda^k}. \tag{20}$$

Furthermore, for every $j \in \mathbb{I}$, we either have

$$\begin{aligned} &(A(q)\, x_\lambda - b(q))_j \ge 0,\ \forall q \in Q \\ \text{and } &(A(q_1)\, x_\lambda - b(q_1))_j = 0 \le \frac{C}{\lambda^k}, \end{aligned}$$

or $\exists q_{i(j)} \in \mathcal{Q}$ associated with j, such that $\left(b(q_{i(j)}) - A(q_{i(j)})x_\lambda\right)_j \ge 0$, which, based on (20), gives

$$\begin{aligned} &\left(A\left(q_{i(j)}\right) x_\lambda - b\left(q_{i(j)}\right)\right)_j = -\left(b\left(q_{i(j)}\right) - A\left(q_{i(j)}\right) x_\lambda\right)_j \ge -\frac{C}{\lambda^k}, \\ \text{and } &(A(q_1)\, x_\lambda - b(q_1))_j > 0. \end{aligned}$$

Hence, both cases reduce to

$$\left\|\min_{q\in\mathcal{Q}} \{A(q)\, x_\lambda - b(q)\}\right\|_\infty \le \frac{C}{\lambda^k}.$$

□

With the above error estimation, we are now ready to show that the solution of Problem 2.1 converges to that of Problem 1.1 exponentially with respect to the penalty parameter.

Theorem 3.3. *Assume that x_λ and x^* are the solution of Problem 2.1 and that of Problem 1.1, respectively. Then for sufficiently large λ, we have*

$$\|x^* - x_\lambda\|_\infty \le \frac{C}{\lambda^k}, \tag{21}$$

where C is a positive constant, independent of x^*, x_λ and λ.

Proof. For $\lambda > 0$, since $A(q_1)x_\lambda - b(q_1) = \lambda \sum_{i=2}^{M} \left[b\left(q_i\right) - A\left(q_i\right) x_\lambda\right]_+^{1/k} \geq 0$, we may define $q_j^* \in \mathcal{Q}$ to be such that for every $j \in \mathbb{I}$,

$$q_{j,\lambda}^* = \begin{cases} \arg\min\limits_{q_{i(j)} \in \mathcal{Q}} \left[A\left(q_{i(j)}\right) x_\lambda - b\left(q_{i(j)}\right)\right]_j, & \text{if } \left(A\left(q_1\right) x_\lambda - b\left(q_1\right)\right)_j > 0, \\ q_1, & \text{if } \left(A\left(q_1\right) x_\lambda - b\left(q_1\right)\right)_j = 0, \end{cases} \tag{22}$$

which means, as seen in Theorem 3.2, that

$$\left|\left(A\left(q_{j,\lambda}^*\right) x_\lambda - b\left(q_{j,\lambda}^*\right)\right)_j\right| = \left|\min_{q_{i(j)} \in \mathcal{Q}} \left(A\left(q_{i(j)}\right) x_\lambda - b\left(q_{i(j)}\right)\right)_j\right| \leq \frac{C_1}{\lambda^k} \tag{23}$$

for some constant $C_1 > 0$ independent of λ. This implies

$$\left(A\left(q_j^*\right) x^* - b\left(q_j^*\right)\right)_j = \min_{q_j \in \mathcal{Q}} \left(A\left(q_j\right) x^* - b\left(q_j\right)\right)_j = 0, \tag{24}$$

where $\lim_{\lambda\to\infty} q_{j,\lambda}^* = q_j^*$ and $\lim_{\lambda\to\infty} x_\lambda = x^*$.

It follows from (24) that

$$\left(A\left(q_{j,\lambda}^*\right) x^* - b\left(q_{j,\lambda}^*\right)\right)_j \geq \min_{q \in \mathcal{Q}} \left(A\left(q\right) x - b\left(q\right)\right)_j = \left(A\left(q_j^*\right) x^* - b\left(q_j^*\right)\right)_j = 0.$$

Hence,

$$\begin{aligned} \left(A\left(q_{j,\lambda}^*\right)\left(x_\lambda - x^*\right)\right)_j &= \left(A\left(q_{j,\lambda}^*\right) x_\lambda - b\left(q_{j,\lambda}^*\right)\right)_j - \left(A\left(q_{j,\lambda}^*\right) x^* - b\left(q_{j,\lambda}^*\right)\right)_j \\ &\leq \left(A\left(q_{j,\lambda}^*\right) x_\lambda - b\left(q_{j,\lambda}^*\right)\right)_j. \end{aligned}$$

Now, using (23), we get

$$\left(A\left(q_{j,\lambda}^*\right)\left(x_\lambda - x^*\right)\right)_j \leq \frac{C_1}{\lambda^k}.$$

