An Analysis of “Buy X, Get One Free” Reward Programs

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We study the effects of redemption hurdles on reward program members’ decision-making and firm profitability. We focus on the popular “Buy X, Get One Free” (BXGO) programs, which set a redemption threshold (X), and possibly an expiration term for the reward. In our model, forward-looking consumers interact with a monopolistic firm, and strategically make purchase and redemption decisions over an infinite time horizon. Our analysis leads to the following results. First, a consumer’s purchase utility and purchase probability increase as her reward point inventory approaches the redemption threshold or expiration. These patterns are consistent with the “point pressure” phenomenon documented in the empirical literature. Second, a redemption threshold alone cannot improve the firm’s profit, unless it is coupled with a finite expiration term, or a positive transaction utility that consumers may derive from reward redemption. Third, setting the optimal redemption threshold requires the program to strike a balance between the effective price paid by consumers and their purchase probabilities. These results have rich managerial implications for effectively designing reward programs.

Key words: customer reward program, redemption threshold, finite expiration term, purchase acceleration, forward-looking consumer, dynamic programming, transaction utility.

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1. Introduction

Customer reward programs are widely used as a means to foster customer loyalty, increase acquisition and retention, and boost long-term profitability. Reward program memberships in the US alone have topped 3.8 billion, according to a 2017 loyalty program census by Colloquy. Over time, these programs have become increasingly complex, with numerous formats and types of administration (Berman 2006). Yet the key premise of most reward programs today remains the same as the classic S&H Green Stamp program introduced more than a century ago: consumers purchase to earn “points” to be redeemed for future rewards (Kopalle and Neslin 2003). Different from cash rebate programs, reward programs often require consumers to accumulate points up to a *redemption threshold* in order to qualify for the reward. Many programs also stipulate a *finite expiration term* before which points must be used or otherwise lost. We refer to these requirements together as *redemption hurdles*. A survey of 32 active coffee chains in North America finds divergent practices regarding whether a reward program is offered, and, if it is, how the redemption hurdles are set.[1] Specifically, 23 (72%) coffee chains offer reward programs, while 9 (28%) do not. Among these 23 programs, 19 (83%) set nontrivial redemption thresholds, ranging from 7 purchases (McCafe) to 20 purchases (Costa Coffee).[2] Interestingly, 9 (39%) programs choose to set a finite expiration term on their reward or reward points, whereas the others choose not to.

Redemption hurdles are often nontrivial to overcome. Consumers may fail to redeem these rewards due to a high redemption threshold, a stringent expiration term, or both. In fact, a third of all reward points issued in the US are not redeemed (Hlavinka and Sullivan 2011). Naturally, redemption hurdles are highly controversial, having been cited as a main source of consumer frustration (Stauss et al. 2005) and a top reason for reward program abandonment (Friedman 2003). Some scholars even argue that redemption hurdles should be completely abandoned (Dorotic et al. 2014). Yet others argue that reward expiration and redemption thresholds can create point pressure and therefore accelerate consumer purchases (Kopalle et al. 2012). Given this controversy, we are interested in how such redemption hurdles would affect consumers' purchase and redemption decisions. Moreover, given that point pressure (purchase acceleration) behavior is often attributed to psychological considerations in the literature (Inman and McAlister 1994, Kivetz et al. 2006), we would like to see whether psychological factors are necessary for explaining purchase acceleration.

On the sellers' side, the use of redemption thresholds may appear to be at odds with the hope to leverage reward benefits to acquire and retain consumers, and to charge higher prices. If consumers face substantial challenges in reward redemption, they would value future rewards less and would rationally adjust their purchase decisions. In particular, a higher redemption threshold would delay consumers' redemption, and a stringent expiration term would put consumers' reward points at a higher risk. Consequently, the seller must compensate by lowering the prices. Do reward programs

[1] The list of coffee shops is compiled from five sources: (1) Twelve Best Chain Coffee Shops in America (http://coffeemakersusa.com/ranked-chain-coffee-shops/); (2) 2016 Top 500 Coffee-Cafe Chains (http://www.restaurantbusinessonline.com/2016-top-500-coffee-cafe-chains); (3) Top 12 Largest Coffee Chains in the World (https://listaka.com/12-largest-coffee-chains-world/); (4) The 10 Best Espressos from Chain Coffee Shops, Ranked (https://vinepair.com/articles/10-best-chain-espressos/); and (5) Coffee Shop Chains That Make Mornings Bearable (https://www.ranker.com/list/best-coffee-shop-chains/chef-jen).

[2] Some programs explicitly state the number of paid purchases required for a free beverage. For the remaining programs, we calculate the implicit redemption thresholds by dividing the dollar amount required for a free beverage by the average price of the coffee.

with a redemption threshold really benefit firms? Should reward programs remove reward expiration, i.e., by never letting the rewards expire? How should redemption hurdles be set in order to balance consumers' purchase probabilities and effective prices? Surprisingly, the literature has so far remained silent in response to these questions.

The current paper sets out to answer these questions. Our research context involves the classic "Buy X, Get One Free" (BXGO) programs (Blattberg et al. 2008) that offer customers free products/services as rewards. The popularity of BXGO programs can be attributed to their simplicity and ease of implementation — they can be self-administered by consumers, for whom punch cards (physical or virtual) suffice. Furthermore, the reward benefits (free products) can easily be communicated to consumers. Examples of rewards offered by BXGO programs include free coffee (Kivetz et al. 2006), golf rounds (Hartmann and Viard 2008), airline tickets (Lu and Su 2015), hotel nights (Kopalle et al. 2012), and grocery items (Lal and Bell 2003).

In our model, a single consumer type interacts with a monopolistic firm over an infinite time horizon. The reward program offered by the firm is characterized by a price p, a redemption threshold X, and possibly an expiration term T. In each period, there is a fixed probability that a consumer comes to the market, in which case she buys at most one unit of the product. Conditional on being in the market, the consumer chooses between the focal product, for which she has a fixed valuation, and an outside option. The outside option is stochastic, capturing temporal variations in the consumer's preference due to factors outside the firm's control. Informed by the empirical findings of Shampanier et al. (2007), the consumer may derive a nonnegative transaction utility from the free product. The consumer decides whether to purchase or redeem, accounting for her reward point inventory, attractiveness of the outside option, and design of the reward program.

