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A Generalized Newton Method for a Class of Discrete-Time Linear Complementarity Systems*

Zhe Sun[†] Xiaoqi Yang[‡]

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Abstract

In this paper, we propose a generalized Newton method for solving a class of discrete-time linear complementarity systems consisting of a system of linear equations and a linear complementarity constraints with a Z-matrix. We obtain a complete characterization of the least element solution of a linear complementarity problem with a Z-matrix that a solution is the least element solution if and only if the principal submatrix corresponding to the nonzero components of the solution is an M-matrix. We present a Newton method for solving a linear complementarity problem with a Z-matrix. We propose a generalized Newton method for solving the discrete-time linear complementarity system where the linear complementarity problem constraint is solved by the proposed Newton method. Under suitable conditions, we show that the generalized Newton method converges globally and finds a solution in finitely many iterations. Preliminary numerical results show the efficiency of the proposed method.

Key Words: linear complementarity system; Z-matrix; least element solution; generalized Newton method; finite termination; linear rate of convergence

Mathematics Subject Classification (2010): 90C33; 49M15

1 Introduction

Consider the following differential linear complementarity system:

$$\begin{aligned} &\dot{x}(t) = Qx(t) + Cy(t) + f(t), \quad t \in [0, T], \\ &y(t) \in \mathrm{SOL}(Bx(t) + g(t), A), \quad t \in [0, T], \\ &x(0) = x_0, \end{aligned} \tag{1.1}$$

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[†]College of Mathematics and Information Science, Jiangxi Normal University, Nanchang, Jiangxi, P.R. China

[‡]Department of Applied Mathematics, The Hong Kong Polytechnic University, Kowloon, Hong Kong, P.R. China

where $Q \in \mathbb{R}^{m\times m}$, $C \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times m}$, $A \in \mathbb{R}^{n\times n}$ is an Z-matrix (i.e., $a_{ij} \leq 0$ for all $i \neq j$ (e.g., see [1])), $f : \mathbb{R} \to \mathbb{R}^m$ and $g : \mathbb{R} \to \mathbb{R}^n$ are two Lipschitz continuous functions with Lipschitz constants $L_f > 0$ and $L_g > 0$, respectively. Throughout this paper, SOL(q, A) denotes the solution set of a linear complementarity problem with a Z-matrix (abbreviated as ZLCP) in the form of

$$y \geq 0, \qquad Ay + q \geq 0, \qquad y^T(Ay + q) = 0, \tag{1.2}$$

where $q = (q_1, q_2, \cdots, q_n)^T \in \mathbb{R}^n$. The set SOL$(q, A)$ is nonempty if the feasible region FEA$(A, q) \triangleq \{y \in \mathbb{R}^n | y \geq 0, Ay + q \geq 0\}$ is nonempty and the least element of FEA(A, q) is a solution of ZLCP (1.2), which is called the least element solution of (1.2) (e.g., see [19, 9]).

The differential linear complementarity system (1.1) has many applications in the scientific and engineering fields (e.g., see [11, 14, 15, 17]). Some systematic-theoretic results of this system on the existence and stabilizability of solutions and how they depend on initial conditions have been studied (e.g., see [14, 15, 25, 26]). It is worth noting that comprehensive study of a closely related topic - differential variational inequalities (abbreviated as DVI) with applications in nonsmooth dynamical systems - has been carried out in [20, 21] and the references therein.

The time-stepping scheme has been widely used for solving the differential linear complementarity system (1.1), in which the time interval $[0, T]$ is subdivided into N subintervals $[t_{l-1}, t_l]$ and a sequence of (discrete-time) linear complementarity systems in the form of

$$\begin{aligned} x = h_{l+1}[(1-\theta)Qx + Cy] + x^{h,l} + h_{l+1}[\theta Q x^{h,l} + f(t_{l+1})], \\ \min(y, Ay + Bx + g(t_{l+1})) = 0, \quad l = 0, 1, \cdots, N-1, \end{aligned}$$

is solved, where $x^{h,0} = x(0)$, $h_{l+1} = t_{l+1} - t_l$. The parameter $\theta \in [0, 1]$ is a scalar to distinguish an explicit scheme ($\theta = 1$), an implicit one ($\theta = 0$), or a semi-implicit one ($\theta \in (0, 1)$), respectively. For detailed discussions of the convergence of the time-stepping scheme, we refer to [13, 21]. For simplicity, we set $\theta = 0$ and assume that $h_l = h = T/N_h$. At each time step one solves a discrete-time linear complementarity system (abbreviated as DLCS) in the form of

$$(I - hQ)x - hCy - [x^{h,l} + hf(t_{l+1})] = 0, \tag{1.3}$$

$$\min(y, Ay + Bx + g(t_{l+1})) = 0, \tag{1.4}$$

where I stands for the identity matrix. A critical part for solving DLCS (1.3)-(1.4) is to deal with (1.4) efficiently and accurately.

To the best of our knowledge, the numerical study of DVI and DLCS (1.3)-(1.4) is very limited. In [6, 7], linear complementarity problem (1.4) was viewed as a constraint of (1.3) and the variable y as a function of the variable x determined by (1.4) respectively. Properties of the least element solution of general ZLCP (1.2) was studied [6, 7]. In particular, it was proved in [7] that if $y(q) \in$ SOL(q, A) is the least element solution of ZLCP (1.2), then the matrix $I - D + DA$ is nonsingular and $y(q) = -(I - D + DA)^{-1}Dq$, where $D = \mathrm{diag}(d_1, \cdots, d_n)$ is a diagonal matrix with diagonals

$$d_i = \begin{cases} 1, & y_i(q) > 0, \\ 0, & \text{otherwise}. \end{cases} \tag{1.5}$$

In addition, it was verified that $y(q)$ is Lipschitz continuous as a function of q and $-(I-D+DA)^{-1}D \in \partial y(q)$, where $\partial y(q)$ is the generalized Jacobian of $y(q)$, see Clarke [8]. By using these properties, DLCS (1.3)-(1.4) were reformulated in [7] as a nonsmooth equation in the form of

$$H(x) \triangleq (I - hQ)x - hCy(q(x)) - [x^{h,l} + hf(t_{l+1})] = 0, \tag{1.6}$$

where $q(x) = Bx + g(t_{l+1})$, and the following generalized Newton iteration was proposed for solving this equation:

$$u^{k+1} = u^k - V_k^{-1}H(u^k) \tag{1.7}$$

with

$$V_k = I - h[Q - C(I - D_k + D_kA)^{-1}D_kB] \in \partial H(u^k),$$

where $D_k = \mathrm{diag}(d_1, \cdots, d_n)$ and d_i is given by (1.5) with $q = q(u^k)$. Under suitable conditions, the iteration (1.7) was shown to converge superlinearly to a solution $x^{h,l+1}$ of (1.3)-(1.4) from the starting point $u^0 = x^{h,l}$. However, the iteration (1.7) encounters some practical issues. Indeed, given a current iterate u^k, it needs to find the least element solution $y(q)$ of a ZLCP in the form of (1.2) with $q = Bu^k + g(t_{l+1})$ in order to define $H(u^k)$ and D_k, which is normally time-consuming. Additionally, it needs to compute the inverse of the matrix $I - D_k + D_kA$ so as to compute V_k.

Generalized Newton methods have been extensively studied for solving piecewise linear systems, such as linear complementarity problems arising from the discretization of American options pricing problems [23] and obstacle problems [16], the discrete HJB equation [2, 29] and piecewise linear systems arising in the numerical solution of the free-surface hydrodynamics models [3, 4, 28]. It has been verified that this type of methods possess a finite termination property, i.e., they are able to find a solution in a finite number of iterations under suitable conditions [2, 3, 10, 12, 27, 28]. In addition, if the generalized Jacobi matrix is an M-matrix, these methods converge globally.

In this paper, we will view DLCS (1.3)-(1.4) as a piecewise linear system with respect to variables x and y and propose a generalized Newton method for solving it.

We first study some new characterizations of solutions of ZLCP (1.2). In particular, we establish a complete characterization of the least element solution of ZLCP (1.2) that a solution of ZLCP (1.2) is the least element solution if and only if the principal submatrix of A corresponding to the nonzero components of the solution is an M-matrix, see Theorem 2.1. In addition, we show that an x-component dominated reduced matrix is also an M-matrix, see Proposition 2.1. By virtue of this latter property, we then propose a Newton method for solving ZLCP (1.2) and show that the method terminates finitely under some additional conditions.

We next propose a generalized Newton method for solving DLCS (1.3)-(1.4). To this end, define a mapping $F : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$ by

$$F(u,v) = \left(\begin{array}{c} (I - hQ)u - hCv - x^{h,l} - hf(t_{l+1}) \\ \min(v, Av + \underbrace{Bu + g(t_{l+1})}_{q(u)}) \end{array} \right) \tag{1.8}$$

and a mapping $G : \mathbb{R}^{m+n} \to \mathbb{R}^{(m+n)\times(m+n)}$ by

$$G(u,v) = \left[\begin{array}{cc} I - hQ & -hC \\ DB & I - D + DA \end{array} \right], \tag{1.9}$$

where

$$D = \mathrm{diag}(d_1, \cdots, d_n), \qquad d_i = \begin{cases} 1, & v_i > [Av + q(u)]_i, \\ 0, & \text{otherwise}. \end{cases} \tag{1.10}$$

It follows from [8] that $G(u,v)$ is a generalized Jacobi matrix of F at (u,v). Instead of computing the least element solution of the following ZLCP as in [6, 7]

$$v \ge 0, \qquad Av + q(u^k) \ge 0, \qquad v^T(Av + q(u^k)) = 0,$$

we find an approximate solution $\bar{v}^k$ and then compute Δu and Δv by solving the following system of linear equations

$$G(u^k, \bar{v}^k) \left(\begin{array}{c} \Delta u \\ \Delta v \end{array} \right) = -F(u^k, \bar{v}^k),$$

and the corresponding new iterates by

$$u^{k+1} = u^k + \Delta u \quad \text{and} \quad v^{k+1} = \bar{v}^k + \Delta v.$$

We will show that under proper conditions, the proposed method converges globally and finds a solution of DLCS (1.3)-(1.4) in finitely many iterations. Moreover, if the ZLCP is solved exactly at each iteration, it converges at a linear convergence rate.

The rest of this paper is organized as follows. In Subsection 1.1, we introduce notation and preliminary lemmas that are used in the paper. In Section 2, we study some characterizations of the least element solution of ZLCP (1.2) and present a Newton method for solving the least element solution of ZLCP (1.2). In Section 3, we propose a novel generalized Newton method for solving DLCS (1.3)-(1.4) and study its convergence. Finally, in Section 4, we give numerical experiments to illustrate the efficiency of the proposed method.