Meanwhile,

$$\begin{aligned} \left(A\left(q_j^*\right)\left(x^* - x_\lambda\right)\right)_j &= \left[\left(A\left(q_j^*\right) x^* - b\left(q_j^*\right)\right) - \left(A\left(q_{j,\lambda}^*\right) x_\lambda - b\left(q_{j,\lambda}^*\right)\right)\right]_j \\ &\quad + \left[\left(A\left(q_{j,\lambda}^*\right) x_\lambda - b\left(q_{j,\lambda}^*\right)\right) - \left(A\left(q_j^*\right) x_\lambda - b\left(q_j^*\right)\right)\right]_j \\ &\leq \left[\left(A\left(q_j^*\right) x^* - b\left(q_j^*\right)\right) - \left(A\left(q_{j,\lambda}^*\right) x_\lambda - b\left(q_{j,\lambda}^*\right)\right)\right]_j, \end{aligned}$$

since the definition $q_{j,\lambda}^*$ in (22) implies $[(A(q_{j,\lambda}^*)x_\lambda - b(q_{j,\lambda}^*)) - (A(q_j^*)x_\lambda - b(q_j^*))] \leq 0$. Moreover, it follows from (23) and (24) that

$$\begin{aligned} &\left[\left(A\left(q_j^*\right) x^* - b\left(q_j^*\right)\right) - \left(A\left(q_{j,\lambda}^*\right) x_\lambda - b\left(q_{j,\lambda}^*\right)\right)\right]_j \\ &= -\left(A\left(q_{j,\lambda}^*\right) x_\lambda - b\left(q_{j,\lambda}^*\right)\right)_j \leq \frac{C_1}{\lambda^k}. \end{aligned}$$

Hence,

$$\left(A\left(q_j^*\right)\left(x^* - x_\lambda\right)\right)_j \leq \frac{C_1}{\lambda^k}.$$

Denoting by $A_1^*, A_2^* \in \mathcal{M}$ the matrices having the jth rows, $j \in \mathbb{I}$, as those of of $A(q_{j,\lambda}^*)$ and $A(q_j^*)$, respectively, we obtain that

$$x_\lambda - x^* \leq \frac{C_1 \left\| A_1^{*-1} \right\|_\infty}{\lambda^k}, \text{ and } x^* - x_\lambda \leq \frac{C_1 \left\| A_2^{*-1} \right\|_\infty}{\lambda^k},$$

since it follows from the fact that both A_1^* and A_2^* are strictly diagonally dominant M-matrices. Now, noting Remark 2.1, we infer that

$$\|x^* - x_\lambda\|_\infty \leq \frac{C}{\lambda^k},$$

for some constant $C > 0$ independent of λ, x_λ and x^*. □

4 Examples

In this section we illustrate the theoretical rates of convergence obtained in (21) and show that the assumption (A) is only sufficient using two examples. The first one is an obstacle problem and the second one is a generalized complementarity problem. We also use a third example to show the efficiency of the new method.

Example 4.1. Consider the following discrete HJB equation

$$\max_{q \in \mathcal{Q}} \{A(q)x - b(q)\} = 0,$$

with $\mathcal{Q} = \{q_1, q_2\}$, and

$$A(q_1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A(q_2) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad b(q_1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad b(q_2) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

This example is from [10] and the exact solution is $x = (-1, 0)^\top$. Clearly, the assumption A1 is not satisfied. The power penalty approach to this example is stated as

$$A(q_1)x - b(q_1) + \lambda \left[A(q_2)x - b(q_2)\right]_+^{1/k} = 0.$$

When $k = 1$, the solution of the penalized equation is $x_\lambda = (-\frac{2\lambda}{1+2\lambda}, 0)^\top$. Thus,

$$\|x - x_\lambda\|_2 = \left\| \begin{bmatrix} -\frac{2\lambda}{1+2\lambda} \\ 0 \end{bmatrix} \right\|_2 = \frac{1}{1+2\lambda} \leq \frac{0.5}{\lambda}.$$

When $k = 2$, the solution of the penalized equation is $x_\lambda = (\sqrt{\lambda^4 + 2\lambda^2} - 1 - \lambda^2, 0)^\top$. Thus,

$$\|x - x_\lambda\|_2 = \left\| \begin{bmatrix} \sqrt{\lambda^4 + 2\lambda^2} - \lambda^2 - 1 \\ 0 \end{bmatrix} \right\|_2 = \sqrt{\lambda^4 + 2\lambda^2} - \lambda^2 - 1 \leq \frac{1}{\lambda^2}.$$

Both results coincide with the theoretical convergence rate $\mathcal{O}(\frac{1}{\lambda^k})$ in (21).