Several important insights emerge from the consumer model. First, a redemption threshold may induce *purchase acceleration* for consumers. Specifically, there may exist a threshold level of reward points inventory, such that a consumer would forgo an attractive outside option and purchase from the focal firm if her point inventory level exceeds the threshold. Second, there may also exist a threshold level with respect to the time until point expiration, such that a consumer would only make a purchase when her reward points are sufficiently close to expiration. These findings are consistent with the *point pressure* phenomenon, which is well documented in the empirical literature. This suggests that our analytical model successfully captures an essential aspect of consumer decision-making due to redemption hurdles. The results are also in line with the empirical evidence that a reward program may boost the overall demand, at least in the short run.[3] Third, these two purchase acceleration effects also interact. To our knowledge, this interesting

[3] For example, Chun and Ovchinnikov (2019) show that a mileage-based reward program can induce "flying-up" by strategic consumers.

interaction effect has never been studied in the empirical literature. Importantly, we also reveal the underlying rationale behind the point pressure: for a forward-looking consumer, closeness to redemption hurdles creates additional incentives to purchase.

Building on the insights from the consumer model, we proceed to study the profitability of the reward program with redemption hurdles. The firm faces its own trade-off between the long-run purchase probability of (i.e., demand) and the effective price paid by consumers. Higher redemption hurdles make it harder for a consumer to qualify for the rewards, and may therefore reduce the point pressure for the consumer; however, high hurdles may help increase the effective price because fewer free products are given away. Our main findings are as follows. First, a reward program with only a redemption threshold cannot improve the firm's profit in the absence of transaction utility. The reason is that, although the purchase acceleration increases a consumer's purchase probability, the consumer also discounts her future utility more, and the firm must lower the price significantly in order to retain the consumer. However, if the consumer derives transaction utility for the free product, a BXGO program can be profitable even when reward points never expire, which explains why many BXGO programs choose not to stipulate an expiration date.

Second, a redemption threshold, along with a finite reward expiration term, can strictly increase the firm's profit, even without any transaction utility. Importantly, a BXGO program can be profit-enhancing even for a single consumer type. The reason is that the expiration term gives the consumer an additional incentive to make a purchase in order to renew the expiration term. The more points she has accumulated, the higher the incentive is, given that the potential loss due to expiration grows as her point inventory increases. Furthermore, an analysis of a "Buy One, Get One Free" program (a special case of a BXGO program) finds that the BXGO program has the potential to implement price discrimination over time within the same consumer, leading to the BXGO program's profitability.

Third, we use extensive numerical studies to investigate the optimal design of BXGO reward programs. We show that for a redemption threshold to be effective, it must strike a delicate balance between the effective price and purchase probability. As such, setting the redemption threshold must account for consumers' arrival rate, their preference for the outside option, and the transaction utility. In addition, the synergy between the redemption threshold and the price level is nontrivial and may defy intuition.

The rest of the paper is organized as follows. Section 2 reviews the relevant literature. Section 3 introduces the main model and analyzes the consumer's decision problem in a BXGO program without a finite expiration term. Section 4 studies the profitability of such a BXGO program.

Section 5 presents the results from extensive numerical studies on the design of the redemption threshold. Section 6 incorporates the expiration term into the model and investigates its effect on consumer behavior and the firm's profit. Section 7 concludes and summarizes future research directions. The Appendix contains all proofs and additional technical details.

2. Literature Review

Our paper is related to several streams of literature. First, we contribute to the recent ongoing research on reward programs in the broad context of revenue management. Lu and Su (2015) build on the classic Littlewood's model (Littlewood 1972), and show that reward points can be used as virtual currencies to extract higher value from low-type consumers. Chung et al. (2018) extend the classic model by Gallego and van Ryzin (1994) and consider a monopolistic firm that balances the revenue from cash sales and reward sales. They show that reward sales have a nontrivial impact on the seller's optimal dynamic pricing policy. These papers consider capacitated settings that face uncertain consumer demand. Our study does not consider the capacity issue; instead, it focuses on the impact of redemption hurdles on consumer behavior and firm profit. A somewhat different perspective is provided by Chun et al. (2019), who consider the financial accounting of reward points and show various policies to manage reward point values, depending on whether the firm maximizes the profit or cash flow. Chun and Ovchinnikov (2019) examine an interesting case where airlines switch from a frequency-based reward to a revenue-based one, and show that such a switch can create a win-win situation. This finding highlights the importance of the interplay between setting prices and designing reward programs. Sun and Zhang (2019) propose an analytical model of a cash reward program that sets a finite expiration term. They show that the program can be used to implement price discrimination among consumers with heterogeneous shopping frequency and valuation. In contrast, we show that a reward program can be profit-enhancing even for a single consumer type. None of the aforementioned papers model reward accumulation.

Second, our work contributes to the empirical literature on point pressure, i.e., the consumers' purchase utilities and/or purchase probabilities may increase as the consumers' point inventory accumulates or approaches expiration. Kivetz et al. (2006) show that a BXGO program can benefit from leveraging consumers' "illusion of progress towards the goal." Specifically, they show that consumers who received 12-stamp coffee cards with two preexisting "bonus" stamps completed the ten required purchases faster than those who received "regular" 10-stamp cards with no bonus stamps. Inman and McAlister (1994) use anticipated regret theory to explain why consumers are more likely to redeem right before expiration. While these two papers attribute the point pressure phenomenon to psychological considerations, we follow the rationality paradigm (e.g., Lewis 2004, Hartmann and Viard 2008, Kopalle et al. 2012), explicitly characterizing the intertemporal trade-off for consumers. To the best of our knowledge, we are the first to analytically show why and how point pressure arises due to redemption hurdles.

Third, our work complements the empirical literature that examines the effectiveness of reward programs on sales, customer retention, and firm profitability. The existing empirical studies find mixed results. Sharp and Sharp (1997) find no evidence that reward programs increase aggregate-level profits; likewise, Liu (2007) finds no effect of reward programs on consumers who are already buying heavily from the seller. On the other hand, Lewis (2004) and Lal and Bell (2003) show that reward programs do increase sales and profits for grocery retailers. Our analytical work complements these aforementioned empirical studies. Due to the nature of these empirical studies, they take reward program parameters as *exogenously* determined. In contrast, our analytical framework allows us to treat program parameters as the firm's *endogenous* decisions. Another limitation of these research is the predominant focus on existing members, i.e., consumers who self-select into reward programs, leaving out the important issue of consumer acquisition and retention. We examine how a firm can leverage redemption thresholds and/or expiration terms in order to strike a balance between customer acquisition and retention, and eventually increase firm profitability.

Finally, our paper contributes to the research stream on strategic consumers. The consumers in our model are forward-looking and rationally account for the points' expected value in their purchases. This type of consumer behavior has been extensively studied in the operations management literature (see, e.g., Su 2007, Su and Zhang 2008, 2009, Aviv and Pazgal 2008, Yin et al. 2009, Aviv and Wei 2014, Besbes and Lobel 2015, Su 2010). Many papers in this literature stream consider consumers' purchase timing decisions, where consumers make at most one purchase. In contrast, our modeling framework captures the fact that consumers interact with the seller over a long time horizon and make repeated purchases. Indeed, one important aspect of our model, reward accumulation, makes sense only when there are such repeated interactions. Because the selling price and the reward program structure stay constant over time, the purchase timing does not help consumers. One exception in this stream of literature is Su (2010), who considers consumers' repeated purchases/consumption and stockpiling. We focus on products/services that cannot be inventoried, including most services and perishable products, for which BXGO programs are frequently offered.

