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GLOBAL WEAK SOLUTIONS TO THE GENERALIZED MCH EQUATION VIA CHARACTERISTICS

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Abstract. In this paper, we study the generalized modified Camassa-Holm (gmCH) equation via characteristics. We first change the gmCH equation for unknowns (u, m) into its Lagrangian dynamics for characteristics $X(\xi, t)$, where $\xi \in \mathbb{R}$ is the Lagrangian label. When $X_\xi(\xi, t) > 0$, we use the solutions to the Lagrangian dynamics to recover the classical solutions with $m(\cdot, t) \in C_0^k(\mathbb{R})$ $(k \in \mathbb{N},\ k \geq 1)$ to the gmCH equation. The classical solutions (u, m) to the gmCH equation will blow up if $\inf_{\xi\in\mathbb{R}} X_\xi(\cdot, T_{\max}) = 0$ for some $T_{\max} > 0$. After the blow-up time $T_{\max}$, we use a double mollification method to mollify the Lagrangian dynamics and construct global weak solutions (with m in space-time Radon measure space) to the gmCH equation by some space-time BV compactness arguments.

1. **Introduction.** In this paper, we are going to study the following family of nonlinear partial differential equations in $\mathbb{R}$:

$$m_t + [(u^2 - u_x^2)^p m]_x = 0, \quad m = u - u_{xx}, \quad x \in \mathbb{R},\ t > 0, \tag{1}$$

subject to the initial condition

$$m(x, 0) = m_0(x), \quad x \in \mathbb{R}. \tag{2}$$

Here p is a positive integer. This equation is known as the generalized modified Camassa-Holm equation (gmCH). It first appeared in [1], where a family of nonlinear dispersive wave equations with peakon solutions was introduced in the following form:

$$m_t + f(u, u_x)m + (g(u, u_x)m)_x = 0, \tag{3}$$

where $f(u, u_x)$ and $g(u, u_x)$ are arbitrary non-singular functions of the wave amplitude and the wave gradient. They called equation (3) as the fg-family equation. When $f(u, u_x) = 0$ and $g(u, u_x) = (u^2 - u_x^2)^p$, the fg-family equation (3) becomes

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the gmCH equation (1). When $p = 1$, equation (1) is known as the Fokas-Olver-Rosenau-Qiao or modified Camassa-Holm (mCH) equation [3, 4, 11, 12]. Since the fundamental solution of the Helmholtz operator $I - \partial_{xx}$ is $G(x) = \frac{1}{2}e^{-|x|}$, function u can be expressed as a convolution of m with the kernel G:

$$u(x,t) = (G * m)(x,t) = \int_{\mathbb{R}} G(x-y)m(y,t)\,\mathrm{d}y.$$

The local well-posedness of the equation (1) in Besov spaces $B^s_{p,r}$ with $s > \max\{2+\frac{1}{p}, \frac{5}{2}\}$ was studied in [13], where two blow-up criterions were also provided. The strong solutions to the gmCH equation (1) will blow up in finite time for some initial data; see [14]. Hence, it is important to find some proper spaces and study global weak solutions to the gmCH equation (1), which is also one of our purposes in this paper. As we know, there are some special weak solutions to the gmCH equation known as N-peakon (or multi-peakon) solutions in the form (see [1, 8, 7, 9, 6]):

$$u(x,t) = \sum_{i=1}^{N} p_i G(x - x_i(t)), \quad m(x,t) = \sum_{i=1}^{N} p_i \delta(x - x_i(t))$$

for some constants p_i, $i = 1, 2, \cdots, N$. Global existence of one peakon (solitary wave) was obtained in [1, 7, 5]. When $N > 1$, global N-peakon weak solutions were also constructed in [5] by a double mollification method. The results on the stability of peakons can be found in [8, 7, 9].

In this paper, we are going to study both local classical solutions and global weak solutions to the gmCH equation (1) via characteristics. Let $X(\xi,t)$ be the related characteristic flow map for the gmCH equation, the gmCH equation (1) in Lagrangian coordinate is then given by:

$$\begin{cases} \partial_t X(\xi,t) = (u^2 - u_x^2)^p (X(\xi,t),t), \ X(\xi,0) = \xi \in \mathbb{R}, \ t > 0, \\ m(\cdot,t) = X(\cdot,t)\# m_0(\cdot), \\ u(x,t) = (G * m)(x,t) = \displaystyle\int_{\mathbb{R}} G(x - X(\theta,t)) m_0(\theta)\,\mathrm{d}\theta. \end{cases} \tag{4}$$

For classical solutions (u,m), taking derivative of (4) with respect to ξ gives

$$\partial_t X_\xi(\xi,t) = 2p[(u^2 - u_x^2)^{p-1} m u_x](X(\xi,t),t) X_\xi(\xi,t),$$

which implies

$$X_\xi(\xi,t) = \exp\left\{ \int_0^t 2p[(u^2 - u_x^2)^{p-1} m u_x](X(\xi,s),s)\,\mathrm{d}s \right\} > 0.$$

Hence $X(\cdot,t) : \mathbb{R} \to \mathbb{R}$ is a homeomorphism for classical solutions, and we have $X(\xi,t) \to \pm\infty$ as $\xi \to \pm\infty$. If we try to solve (4) directly, it is not easy to find a proper Banach space for $X(\cdot,t)$. For $p = 1$, the local well-posedness for (4) was obtained in [6] for compactly supported initial data m_0. In this paper, we are going to take advantage of the far field estimates to transform (4) into another dynamics and remove the condition for compact support in [6]. Since we are interested in the solutions (u,m) decaying to zero when spacial variable x goes to infinity, we have the following a-priori estimate at far fields:

$$\lim_{\xi \to \pm\infty} \partial_t X(\xi,t) = \lim_{\xi \to \pm\infty} (u^2 - u_x^2)^p (X(\xi,t),t) = 0,$$

with initial data $\lim_{\xi\to\pm\infty}[X(\xi,0)-\xi]=0$, which formally implies

$$\lim_{\xi\to\pm\infty}[X(\xi,t)-\xi]=0. \tag{5}$$

Instead of solving (4) directly, we are going to work on the corresponding dynamics for $Y(\xi,t)=X(\xi,t)-\xi$ in Banach space $C_0^k(\mathbb{R})$ $(k\geq 0,\ k\in\mathbb{N})$. We obtain local existence and uniqueness for Y and then use it to recover X and the classical solution (u,m) to the gmCH equation (1) (see Section 2).