Example 4.2. By setting $\mathcal{Q} = \{q_1, q_2\}$ and

$$A(q_1) = \begin{bmatrix} 3 & -2 \\ -4 & 5 \end{bmatrix}, \quad A(q_2) = \begin{bmatrix} 4 & -4 \\ -1 & 2 \end{bmatrix}, \quad b(q_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad b(q_2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

we consider the following discrete HJB equation

$$\min_{q \in \mathcal{Q}} \{A(q)x - b(q)\} = 0.$$

This example is from [7] and the exact solution is $x = (9/4, 2)^\top$. Noting that the assumption (A) is not satisfied as well. The power penalty approach to this example is

$$A(q_1)x - b(q_1) - \lambda \left[A(q_2)x - b(q_2)\right]_+^{1/k} = 0.$$

When $k = 1$, the solution of the penalized equation is $x_\lambda = (\frac{9\lambda+3}{4\lambda+23}, \frac{8\lambda+7}{4\lambda+23})^\top$. Thus,

$$\|x - x_\lambda\|_2 = \left\| \begin{bmatrix} \frac{9}{4} - \frac{9\lambda+3}{4\lambda+23} \\ 2 - \frac{8\lambda+7}{4\lambda+23} \end{bmatrix} \right\|_2 = \frac{39\sqrt{26}}{23 + 4\lambda} \le \frac{50}{\lambda}.$$

When $k = 2$, the solution of the penalized equation is

$$x_\lambda = \left(\frac{1}{529}(5\sqrt{4\lambda^4 + 897\lambda^2} - 10\lambda^2 + 69), \frac{1}{529}(4\sqrt{4\lambda^4 + 897\lambda^2} - 8\lambda^2 + 161) \right)^\top.$$

Thus,

$$\begin{aligned} \|x - x_\lambda\|_2 &= \left\| \begin{bmatrix} \frac{9}{4} - \frac{1}{529}(5\sqrt{4\lambda^4 + 897\lambda^2} - 10\lambda^2 + 69) \\ 2 - \frac{1}{529}(4\sqrt{4\lambda^4 + 897\lambda^2} - 8\lambda^2 + 161) \end{bmatrix} \right\|_2 \\ &= \frac{\sqrt{41}}{2116}(8\lambda^2 - 4\sqrt{\lambda^2(4\lambda^2 + 897)} + 897) \\ &\le \frac{1605.64}{8\lambda^2 + 4\sqrt{\lambda^2(4\lambda^2 + 897)} + 897} \\ &\le \frac{101}{\lambda^2} \end{aligned}$$

These results again confirms the theoretical convergence rate $\mathcal{O}(\frac{1}{\lambda^k})$ in (21).

Example 4.3. Consider the Markovian dynamic programming (MDP) problem in [8] which can be written as

$$V_i = \max\left\{V_{i-1} + f_i^1, V_{i+1} + f_i^2\right\}, \quad i = 0, \ldots, M,$$

where $f_0^1 = f_0^2 = f_M^1 = f_M^2 = 0$, $f_i^1 = -1, f_i^2 = -2$ for all $i = 1, \ldots, M-1$, and $f_{M-1}^1 = -1$, $f_{M-1}^2 = 2M$. We apply the l_1 penalty method to solve this problem, which results in

$$A^1 V_\lambda - \lambda[b^2 - A^2 V_\lambda]_+ = b^1, \tag{25}$$

with

$$A^1 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

are two $(M+1) \times (M+1)$ matrices and

$$b^1 = \begin{bmatrix} 0 & -2 & \cdots & -2 & 2M & 0 \end{bmatrix}^\top, \quad b^2 = \begin{bmatrix} 0 & -1 & \cdots & -1 & -1 & 0 \end{bmatrix}^\top$$

are two $M+1$ vectors.

In our numerical tests we increase M from 100 to 2000. We use the generalized Newton method to solve the semismooth equation (25). It is worth noting that though both A^1 and A^2 are not strictly diagonally dominant M-matrices, the penalty method works very well. All the numerical results show that the number of iterations of the l_1 penalty method stays between 1 and 2 when the initial guess is set to be $V_0 = 0$. However, as stated in [3, 8], with the same initial guess ($V_0 = 0$), the number of iteration of the policy iteration is $M-1$ since it will correct the optimal control one by one, from grid $M-1$ to grid 1. This example shows that comparing the popular used policy iteration method, the proposed penalty method works efficiently.

5 Conclusions

A power penalty approach to the discrete HJB equations with a finite control set was developed. An exponential convergence rate estimate was obtained for the solution of the power penalized nonlinear equation to that of the discrete HJB equation. Examples were examined to confirm the theoretical results and show its efficiency. The convergence rate estimates imply one advantage, that is, to achieve the same level accuracy of the approximation solution to that of the discrete HJB equation, the penalty parameter required for $k > 1$ is smaller than that required for $k = 1$. Moreover, we also showed that under some circumstance, the penalty method is much more efficient than the popular policy iteration method.

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