3. Model Setup

We consider a monopolistic firm offering a nondurable product to a group of consumers over an infinite time horizon. Without loss of generality, we normalize the firm's marginal cost to 0; thus, we use "revenue" and "profit" interchangeably throughout the paper. To separate out the effects of the two redemption hurdles, our main model focuses on a BXGO program with no expiration date. In Section 6, we incorporate an expiration term into the reward program and explore its impact on consumer behavior and the firm's profitability.

The BXGO program consists of two components: a price p and a redemption threshold X. We assume that the price p is fixed over time, and $X \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of positive integers. Informed by industry practices, the reward program is assumed to work as follows: in each period, a consumer earns one point if she makes a purchase, and zero points otherwise. A consumer with X points is eligible for a free product. These assumptions are in line with many sellers that adopt BXGO programs, such as coffee shops (e.g., Kivetz et al. 2006), golf clubs (e.g., Hartmann and Viard 2008), and hotels (e.g., Kopalle et al. 2012). Apparently, this program captures both reward accumulation and a redemption threshold.[4]

Different from the existing research that examines how reward program can enhance profit by exploiting consumer heterogeneity (e.g., Kim et al. 2001, Sun and Zhang 2019), we consider a *homogeneous* consumer population and focus on a representative consumer. In each period, the consumer comes to the market with a probability λ, and she is off the market with a probability $1-\lambda$. Conditional on being in the market (e.g., she has free time to play golf), the consumer has a fixed valuation v for the focal product. At the same time, she faces a stochastic outside option v_0 that takes a value in $\{0, \bar{v}\}$, where $v \geq \bar{v} \geq 0$. We use q to denote the probability that the outside option is 0. The stochastic outside option explicitly captures temporal variation in the consumer's preference, which can be attributed to many factors outside the firm's control, including variety-seeking behavior (McAlister and Pessemier 1982), promotional activities (Gönül and Srinivasan 1996), and the availability of competitive offerings (Kim et al. 2001). Based on the empirical evidence that consumers value "free" products over and beyond pure economic considerations (Thaler 1983, Shampanier et al. 2007, Nicolau and Sellers 2012), we allow the possibility that she derives a nonnegative *transaction utility*, k, from redemption. Our analysis starts with the special case $k = 0$, i.e., the consumer does not derive any transaction utility.

Conditional on being in the market and the value of the outside option, a consumer decides whether to make a purchase or redeem for the free reward, accounting for her reward point status, and the terms of the reward program set by the firm. The consumer is forward-looking and seeks to maximize her total discounted surplus. Informed by the existing empirical literature of reward programs (Lewis 2004, Kopalle et al. 2012), we assume that the consumer discounts her utility with a per-period discount factor $\delta \in (0, 1)$.

[4] For simplicity and analytical tractability, we abstract away from other possible reward program components, such as customer reward tiers, in the main model. We leave models that consider reward tiers for future research.

3.1 The Consumer's Decision Problem

The consumer's decision problem can be formulated as an infinite-horizon dynamic program (see, e.g., Puterman 1994). The state of the dynamic program is x, which represents the number of points accumulated by the consumer. Let $J(\cdot)$ denote the value function. The optimality equations can be written as

$$\begin{aligned} J(x) = {} & \lambda q \max\Big\{v - p + \delta J(x+1), \delta J(x)\Big\} + \lambda(1-q)\max\Big\{v - p + \delta J(x+1), \bar{v} + \delta J(x)\Big\} \\ & + (1-\lambda)\delta J(x), \qquad \forall x = 0, 1, \ldots, X-1, \qquad (1) \\ J(X) = {} & \lambda q \max\Big\{v + k + \delta J(0), \delta J(X)\Big\} + \lambda(1-q)\max\Big\{v + k + \delta J(0), \bar{v} + \delta J(X)\Big\} \\ & + (1-\lambda)\delta J(X). \qquad (2) \end{aligned}$$

Equations (1) and (2) jointly characterize the consumer's decision calculus over the reward accumulation cycle. Equation (1) refers to the cases when the consumer does not have a sufficient number of reward points for redemption. The first and second terms on the right-hand side correspond to the scenarios where the consumer comes to the market and faces a low and a high outside option, respectively. If she makes a purchase, she earns one point and receives a utility $v - p + \delta J(x+1)$. Otherwise, she earns no points and receives a utility $0 + \delta J(x)$, or $\bar{v} + \delta J(x)$, depending on the value of the outside option. The third term corresponds to the scenario where the consumer does not come to the market. The discount factor δ implies the potential loss of utility due to a delay in redemption. Similarly, Equation (2) characterizes the consumer's decision-making when she is eligible for a free product. The three terms can be interpreted similarly as those in Equation (1), but the consumer focuses on deciding whether to redeem or not. For each redemption, the consumer receives a nonnegative transaction utility k.

To better understand the consumer's decision problem, we analyze the structural properties of the value function $J(\cdot)$. We have the following results.

Proposition 1. *The value function is increasing and convex. Specifically,* $J(x+1) - J(x) \leq J(x+2) - J(x+1)$ *for all* $0 \leq x \leq X-2$.

Proposition 1 implies that as the consumer's point inventory approaches the redemption threshold, each additional reward point becomes more valuable to her. This result can be rationalized by consumer discounting: she prefers redemption for the free product as early as possible. Hence, the value of a purchase, relative to no purchase, increases as she approaches the redemption threshold.

Proposition 1 immediately implies the following two corollaries.

Corollary 1. *For any price* p *and* $0 \leq x \leq X-2$, *(a) if* $v - p + \delta J(x+1) \geq \delta J(x)$*, then* $v - p + \delta J(x+2) \geq \delta J(x+1)$*; that is, when* $v_0 = 0$*, if the consumer makes a purchase in state* x*, then she will also make a purchase in state* $x+1$*;*

(b) if $v - p + \delta J(x+1) \geq \bar{v} + \delta J(x)$*, then* $v - p + \delta J(x+2) \geq \bar{v} + \delta J(x+1)$*; that is, when* $v_0 = \bar{v}$*, if the consumer makes a purchase in state* x*, then she will also make a purchase in state* $x+1$*.*

Part (a) covers the case when the outside option is 0. It states that if a consumer with x reward points is willing to make a purchase, then she prefers to make a purchase at any level of a point inventory larger than x. This implies that (1) the consumer has the lowest willingness to pay for the focal product in the state with no reward points (i.e., state 0) and (2) if the consumer makes a purchase in state 0, then she will make a purchase whenever she is on the market. Part (b) shows a similar result for the case when the outside option is $\bar{v}$.