For the global existence of weak solutions, we are going to use a double mollification method proposed in [6] to study the Lagrangian dynamics (4). In [6], the global weak solutions to the mCH equation were obtained for compactly supported initial data m_0. Here, we will take advantage of (5) and remove the condition for compact support. See details in Section 3.

The rest of this paper is organized as follows. In Section 2, we are going to consider the local well-posedness of classical solutions to the gmCH equation with initial data $m_0\in C_0^k(\mathbb{R})\cap L^1(\mathbb{R})(k\geq 1,\ k\in\mathbb{N})$. We will first transform system (4) into an equation for Y as stated before, and then we will use the Picard theorem for ODE on a Banach space to get the local existence and uniqueness of Y. At last, we will recover classical solutions for the gmCH equation (1) by using local solutions to (4). In Section 3, we are going to study global weak solutions to the gmCH equation with initial data $m_0\in\mathcal{M}(\mathbb{R})$. We will use a double mollification mehtod to study the Lagrangian dynamics (4) and then derive the global existence of weak solutions to the gmCH equation.

2. **Lagrangian dynamics and local well-posedness.** In this section, for initial data $m_0\in C_0^k\cap L^1(\mathbb{R})$ $(k\geq 1,\ k\in\mathbb{N})$, we use the Lagrangian dynamics to prove local well-posedness for classical solutions to the gmCH equation. Notice that we have (5) for classical solutions in $C^k(\mathbb{R})$. Hence, as stated in the Introduction, we are going to consider the dynamic for the following function:

$$Y(\xi,t):=X(\xi,t)-\xi.$$

Equation (4) is transformed into:

$$\begin{cases}\partial_t Y(\xi,t)=F(Y(\xi,t)), & \xi\in\mathbb{R},\ t>0,\\ Y(\xi,0)=0,\end{cases} \tag{6}$$

where the vector field

$$F(Y(\xi,t)):=\Bigg[\left(\int_{\mathbb{R}}G(Y(\xi,t)-Y(\theta,t)+\xi-\theta)m_0(\theta)\,\mathrm{d}\theta\right)^2 - \left(\int_{\mathbb{R}}G'(Y(\xi,t)-Y(\theta,t)+\xi-\theta)m_0(\theta)\,\mathrm{d}\theta\right)^2\Bigg]^p. \tag{7}$$

We view Lagrangian dynamic (6) as an ODE on Banach space $\mathcal{B}=C_0^k(\mathbb{R})$ for $k\in\mathbb{N}^+$. Recall the Picard theorem on a Banach space (see e.g. [10, Theorem 3.1]), and we need to choose a suitable open subset $\mathcal{O}\subset\mathcal{B}$ to solve (6). Here, we define

$$\mathcal{O}:=\{Y\in\mathcal{B};\ Y_\xi+1>0\}. \tag{8}$$

We have the following local existence and uniqueness for (6):

Theorem 2.1. *Let $m_0 \in C_0(\mathbb{R}) \cap L^1(\mathbb{R})$. Then, there exists a time $T > 0$ such that the Lagrangian dynamic (6) has a unique solution $Y \in C^1([0,T);\mathcal{O})$. Moreover, if we assume $m_0 \in C_0^k(\mathbb{R}) \cap L^1(\mathbb{R})$ for some integer $k \geq 1$, then we have:*

$$Y \in C^1([0,T); C_0^{k+1}(\mathbb{R})).$$

Proof. Consider the Banach space $\mathcal{B} := C_0^1(\mathbb{R})$ with the norm

$$\|f\|_{\mathcal{B}} := \sup_{\xi\in\mathbb{R}} |f(\xi)| + \sup_{\xi\in\mathbb{R}} |f'(\xi)|, \quad \forall f \in \mathcal{B}.$$

Let $\mathcal{O}$ be defined by (8). Obviously, the set $\mathcal{O}$ is an open subset of $\mathcal{B}$.

Step 1. For any $Y \in \mathcal{O}$, we prove that $F(Y) \in \mathcal{B}$ for F given by (7). We first prove that

$$\lim_{\xi\to\pm\infty} F(Y(\xi)) = 0. \tag{9}$$

Due to $Y \in \mathcal{O} \subset C_0^1(\mathbb{R})$, we have

$$\lim_{\xi\to\pm\infty} [Y(\xi) + \xi] = \pm\infty.$$

Hence,

$$\lim_{\xi\to\pm\infty} G\left(Y(\xi) - Y(\theta) + \xi - \theta\right) = 0 \text{ for any } \theta \in \mathbb{R}$$

and

$$\lim_{\xi\to\pm\infty} G'\left(Y(\xi) - Y(\theta) + \xi - \theta\right) = 0 \text{ for any } \theta \in \mathbb{R}.$$