1.1 Notation and preliminary lemmas

Given a vector $x \in \mathbb{R}^n$, a matrix $B \in \mathbb{R}^{n\times n}$, two index sets $\mathcal{I} \subseteq \{1, 2, \cdots, n\}$ and $\mathcal{J} \subseteq \{1, 2, \cdots, n\}$, $B_{\mathcal{I}\mathcal{J}}$ denotes the submatrix of B consisting of rows and columns indexed by $\mathcal{I}$ and $\mathcal{J}$ respectively, and $x_{\mathcal{I}}$ denotes the subvector of x consisting of components indexed by $\mathcal{I}$. For an index i, B_i denotes the ith row of B, and in particular e_i denotes the ith row of I. A vector x is nonnegative (resp. positive), denoted by $x \geq 0$ (resp. $x > 0$), if its components are nonnegative (resp. positive). A matrix B is nonnegative, denoted by $B \geq 0$, if its entries are nonnegative. We write that $x \geq y$ (resp. $B \geq C$) if x (resp. B) and y (resp. C) satisfy that $x - y \geq 0$ (resp. $B - C \geq 0$). Denote by $U(x, r)$ the open ball centered by x with the radius of r in the ℓ_2 norm.

We end this section by introducing two lemmas, the first of which is from [1].

Lemma 1.1. *Let $B \in \mathbb{R}^{m\times m}$ be a Z-matrix. Then, B is an M-matrix if and only if one of the following two statements is true.*
(i) B is monotone, i.e., if $Bv \geq 0$, then $v \geq 0$.
(ii) There exists $v > 0$ with $Bv > 0$.

Lemma 1.2. *Suppose that $B \in \mathbb{R}^{m\times m}$ is a Z-matrix. Let $\mathcal{I} \subseteq \{1, 2, \cdots, m\}$ and $\mathcal{I} \neq \emptyset$. Then, the following statements are true.*
(i) Let B be an M-matrix. Then, $B_{\mathcal{I}\mathcal{I}}$ is an M-matrix.
(ii) Let $B_{\mathcal{I}\mathcal{I}}$ be an M-matrix and $W \in \mathbb{R}^{m\times m}$ be defined as follows:

$$W_i = \begin{cases} B_i, & \text{if } \; i \in \mathcal{I}, \\ e_i, & \text{otherwise.} \end{cases}$$

Then, W is an M-matrix.

Proof. Let B be an M-matrix. Then, $B_{\mathcal{I}\mathcal{I}}$ is a Z-matrix and $B_{\mathcal{I}\mathcal{I}^c} \leq 0$, where $\mathcal{I}^c = \{1, 2, \cdots, m\}/\mathcal{I}$. Moreover, one obtains from Lemma 1.1 that there is a vector $v \in \mathbb{R}^m$ such that $v > 0$ and $Bv > 0$. Note that $(Bv)_{\mathcal{I}} = B_{\mathcal{I}\mathcal{I}}v_{\mathcal{I}} + B_{\mathcal{I}\mathcal{I}^c}v_{\mathcal{I}^c}$. It follows that $B_{\mathcal{I}\mathcal{I}}v_{\mathcal{I}} = (Bv)_{\mathcal{I}} - B_{\mathcal{I}\mathcal{I}^c}v_{\mathcal{I}^c} > 0$, where the strict inequality is due to the facts that $(Bv)_{\mathcal{I}} > 0$, $B_{\mathcal{I}\mathcal{I}^c} \leq 0$, and $v_{\mathcal{I}^c} > 0$. Recall that $B_{\mathcal{I}\mathcal{I}}$ is a Z-matrix and $v_{\mathcal{I}} > 0$. We conclude from Lemma 1.1 that $B_{\mathcal{I}\mathcal{I}}$ is an M-matrix.

Since $B_{\mathcal{I}\mathcal{I}}$ is an M-matrix, it follows from Lemma 1.1 that there is a positive vector $v \in \mathbb{R}^{|\mathcal{I}|}$ such that $B_{\mathcal{I}\mathcal{I}}v > 0$. Define a vector $v(\epsilon) \in \mathbb{R}^m$ such that $v_i(\epsilon) = v_i$ for $i \in \mathcal{I}$ and $v_i(\epsilon) = \epsilon$ for $i \notin \mathcal{I}$. Then, we get by a simple calculation that $v(\epsilon) > 0$ and $Wv(\epsilon) > 0$ for any $\epsilon > 0$ sufficiently small. Recall that W is a Z-matrix. We conclude from Lemma 1.1 that W is an M-matrix. $\square$

2 On the least element solution of ZLCP (1.2)

In this section, we shall obtain some new characterizations for the least element solution of ZLCP (1.2) and propose a Newton method for solving ZLCP (1.2).

2.1 Characterizations for the least element solution of ZLCP (1.2)

In this subsection, we study characterizations of the least element solution of ZLCP (1.2). We first give a lemma.

Lemma 2.1. *Let $x^* \in \mathbb{R}^n$ be the least element solution of ZLCP (1.2). Then, the following statements are true.*
(i) If there exists an index i such that $a_{ii} > 0$ and $q_i < 0$, then $x_i^* \geq -q_i/a_{ii}$.
(ii) If there exists an index i such that $a_{ii} \leq 0$, then $x_i^* = 0$.

Proof. Statement (i) is from [5, 9] and we omit the proof here.

(ii) Assume for contradiction that $x_i^* > 0$. Since x^* is the least element solution of (1.2), one has

$$x_i^* > 0, \quad (Ax^* + q)_i = 0 \qquad \text{and} \qquad x_l^* \geq 0, \quad (Ax^* + q)_l \geq 0, \quad \forall l \neq i. \tag{2.1}$$

Let $\bar{x} = x^* - x_i^* e_i^T$. Then, it follows that $\bar{x}_i = 0 < x_i^*$ and $\bar{x}_l = x_l^* \geq 0$ for any $l \neq i$, which mean that $0 \leq \bar{x} \leq x^*$ and $\bar{x} \neq x^*$. Recall that $a_{ii} \leq 0$ and $a_{li} \leq 0$. One can obtain from (2.1) that

$$\begin{aligned}
(A\bar{x} + q)_i &= a_{ii}\bar{x}_i + \sum_{j\neq i} a_{ij}\bar{x}_j + q_i = a_{ii}x_i^* + \sum_{j\neq i} a_{ij}x_j^* + q_i - a_{ii}x_i^* \geq (Ax^* + q)_i = 0,\\
(A\bar{x} + q)_l &= a_{li}\bar{x}_i + \sum_{j\neq i} a_{lj}\bar{x}_j + q_l = a_{li}x_i^* + \sum_{j\neq i} a_{lj}x_j^* + q_l - a_{li}x_i^* \geq (Ax^* + q)_l \geq 0, \quad \forall l \neq i.
\end{aligned}$$

Therefore, it holds that $A\bar{x} + q \geq 0$. This, together with the fact that $\bar{x} \geq 0$, imply $\bar{x} \in \mathrm{FEA}(A, q)$. Notice that $\bar{x} \leq x^*$ and $\bar{x} \neq x^*$. One can deduce that x^* is not the least element solution of ZLCP (1.2), which yields a contradiction. Therefore, it must hold that $x_{i'}^* = 0$. The proof is complete. □

Without loss of generality, we make assumptions on $A \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$ as follows.

Assumption 2.1. *All diagonal elements of $A \in \mathbb{R}^{n\times n}$ are positive and there exists an index i such that $q_i < 0$.*

Let $x \in \mathbb{R}^n$. In the remaining part of this section, define

$$\mathcal{I}(x) \triangleq \{i \mid x_i > 0\}, \qquad \mathcal{J}(x) \triangleq \{i \mid x_i = 0\}.$$

It has been verified by Chen and Xiang [7] that if $x^* \in \mathbb{R}^n$ is the least element solution of ZLCP (1.2), then the principal submatrix $A_{\mathcal{I}(x^*)\mathcal{I}(x^*)}$ is nonsingular. However, the converse of this result is not necessarily true. The next theorem presents a sufficient and necessary condition for the least element solution of ZLCP (1.2).

Theorem 2.1. *Let A and q in ZLCP* (1.2) *satisfy Assumption 2.1. Suppose that $x^* \in \mathbb{R}^n$ is a solution of (1.2). Then, x^* is the least element solution of ZLCP (1.2) if and only if $A_{\mathcal{I}(x^*)\mathcal{I}(x^*)}$ is an M-matrix.*

Proof. Suppose that x^* is the least element solution of ZLCP (1.2). Notice that A and q satisfy Assumption 2.1 and that A is a Z-matrix. One can obtain by Lemma 2.1 (i) that $\mathcal{I}(x^*) \neq \emptyset$, which, together with the fact that A is a Z-matrix, means that $A_{\mathcal{I}(x^*)\mathcal{I}(x^*)}$ is also a Z-matrix. Therefore, it follows from Lemma 1.1 that to prove $A_{\mathcal{I}(x^*)\mathcal{I}(x^*)}$ is an M-matrix, it suffices to show that $A_{\mathcal{I}(x^*)\mathcal{I}(x^*)}$ is monotone. Assume for contradiction that $A_{\mathcal{I}(x^*)\mathcal{I}(x^*)}$ is not monotone. Then, we can obtain by

Lemma 1.1 that there exist a vector $w \in \mathbb{R}^{|\mathcal{I}(x^*)|}$, where $|\mathcal{I}(x^*)|$ denotes the cardinality of $\mathcal{I}(x^*)$, and an index i_1 such that

$$A_{\mathcal{I}(x^*)\mathcal{I}(x^*)} w \geq 0, \qquad w_{i_1} < 0. \tag{2.2}$$

Consider the following ZLCP

$$v \geq 0, \qquad A_{\mathcal{I}(x^*)\mathcal{I}(x^*)} v + q_{\mathcal{I}(x^*)} \geq 0, \qquad v^T [A_{\mathcal{I}(x^*)\mathcal{I}(x^*)} v + q_{\mathcal{I}(x^*)}] = 0. \tag{2.3}$$

Recall that x^* is the least element solution of ZLCP (1.2). Then, $x^*_{\mathcal{I}(x^*)}$ satisfies that

$$x^*_{\mathcal{I}(x^*)} > 0, \qquad A_{\mathcal{I}(x^*)\mathcal{I}(x^*)} x^*_{\mathcal{I}(x^*)} + q_{\mathcal{I}(x^*)} = 0,$$

which, together with (2.2), imply that there exists a constant $\delta > 0$ sufficiently small such that

$$x^*_{\mathcal{I}(x^*)} + \delta w \geq 0, \qquad A_{\mathcal{I}(x^*)\mathcal{I}(x^*)} [x^*_{\mathcal{I}(x^*)} + \delta w] + q_{\mathcal{I}(x^*)} = \delta A_{\mathcal{I}(x^*)\mathcal{I}(x^*)} w \geq 0.$$