COROLLARY 2. *The consumer's willingness to make a purchase for the focal product increases with her point inventory. Specifically, for any price* p *and* $0 \leq x \leq X - 2$*,*

$$v - p + \delta J(x+2) - \big(\bar{v} + \delta J(x+1)\big) \geq v - p + \delta J(x+1) - \big(\bar{v} + \delta J(x)\big). \tag{3}$$

The left (right) side of (3) is the utility difference between a purchase and no purchase when the consumer has $x+1$ (x) points in hand. The inequality implies that, conditional on a high outside option, making a purchase becomes increasingly attractive as the consumer approaches the redemption threshold.

Corollaries 1 and 2 show that, conditional on a high outside option, the consumer's behavior follows a threshold strategy. When the consumer has a low point inventory, she prefers not to make a purchase. However, the purchase option becomes increasingly attractive as her point inventory accumulates. Hence, there exists a threshold level of point inventory, beyond which the consumer starts to purchase, and keeps purchasing thereafter. This result is formalized in Proposition 2(b) below. This finding is consistent with the "point pressure" phenomenon documented in the empirical literature (e.g., Kopalle et al. 2012). To the best of our knowledge, we are the first to analytically show consumers' point pressure behavior. Moreover, our result is obtained in a fully rational model. In contrast, point pressure is often attributed to psychological considerations in the existing literature.

Proposition 2 characterizes the consumer's purchase/redemption decisions. Conditional on a low outside option, the consumer always makes a purchase when she has no points in hand; otherwise, the consumer leaves the market in the long run. This implies that when the outside option is low, the consumer always makes a purchase in states $\{1, 2, \ldots, X-1\}$ (see Corollary 1(a)) and redeems when she has X points in hand.[5] Hence, we focus on the more interesting case when the value of the outside option is high.

To facilitate the presentation, let

$$A = \frac{\lambda q \delta}{1-\delta+\lambda q \delta}, \qquad A_1 = \frac{\lambda \delta}{1-\delta+\lambda\delta}.$$

Also, for $0 \le \tau \le X$, let

$$p^B(\tau) = \frac{q(1-A^\tau A_1^{X-\tau+1})v + (1-q)A^\tau(1-A_1^{X-\tau+1})(v-\bar{v}) + A^\tau A_1^{X-\tau}(1-A_1)k}{A^\tau(1-A_1^{X-\tau}) + q(1-A^\tau)},$$

$$p^D(\tau) = \frac{\left(1-A_1^{X-\tau+1} + q(1-A^\tau)A_1^{X-\tau+1}\right)(v-\bar{v}) + A_1^{X-\tau}(1-A_1)k}{1-A_1^{X-\tau} + q(1-A^\tau)A_1^{X-\tau+1}}.$$

Proposition 2. *Conditional on being in the market and facing a high outside option (i.e., $v_0 = \bar{v}$), the consumer's purchase decision rule follows one of the following four cases.*

(a) If $p \le p^D(0)$, then the consumer always makes a purchase/redeems;

(b) If $p^D(\tau-1) < p \le \min\{p^B(\tau), p^D(\tau)\}$ for any threshold $\tau \in \{1,\dots,X-1\}$, then the consumer does not make a purchase in states $\{0,1,\dots,\tau-1\}$; however, she makes a purchase in states $\{\tau,\tau+1,\dots,X-1\}$, and redeems in state X;

(c) If $p^D(X-1) < p \le \min\{p^B(X), p^D(X)\}$, then the consumer does not make a purchase in any state, but redeems in state X;

(d) If $p^D(X) < p \le \frac{(1-A^{X+1})v + A^X(1-A)k}{1-A^X}$, then the consumer neither makes a purchase nor redeems in any state.

Part (a) shows that if the price is sufficiently low, the consumer always makes a purchase/redeems, regardless of the outside option's value. Part (b) characterizes the situation where there is a threshold $\tau < X$ such that the consumer (facing a high outside option) only makes a purchase when she has accumulated more than τ points. That is, the consumer accelerates her purchase as she approaches the redemption threshold. The boundary price $p^B(\tau)$ ensures that, conditional on a low outside option, the consumer always makes a purchase when she has no points in hand. Similarly, the boundary prices $p^D(\tau)$ and $p^D(\tau-1)$ ensure that, conditional on a high outside option, the consumer makes a purchase in state τ (and above) but does not make a purchase in state $\tau-1$ (and below), respectively. Observe that the boundary prices $p^D(\tau-1)$, $p^B(\tau)$, and $p^D(\tau)$ all increase in the threshold τ. This means that the lower the price is, the smaller the threshold τ is, and thus, the earlier the consumer starts to accelerate her purchases. Part (c) shows

[5] See Lemma 4 in the Appendix.

that the consumer does not make a purchase in any state but redeems when the outside option is high. Part (d) implies that when the price is high, the consumer does not make a purchase/redeem when the outside option is high, where $\frac{(1-A^{X+1})v+A^X(1-A)k}{1-A^X}$ is the boundary price beyond which the consumer will not make a purchase, even if the value of the outside option is low.

Before proceeding, we make a comment on the technical conditions in Proposition 2. Note that if $p^D(\tau) \le p^B(\tau)$ for $1 \le \tau \le X$, then the conditions in Parts (b) and (c) reduce to $p^D(\tau-1) < p \le p^D(\tau)$ and $p^D(X-1) < p \le p^D(X)$, respectively. Thus, there is no price gap in Parts (b) and (c). Otherwise, there exist price gaps among the boundary prices in Proposition 2. However, we can verify that if the price falls in the gaps, then the consumer will not participate in the program. Hence, the prices in the gaps will never be selected by the firm.

One may expect that the consumer would only redeem when the value of the outside option is low; that is, she will take the high outside option, withhold her points, and wait until the outside option is low. However, Parts (a)-(c) show that this may not be the case because when the consumer discounts her utility, the discounted expected utility of waiting to redeem in the future with a low outside option is outweighed by the instantaneous gain in utility from redeeming in the current period. Hence, the consumer prefers redemption even with a high outside option.

4. The Profitability of BXGO Programs

In this section, we build on the insights from the consumer's optimal responses and analyze the profitability of BXGO programs. Section 4.1 analyzes the case with no transaction utility, while Section 4.2 investigates the case with a positive transaction utility. We show that the BXGO program cannot improve the firm's profit when the transaction utility is strictly zero; however, a small positive transaction utility is sufficient to make the program profitable.