By the Lebesgue dominated convergence theorem, we know that (9) holds, which means

$$F(Y) \in C_0(\mathbb{R}).$$

Next, we estimate the derivatives of $F(Y)$. Due to properties of G and the monotonicity of $Y(\xi) + \xi$, we have

$$\begin{aligned}
&\int_{\mathbb{R}} G(Y(\xi) - Y(\theta) + \xi - \theta) m_0(\theta)\, \mathrm{d}\theta \\
=&\int_{-\infty}^{\xi} G(Y(\xi) - Y(\theta) + \xi - \theta) m_0(\theta)\, \mathrm{d}\theta + \int_{\xi}^{\infty} G(Y(\xi) - Y(\theta) + \xi - \theta) m_0(\theta)\, \mathrm{d}\theta \\
=:&H_1(Y(\xi)) + H_2(Y(\xi))
\end{aligned}$$

and

$$\int_{\mathbb{R}} G'(Y(\xi) - Y(\theta) + \xi - \theta) m_0(\theta)\, \mathrm{d}\theta = -H_1(Y(\xi)) + H_2(Y(\xi)).$$

Direct calculation shows

$$F(Y(\xi)) = 4^p [H_1(Y(\xi))]^p [H_2(Y(\xi))]^p. \tag{10}$$

Since

$$\partial_\xi H_1(Y(\xi)) = \frac{1}{2} m_0(\xi) - H_1(Y(\xi))(\partial_\xi Y(\xi) + 1) \tag{11}$$

and

$$\partial_\xi H_2(Y(\xi)) = -\frac{1}{2} m_0(\xi) + H_2(Y(\xi))(\partial_\xi Y(\xi) + 1), \tag{12}$$

take derivative of $F(Y(\xi))$ with respect to ξ and we obtain

$$\partial_\xi F(Y(\xi)) = p 2^{2p-1} m_0(\xi) [H_1(Y(\xi))]^{p-1} [H_2(Y(\xi))]^{p-1} [H_2(Y(\xi)) - H_1(Y(\xi))]. \tag{13}$$

It is easy to compute that

$$|H_1(Y)(\xi)| \le \frac{1}{2}\|m_0\|_{L^1}, \quad |H_2(Y)(\xi)| \le \frac{1}{2}\|m_0\|_{L^1}. \tag{14}$$

Then by (13) and $m_0 \in C_0(\mathbb{R})$, we obtain

$$\lim_{\xi\to\pm\infty} \partial_\xi F(Y(\xi)) = 0.$$

Hence $F(Y) \in C_0^1(\mathbb{R})$.

Step 2. We proof that functional F is Lipschitz continuous in $\mathcal{O}$. For any given $Y_1, Y_2 \in \mathcal{O}$ and $\xi \in \mathbb{R}$, we have

$$\begin{aligned}
&|F(Y_1(\xi)) - F(Y_2(\xi))| \\
&= 4^p\, |[H_1(Y_1(\xi))]^p[H_2(Y_1(\xi))]^p - [H_1(Y_2(\xi))]^p[H_2(Y_2(\xi))]^p| \\
&\le 4^p\, |[H_1(Y_1(\xi))]^p|\, |[H_2(Y_1(\xi))]^p - [H_2(Y_2(\xi))]^p| \\
&\quad + 4^p\, |[H_2(Y_2(\xi))]^p|\, |[H_1(Y_1(\xi))]^p - [H_1(Y_2(\xi))]^p| \\
&\le 4^p|[H_1(Y_1(\xi))]^p||H_2(Y_1(\xi)) - H_2(Y_1(\xi))||\sum_{k=0}^{p-1}[H_2(Y_1(\xi))]^k[H_2(Y_1(\xi))]^{p-1-k}| \\
&\quad + 4^p|[H_2(Y_2(\xi))]^p||H_1(Y_1(\xi)) - H_1(Y_2(\xi))||\sum_{k=0}^{p-1}[H_1(Y_1(\xi))]^k[H_1(Y_2(\xi))]^{p-1-k}| \\
&\le p\|m_0\|_{L^1}^{2p}\|Y_1 - Y_2\|_{\mathcal{B}}.
\end{aligned}$$

Similarly, we also have

$$|\partial_\xi F(Y_1(\xi)) - \partial_\xi F(Y_2(\xi))| \le 2p(2p-1)\|m_0\|_{\sup}\|m_0\|_{L^1}^{2p-1}\|Y_1 - Y_2\|_{\mathcal{B}}.$$

Combining the above two inequalities gives

$$\|F(Y_1) - F(Y_2)\|_{\mathcal{B}} \le (p\|m_0\|_{L^1} + 2p(2p-1)\|m_0\|_{\sup})\, \|m_0\|_{L^1}^{2p-1}\|Y_1 - Y_2\|_{\mathcal{B}}.$$

Hence, functional F is Lipschitz continuous in $\mathcal{O}$.

Step 3. Due to $Y_0 = 0 \in \mathcal{O}$, by Picard Theorem [10, Theorem 3.1], there exists a time T such that Lagrangian dynamic (6) has a unique local solution $Y \in C^1([0,T);\mathcal{O})$.