Therefore, $x^*_{\mathcal{I}(x^*)} + \delta w \in \text{FEA}(A_{\mathcal{I}(x^*)\mathcal{I}(x^*)}, q_{\mathcal{I}(x^*)})$, where $\text{FEA}(A_{\mathcal{I}(x^*)\mathcal{I}(x^*)}, q_{\mathcal{I}(x^*)})$ is a feasible region of ZLCP (2.3). Moreover, $x^*_{\mathcal{I}(x^*)} \in \text{FEA}(A_{\mathcal{I}(x^*)\mathcal{I}(x^*)}, q_{\mathcal{I}(x^*)})$. Observe from (2.2) that $w_{i_1} < 0$. It follows that

$$[x^*_{\mathcal{I}(x^*)} + \delta w]_{i_1} = [x^*_{\mathcal{I}(x^*)}]_{i_1} + \delta w_{i_1} < [x^*_{\mathcal{I}(x^*)}]_{i_1}.$$

Recall that $A_{\mathcal{I}(x^*)\mathcal{I}(x^*)}$ is a Z-matrix and $\text{FEA}(A_{\mathcal{I}(x^*)\mathcal{I}(x^*)}, q_{\mathcal{I}(x^*)}) \neq \emptyset$. It follows from Proposition 3.11.3 of [9] that $\min(x^*_{\mathcal{I}(x^*)} + \delta w, x^*_{\mathcal{I}(x^*)}) \in \text{FEA}(A_{\mathcal{I}(x^*)\mathcal{I}(x^*)}, q_{\mathcal{I}(x^*)})$ and ZLCP (2.3) has a solution $\bar{v} \in \mathbb{R}^{|\mathcal{I}(x^*)|}$ that satisfies $\bar{v} \leq x^*_{\mathcal{I}(x^*)}$ and $\bar{v} \neq x^*_{\mathcal{I}(x^*)}$. Define a vector $\bar{x} \in \mathbb{R}^n$ by

$$\bar{x}_i = \begin{cases} \bar{v}_i, & \text{if } i \in \mathcal{I}(x^*), \\ 0, & \text{otherwise}. \end{cases} \tag{2.4}$$

Then, $0 \leq \bar{x} \leq x^*$ and $\bar{x} \neq x^*$. Moreover, by (2.4), one has

$$(A\bar{x} + q)_{\mathcal{I}(x^*)} = A_{\mathcal{I}(x^*)\mathcal{I}(x^*)} \bar{x}_{\mathcal{I}(x^*)} + A_{\mathcal{I}(x^*)\mathcal{J}(x^*)} \bar{x}_{\mathcal{J}(x^*)} + q_{\mathcal{I}(x^*)} = A_{\mathcal{I}(x^*)\mathcal{I}(x^*)} \bar{v} + q_{\mathcal{I}(x^*)} \tag{2.5}$$

and

$$(A\bar{x} + q)_{\mathcal{J}(x^*)} = A_{\mathcal{J}(x^*)\mathcal{I}(x^*)} \bar{x}_{\mathcal{I}(x^*)} + A_{\mathcal{J}(x^*)\mathcal{J}(x^*)} \bar{x}_{\mathcal{J}(x^*)} + q_{\mathcal{J}(x^*)} = A_{\mathcal{J}(x^*)\mathcal{I}(x^*)} \bar{v} + q_{\mathcal{J}(x^*)}. \tag{2.6}$$

On the other hand, noticing that $\bar{v}$ is a solution of ZLCP (2.3), $0 \leq \bar{v} \leq x^*_{\mathcal{I}(x^*)}$, $A_{\mathcal{J}(x^*)\mathcal{I}(x^*)} \leq 0$, and $x^*_{\mathcal{J}(x^*)} = 0$, one derives

$$A_{\mathcal{I}(x^*)\mathcal{I}(x^*)} \bar{v} + q_{\mathcal{I}(x^*)} \geq 0 \tag{2.7}$$

and

$$A_{\mathcal{J}(x^*)\mathcal{I}(x^*)} \bar{v} + q_{\mathcal{J}(x^*)} \geq A_{\mathcal{J}(x^*)\mathcal{I}(x^*)} x^*_{\mathcal{I}(x^*)} + q_{\mathcal{J}(x^*)} = (Ax^* + q)_{\mathcal{J}(x^*)} \geq 0, \tag{2.8}$$

where the last inequality in (2.8) follows from the fact that x^* is a solution of ZLCP (1.2). By combining (2.5), (2.6), (2.7), and (2.8), one further obtains that $A\bar{x} + q \geq 0$, which, together with $\bar{x} \geq 0$, implies that $\bar{x} \in \text{FEA}(A, q)$. Recall that $\bar{v} \leq x^*_{\mathcal{I}(x^*)}$ and $\bar{v} \neq x^*_{\mathcal{I}(x^*)}$. Then, x^* cannot be the least element of ZLCP (1.2), which yields a contradiction. Therefore, $A_{\mathcal{I}(x^*)\mathcal{I}(x^*)}$ must be monotone. Recall that $A_{\mathcal{I}(x^*)\mathcal{I}(x^*)}$ is also a Z-matrix. One can obtain by Lemma 1.1 that $A_{\mathcal{I}(x^*)\mathcal{I}(x^*)}$ is an M-matrix.

Conversely, suppose that $A_{\mathcal{I}(x^*)\mathcal{I}(x^*)}$ is an M-matrix. For any $x \in \mathrm{FEA}(A, q)$, one has

$$x_{\mathcal{J}(x^*)} \geq 0 = x^*_{\mathcal{J}(x^*)}, \qquad (Ax+q)_{\mathcal{I}(x^*)} \geq 0 = (Ax^*+q)_{\mathcal{I}(x^*)},$$

which imply that

$$x_{\mathcal{J}(x^*)} \geq x^*_{\mathcal{J}(x^*)} \tag{2.9}$$

and

$$A_{\mathcal{I}(x^*)\mathcal{I}(x^*)}x_{\mathcal{I}(x^*)} + A_{\mathcal{I}(x^*)\mathcal{J}(x^*)}x_{\mathcal{J}(x^*)} + q_{\mathcal{I}(x^*)} \geq A_{\mathcal{I}(x^*)\mathcal{I}(x^*)}x^*_{\mathcal{I}(x^*)} + A_{\mathcal{I}(x^*)\mathcal{J}(x^*)}x^*_{\mathcal{J}(x^*)} + q_{\mathcal{I}(x^*)}.$$

Recall that $A_{\mathcal{I}(x^*)\mathcal{J}(x^*)} \leq 0$. It follows from (2.9) that

$$A_{\mathcal{I}(x^*)\mathcal{I}(x^*)}[x_{\mathcal{I}(x^*)} - x^*_{\mathcal{I}(x^*)}] \geq -A_{\mathcal{I}(x^*)\mathcal{J}(x^*)}[x_{\mathcal{J}(x^*)} - x^*_{\mathcal{J}(x^*)}] \geq 0.$$

Since $A_{\mathcal{I}(x^*)\mathcal{I}(x^*)}$ is an M-matrix, one has by Lemma 1.1 that $x_{\mathcal{I}(x^*)} \geq x^*_{\mathcal{I}(x^*)}$, which, together with (2.9), implies that $x \geq x^*$. Therefore, x^* is the least element of $\mathrm{FEA}(A, q)$, that is, x^* is the least element solution of ZLCP (1.2). □

For any $x \in \mathbb{R}^n$, denote

$$\alpha(x) \triangleq \{i \mid (Ax+q)_i < x_i\}, \quad \beta(x) \triangleq \{i \mid (Ax+q)_i = x_i\}, \quad \gamma(x) \triangleq \{i \mid (Ax+q)_i > x_i\}, \tag{2.10}$$

and

$$\alpha_{<0}(x) = \{i \in \alpha(x) \mid (Ax+q)_i < 0\}.$$

Let $i_0 \in \beta(x)$ and $i_L \in \alpha_{<0}(x)$. If there exist indices $i_1, i_2, \cdots, i_{L-1}$, where $i_l \in \alpha(x) \cup \beta(x)$ for $l = 1, 2 \cdots, L-1$, such that $a_{i_l i_{l+1}} < 0$ for all $l = 0, 1, \cdots, L-1$, then we say that i_0 *is connected to* i_L. We say that i_0 *is connected to* $\alpha_{<0}(x)$, if there is an index $i \in \alpha_{<0}(x)$ such that i_0 *is connected to* i. Define a subset $\bar{\beta}(x)$ of $\beta(x)$ by

$$\bar{\beta}(x) = \{i \in \beta(x) \mid i \text{ is connected to } \alpha_{<0}(x)\}.$$

Let $x^* \in \mathbb{R}^n$ be the least element solution of ZLCP (1.2). Define a set $\mathcal{S}(A, q) \subset \mathbb{R}^n$ by

$$\mathcal{S}(A, q) = \{x \in \mathbb{R}^n \mid x \leq x^*,\ x_i(Ax+q)_i \leq 0, \quad \forall i = 1, 2, \cdots, n\}. \tag{2.11}$$

Proposition 2.1. *Let A and q in ZLCP (1.2) satisfy Assumption 2.1. Suppose that ZLCP (1.2) has a solution and that $x^* \in \mathbb{R}^n$ is the least element solution of ZLCP (1.2). Let $x \in \mathcal{S}(A, q)$. Then, $\alpha(x) \cup \bar{\beta}(x) \subseteq \mathcal{I}(x^*)$ and $A_{\alpha(x)\cup\bar{\beta}(x)\alpha(x)\cup\bar{\beta}(x)}$ is an M-matrix. Moreover, if $x \in \mathcal{S}(A, q)$ is a solution of ZLCP (1.2), then x is the least element solution.*

Proof. Let $x \in \mathcal{S}(A, q)$. Without loss of generality, let us assume that $x_i \leq 0$ (otherwise, one has by the fact $x \leq x^*$ that $x_i^* \geq x_i > 0$ and thus $i \in \mathcal{I}(x^*)$). Notice that $x_i(Ax+q)_i \leq 0$. One has $x_i = 0$ or $x_i < 0$ and $(Ax+q)_i \geq 0$.