4.1 The Analysis with No Transaction Utility

As stated in Proposition 2, a consumer's optimal response under a high outside option can be divided into four cases: (a) the consumer always makes a purchase/redeems whenever she comes to the market; (b) the consumer only makes a purchase when she has more than τ points in hand, where $1 \le \tau \le X-1$; (c) the consumer does not make a purchase in any state but redeems in state X; and (d) the consumer neither makes a purchase nor redeems. We use π_a^*, π_b^*, π_c^*, and π_d^* to denote the profit arising from an optimally designed reward program for each of the four cases, respectively.

In Case (a), the consumer always makes a purchase/redeems, regardless of the outside option, and the firm collects a revenue no more than $\lambda(v-\bar{v})$, given that $v-\bar{v}$ is the maximum long-run average price that a rational consumer is willing to pay in this case. Thus, $\pi_a^* \le \lambda(v - \bar{v})$. Similarly, in Case (d), the consumer makes a purchase/redeems only when she draws a low value for the outside option, and the firm's revenue cannot exceed λqv. Thus, $\pi_d^* \le \lambda qv$. Hence, it remains to investigate the profitability of Cases (b) and (c).

We now analyze Case (b) to check whether purchase acceleration behavior induced by a reward program can increase the firm's profit.[6] The state of a consumer who makes a purchase or redeems follows a Markov chain, as illustrated in Figure 1. When the consumer's point inventory x is below the threshold τ, she makes a purchase only with a low outside option, and the transition probability from state x to $x+1$ is λq. When her point inventory is above τ, she always makes a purchase, and the transition probability from state x to $x+1$ is λ. When the consumer has X points in hand, she always redeems, and the the transition probability from state X to 0 is λ as well.

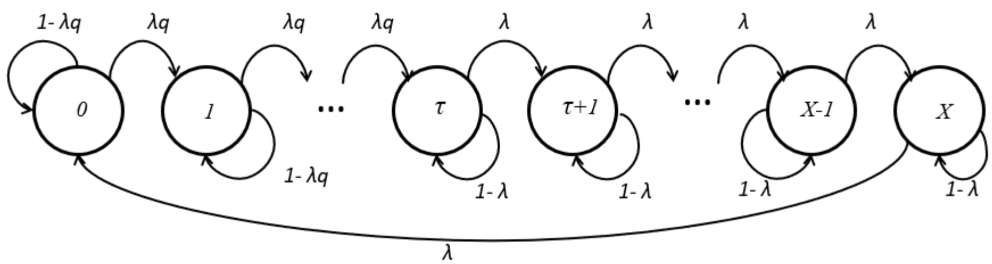


Figure 1 State transition of a consumer in Case (b)

Let q_i denote the stationary probability of state i, where $0 \le i \le X$. Writing down and solving the balance equations yield the following result.

LEMMA 1. *The stationary probability q_i is characterized as follows:*

$$q_i = \begin{cases} \frac{1}{\tau + q(X-\tau+1)}, & \text{if } i = 0, 1, \ldots, \tau - 1, \\ \frac{q}{\tau + q(X-\tau+1)}, & \text{if } i = \tau, \ldots, X. \end{cases}$$

The customer's purchase probability, or the demand for the firm, is given by

$$\lambda\left(q\sum_{i=0}^{\tau-1} q_i + \sum_{i=\tau}^{X-1} q_i\right) = \frac{\lambda qX}{\tau + q(X-\tau+1)}.$$

Note that the firm does not have direct control over τ, which is jointly determined by the consumer parameters (λ, q, δ, v, $\bar{v}$) and the firm's decisions (p, X). The firm's optimization problem can be written as

$$\max_{p,X} \quad \frac{\lambda qX}{\tau + q(X-\tau+1)} p \tag{4}$$

[6] In the interest of space, we relegate the detailed analysis of Case (c) to the proof of Lemma 3 in the Appendix, and only present the key result in the main text.

$$\text{s.t. } p^D(\tau-1) < p \le \min\left\{p^B(\tau), p^D(\tau)\right\}, \text{ where } \tau \in \{1,\ldots,X-1\}.$$

We identify a set of sufficient conditions under which offering a BXGO program cannot improve the firm's profit in the absence of transaction utility.[7] The result is summarized in Lemma 2.

LEMMA 2. *Suppose $k=0$. We have $\pi_b^* \le \max\{\lambda qv, \lambda(v-\bar{v})\}$ if either of the following conditions holds: (a) $q \le 1/2$; (b) $q > 1/2$ and $v - \bar{v} \le \frac{v}{1+A_1}$.*

Conditions (a) and (b) of Lemma 2 state that the profit-enhancing capability of the reward program is limited by the competition from the outside option, characterized by q and $v-\bar{v}$. Recall that q denotes the probability that the outside option takes the value 0. Condition (a) says that the consumer is more likely to draw a high outside option, while condition (b) says that if the consumer is less likely to draw a high value, the magnitude of the high value should be large enough. In either case, the reward program cannot improve the firm's profit.

A similar analysis gives the conditions under which the BXGO program with consumer's behavior in Case (c) does not improve the firm's profit.

LEMMA 3. *Suppose $k=0$. We have $\pi_c^* \le \max\{\lambda qv, \lambda(v-\bar{v})\}$ if either of the following sufficient conditions holds: (a) $q \le A$; (b) $q > A$ and $v-\bar{v} \le Av$.*

Summarizing Lemmas 2 and 3 gives the following proposition.

PROPOSITION 3. *In the absence of transaction utility, offering a BXGO program cannot improve the firm's profit if either of the following conditions holds:*[8]

(a) $q \le 1/2$ and $\delta \ge \frac{2}{2+\lambda}$;

(b) $q > 1/2$ and $v - \bar{v} \le \min\{Av, \frac{v}{1+A_1}\}$.

To summarize, Proposition 3 shows that while a BXGO program can induce consumers' purchase acceleration, it cannot increase firm profitability. In a BXGO program without an expiration term, consumer discounting has two opposite effects on the firm's profit. On the one hand, consumers may accelerate their purchases. Hence, their overall purchase probability is higher than those with no purchase acceleration (i.e., Cases (c) and (d)), which has a positive effect on the firm's profit. On the other hand, because the consumers discount future utility, the firm must lower the price sufficiently in order to retain them. The latter effect dominates; therefore, a BXGO reward program cannot improve the firm's profit. In Section 6, we show that when consumers' purchase acceleration is driven by both discounting and expiration, a BXGO program can benefit the firm even in the absence of transaction utility.

[7] In addition to the set of sufficient conditions, we also numerically verified that the optimal profit is smaller than that with no reward program when these conditions are not satisfied.

[8] We also conducted extensive numerical studies and did not find any parameter set under which the BXGO program improves the firm's profit.