Step 4. (Regularity) When $m_0 \in C_0^k(\mathbb{R})$ for some integer $k \ge 1$, we solve Lagrangian dynamic (6) in Banach space $\tilde{\mathcal{B}} := C_0^{k+1}(\mathbb{R})$ with norm

$$\|f\|_{\mathcal{B}} := \sum_{i=0}^{k+1} \sup_{\xi\in\mathbb{R}} |f^{(i)}(\xi)|, \quad \forall f \in \mathcal{B}$$

and open subset

$$\tilde{\mathcal{O}} := \left\{ Y \in \tilde{\mathcal{B}};\ Y_\xi + 1 > 0 \ \text{ for } \ \xi \in \mathbb{R} \right\}.$$

For any $Y \in C_0^{k+1}(\mathbb{R})$, recalling equalities (11), (12), (13) and the estimates (14), we can show that

$$\lim_{\xi\to\pm\infty} \frac{\partial^r}{\partial\xi^r} F(Y(\xi)) = 0, \quad 1 \le r \le k+1,$$

and hence $F(Y) \in C_0^{k+1}(\mathbb{R})$. On the other hand, as in the proof in Step 2, we can prove that functional F is Lipschitz continuous in $\tilde{\mathcal{O}}$. Therefore, by Picard theorem we can obtain $Y \in C^1([0,T); C_0^{k+1}(\mathbb{R}))$ for some $T > 0$. $\square$

Next, we recover the local solutions to the gmCH equation (1) in Eulerian coordinate by the solutions to the Lagrangian dynamic (6) given by Theorem 2.1. We have the following local well-posedness theorem for the gmCH equation(1):

Theorem 2.2. *Let $m_0 \in C_0^k(\mathbb{R}) \cap L^1(\mathbb{R})$ be an initial datum for the gmCH equation (1) for some integer $k \geq 1$. Then, there is a unique classical solution (u, m) to gmCH equation (1) and we have*

$$u \in C^1([0,T); C_0^{k+2}(\mathbb{R})),\ \ m \in C^1([0,T); C_0^k(\mathbb{R}) \cap L^1(\mathbb{R})), \tag{15}$$

$$u(\cdot,t) = G * m(\cdot,t) \in W^{1,p}(\mathbb{R}) \ \ \textit{for any} \ \ t \in [0,T),\ p \geq 1. \tag{16}$$

Proof. For $m_0 \in C_0^k(\mathbb{R}) \cap L^1(\mathbb{R})$, let $Y \in C^1([0,T); C_0^{k+1}(\mathbb{R}))$ be a solution to Lagrangian dynamic (6) given by Theorem 2.1. Set $X(\xi,t) := Y(\xi,t) + \xi$ for $\xi \in \mathbb{R}$, so it satisfies the following equations:

$$\begin{aligned} \partial_t X(\xi,t) = \Bigg[\Big(\int_{\mathbb{R}} G(X(\xi,t) - X(\theta,t)) m_0(\theta)\, \mathrm{d}\theta \Big)^2 & \\ - \Big(\int_{\mathbb{R}} G'(X(\xi,t) - X(\theta,t)) m_0(\theta)\, \mathrm{d}\theta \Big)^2 \Bigg]^p &, \end{aligned} \tag{17}$$

and

$$\partial_\xi X(\xi,t) > 0, \ \text{for all } \xi \in \mathbb{R} \text{ and } t \in [0,T).$$

Define (u,m) by

$$u(x,t) := \int_{\mathbb{R}} G(x - X(\theta,t)) m_0(\theta)\, \mathrm{d}\theta, \quad m(x,t) := \int_{\mathbb{R}} \delta(x - X(\theta,t)) m_0(\theta)\, \mathrm{d}\theta. \tag{18}$$

Similarly to [6, Theorem 2.6], we can proof (15) and check that (u,m) defined by (18) is a unique classical solution to (1).

Next, we only need to prove (16). Because $m(\cdot,t) \in C_0(\mathbb{R}) \cap L^1(\mathbb{R})$, we know $m(\cdot,t) \in L^p(\mathbb{R})$ for any $t \in [0,T)$ and $p \geq 1$. Due to Young's inequality, we have $u(\cdot,t) = G * m(\cdot,t) \in L^p(\mathbb{R})$ and $u_x(\cdot,t) = G_x * m(\cdot,t) \in L^p(\mathbb{R})$ for any $t \in [0,T)$ and $p \geq 1$. Hence, (16) holds.

□

Remark 1. Due to the continuation of an autonomous ODE on a Banach space (see [10, Theorem 3.3]), we can extend our local solutions to the Lagrangian dynamic (6) or the gmCH equation (1) to some time $T_{\max}$. If $T_{\max}$ is finite, then we have $\inf_{(\xi,t)\in\mathbb{R}\times[0,T_{\max})} \partial_\xi Y(\xi,t) + 1 = 0$, which means $\inf_{(\xi,t)\in\mathbb{R}\times[0,T_{\max})} \partial_\xi X(\xi,t) = 0$. This finite $T_{\max}$ corresponds to the blow-up time for the classical solutions (see [6, Theorem 1.1] for the case of $p = 1$).

For some initial data $m_0 \in H^s(\mathbb{R})$ with $s > \frac{1}{2}$, the classical solutions will blow up in finite time; see [14, Theorem 3.3]. Hence, the nature questions are in which space we can extend and how to extend the solutions globally after $T_{\max}$. In the next section, we are going to solve the above questions.

3. **Global weak solutions via mollified characteristics.** In this section, we are going to obtain global weak solutions to the gmCH equation (1). Assume initial data m_0 belongs to the Radon measure space $\mathcal{M}(\mathbb{R})$. The main ideas come from [6], where the global weak solutions to the mCH equation were obtained for m_0 with compact support. Here, we take advantage of the far field estimate (5) and obtain global weak solutions to the gmCH equation (1) without compact support for the initial datum m_0.