Let $i \in \alpha(x)$ be chosen arbitrarily. Then, $x_i = 0$ and $(Ax+q)_i < 0$. Since x^* is the least element solution of ZLCP (1.2), one has $Ax^* + q \geq 0$, which, together with $(Ax+q)_i < 0$, yields

$$a_{ii}x_i^* + \sum_{j \neq i} a_{ij}x_j^* + q_i = (Ax^* + q)_i \geq 0 > (Ax+q)_i = a_{ii}x_i + \sum_{j \neq i} a_{ij}x_j + q_i.$$

Recall that $a_{ii} > 0$, $a_{ij} \leq 0$ with $j \neq i$, $x_i = 0$ and $x^* \geq x$. One can deduce from the last formula that $x_i^* > 0$, thereby yielding $i \in \mathcal{I}(x^*)$. It follows from the arbitrariness of $i \in \alpha(x)$ that $\alpha(x) \subseteq \mathcal{I}(x^*)$.

Let $i_0 \in \bar{\beta}(x)$ be chosen arbitrarily. Then, i_0 is connected to $\alpha_{<0}(x)$, which means that there exist indices $i_l \in \alpha(x) \cup \beta(x)$ $(l = 1, 2, \cdots, L-1)$ and $i_L \in \alpha_{<0}(x)$ such that $a_{i_l i_{l+1}} < 0$ for all $l = 0, 1, \cdots, L-1$. Recall from the facts $x \in \mathcal{S}(A, q)$ and $i_L \in \alpha_{<0}(x)$ that $(Ax+q)_{i_l} \leq 0$ for $l = 0, 1, \cdots, L-1$ and $(Ax+q)_{i_L} < 0$, which, together with $Ax^* + q \geq 0$, mean that

$$a_{i_l i_l}(x_{i_l}^* - x_{i_l}) \geq -\sum_{j \neq i_l} a_{i_l j}(x_j^* - x_j), \quad \forall l = 0, 1, \cdots, L-1$$

and

$$a_{i_L i_L}(x_{i_L}^* - x_{i_L}) > -\sum_{j \neq i_L} a_{i_L j}(x_j^* - x_j).$$

Notice that $a_{ii} > 0$ and $a_{ij} \leq 0$ for any i and $j \neq i$ and that $x^* \geq x$. One further obtains

$$x_{i_l}^* - x_{i_l} \geq -\frac{a_{i_l i_{l+1}}}{a_{i_l i_l}}(x_{i_{l+1}}^* - x_{i_{l+1}}), \qquad \forall l = 0, 1, \cdots, L-1,$$

and

$$x_{i_L}^* - x_{i_L} > 0.$$

Since $i_0 \in \bar{\beta}(x)$ and $x \in \mathcal{S}(A, q)$, one has $x_{i_0} = 0$, which, along with $a_{i_l i_{l+1}} < 0$ for $l = 0, 1, \cdots, L-1$, imply that $x_{i_0}^* > 0$. That is, $i_0 \in \mathcal{I}(x^*)$. By the arbitrariness of i_0, we deduce that $\bar{\beta}(x) \subseteq \mathcal{I}(x^*)$.

The above discussion shows that $\alpha(x) \cup \bar{\beta}(x) \subseteq \mathcal{I}(x^*)$. This, along with Lemma 1.2 and Theorem 2.1, means that $A_{\alpha(x)\cup\bar{\beta}(x)\alpha(x)\cup\bar{\beta}(x)}$ is an M-matrix. If $x \in \mathcal{S}(A, q)$ is a solution of (1.2), then $\bar{\beta}(x) = \emptyset$ and one can deduce from Theorem 2.1 that x is the least element solution. The proof is complete. □

2.2 A Newton method for solving the least element solution of ZLCP (1.2)

In this subsection, we propose a Newton method for solving the least element solution of ZLCP (1.2). Under suitable conditions, we study the convergence of the method. In particular, we show that the method can find a vector $\bar{x} \in \mathbb{R}^n$ such that $\bar{x} \in \mathcal{S}(A, q)$ and $\|\min(\bar{x}, A\bar{x} + q)\| \leq \eta$, where $\eta \geq 0$.

The details of the Newton method are presented as follows.

Algorithm 2.1.

0. Input $\eta \geq 0$ and $x^{(0)} \in \mathcal{S}(A,q)$. Let $\alpha^0 = \alpha(x^{(0)})$, $\beta^0 \subseteq \beta(x^{(0)})$ and $\gamma^0 = \{1,2,\cdots,n\}/(\alpha^0 \cup \beta^0)$, and set $k := 0$.

1. Find $x^{(k+1)} \in \mathbb{R}^n$ by solving the following system of linear equations

$$\underbrace{\begin{pmatrix} A_{\alpha^k\alpha^k} & A_{\alpha^k\beta^k} & A_{\alpha^k\gamma^k} \\ A_{\beta^k\alpha^k} & A_{\beta^k\beta^k} & A_{\beta^k\gamma^k} \\ 0 & 0 & I \end{pmatrix}}_{A^k} \begin{pmatrix} x_{\alpha^k} \\ x_{\beta^k} \\ x_{\gamma^k} \end{pmatrix} + \underbrace{\begin{pmatrix} q_{\alpha^k} \\ q_{\beta^k} \\ 0 \end{pmatrix}}_{q^k} = 0. \tag{2.12}$$

2. Stop if $\|\underbrace{\min(x^{(k+1)}, Ax^{(k+1)} + q)}_{F(x^{(k+1)})}\| \leq \eta$; otherwise, choose

$$\alpha^{k+1} = \alpha(x^{(k+1)}), \quad \beta^{k+1} \subseteq \beta(x^{(k+1)}) \quad \text{and} \quad \gamma^{k+1} = \{1,2,\cdots,n\}/(\alpha^{k+1} \cup \beta^{k+1}).$$

3. Set $k := k+1$, and go to Step 1.

End.

Proposition 2.2. *Let A and q in ZLCP* (1.2) *satisfy Assumption 2.1. Suppose that ZLCP* (1.2) *has a solution and that $x^* \in \mathbb{R}^n$ is the least element solution. Let $x^{(k)}$ be the current iterate satisfying that $F(x^{(k)}) \neq 0$ and $x^{(k)} \in \mathcal{S}(A,q)$. Assume that $\beta^k \subseteq \bar{\beta}(x^{(k)})$. Then the system* (2.12) *has a unique solution $x^{(k+1)}$. Moreover, $x^{(k)} \leq x^{(k+1)} \leq x^*$, $\alpha^k \subseteq \alpha^{k+1}$, and $x^{(k+1)} \in \mathcal{S}(A,q)$.*

Proof. Suppose that $x^{(k)}$ is the current iterate satisfying that $F(x^{(k)}) \neq 0$ and $x^{(k)} \in \mathcal{S}(A,q)$. Since $\alpha^k = \alpha(x^{(k)})$ and $\beta^k \subseteq \bar{\beta}(x^{(k)})$, it follows from Proposition 2.1 and Lemma 1.2 that $A_{\alpha^k\cup\beta^k\alpha^k\cup\beta^k}$ and A^k in (2.12) are M-matrices and the system (2.12) has a unique solution $x^{(k+1)}$.

It follows from (2.12) that $x^{(k+1)}_{\gamma^k} = 0$. Observe that $x^*_{\gamma^k} \geq 0$. If $\alpha^k \cup \beta^k = \emptyset$, then $x^* \geq x^{(k+1)}$; otherwise, $(Ax^* + q)_{\alpha^k\cup\beta^k} \geq (Ax^{(k+1)} + q)_{\alpha^k\cup\beta^k}$ and then, $A_{\alpha^k\cup\beta^k\alpha^k\cup\beta^k}(x^* - x^{(k+1)})_{\alpha^k\cup\beta^k} \geq -A_{\alpha^k\cup\beta^k\gamma^k}(x^* - x^{(k+1)})_{\gamma^k} \geq 0$, which, along with the fact that $A_{\alpha^k\cup\beta^k\alpha^k\cup\beta^k}$ is an M-matrix and Lemma 1.1, means that $x^*_{\alpha^k\cup\beta^k} \geq x^{(k+1)}_{\alpha^k\cup\beta^k}$ and hence $x^* \geq x^{(k+1)}$. On the other hand, since $x^{(k)} \in \mathcal{S}(A,q)$, we have $x^{(k)}_i \leq 0$ for any $i \in \gamma^k$ and $(Ax^{(k)} + q)_i \leq 0$ for any $i \in \alpha^k \cup \beta^k$. Observe from (2.12) that $x^{(k+1)}_i = 0$ for any $i \in \gamma^k$ and $(Ax^{(k+1)} + q)_i = 0$ for any $i \in \alpha^k \cup \beta^k$. If $\alpha^k \cup \beta^k = \emptyset$, then $x^{(k+1)} \geq x^{(k)}$; otherwise, it follows that $(Ax^{(k+1)} + q)_{\alpha^k\cup\beta^k} \geq (Ax^{(k)} + q)_{\alpha^k\cup\beta^k}$ and hence, $A_{\alpha^k\cup\beta^k\alpha^k\cup\beta^k}(x^{(k+1)} - x^{(k)})_{\alpha^k\cup\beta^k} \geq -A_{\alpha^k\cup\beta^k\gamma^k}(x^{(k+1)} - x^{(k)})_{\gamma^k} \geq 0$, which, along with Lemma 1.1, means that $x^{(k+1)}_{\alpha^k\cup\beta^k} \geq x^{(k)}_{\alpha^k\cup\beta^k}$ and hence $x^{(k+1)} \geq x^{(k)}$. The above discussion shows that $x^{(k)} \leq x^{(k+1)} \leq x^*$.

Let $\alpha^k \neq \emptyset$ and $i \in \alpha^k$ be chosen arbitrarily. It follows from $x^{(k)} \in \mathcal{S}(A,q)$ that $x^{(k)}_i \geq 0$. If $x^{(k)}_i > 0$, one has $x^{(k+1)}_i > 0$, which, along with $(Ax^{(k+1)} + q)_i = 0$ (see (2.12)), implies that $i \in \alpha^{k+1}$; otherwise, one has by (2.10) and (2.12) that $(Ax^{(k+1)} + q)_i = 0 > (Ax^{(k)} + q)_i$, which means that $x^{(k+1)}_i > -\sum_{j\neq i} a_{ij}(x^{(k+1)}_j - x^{(k)}_j)/a_{ii} \geq 0$ and hence, $i \in \alpha^{k+1}$. By the arbitrariness of i, one concludes that $\alpha^k \subseteq \alpha^{k+1}$.