4.2 The Impact of Transaction Utility

A distinct feature of a BXGO program is that it offers free products as rewards. Other than being an "efficient reward" (Kim et al. 2001), the study by Shampanier et al. (2007) revealed another reason why free products are attractive options for a reward. Specifically, the authors documented a "zero-price" effect that is robust in both the lab and the field: consumers have strong preferences for free products over and beyond the pure economic benefits. We formalize this effect by including the transaction utility that arises from redemption for the free product. Intuitively, the transaction utility may increase the effectiveness of the reward program, as consumers take it into account in their purchase and redemption decisions. The following analysis confirms such an intuition and also investigates the magnitude of the transaction utility needed for a BXGO program to be profitable.

Recall that when there is no transaction utility, $\pi_a^* \leq \lambda(v - \bar{v})$ because the maximum long-run average price that a rational consumer is willing to pay cannot exceed $v - \bar{v}$ in Case (a). Moreover, because consumers discount future utility, the effective price they pay is strictly smaller than $v - \bar{v}$. In the presence of transaction utility, consumers obtain a higher surplus from each redemption, which enables the firm to charge an effective price higher than $v - \bar{v}$, leading to a profit increase. The same logic applies to the other cases. However, because consumers redeem more frequently in Case (a), one can expect that the threshold of the transaction utility for the BXGO program to be profitable in Case (a) is smaller than those in other cases. For example, in Case (d), consumers purchase/redeem only when the outside option is low; hence, for the same (p, X), it takes longer for consumers to obtain the transaction utility, compared to Case (a), where consumers always purchase/redeem, regardless of the outside option. In other words, the consumers in Case (a) redeem more frequently than in Case (d), which requires a smaller transaction utility for the BXGO program to be profitable.

Proposition 4. *Offering a BXGO program can improve the firm's profit if either of the following conditions holds,*

(a) $v - \bar{v} \geq qv$ *and* $k \geq \frac{1 - A_1^X - XA_1^X(1-A_1)}{XA_1^X(1-A_1)}(v - \bar{v})$;

(b) $v - \bar{v} < qv$ *and* $k \geq \frac{1 - A^X - XA^X(1-A)}{XA^X(1-A)}v$.

Proposition 4 shows that a BXGO program can be more profitable than without a reward program in the presence of transaction utility. Note that the threshold value of the transaction utility can be rather small, compared to the consumer's valuations v and $v - \bar{v}$. For example, let $(\lambda, v, \bar{v}, q, \delta) = (0.5, 1, 0.3, 0.5, 0.99)$; then, the threshold is only 1.41%, 2.14%, 2.87%, 3.61%, and 4.36% of v when X varies from 1 to 5, respectively. Thus, a small level of transaction utility is sufficient to ensure the profitability of a BXGO program, which provides an explanation for why so many BXGO programs choose not to stipulate an expiration date in practice.

Table 1 Consumer's purchase decision when $v_0 = \bar{v}$

x \ k	0	0.01	0.02	0.03	0.04	0.05	0.06
0	0	0	0	0	0	0	1
1	0	0	0	0	1	1	1
2	0	0	1	1	1	1	1
3	1	1	1	1	1	1	1
4	1	1	1	1	1	1	1

5. Numerical Studies

This section numerically investigates several questions with managerial importance. Section 5.1 illustrates that the redemption threshold leads to consumers' purchase acceleration behavior. Section 5.2 analyzes how the redemption threshold changes with consumer characteristics. Section 5.3 investigates the joint effect of the price and redemption threshold by studying BXGO programs with parameters $(p/n, nX)$ under different n. Throughout this section, the parameters $(\lambda, v, \bar{v}, q, \delta, k)$ are set at $(0.5, 1, 0.3, 0.5, 0.99, 0)$, unless specified otherwise.

5.1 Consumer Behavior

Table 1 shows an example where, under the price and redemption threshold $(p, X) = (0.88, 4)$, a consumer accelerates her purchase when she approaches the redemption threshold. An element 1 (0) denotes that the consumer does (does not) make a purchase. Because the consumer always makes a purchase when $v_0 = 0$, we show the consumer's purchase decision when $v_0 = \bar{v}$. Moreover, to examine the effect of transaction utility on a consumer's purchase acceleration behavior, we also compute the consumer's decisions for different levels of transaction utility $k > 0$.

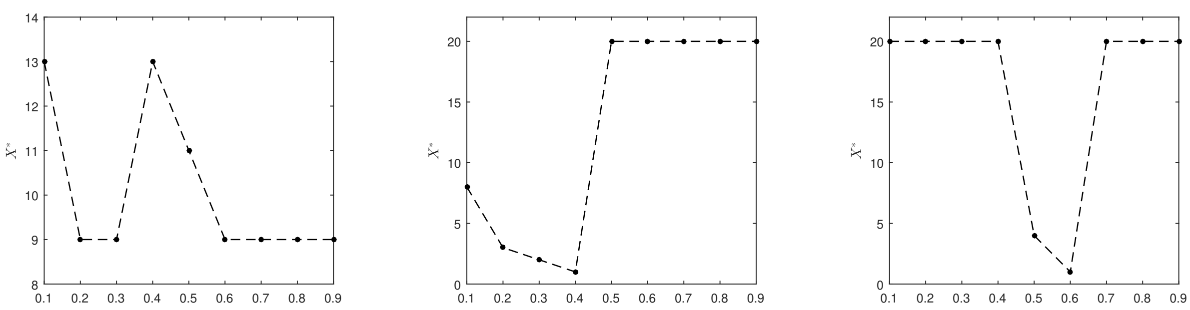
We make two observations from Table 1. First, the column with $k = 0$ confirms purchase acceleration behavior when there is no transaction utility. In particular, a consumer does not make a purchase until she has 3 or more points. Second, the consumer's purchase acceleration behavior persists for different values of the transaction utility k. Interestingly, the point threshold for purchase acceleration decreases as the transaction utility k increases. As the transaction utility k increases from 0 to 0.05, τ decreases from 3 to 1. When $k = 0.06$, the customer purchases in each state. This result can be explained by the fact that a lower point threshold for purchase allows the consumer to earn the transaction utility earlier and more frequently.

5.2 The Optimal Choice of Redemption Threshold X

We are also interested in the design aspects of the reward program: under what conditions should the firm set a high or low redemption threshold? How does the optimal redemption threshold change with respect to consumer characteristics, such as the arrival rate and the attractiveness of the outside option?

We first examine the consumer's response when the firm optimizes over both the price and the redemption threshold. At optimality, consumers either always make a purchase, regardless of the realized value of the outside option, or only make a purchase when the value of the outside option is low. As a result, there is *no* purchase acceleration. To understand this result, note that by adjusting the program parameters, the firm trades off between a higher purchase probability and a higher effective price. When the benefit from increasing the purchase probability outweighs that from increasing the effective price, the firm would lower the price to induce consumers to purchase in all states. This happens when $v-\bar{v}$ is relatively large, compared to qv. When maintaining a high effective price is more beneficial to the firm, the firm would raise the price as long as consumers are willing to make a purchase when the value of the outside option is low. This happens when $v-\bar{v}$ is relatively small, compared to qv. Therefore, the optimality belongs to either of these two extremes.