The gmCH equation (1) is rewritten as

$$
\begin{aligned}
&(1-\partial_{xx})u_t + \left[(u^2-u_x^2)^p(u-u_{xx})\right]_x \\
&= (1-\partial_{xx})u_t + \sum_{k=0}^{p}(-1)^k\binom{p}{k}\left[u^{2(p-k)+1}u_x^{2k} + \frac{2(p-k)}{2k+11}u^{2(p-k)-1}u_x^{2k+2}\right]_x \\
&\quad - \sum_{k=0}^{p}\frac{(-1)^k}{2k+1}\binom{p}{k}\left[u^{2(p-k)}u_x^{2k+1}\right]_{xx} \\
&=: (1-\partial_{xx})u_t + A(u,u_x)_x + B(u,u_x)_x - C(u,u_x)_{xx} = 0,
\end{aligned}
\tag{19}
$$

where

$$
\begin{aligned}
A(u,u_x) &:= \sum_{k=0}^{p}(-1)^k\binom{p}{k}u^{2(p-k)+1}u_x^{2k}, \\
B(u,u_x) &:= \sum_{k=0}^{p}(-1)^k\binom{p}{k}\frac{2(p-k)}{2k+1}u^{2(p-k)-1}u_x^{2k+2}, \\
C(u,u_x) &:= \sum_{k=0}^{p}\frac{(-1)^k}{2k+1}\binom{p}{k}u^{2(p-k)}u_x^{2k+1}.
\end{aligned}
\tag{20}
$$

For any $\phi \in C_c^\infty(\mathbb{R}\times[0,\infty))$, we denote the functional

$$
\begin{aligned}
\mathcal{L}(u,\phi) &:= \int_0^\infty\int_{\mathbb{R}} u(x,t)[\phi_t(x,t)-\phi_{txx}(x,t)]\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^\infty\int_{\mathbb{R}}[A(u,u_x)+B(u,u_x)]\phi_x(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_0^\infty\int_{\mathbb{R}}C(u,u_x)\phi_{xx}(x,t)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{21}
$$

Now the definition of weak solutions to the gmCH equation (1)-(2) is given as follows:

Definition 3.1. For $m_0 \in \mathcal{M}(\mathbb{R})$, a function

$$u \in C([0,+\infty);H^1(\mathbb{R})) \cap L^\infty(0,+\infty;W^{1,\infty}(\mathbb{R})) \cap W^{1,\infty}(0,+\infty;L^\infty(\mathbb{R}))$$

is said to be a global weak solution to the gmCH equation (1)-(2) if

$$\mathcal{L}(u,\phi) + \int_{\mathbb{R}}\phi(x,0)m_0(\mathrm{d}x) = 0 \tag{22}$$

holds for all $\phi \in C_c^\infty(\mathbb{R}\times[0,\infty))$.

Next, we describe the double mollification method for (4). Let $\{\rho_\epsilon\}_{\epsilon>0}$ be a family of standard even mollifiers. We define $G^\epsilon(x) := (\rho_\epsilon * G)(x)$. Obviously, both G^ϵ and $G_x^\epsilon \in C_0^\infty(\mathbb{R})$ are global Lipschitz. For any measurable function $X:\mathbb{R}\to\mathbb{R}$, we define

$$U_\epsilon(x;X) := \left[\left(\int_{\mathbb{R}}G^\epsilon(x-X(\theta))m_0(\mathrm{d}\theta)\right)^2 - \left(\int_{\mathbb{R}}G_x^\epsilon(x-X(\theta))m_0(\mathrm{d}\theta)\right)^2\right]^p.$$

Take mollification one more time and we set

$$U^\epsilon(x;X) := (\rho_\epsilon * U_\epsilon)(x;X).$$

Then the regularized Lagrangian dynamic is given by

$$\begin{cases} \partial_t X(\xi,t) = U^\epsilon(X(\xi,t);X(\cdot,t)), \\ X(\xi,0) = \xi \in \mathbb{R}. \end{cases} \tag{23}$$

Notice that with one time mollification, vector field U_ϵ is already global Lipschitz and we can construct global approximated solutions with vector field U_ϵ. However, the solutions corresponding to the vector field U_ϵ do not satisfy weak consistency (one can check this by following Step 3 in the proof of Theorem 3.3). This is the reason that we choose double mollification.

As we will use some space-time BV compactness arguments to develop our results, we recall the concept of space $BV(\mathbb{R}^d)$.

Definition 3.2. (i) For dimension $d \geq 1$ and an open set $\Omega \subseteq \mathbb{R}^d$, a function $f \in L^1(\Omega)$ belongs to $BV(\Omega)$ if

$$Tot.Var.\{f\} := \sup\left\{\int_\Omega f(x)\nabla\cdot\phi(x)\,\mathrm{d}x : \phi \in C_c^1(\Omega;\mathbb{R}^d), \quad \|\phi\|_{L^\infty} \leq 1\right\} < \infty.$$

(ii) (Equivalent definition for one dimension case) A function f belongs to $BV(\mathbb{R})$ if for any $\{x_i\} \subseteq \mathbb{R}$, $x_i < x_{i+1}$, the following statement holds:

$$Tot.Var.\{f\} := \sup_{\{x_i\}}\left\{\sum_i |f(x_i) - f(x_{i-1})|\right\} < \infty.$$

Next, we use (23) to obtain global weak solutions to the gmCH equation (1). The main theorem is as follows:

Theorem 3.3. *Let the initial data $m_0 \in \mathcal{M}(\mathbb{R})$ satisfy*

$$M_1 := |m_0|(\mathbb{R}) < +\infty. \tag{24}$$

Then there exists a global weak solution (u, m) to the gmCH equation (1)-(2). Furthermore, for any $T > 0$, we have

$$u \in BV(\mathbb{R}\times[0,T)), \quad m = (1-\partial_{xx})u \in \mathcal{M}(\mathbb{R}\times[0,T)).$$