Observe that $(Ax^{(k+1)}+q)_i = 0$ or $x_i^{(k+1)} = 0$ for any i and $x^{(k+1)} \le x^*$. It follows from (2.11) that $x^{(k+1)} \in \mathcal{S}(A,q)$. The proof is complete. □

Proposition 2.2 shows that Algorithm 2.1 is well-defined as long as ZLCP (1.2) has a solution and $\beta^k \subseteq \bar{\beta}(x^{(k)})$ for each k. Next, we establish a global convergence theorem of Algorithm 2.1.

Theorem 2.2. *Let A and q in ZLCP (1.2) satisfy Assumption 2.1. Suppose that ZLCP (1.2) has a solution and that $x^* \in \mathbb{R}^n$ is the least element solution. Assume that $\{x^{(k)}\}$ is a sequence of iterates generated by Algorithm 2.1 and that β^k satisfies that $\beta^k \subseteq \bar{\beta}(x^{(k)})$ for each k. Then, there exists some $K \le n$ such that $x^{(K+1)} = x^*$.*

Proof. Since $\{x^{(k)}\}$ is generated by Algorithm 2.1 and $\beta^k \subseteq \bar{\beta}(x^{(k)})$, one can get by Proposition 2.2 that $\alpha^k \subseteq \alpha^{k+1} \subseteq \{1, 2, \cdots, n\}$ for any k, which implies that there is a positive integer $K \le n$ such that $\alpha^{K+1} = \alpha^K$. Therefore, one has

$$A_i x^{(K+1)} + q_i = 0, \qquad x_i^{(K+1)} > 0, \quad \forall i \in \alpha^{K+1}$$

and

$$A_i x^{(K+1)} + q_i \ge 0, \qquad x_i^{(K+1)} = 0, \quad \forall i \in \beta^{K+1} \cup \gamma^{K+1},$$

which mean that $F(x^{(K+1)}) = 0$, that is, $x^{(K+1)}$ is a solution of ZLCP (1.2). Observe from Proposition 2.2 that $x^{(K+1)} \le x^*$. Therefore, one has $x^{(K+1)} = x^*$. The proof is complete. □

3 A generalized Newton method for solving DLCS (1.3)-(1.4)

In this section, we propose a novel generalized Newton method for solving DLCS (1.3)-(1.4) and study its convergence. The details of the method are presented as follows.

Algorithm 3.1.

0. Input $\epsilon \ge 0$ and a sequence $\{\eta_k\} \subset \mathbb{R}_+$ such that $\eta_k \to 0$. Let $u^0 = x^{h,l}$ and $q^0 = Bu^0 + g(t_{l+1})$. Set $k := 0$.

1. Find a $\bar{v}^k \in \mathcal{S}(A, q^k)$ by solving the following ZLCP

$$v \ge 0, \qquad Av + q^k \ge 0, \qquad v^T(Av + q^k) = 0 \tag{3.1}$$

   such that

$$\|\min(\bar{v}^k, A\bar{v}^k + q^k)\| \le \eta_k. \tag{3.2}$$

2. Compute Δu and Δv by solving the following system of linear equations

$$G(u^k, \bar{v}^k) \begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix} = -F(u^k, \bar{v}^k), \tag{3.3}$$

   where $G(u^k, \bar{v}^k)$ and $F(u^k, \bar{v}^k)$ are defined by (1.9)-(1.10) and (1.8), respectively.

3. Let

$$u^{k+1} = u^k + \Delta u, \quad v^{k+1} = \bar{v}^k + \Delta v \quad \text{and} \quad q^{k+1} = Bu^{k+1} + g(t_{l+1}). \tag{3.4}$$

4. Stop if $\|\min(v^{k+1}, Av^{k+1} + q^{k+1})\| \leq \epsilon$. Otherwise, set $k := k+1$ and go to Step 1.

End.

Remark 3.1. *Compared with the existing Newton method (e.g., see [2, 3, 4, 16, 22, 23, 24, 28, 29]), we have added Step 1 in Algorithm 3.1, which, as seen later, turns out to be crucial in generating a nonsingular generalized Jacobi matrix and globalizing the proposed method.*

Let

$$\kappa = \|Q\| + L_A\|C\|\|B\|, \tag{3.5}$$

where

$$L_A = \max\{\|A_{\alpha\alpha}^{-1}\| \mid A_{\alpha\alpha} \text{ is nonsingular for } \alpha \subseteq \{1, \cdots, n\}\}. \tag{3.6}$$

For convenience, we denote the diagonal matrix D defined in (1.10) by D_k for the vector pair $(u^k, \bar{v}^k)$, i.e.,

$$D_k = \operatorname{diag}(d_1^k, \cdots, d_n^k), \qquad d_i^k = \begin{cases} 1, & \bar{v}_i^k > (A\bar{v}^k + q^k)_i, \\ 0, & \text{otherwise.} \end{cases} \tag{3.7}$$

The following lemma follows directly from (3.7), (2.10), Proposition 2.1 and Lemma 1.2.

Lemma 3.1. *Let $k \geq 0$. Suppose that ZLCP (3.1) has a solution and $\bar{v}^k \in \mathcal{S}(A, q^k)$. Then, $I - D_k + D_kA$ is an M-matrix.*

The following proposition shows that Algorithm 3.1 is well-defined.

Proposition 3.1. *Assume that h and κ satisfy that $h\kappa < 1$. Let $k \geq 0$. Suppose that ZLCP (3.1) has a solution and $\bar{v}^k \in \mathcal{S}(A, q^k)$. Then, the following statements are true.*
(i) The following matrix V_k is well-defined

$$V_k = I - h(Q - C(I - D_k + D_kA)^{-1}D_kB). \tag{3.8}$$

Moreover, V_k is a nonsingular matrix satisfying $\|V_k^{-1}\| \leq 1/(1 - h\kappa)$.
(ii) The $G(u^k, \bar{v}^k)$ in (3.3) is nonsingular, and

$$G(u^k, \bar{v}^k) = \begin{bmatrix} I & -hC(I - D_k + D_kA)^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} V_k & 0 \\ D_kB & I - D_k + D_kA \end{bmatrix}. \tag{3.9}$$

Proof. (i) Since ZLCP (3.1) has a solution and $\bar{v}^k \in \mathcal{S}(A, q^k)$, one can obtain from Lemma 3.1 that $I - D_k + D_kA$ is an M-matrix, which means that V_k given in (3.8) is well-defined. Notice from (3.5) and (3.6) that $\|Q - C(I - D_k + D_kA)^{-1}D_kB\| \leq \kappa$. In view of $h\kappa < 1$, it follows directly from Lemma 2.3.2 of [18] that V_k is a nonsingular matrix satisfying $\|V_k^{-1}\| \leq 1/(1 - h\kappa)$.

(ii) Recall that $I - D_k + D_k A$ is an M-matrix. One can derive from (1.9) that

$$G(u^k, \bar{v}^k) = \begin{bmatrix} I & -hC(I - D_k + D_k A)^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} I - hQ + hC(I - D_k + D_k A)^{-1} D_k B & 0 \\ D_k B & I - D_k + D_k A \end{bmatrix}.$$

Notice that $V_k = I - hQ + hC(I - D_k + D_k A)^{-1} D_k B$ and V_k is nonsingular. Therefore, $G(u^k, \bar{v}^k)$ is nonsingular and it can be expressed in terms of (3.9). The proof is complete. □

Proposition 3.2. *Let $k \geq 0$. Assume that ZLCP* (3.1) *has a solution. Suppose that v^{k+1} and q^{k+1} are generated by Step 3 of Algorithm 3.1. Then, $v^{k+1} \in \mathcal{S}(A, q^{k+1})$.*

Proof. Let $k \geq 0$. It then follows from (3.7) that

$$\min(\bar{v}^k, A\bar{v}^k + q^k) = (I - D_k)\bar{v}^k + D_k(A\bar{v}^k + q^k), \tag{3.10}$$

which, together with (3.3) and (3.4), implies that

$$\begin{aligned} &(I - D_k)v^{k+1} + D_k(Av^{k+1} + q^{k+1}) \\ &= (I - D_k)\bar{v}^k + (I - D_k)\Delta v + D_k(A\bar{v}^k + q^k + A\Delta v + B\Delta u) \\ &= (I - D_k)\bar{v}^k + D_k(A\bar{v}^k + q^k) + (I - D_k + D_k A)\Delta v + D_k B\Delta u \\ &= (I - D_k)\bar{v}^k + D_k(A\bar{v}^k + q^k) - \min(\bar{v}^k, A\bar{v}^k + q^k) \\ &= 0. \end{aligned} \tag{3.11}$$

Therefore, one can obtain from (3.7) that

$$v_i^{k+1}(Av^{k+1} + q^{k+1})_i = 0, \qquad \forall i = 1, 2, \ldots, n. \tag{3.12}$$

On the other hand, let $y(u^{k+1}) \in \mathrm{SOL}(q^{k+1}, A)$ be the least element solution. Then, one has

$$(I - D_k)y(u^{k+1}) + D_k(Ay(u^{k+1}) + q^{k+1}) \geq 0,$$

which along with (3.11) yields

$$(I - D_k + D_k A)y(u^{k+1}) \geq (I - D_k + D_k A)v^{k+1}.$$

Notice that ZLCP (3.1) has a solution and $\bar{v}^k \in \mathcal{S}(A, q^k)$. One can derive from Lemma 3.1 that $I - D_k + D_k A$ is an M-matrix. Therefore, one gets by Lemma 1.1 that $y(u^{k+1}) \geq v^{k+1}$. Combining this with (3.12) and (2.11) yields that $v^{k+1} \in \mathcal{S}(A, q^{k+1})$. The proof is then complete. □

Remark 3.2. *Proposition 3.2 shows that v^k ($k \geq 1$) can be chosen as an initial iterate when Algorithm 2.1 is applied to solve ZLCP* (3.1).