Next, we investigate the sensitivity of the optimal redemption threshold with respect to the consumer parameters, including $\bar{v}$ (the high value of her outside option), q (the probability that $v_0=0$), the arrival rate λ, and the transaction utility k. In our analysis, we impose an upper bound of 20 on the redemption threshold.[9]



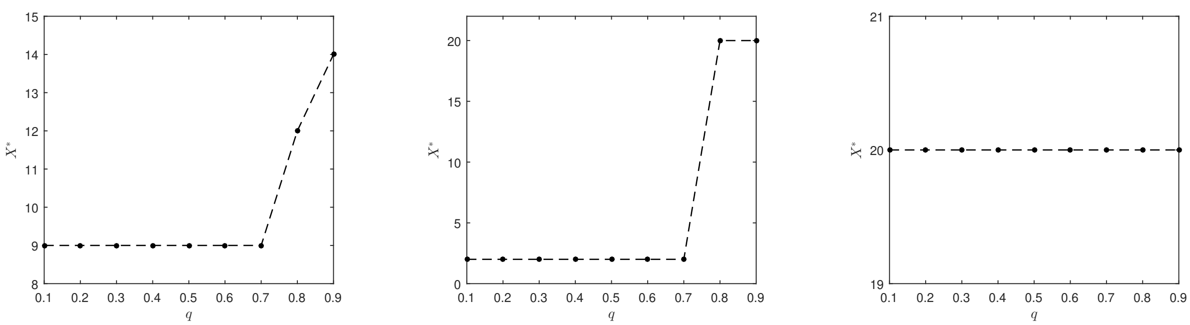
(a) Endogenous p (b) Exogenous $p=1$ (c) Exogenous $p=0.6$

Figure 2 The impact of $\bar{v}$ on the redemption threshold.

Figure 2 shows how the optimal redemption threshold changes with respect to $\bar{v}$ for an endogenous price and two exogenous prices. As $\bar{v}$ increases, the outside option becomes more attractive, so one may expect that the firm would "compensate" by lowering the redemption threshold. However,

[9] Excessively large redemption thresholds are rarely used in practice. For example, our survey of coffee chains shows that the maximum redemption threshold is 20.

this is *not* the case. Figure 2(a) indicates that the optimal redemption threshold is not monotone in $\bar{v}$ when the price is endogenously set. The joint effect of optimizing the price and changing $\bar{v}$ is nuanced. Figures 2(b) and (c) show that even under an exogenous price, the optimal redemption threshold is not monotone in $\bar{v}$. Rather, it first decreases and then increases. This can be explained as follows. When $\bar{v}$ is small (i.e., $v - \bar{v}$ is relatively large), the firm would set the redemption threshold such that consumers always make a purchase. As $\bar{v}$ increases, the outside option becomes more attractive, and the firm must lower the redemption threshold (given that the price is fixed) to retain consumers. When $\bar{v}$ is sufficiently large (i.e., $v - \bar{v}$ is sufficiently small), it is no longer optimal to induce consumers to make a purchase in all states. Rather, the firm should switch to inducing a purchase only when the outside option is low by adopting a larger redemption threshold. This is indicated by the upward jumps in Figures 2(b) and 2(c). This pattern is consistent under both high and low exogenous prices.

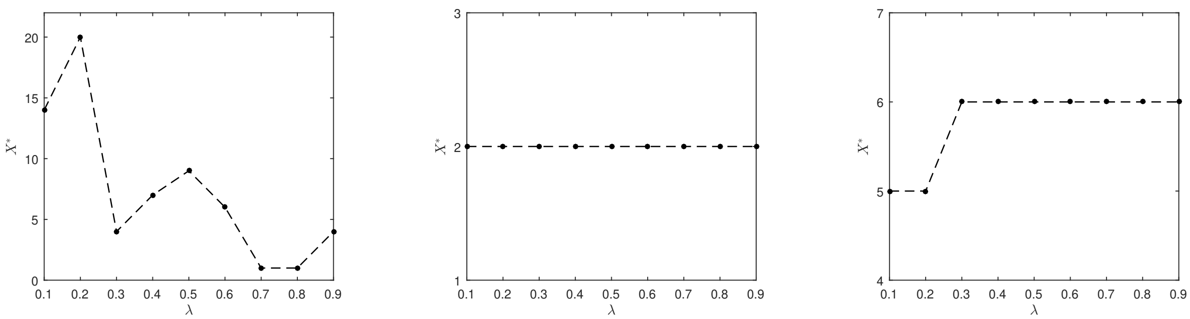


(a) Endogenous p (b) Exogenous $p = 1$ (c) Exogenous $p = 0.6$

Figure 3 The impact of q on the redemption threshold.

Figure 3 depicts how the optimal redemption threshold changes with respect to q, the probability that $v_0 = 0$, which varies from 0.1 to 0.9. As q increases, the outside option becomes *less* likely to be attractive. Thus, one may expect the firm to exploit this situation by raising the redemption threshold. Although Figure 3(a) indicates that X^* indeed weakly increases in q, the underlying logic is somewhat more convoluted. When q is small, consumers purchase in all states. As q increases, it does not affect the optimal choice of threshold, as long as consumers continue to make a purchase in all states. This explains why X^* is constant when $q \leq 0.7$. When q is large enough (i.e., qv is large), it is no longer optimal to induce consumers to make a purchase in all states. Rather, the firm should increase the redemption threshold substantially such that consumers only make a purchase when the outside option is low. This explains the upward jump in X^* when $q = 0.8$. The logic is similar for the case where the price is exogenously set at a high level, as in Figure 3(b).

When the price is exogenous and low, consumers make a purchase in all states, even if the firm chooses a large redemption threshold. This explains why the redemption threshold is constant in Figure 3(c).



(a) Endogenous p (b) Exogenous $p=1$ (c) Exogenous $p=0.8$

Figure 4 The impact of λ on the redemption threshold.