Proof. **Step 1. Regularized Lagrangian dynamics and approximated solutions.** Instead of studying (23) directly, we will use the equation for $Y(\xi,t) = X(\xi,t) - \xi$. Let

$$F^\epsilon(Y(\xi,t)) := U^\epsilon(X(\xi,t);X(\cdot,t)). \tag{25}$$

From (23), we have the following equation for Y :

$$\begin{cases} \partial_t Y(\xi,t) = F^\epsilon(Y(\xi,t)), \quad \xi\in\mathbb{R}, \quad t>0, \\ Y(\xi,0) = 0. \end{cases} \tag{26}$$

Consider equation (26) on Banach space $C_0^1(\mathbb{R})$. Direct checking shows that the map $F^\epsilon : C_0^1(\mathbb{R}) \to C_0^1(\mathbb{R})$ is globally Lipschitz continuous. Using the Picard theorem again, we obtain a unique global solution $Y^\epsilon \in C^1([0,+\infty);C_0^1(\mathbb{R}))$ to equation (26) for $\epsilon > 0$. The solution to (23) is recovered by $X^\epsilon(\xi,t) = Y^\epsilon(\xi,t) + \xi$, and $X^\epsilon \in C^1([0,+\infty);C^1(\mathbb{R}))$.

We define the approximated solutions by the global characteristics:

$$u^\epsilon(x,t) := \int_{\mathbb{R}} G^\epsilon(x - X^\epsilon(\theta,t))m_0(\mathrm{d}\theta) \tag{27}$$

and

$$m^\epsilon(x,t) := (1-\partial_{xx})u^\epsilon(x,t), \quad m_\epsilon(\cdot,t) := X^\epsilon(\cdot,t)\#m_0(\cdot). \tag{28}$$

Since m_ϵ is the push forward of m_0 by the flow map $X^\epsilon(\cdot,t)$, function m_ϵ is a weak solution of

$$m_t + [U^\epsilon(X)m]_x = 0, \tag{29}$$

and the following relation between u^ϵ and m_ϵ holds:

$$u^\epsilon(x,t) = \int_{\mathbb{R}} G^\epsilon(x-y)m_\epsilon(\mathrm{d}y,t).$$

Moreover, we have the relation between m_ϵ and m^ϵ:

$$m^\epsilon(x,t) = (1-\partial_{xx})\int_{\mathbb{R}} G^\epsilon(x-y)m_\epsilon(\mathrm{d}y,t) = \int_{\mathbb{R}} \rho_\epsilon(x-y)m_\epsilon(\mathrm{d}y,t). \tag{30}$$

Step 2. Compactness argument. In this step, we prove that for any $T>0$ there exist subsequences of u^ϵ and u^ϵ_x (still denoted as u^ϵ and u^ϵ_x), and functions $u,\ u_x \in BV(\mathbb{R}\times[0,T))$ such that

(i) convergence:

$$u^\epsilon \to u,\ u^\epsilon_x \to u_x \ \text{ in } \ L^1_{loc}(\mathbb{R}\times[0,+\infty)) \ \text{ as } \ \epsilon \to 0 \tag{31}$$

and

$$u^\epsilon_t \overset{*}{\rightharpoonup} u_t \ \text{ in } \ L^\infty(\mathbb{R}\times[0,+\infty)) \ \text{ as } \ \epsilon\to 0; \tag{32}$$

(ii) boundedness:

$$\|u\|_{L^\infty} \le \frac{1}{2}M_1, \quad \|u_x\|_{L^\infty} \le \frac{1}{2}M_1 \tag{33}$$

and

$$\|u_t\|_{L^\infty} \le \frac{1}{2}M_1; \tag{34}$$

(iii) time Lipschitz:

$$\int_{\mathbb{R}} |u(x,t)-u(x,s)|\,\mathrm{d}x \le \frac{1}{2^p}M_1^{2p+1}|t-s| \tag{35}$$

and

$$\int_{\mathbb{R}} |u_x(x,t)-u_x(x,s)|\,\mathrm{d}x \le \frac{1}{2^p-1}M_1^{2p+1}|t-s| \tag{36}$$

for any $t,s\in[0,+\infty)$;

(iv) space of u:

$$u \in C([0,+\infty);H^1(\mathbb{R})) \cap L^\infty(0,+\infty;W^{1,\infty}(\mathbb{R})) \cap W^{1,\infty}(0,+\infty;L^\infty(\mathbb{R})). \tag{37}$$

In fact, we have $\text{Tot.Var.}\{G^\epsilon\} = 1$, $\quad \text{Tot.Var.}\{G^\epsilon_x\} = 2$ and

$$\|G^\epsilon\|_{L^\infty} \le \frac{1}{2}, \quad \|G^\epsilon_x\|_{L^\infty} \le \frac{1}{2} \quad \|G^\epsilon\|_{L^1} \le 1, \quad \|G^\epsilon_x\|_{L^1} \le 1.$$

From the definition of u^ϵ (27), we have

$$\|u^\epsilon\|_{L^\infty} \le \frac{1}{2}M_1 \ \text{ and } \ \|u^\epsilon_x\|_{L^\infty} \le \frac{1}{2}M_1 \ \text{ for any } \ \epsilon>0. \tag{38}$$

By [2, Lemma 2.3], we have the time Lipschitz estimate for u^ϵ:

$$\begin{aligned}\int_{\mathbb{R}}|u^\epsilon(x,t)-u^\epsilon(x,s)|\,\mathrm{d}x &\le \int_{\mathbb{R}}\int_{\mathbb{R}}|G^\epsilon(x-X^\epsilon(\theta,t))-G^\epsilon(x-X^\epsilon(\theta,s))|m_0(\mathrm{d}\theta)\,\mathrm{d}x\\ &\le \int_{\mathbb{R}}\text{Tot.Var.}\{G^\epsilon\}|X^\epsilon(\theta,t)-X^\epsilon(\theta,s)|m_0(\mathrm{d}\theta)\\ &\le \int_{\mathbb{R}}\|U^\epsilon(X^\epsilon)\|_{L^\infty}|t-s|m_0(\mathrm{d}\theta)\le \frac{1}{2^p}M_1^{2p+1}|t-s|.\end{aligned}$$