Proposition 3.3. *Assume that h and κ satisfy that $h\kappa < 1$. Suppose that ZLCP* (3.1) *has a solution. Let u^{k+1} be generated by Step 3 of Algorithm 3.1. Then,*

$$u^{k+1} = u^k - V_k^{-1}[(I - hQ)u^k - hC\hat{v}^k - (x^{h,l} + hf(t_{l+1}))], \tag{3.13}$$

where V_k *is defined by* (3.8) *and*

$$\hat{v}^k = -(I - D_k + D_k A)^{-1} D_k q^k. \tag{3.14}$$

Moreover, for any $x \in \mathbb{R}^m$,

$$u^{k+1} - x = -V_k^{-1}[(I - hQ)x + hC(I - D_k + D_k A)^{-1} D_k(Bx + g(t_{l+1})) - x^{h,l} - hf(t_{l+1})]. \tag{3.15}$$

In particular, if $\eta_k = 0$ *in* (3.2)*, then* $\hat{v}^k$ *is the least element solution of ZLCP* (3.1)*.*

Proof. Let $h\kappa < 1$. It follows from (3.3) and (3.9) that

$$\begin{bmatrix} V_k & 0 \\ D_k B & I - D_k + D_k A \end{bmatrix} \begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix}$$
$$= - \begin{pmatrix} (I - hQ)u^k - hC\bar{v}^k - x^{h,l} - hf(t_{l+1}) + hC(I - D_k + D_k A)^{-1} \min(\bar{v}^k, A\bar{v}^k + q^k) \\ \min(\bar{v}^k, A\bar{v}^k + q^k) \end{pmatrix},$$

which, together with (3.10), means that

$$V_k \Delta u = -[(I - hQ)u^k + hC(I - D_k + D_k A)^{-1} D_k q^k - x^{h,l} - hf(t_{l+1})]. \tag{3.16}$$

Observe from Proposition 3.1 that V_k is nonsingular. This, along with (3.16), Step 3 of Algorithm 3.1 and (3.14), yields (3.13).

Let $x \in \mathbb{R}^m$. Then, one has

$$\begin{aligned} u^{k+1} - x &= u^k - x - V_k^{-1}[(I - hQ)u^k - hC\hat{v}^k - (x^{h,l} + hf(t_{l+1}))] \\ &= V_k^{-1}[V_k u^k - V_k x - (I - hQ)u^k + hC\hat{v}^k + x^{h,l} + hf(t_{l+1})]. \end{aligned}$$

This, together with (3.8) and (3.14), implies that

$$\begin{aligned} u^{k+1} - x &= -V_k^{-1}[V_k x + hC(I - D_k + D_k A)^{-1} D_k g(t_{l+1}) - x^{h,l} - hf(t_{l+1})] \\ &= -V_k^{-1}[(I - hQ)x + hC(I - D_k + D_k A)^{-1} D_k(Bx + g(t_{l+1})) - x^{h,l} - hf(t_{l+1})]. \end{aligned}$$

In particular, if $\eta_k = 0$ in (3.2), one can derive from Proposition 2.1 that $\bar{v}^k$ is the least element solution of ZLCP (3.1), which, together with (3.10) and (3.14), means that $\hat{v}^k = \bar{v}^k$. The proof is then complete. □

Let

$$\gamma \triangleq \frac{h\left(\|Qx^{h,l} + f(t_l)\| + L_A\|C\|\|Bx^{h,l} + g(t_l)\| + L_f h + L_g\|B\|^{-1}\right)}{1 - h\kappa}. \tag{3.17}$$

Proposition 3.4. *Let* h *and* κ *satisfy that* $h\kappa < 1$*. Assume that there exists a vector* $y(u) \in \mathrm{SOL}(q(u), A)$ *for each* $u \in U(x^{h,l}, \gamma)$*. Then, Algorithm 3.1 generates a sequence of iterates* $\{u^k\}$ *which belongs to* $U(x^{h,l}, \gamma)$*.*

Proof. Note that $u^0 = x^{h,l} \in U(x^{h,l}, \gamma)$. Without loss of generality, let $u^k \in U(x^{h,l}, \gamma)$. Then, we can obtain by the assumption that ZLCP (3.1) has a solution. Recall that $h\kappa < 1$. It follows from Proposition 3.3 that u^{k+1} can be expressed in the form (3.13). Moreover, one has by (3.15)

$$u^{k+1} - x^{h,l} = hV_k^{-1}\left(Qx^{h,l} + f(t_{l+1}) - C(I - D_k + D_kA)^{-1}D_k[Bx^{h,l} + g(t_{l+1})]\right). \tag{3.18}$$

Recall that f and g are Lipschitz continuous functions with Lipschitz constants L_f and L_g, respectively. One has

$$\|Qx^{h,l} + f(t_{l+1})\| \le \|Qx^{h,l} + f(t_l)\| + \|f(t_{l+1}) - f(t_l)\| \le \|Qx^{h,l} + f(t_l)\| + L_f h$$

and

$$\|Bx^{h,l} + g(t_{l+1})\| \le \|Bx^{h,l} + g(t_l)\| + \|g(t_{l+1}) - g(t_l)\| \le \|Bx^{h,l} + g(t_l)\| + L_g h.$$

Observe from (3.6) and (3.7) that $\|(I - D_k + D_kA)^{-1}D_k\| \le L_A$. In view of this, the above inequalities and (3.18), one can easily see that

$$\|u^{k+1} - x^{h,l}\| \le h\|V_k^{-1}\|\left(\|Qx^{h,l} + f(t_l)\| + L_A\|C\|\|Bx^{h,l} + g(t_l)\| + L_f h + L_A L_g\|C\|h\right). \tag{3.19}$$

Notice that $h\kappa < 1$ and $B \neq 0$. It follows from (3.5) that $L_A\|C\|h < \|B\|^{-1}$. In addition, we can get by Proposition 3.1 that $\|V_k^{-1}\| \le 1/(1 - h\kappa)$. These, together with (3.19) and (3.17), imply that

$$\|u^{k+1} - x^{h,l}\| < \frac{h\left(\|Qx^{h,l} + f(t_l)\| + L_A\|C\|\|Bx^{h,l} + g(t_l)\| + L_f h + L_g\|B\|^{-1}\right)}{1 - h\kappa} = \gamma.$$

Thus, $u^{k+1} \in U(x^{h,l}, \gamma)$. Since $u^0 = x^{h,l} \in U(x^{h,l}, \gamma)$, we can conclude by induction that Algorithm 3.1 generates a sequence of iterates $\{u^k\}$ which belongs to $U(x^{h,l}, \gamma)$. The proof is complete. □

Remark 3.3. *The parameter γ obtained here is slightly different from the one given in [7]. Indeed, the authors in the proof of Lemma 3.1 of [7] incorrectly viewed $y^{h,i} = v(x^{h,i})$, where $y^{h,i} \in \mathrm{SOL}(Nx^{h,i} + g(t_{h,i}), M)$ and $v(x^{h,i}) \in \mathrm{SOL}(Nx^{h,i} + g(t_{h,i+1}), M)$, and $\|H(u) - x^{h,i}\| \le h(\|Ax^{h,i} + By^{h,i} + f(t_{h,i+1})\| + (\|A\| + \mathcal{L}\|B\|\|N\|)\gamma + \mathcal{L}L_g\|B\|h)$ if $v(x^{h,i})$ is replaced by $y^{h,i}$. Additionally, $y^{h,l} = -(I - D_k + D_kA)^{-1}D_k[Bx^{h,l} + g(t_{l+1})]$ if $\eta_k = 0$ in (3.2). Finally, noting that the value of $f(t_{l+1})$ is dependent on h, we replace $f(t_{l+1})$ by $f(t_l)$ via using the Lipschitz continuity of f.*

The following lemma is from [7].

Lemma 3.2. *Let h and κ satisfy that $h\kappa < 1$. Assume that there is a vector $y(u) \in \mathrm{SOL}(q(u), A)$ for each $u \in U(x^{h,l}, \gamma)$. Then, the nonsmooth equation (1.6) has a unique solution $x^{h,l+1} \in U(x^{h,l}, \gamma)$.*

Theorem 3.1. *Assume that h and κ satisfy that $h\kappa < 1$ and that there is a vector $y(u) \in \mathrm{SOL}(q(u), A)$ for each $u \in U(x^{h,l}, \gamma)$. Let*

$$\tau = \frac{2hL_M\|C\|\|B\|}{1 - h\kappa}, \tag{3.20}$$

where

$$L_M = (1 + L_A)(1 + \|A\|). \tag{3.21}$$

Suppose that $0 \leq \tau < 1$. Then, the sequence $\{u^k\}$ generated by Algorithm 3.1 converges to $x^{h,l+1}$. Moreover, if $\eta_k = 0$ for any k, $\{u^k\}$ converges Q-linearly.

Proof. Since $h\kappa < 1$ and there is a vector $y(u) \in \mathrm{SOL}(q(u), A)$ for each $u \in U(x^{h,l}, \gamma)$, we can derive from Lemma 3.2 that equation (1.6) has a unique solution $x^{h,l+1} \in U(x^{h,l}, \gamma)$, or equivalently, DLCS (1.3)-(1.4) has a solution $(x^{h,l+1}, y(x^{h,l+1}))$. This, together with Proposition 3.3, yields that

$$u^{k+1} - x^{h,l+1} = -V_k^{-1}((I - hQ)x^{h,l+1} - x^{h,l} - hf(t_{l+1}) + hC(I - D_k + D_kA)^{-1}D_k[Bx^{h,l+1} + g(t_{l+1})]).$$

Recall that $hCy(x^{h,l+1}) = (I - hQ)x^{h,l+1} - x^{h,l} - hf(t_{l+1})$. It follows that

$$\begin{aligned} &u^{k+1} - x^{h,l+1} \\ &= -hV_k^{-1}C[y(x^{h,l+1}) + (I - D_k + D_kA)^{-1}D_k(Bx^{h,l+1} + g(t_{l+1}))] \\ &= -hV_k^{-1}C(I - D_k + D_kA)^{-1}[(I - D_k + D_kA)y(x^{h,l+1}) + D_k(Bx^{h,l+1} + g(t_{l+1}))] \\ &= -hV_k^{-1}C(I - D_k + D_kA)^{-1}[(I - D_k)y(x^{h,l+1}) + D_k(Ay(x^{h,l+1}) + Bx^{h,l+1} + g(t_{l+1}))], \end{aligned} \tag{3.22}$$

which means that

$$\begin{aligned} &\|u^{k+1} - x^{h,l+1}\| \\ &\leq h\|V_k^{-1}C\|\|(I - D_k + D_kA)^{-1}[(I - D_k)y(x^{h,l+1}) + D_k(Ay(x^{h,l+1}) + Bx^{h,l+1} + g(t_{l+1}))]\|. \end{aligned} \tag{3.23}$$

Notice that $(x^{h,l+1}, y(x^{h,l+1}))$ is a solution of DLCS (1.3)-(1.4). It holds

$$(I - D_k)y(x^{h,l+1}) + D_k[Ay(x^{h,l+1}) + Bx^{h,l+1} + g(t_{l+1})] \geq 0. \tag{3.24}$$