Similarly, we have

$$\int_{\mathbb{R}}|u_x^\epsilon(x,t)-u_x^\epsilon(x,s)|\,\mathrm{d}x \le \frac{1}{2^{p-1}}M_1^{2p+1}|t-s|.$$

By using [2, Theorem 2.4], we have (31), (33), (35) and (36). Due to [2, Theorem 2.6], we know that both u and u_x are space-time BV functions. Since the definition of u^ϵ, for any $x\in\mathbb{R}$ and $t\ge 0$, we have

$$\begin{aligned}|u_t^\epsilon(x,t)| &= \left|\int_{\mathbb{R}}G_x^\epsilon(x-X^\epsilon(\theta,t))\partial_t X^\epsilon(\theta,t)m_0(\mathrm{d}\theta)\right|\\ &\le \int_{\mathbb{R}}\|G_x^\epsilon\|_{L^\infty}\|U^\epsilon(X^\epsilon)\|_{L^\infty}m_0(\mathrm{d}\theta)\le \frac{1}{2^{p+1}}M_1^{2p+1}.\end{aligned}$$

Hence, u_t^ϵ is uniformly bounded in $L^\infty(\mathbb{R}\times[0,+\infty))$. Consequently, we can find a subsequence of u_t^ϵ (still denoted as u_t^ϵ) and $v\in L^\infty(\mathbb{R}\times[0,+\infty))$ such that

$$u_t^\epsilon \xrightharpoonup{*} v \quad\text{in}\quad L^\infty(\mathbb{R}\times[0,+\infty))\quad\text{as}\quad \epsilon\to 0.$$

It is obvious that v is the weak derivative of u with respect to the time variable t. Therefore

$$u\in W^{1,\infty}(0,+\infty;L^\infty(\mathbb{R})).$$

Moreover, due to (35) we obtain

$$\begin{aligned}\|u(\cdot,t)-u(\cdot,s)\|_{L^2}^2 &= \int_{\mathbb{R}}|u(x,t)-u(x,s)|^2\,\mathrm{d}x\\ &\le M_1\int_{\mathbb{R}}|u(x,t)-u(x,s)|\,\mathrm{d}x \le \frac{1}{2^p}M_1^{2p+2}|t-s|.\end{aligned}$$

Similarly,

$$\|u_x(\cdot,t)-u_x(\cdot,s)\|_{L^2}^2 \le \frac{1}{2^{p-1}}M_1^{2p+2}|t-s|.$$

These two inequalities imply

$$\begin{aligned}\|u(\cdot,t)-u(\cdot,s)\|_{H^1}^2 &\le 2\Big(\|u(\cdot,t)-u(\cdot,s)\|_{L^2}^2+\|u_x(\cdot,t)-u_x(\cdot,s)\|_{L^2}^2\Big)\\ &\le \frac{3}{2^{p-1}}M_1^{2p+2}|t-s|.\end{aligned}$$

Therefore, we have (37).

Step 3. Weak consistency. In this step, we are going to show that for any $\phi\in C_c^\infty(\mathbb{R}\times[0,\infty))$,

$$\left|\mathcal{L}(u^\epsilon,\phi)+\int_{\mathbb{R}}\phi(x,0)m_0(\mathrm{d}x)\right|\le C\epsilon. \tag{39}$$

for some constant C depending on ϕ, but independent of ϵ. Here $\mathcal{L}$ is defined by (21) and u^ϵ is defined by (27). For simplicity of notations, we denote

$$\langle f, g\rangle := \int_0^\infty \int_{\mathbb{R}} f(x,t)g(x,t)\,\mathrm{d}x\,\mathrm{d}t.$$

Since m_ϵ is a weak solution to (29), there holds

$$\langle m_\epsilon, \phi_t\rangle + \langle U^\epsilon m_\epsilon, \phi_x\rangle = -\int_{\mathbb{R}} \phi(x,0)m_0(\mathrm{d}x). \tag{40}$$

On the other hand, we also have

$$\begin{aligned}
\mathcal{L}(u^\epsilon,\phi) &= \int_0^\infty \int_{\mathbb{R}} u^\epsilon[\phi_t - \phi_{txx}]\,\mathrm{d}x\,\mathrm{d}t + \int_0^\infty \int_{\mathbb{R}} [A(u^\epsilon, u^\epsilon_x) + B(u^\epsilon, u^\epsilon_x)]\phi_x\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \int_0^\infty \int_{\mathbb{R}} C(u^\epsilon, u^\epsilon_x)\phi_{xx}\,\mathrm{d}x\,\mathrm{d}t \\
&= \langle \phi_t, (1-\partial_{xx})u^\epsilon\rangle + \langle [(u^\epsilon)^2 - (\partial_x u^\epsilon)^2]^p(1-\partial_{xx})u^\epsilon, \phi_x\rangle \\
&= \langle m^\epsilon, \phi_t\rangle + \langle U_\epsilon m^\epsilon, \phi_x\rangle.
\end{aligned} \tag{41}$$

Combining (40) and (41) yields

$$\left|\mathcal{L}(u^\epsilon,\phi) + \int_{\mathbb{R}} \phi(x,0)m_0(\mathrm{d}x)\right| = |\langle m^\epsilon - m_\epsilon, \phi_t\rangle + \langle U_\epsilon m^\epsilon - U^\epsilon m_\epsilon, \phi_x\rangle|. \tag{42}$$

Since $\phi \in C_c^\infty(\mathbb{R}\times[0,\infty))$, there exists T_ϕ such that $\phi(x,t) = 0$ for $t > T_\phi$ and $x\in\mathbb{R}$. The first term in (42) can be estimated as

$$\begin{aligned}
|\langle m^\epsilon - m_\epsilon, \phi_t\rangle| &= \left|\int_0^\infty \left(\int_{\mathbb{R}} \phi_t(x,t)m^\epsilon(x,t)\,\mathrm{d}x - \int_{\mathbb{R}} \phi_t(x,t)m_\epsilon(\mathrm{d}x,t)\right)\mathrm{d}t\right| \\
&= \left|\int_0^\infty \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \big[\phi_t(x,t) - \phi_t(y,t)\big]\rho_\epsilon(x-y)m_\epsilon(\mathrm{d}y,t)dx\right)\mathrm{d}t\right| \\
&= \int_0^{T_\phi} \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \big|\phi_t(x,t) - \phi_t(X^\epsilon(\theta,t),t)\big|\rho_\epsilon(x - X^\epsilon(\theta,t))m_0(\mathrm{d}\theta)\,\mathrm{d}x\right)\mathrm{d}t \\
&\le M_1\|\phi_{tx}\|_{L^\infty}T_\phi\epsilon.
\end{aligned}$$

For the second term of (42), we can obtain that

$$\begin{aligned}
&|\langle U_\epsilon m^\epsilon - U^\epsilon m_\epsilon, \phi_x\rangle| \\
&= \Bigg|\int_0^\infty \int_{\mathbb{R}}\int_{\mathbb{R}} U_\epsilon(x,t)\phi_x(x,t)\rho_\epsilon(x - X^\epsilon(\theta,t))m_0(\mathrm{d}\theta)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad\qquad - \int_0^\infty \int_{\mathbb{R}} U^\epsilon(X^\epsilon(\theta,t),t)\phi_x(X^\epsilon(\theta,t),t)m_0(\mathrm{d}\theta)\,\mathrm{d}t\Bigg| \\
&= \Bigg|\int_0^{T_\phi} \int_{\mathbb{R}}\int_{\mathbb{R}} U_\epsilon(x,t)\phi_x(x,t)\rho_\epsilon(x - X^\epsilon(\theta,t))m_0(\mathrm{d}\theta)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad - \int_0^{T_\phi} \int_{\mathbb{R}}\int_{\mathbb{R}} U_\epsilon(x,t)\rho_\epsilon(x - X^\epsilon(\theta,t))\phi_x(X^\epsilon(\theta,t),t)m_0(\mathrm{d}\theta)\,\mathrm{d}x\,\mathrm{d}t\Bigg| \\
&\le M_1\|U_\epsilon\|_{L^\infty}\|\phi_{xx}\|_{L^\infty}T_\phi\epsilon \le \frac{1}{2^p}M_1^{2p+1}\|\phi_{xx}\|_{L^\infty}T_\phi\epsilon.
\end{aligned} \tag{43}$$

Hence, inequality (39) holds.

Step 4. Global weak solution. In this step, we prove that the limit function u obtained in Step 2 is a global weak solution to the gmCH equation (1)-(2).

For any $\phi \in C_c^\infty(\mathbb{R}\times[0,\infty))$, we have

$$\begin{aligned}\mathcal{L}(u^\epsilon,\phi) = &\int_0^\infty \int_{\mathbb{R}} u^\epsilon[\phi_t - \phi_{txx}]\,\mathrm{d}x\,\mathrm{d}t + \int_0^\infty \int_{\mathbb{R}} [A(u^\epsilon,u_x^\epsilon) + B(u^\epsilon,u_x^\epsilon)]\phi_x\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^\infty \int_{\mathbb{R}} C(u^\epsilon,u_x^\epsilon)\phi_{xx}\,\mathrm{d}x\,\mathrm{d}t \\ =: &\, I_1^\epsilon + I_2^\epsilon + I_3^\epsilon. \end{aligned} \tag{44}$$

By the properties of u^ϵ and u in Step 2, for any integer $k, l > 0$ it holds that

$$(u^\epsilon)^k (u_x^\epsilon)^l \to u^k u_x^l \quad \text{in} \quad L^1_{loc}(\mathbb{R}\times[0,+\infty)) \quad \text{as} \quad \epsilon \to 0. \tag{45}$$

This implies the following results for the terms in (44) as $\epsilon \to 0$:

$$I_1^\epsilon = \int_0^\infty \int_{\mathbb{R}} u^\epsilon[\phi_t - \phi_{txx}]\,\mathrm{d}x\,\mathrm{d}t \to \int_0^\infty \int_{\mathbb{R}} u[\phi_t - \phi_{txx}]\,\mathrm{d}x\,\mathrm{d}t,$$

$$I_2^\epsilon \to \int_0^\infty \int_{\mathbb{R}} [A(u,u_x) + B(u,u_x)]\phi_x\,\mathrm{d}x\,\mathrm{d}t, \quad I_3^\epsilon \to \int_0^\infty \int_{\mathbb{R}} C(u,u_x)\phi_{xx}\,\mathrm{d}x\,\mathrm{d}t.$$

Combining the above estimates and (39), we finally obtain

$$\mathcal{L}(u,\phi) + \int_{\mathbb{R}} \phi(x,0) m_0(\mathrm{d}x) = 0.$$

Hence, the limiting function u satisfies (22), and u is a weak solution to the gmCH equation as defined by Definition 3.1. $\square$

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