On the other hand, since $\bar{v}^k \in \mathcal{S}(A, q^k)$, one can see from (3.10) that

$$(I - D_k)\bar{v}^k + D_k(A\bar{v}^k + q^k) \leq 0,$$

which, along with (3.24) and $q^k = Bu^k + g(t_{l+1})$, implies

$$\begin{aligned} 0 &\leq (I - D_k)y(x^{h,l+1}) + D_k[Ay(x^{h,l+1}) + Bx^{h,l+1} + g(t_{l+1})] \\ &\leq (I - D_k)y(x^{h,l+1}) + D_k[Ay(x^{h,l+1}) + Bx^{h,l+1} + g(t_{l+1})] - (I - D_k)\bar{v}^k - D_k(A\bar{v}^k + q^k) \\ &\leq (I - D_k + D_kA)[y(x^{h,l+1}) - \bar{v}^k] + D_kB(x^{h,l+1} - u^k). \end{aligned} \tag{3.25}$$

In addition, it follows from Lemma 3.1 that $I - D_k + D_kA$ is an M-matrix, which means that $(I - D_k + D_kA)^{-1} \geq 0$. This, together with (3.25), yields

$$\begin{aligned} 0 &\leq (I - D_k + D_kA)^{-1}[(I - D_k)y(x^{h,l+1}) + D_k(Ay(x^{h,l+1}) + Bx^{h,l+1} + g(t_{l+1}))] \\ &\leq y(x^{h,l+1}) - \bar{v}^k + (I - D_k + D_kA)^{-1}D_kB(x^{h,l+1} - u^k). \end{aligned}$$

Therefore, we can derive from (3.23) that

$$\begin{aligned} &\|u^{k+1} - x^{h,l+1}\| \\ &\leq h\|V_k^{-1}C\|\|y(x^{h,l+1}) - \bar{v}^k + (I - D_k + D_kA)^{-1}D_kB(x^{h,l+1} - u^k)\| \\ &= h\|V_k^{-1}C\|\|y(x^{h,l+1}) - y(u^k) + y(u^k) - \bar{v}^k + (I - D_k + D_kA)^{-1}D_kB(x^{h,l+1} - u^k)\|, \end{aligned} \tag{3.26}$$

where $y(u^k) \in \mathrm{SOL}(q^k, A)$ is the least element solution of ZLCP (3.1). Notice that $y(x^{h,l+1}) \in \mathrm{SOL}(Bx^{h,l+1} + g(t_{l+1}), A)$ is the least element solution. It follows from Theorem 2.3 of [7] that

$$\|y(x^{h,l+1}) - y(u^k)\| \leq L_A \|B\| \|x^{h,l+1} - u^k\|. \tag{3.27}$$

In view of (3.6) and (3.7), one can observe that $\|(I - D_k + D_k A)^{-1} D_k\| \leq L_A$, which, together with (3.26), (3.27) and Proposition 3.1, implies that

$$\|u^{k+1} - x^{h,l+1}\| \leq \frac{2hL_A\|C\|\|B\|}{1 - h\kappa} \|u^k - x^{h,l+1}\| + \frac{h\|C\|}{1 - h\kappa} \|y(u^k) - \bar{v}^k\|. \tag{3.28}$$

Let $D_{q^k} = (d_1, \cdots, d_n)$ be a diagonal matrix with diagonals

$$d_i = \begin{cases} 1, & y_i(u^k) > 0, \\ 0, & \text{otherwise}. \end{cases}$$

Then, we have

$$(I - D_{q^k}) y(u^k) + D_{q^k}[Ay(u^k) + q^k] = 0$$

and

$$(I - D_{q^k}) \bar{v}^k + D_{q^k}(A\bar{v}^k + q^k) \geq \min(\bar{v}^k, A\bar{v}^k + q^k),$$

which mean that

$$-\min(\bar{v}^k, A\bar{v}^k + q^k) \geq (I - D_{q^k} + D_{q^k} A)(y(u^k) - \bar{v}^k).$$

It follows from Lemma 1.2 and Theorem 2.1 that $I - D_{q^k} + D_{q^k} A$ is an M-matrix. This along with the last inequality implies

$$y(u^k) - \bar{v}^k \leq -(I - D_{q^k} + D_{q^k} A)^{-1} \min(\bar{v}^k, A\bar{v}^k + q^k). \tag{3.29}$$

On the other hand, we have by a simple calculation

$$(I - D_{q^k} + D_{q^k} A)^{-1} = \begin{bmatrix} I & 0 \\ 0 & A_{\mathcal{I}\mathcal{I}}^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -A_{\mathcal{J}\mathcal{I}} & I \end{bmatrix},$$

where $\mathcal{I} = \{i \mid y_i(u^k) > 0\}$ and $\mathcal{J} = \{i \mid y_i(u^k) = 0\}$. This together with (3.6) and (3.21) yields $\|(I - D_{q^k} + D_{q^k} A)^{-1}\| \leq L_M$. Recall that $y(u^k) \in \mathrm{SOL}(q^k, A)$ is the least element solution of (3.1) and $\bar{v}^k \in \mathcal{S}(A, q^k)$. One has $y(u^k) - \bar{v}^k \geq 0$. These, along with (3.29) and (3.2), mean that $\|y(u^k) - \bar{v}^k\| \leq L_M \eta_k$. Combining this with (3.28), one further obtains

$$\|u^{k+1} - x^{h,l+1}\| \leq \frac{2hL_A\|C\|\|B\|}{1 - h\kappa} \|u^k - x^{h,l+1}\| + \frac{hL_M\|C\|}{1 - h\kappa} \eta_k. \tag{3.30}$$

Below, we prove that $\{u^k\}$ converges to $x^{h,l+1}$. Consider first the case $\|B\| = 0$, in which $\tau = 0$ and it follows from (3.30) that $\{u^k\}$ converges to $x^{h,l+1}$ since $\eta_k \to 0$. Next, consider the case $\|B\| \neq 0$, in which one can derive from (3.30) and (3.20) that

$$\|u^{k+1} - x^{h,l+1}\| \leq \tau \left(\|u^k - x^{h,l+1}\| + \frac{1}{2\|B\|} \eta_k \right). \tag{3.31}$$

Let $\epsilon > 0$ be chosen arbitrarily. Recall from Algorithm 3.1 that $\eta_k \to 0$. Thus, there is a positive integer K_1 and a positive constant η such that $\eta_i < 2\|B\|(1-\tau)\epsilon/(3\tau)$ for any $i \geq K_1$ and $\eta_i \leq \eta$ for any i. These, along with (3.31), mean that

$$\begin{aligned}
\|u^{k+1} - x^{h,l+1}\| &\leq \tau^{k+1}\|u^0 - x^{h,l+1}\| + \frac{1}{2\|B\|}\sum_{i=0}^{k}\tau^{k+1-i}\eta_i \\
&= \tau^{k+1}\|u^0 - x^{h,l+1}\| + \frac{1}{2\|B\|}\left(\sum_{i=0}^{K_1}\tau^{k+1-i}\eta_i + \sum_{i=K_1+1}^{k}\tau^{k+1-i}\eta_i\right) \\
&\leq \tau^{k+1}\|u^0 - x^{h,l+1}\| + \frac{\tau}{2\|B\|(1-\tau)}\left(\tau^{k-K_1}\eta + \frac{2\|B\|(1-\tau)\epsilon}{3\tau}\right). \qquad (3.32)
\end{aligned}$$

Since $0 \leq \tau < 1$, there is a positive integer $K_2 \geq K_1$ such that $\tau^{k+1} \leq \epsilon/(3\|u^0 - x^{h,l+1}\|)$ and $\tau^{k-K_1} \leq 2\|B\|(1-\tau)\epsilon/(3\tau\eta)$. Combining these with (3.32), we obtain that for any $k \geq K_2$, it holds

$$\|u^{k+1} - x^{h,l+1}\| \leq \epsilon.$$

By the arbitrariness of ϵ, it follows that $\{u^k\}$ converges to $x^{h,l+1}$. The above discussion shows that $\{u^k\}$ converges to $x^{h,l+1}$ as $0 \leq \tau < 1$.

Moreover, if $\eta_k = 0$, we can conclude from (3.31) that $\{u^k\}$ converges Q-linearly. □

Theorem 3.2. *Suppose that the sequence of iterates $\{u^k\}$ generated by Algorithm 3.1 converges to $x^{h,l+1}$. Then, there exists a positive integer K such that $u^K = x^{h,l+1}$. Moreover, if for any k, $G(u^k, \bar{v}^k)$ in* (3.3) *is an M-matrix, then $K \leq n+1$.*

Proof. Suppose that $\{u^k\} \to x^{h,l+1}$. Let $y(u^k) \in \mathrm{SOL}(q^k, A)$ and $y(x^{h,l+1}) \in \mathrm{SOL}(Bx^{h,l+1} + g(t_{l+1}), A)$ be the least element solution. It follows from (3.27) that $\{y(u^k)\} \to y(x^{h,l+1})$. Recall that $\{\bar{v}^k\}$ is a sequence generated by Algorithm 3.1 and $\{\eta_k\} \to 0$. One has $\{\|\bar{v}^k - y(u^k)\|\} \to 0$. Hence,

$$\{\bar{v}^k\} \to y(x^{h,l+1}), \qquad (3.33)$$

which, along with the facts that $\{u^k\} \to x^{h,l+1}$ and $q^k = Bu^k + g(t_{l+1})$, implies that

$$\{A\bar{v}^k + q^k\} \to Ay(x^{h,l+1}) + Bx^{h,l+1} + g(t_{l+1}). \qquad (3.34)$$

Let i be an arbitrary index satisfying that $y_i(x^{h,l+1}) > 0$. In view of $y(x^{h,l+1}) \in \mathrm{SOL}(q(x^{h,l+1}), A)$, we can easily see that

$$[Ay(x^{h,l+1}) + Bx^{h,l+1} + g(t_{l+1})]_i = 0.$$

Therefore, one can derive from (3.33) and (3.34) that for k sufficiently large,

$$\bar{v}_i^k > (A\bar{v}^k + q^k)_i.$$

This, together with (3.7), means that for k sufficiently large, one has $d_i^k = 1$ and

$$[(I - D_k)y(x^{h,l+1}) + D_k(Ay(x^{h,l+1}) + Bx^{h,l+1} + g(t_{l+1}))]_i = 0.$$

Let i be an arbitrary index satisfying that $[Ay(x^{h,l+1})+Bx^{h,l+1}+g(t_{l+1})]_i > 0$. We can observe from the fact $y(x^{h,l+1}) \in \mathrm{SOL}(q(x^{h,l+1}), A)$ that $y_i(x^{h,l+1}) = 0$. It, along with (3.33) and (3.34), implies that for k sufficiently large,

$$\bar{v}_i^k < (A\bar{v}^k + q^k)_i,$$

which, together with (3.7), means that $d_i^k = 0$ and

$$[(I - D_k)y(x^{h,l+1}) + D_k(Ay(x^{h,l+1}) + Bx^{h,l+1} + g(t_{l+1}))]_i = 0.$$

The above discussion shows that for k sufficiently large, one has

$$(I - D_k)y(x^{h,l+1}) + D_k(Ay(x^{h,l+1}) + Bx^{h,l+1} + g(t_{l+1})) = 0.$$

In view of this and (3.22), one can easily conclude that there exists a positive integer K such that $u^K = x^{h,l+1}$.

If $G(u^k, \bar{v}^k)$ in (3.3) is an M-matrix, one can obtain by a similar argument as in the proofs of Proposition 2.2 and Theorem 2.2 that $K \leq n + 1$. The proof is complete. □

Remark 3.4. *Proposition 3.3 shows that Algorithm 3.1 reduces to the generalized Newton iteration* (1.7) *as* $\eta_k = 0$ *in* (3.2). *The convergence results obtained here are stronger than those in [7]. Actually, Algorithm 3.1 converges and it can find the solution in at most finitely many iterations as* $\eta_k \to 0$, *and moreover, Algorithm 3.1 converges globally Q-linearly as* $\eta_k = 0$. *See Theorems 3.1 and 3.2 for details. However, the Newton iteration* (1.7) *was only verified to converge superlinearly in Theorem 3.2 of [7].*

4 Numerical experiments

In this section we conduct some preliminary numerical experiments to test the performance of our proposed algorithm (Algorithm 3.1). All ZLCPs involved in Algorithm 3.1 are solved by Algorithm 2.1. The codes of both algorithms are written in Matlab and all computations are performed on an iFound desktop with a 3.00 GHz Intel Core E5700 processor and 2.00 GB of RAM.

Example 4.1. *Consider a differential linear complementarity system in the form of* (1.1), *where* $T = 4$, $m = n^2$, $x(t) \in \mathbb{R}^m$, $y(t) \in R^n$, $f(t) = C\Psi(t) \in \mathbb{R}^m$, $g(t) = A\Psi(t) \in \mathbb{R}^n$, $x_0 \in \mathbb{R}^m$,

$$A = 2I - \tau^2 W, \quad B = -2(I \otimes e_1), \quad Q = c(I \otimes W + W \otimes I), \quad C = \frac{c}{\tau^2}(I \otimes e_1^T)$$

with $c = 2 \times 10^{-3}$, $n = \frac{1}{\tau} - 1$, $I \in \mathbb{R}^{n \times n}$ *being the identity matrix,* e_1 *being the first row of* I, $\otimes$ *denoting the Kronecker product, and*

$$W = -\frac{1}{\tau^2}\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}_{n \times n}.$$

	$k=1$	$k=2$	$k=3$
$l=1$	2.2175	0.1778	0.0000
$l=2$	8.6956	0.0000	-
$l=3$	4.6609	0.0330	0.0000
$l=4$	3.3617	0.0000	-
$l=5$	2.2778	0.0000	-
$l=6$	1.7678	0.0024	0.0000
$l=7$	1.3180	0.0000	-
$l=8$	1.0075	0	-
$l=9$	0.8098	0.0000	-
$l=10$	0.7239	0.0000	-

Table 4.1. The value of $\|\min(v^k, Av^k + Bu^k + g(t_l))\|$ for $\tau = 1/200$ and $h = 0.4$.

Example 4.1 arises from the spatial semi-discretization of a parabolic Signorini problem in the form of

$$\begin{aligned} &c\Delta u - \partial_t u = 0 && \text{in} \quad \Omega_T := \Omega \times (0,T), \\ &0 \le \partial_\nu u \perp (u - \psi) \ge 0 && \text{on} \quad \mathcal{M}_T := \mathcal{M} \times (0,T), \\ &u = 0 && \text{on} \quad \mathcal{S}_T := \mathcal{S} \times (0,T), \\ &u(\cdot, 0) = u_0 && \text{on} \quad \Omega_0 := \Omega \times 0, \end{aligned}$$

where $\Omega = (0,1) \times (0,1)$, $\mathcal{M} = (0,1) \times \{0\}$, $\mathcal{S} = \partial\Omega \setminus \mathcal{M}$, ∂_ν denotes the outer normal derivative on $\partial\Omega$, ψ and u_0 are defined by

$$\psi(x_1, t) = \begin{cases} 4/(1+t), & \text{if } |x_1 - 1/2| \ge 1/4, \\ \sin(2\pi t), & \text{otherwise}, \end{cases} \qquad u_0(x_1, x_2) = 2x_1x_2(1-x_1)(1-x_2).$$

See [30] for details.

We discretize the above differential linear complementarity system by the implicit Euler scheme with time step-size h and solve the corresponding DLCS at each time step by Algorithm 3.1. It follows from Theorem 2.4.14 of [18] that A and $G(u,v)$ in (1.9) are M-matrices for any u and v, which, together with Theorem 3.2, implies that Algorithm 3.1 terminates in at most n iterations. Below, we shall verify numerically the finite termination of Algorithm 3.1 and study numerically the dependence of the number of iterations of two algorithms (i.e. Algorithm 2.1 and Algorithm 3.1) on the parameters (i.e., n, η_k) and the initial iterate $x^{(1)}$.

We first test the finite termination of Algorithm 3.1. We set $\epsilon = 10^{-10}$ and $\eta_k = 0$ in Algorithm 3.1 and $x^{(0)} = 0$ in Algorithm 2.1, and fix $h = 0.4$. Tables 4.1 and 4.2 list the values of $\|\min(v^k, Av^k + Bu^k + g(t_l))\|$ for $\tau = 1/200$ and $\tau = 1/400$, respectively. We can see from these two tables that Algorithm 3.1 terminates in at most five iterations even if the value of the last but one iterate is large, which shows that Algorithm 3.1 possesses finite termination.

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
$l=1$	2.7535	0.3572	0.0799	0.0125	0.0000
$l=2$	13.7797	0.0000	-	-	-
$l=3$	7.5202	0.2374	0.0568	0.0078	0.0000
$l=4$	5.1764	0.0000	-	-	-
$l=5$	3.7057	0.0000	-	-	-
$l=6$	3.0099	0.0043	0.0000	-	-
$l=7$	2.3691	0.0000	-	-	-
$l=8$	1.8233	0	-	-	-
$l=9$	1.6548	0.0000	-	-	-
$l=10$	1.3925	0.0000	-	-	-

Table 4.2. The value of $\|\min(v^k, Av^k + Bu^k + g(t_l))\|$ for $\tau = 1/400$ and $h = 0.4$.

	$h=0.4$	$h=0.2$	$h=0.1$	$h=0.05$
$n=99$	(2, 1.9)	(3, 1.9)	(2, 1.675)	(3, 1.35)
$n=199$	(3, 2.3)	(3, 2.15)	(3, 1.975)	(3, 1.8375)
$n=399$	(5, 2.7)	(4, 2.35)	(4, 2.15)	(4, 2.0875)

Table 4.3. The (N_m, N_a) for different n and h.

	$h = 0.04$		$h = 0.02$		$h = 0.01$	
	W1	W2	W1	W2	W1	W2
$n = 99$	(1.23, 1.43)	(1.24, 1.09)	(1.065, 1.22)	(1.065, 1.07)	(1.05, 1.205)	(1.0525, 1.05)
$n = 199$	(1.75, 2.39)	(1.75, 1.63)	(1.455, 1.81)	(1.46, 1.065)	(1.205, 1.42)	(1.2075, 1.02)
$n = 399$	(2.03, 2.79)	(2.03, 2.00)	(1.865, 2.56)	(1.875, 1.63)	(1.605, 2.115)	(1.61, 1.0675)

Table 4.4. The (N_{a1}, N_{a2}) for different n and h.

We then study the dependence of Algorithm 3.1 on the parameter n. We set $\epsilon = 10^{-10}$ and $\eta_k = 0$ in Algorithm 3.1 and $x^{(0)} = 0$ in Algorithm 2.1. Table 4.3 lists the maximum number of iterations (abbreviated as N_m) and the average number of iterations (abbreviated as N_a) for different n and h. We can see from Table 4.3 that the maximum number of iterations and the average number of iterations increase at most linearly with n if h is kept fixed.

We finally test the dependence of the number of iterations of two algorithms on the parameter η_k and the initial iterate $x^{(0)}$. We set the parameters (i.e., ϵ, η_k) of Algorithm 3.1 and the initial iterate $x^{(0)}$ of Algorithm 2.1 by the following two different ways:

(W1) $\epsilon = 10^{-10}$, $\eta_k = 0$ and $x^{(0)} = 0$;

(W2) $\epsilon = 10^{-10}$, $\eta_k = \frac{0.1}{k+1}$, $x^{(0)} = 0$ as $k = 0$ and $x^{(0)} = v^{k-1}$ as $k \geq 1$ with v^{k-1} given in (3.4).

Table 4.4 lists the average number of iterations needed by Algorithm 3.1 (abbreviated as N_{a1}) and the average number of iterations needed by Algorithm 2.1 (abbreviated as N_{a2}) per time step. We can see from Table 4.4 that the average number of iterations needed by Algorithm 3.1 is almost the same regardless of whether ZLCP (3.1) is solved exactly or inexactly but Algorithm 2.1 requires less iterations as ZLCP (3.1) is solved inexactly.

5 Concluding remarks

In this paper we proposed a new generalized Newton method for solving a class of discrete-time linear complementarity system in which the coefficient matrix of linear complementarity constraint is a Z-matrix. We first derived some new characterizations of the least element solution of the Z-matrix linear complementarity problem (ZLCP). In particular, we proved that a solution of the ZLCP is the least element solution if and only if the principal submatrix corresponding to the nonzero components of the solution is an M-matrix. This characterization is stronger than that obtained by Chen and Xiang [7]. Then, we proposed a Newton method for solving the least element solution of the ZLCP and study its convergence. Finally, we proposed a new generalized Newton method for solving the discrete linear complementarity system which arises from the implicit time-stepping scheme for differential linear complementarity systems. Under suitable conditions, we proved that the proposed method has a globally linear rate of convergence and a finite-termination property. Preliminary numerical results showed the efficiency of the proposed method.

The current development of this paper is based on the assumption that the matrix A in the linear complementarity constraint is a Z-matrix. It is worthy of a further research whether it can be extended to the case where the matrix A in the linear complementarity constraint is a positive semidefinite matrix and the case where the complementarity constraint is a Z-function complementarity problem.

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