Figure 4 depicts how the optimal redemption threshold changes with respect to the consumer's arrival rate λ, which varies from 0.1 to 0.9. As λ increases, consumers' purchase probability increases. One may expect the firm to increase the redemption threshold. However, this may not be the case. Figure 4(a) indicates that there is no clear pattern in the optimal redemption threshold if the price is being optimized simultaneously. Figures 4(b) and (c) show that, for an exogenous price, the optimal redemption threshold weakly increases in λ. However, the variation is rather moderate, as the lines remain flat for most values of λ. This can be attributed to the fact that the points never expire in our base model; as a result, the impact of the consumer arrival rate is rather limited.[10]

Table 2 The impact of transaction utility

$\bar{v}$	0.3		0.6	
k	0	0.05	0	0.05
X^*	9	1	9	1
p^*	0.77	1.06	1.09	2
π^*	0.347	0.353	0.245	0.25

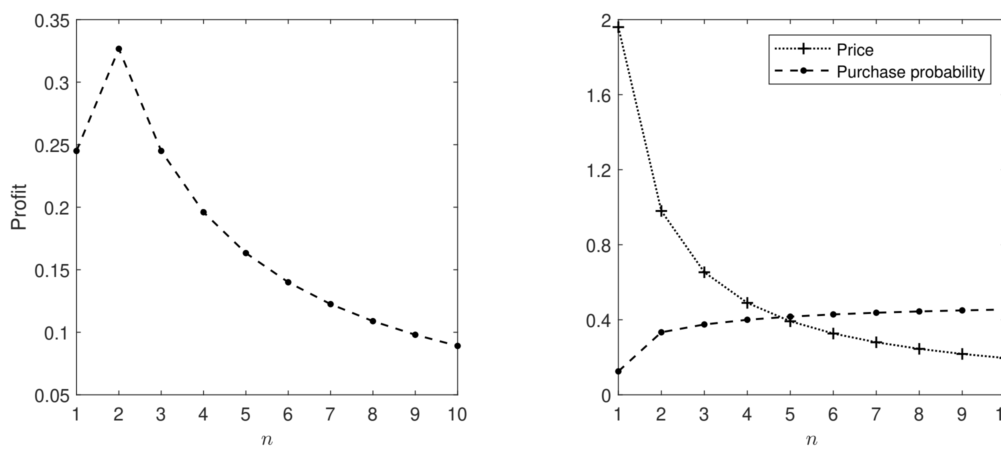
Finally, we study the impact of transaction utility on the firm's profit and the redemption threshold. When the price is endogenous and the transaction utility is high, the firm prefers the consumer to redeem and gain transaction utility more frequently. This would allow the firm to raise

[10] One would expect that in the presence of an expiration term, the impact of the arrival rate on the optimal redemption threshold is more pronounced. We confirmed this intuition with numerical studies.

the price and obtain a higher profit. Therefore, it is reasonable to expect that the firm *lowers* the redemption threshold for higher levels of transaction utility. Table 2 confirms this intuition. Here, we examine a low and a high level of $\bar{v}$ (0.3 and 0.6) and optimize over p and X. When $\bar{v} = 0.3$, consumers always make a purchase, given that $v - \bar{v}$ is relatively large. When k increases from 0 to 0.05, the optimal redemption threshold decreases from 9 to 1, and the optimal price increases from 0.77 to 1.06. Of course, the profit improves as well. A similar result is observed when $\bar{v} = 0.6$, where consumers make a purchase only when the outside option is low. To summarize, even a relatively small transaction utility may lead to a substantial decrease in the optimal redemption threshold.

It is intuitive that the redemption threshold increases in transaction utility under an exogenous price. Since the price is fixed, it is natural for the firm to take advantage of the transaction utility to raise the redemption threshold, which effectively increases the percentage of paid purchases.

5.3 BXGO Programs with $(p/n, nX)$ under Different n



(a) Profit (b) Purchase probability and price

Figure 5 BXGO programs with $(p/n, nX)$ under different n

The price p and the redemption threshold X are interdependent, in the sense that the firm might "compensate" for a high X with a low p, and vice versa. What if we inversely scale these two parameters in a BXGO program? To investigate this question, we consider BXGO programs with parameters $(p/n, nX)$ for different values of n. These programs require consumers to spend the same amount of money in order to earn a free product. It is intriguing to see how the firm's profit changes with n.

We examine this question by assuming a "base" price and redemption threshold $(p, X) = (1.96, 1)$. Figure 5(a) shows the profits when n varies from 1 to 10, while Figure 5(b) further breaks down the profits into the corresponding purchase probabilities and prices. The solution to the consumer problem reveals that consumers (1) make a purchase only when the outside option is low for $n=1$ (corresponding to relatively high prices) and (2) always make a purchase for $n \geq 2$ (corresponding to relatively low prices). For each n, one can verify that the stationary probability q_i for each state i (where $i=0,1,\ldots,nX$) equals $\frac{1}{nX+1}$. Hence, the purchase probability for $n=1$ and $n \geq 2$ is $\lambda q \sum_{i=0}^{nX-1} q_i = \lambda q \frac{nX}{nX+1}$ and $\lambda \sum_{i=0}^{nX-1} q_i = \lambda \frac{nX}{nX+1}$, respectively. Therefore, the profit, denoted by π_n, is given by

$$\pi_n = \begin{cases} \lambda q \frac{nX}{nX+1} \frac{p}{n} = \lambda \frac{1}{4} p, & \text{if } n=1, \\ \lambda \frac{nX}{nX+1} \frac{p}{n} = \lambda \frac{n}{n+1} \frac{p}{n}, & \text{if } n \geq 2. \end{cases}$$

Note that when $n \geq 2$, consumers always make a purchase. As n increases, the price decreases at a rate of $1/n$, while the purchase probability increases at a rate less than n, as illustrated in Figure 5(b). Thus, the profit decreases in n when $n \geq 2$. Interestingly, when $n=1$ and 3, the profit is the same. For $n=3$, the price is a third of that when $n=1$, while the purchase probability is three times that when $n=1$, leading to the same profit.

6. BXGO Programs with an Expiration Date

A commonly observed practice by BXGO programs is to set a finite expiration date for reward points, and allow consumers to renew their point inventories with a purchase or redemption. For example, Gloria Jean's Coffee specifies that the points will expire if the consumers have not engaged in "point activity" (purchase or redemption) for 12 months. When a consumer makes a purchase, she not only earns new points, but also renews the expiration date for all of her points. Similarly, Marriott Hotel specifies that consumers' points will expire after 24 months of "inactivity", but a purchase or redemption resets the expiration date of all points to another 24 months.

Given the prevalence of finite expiration terms, we incorporate it into the base model and investigate its effects on consumer behavior and firm profit. The firm's reward program can then be denoted by the triplet (p,X,T), which consists of a price p, a redemption threshold X, and an expiration term T, where both $X \in \mathbb{N}$ and $T \in \mathbb{N}$. Consistent with common industry practices, we assume that the reward program works as follows: in each period, if a consumer makes a purchase, she earns one point, and the expiration date for her entire point inventory is renewed to be T. Otherwise, no new point is earned, and all current points in hand will be one period closer to expiration. A consumer with X points in hand is eligible for a free product, which can be redeemed before the points expire.

As before, the consumer's decision problem can be formulated as a dynamic program. In order to track point expiration, we need to extend the consumer's state space to (x,t), where x is the number of points in hand, and t is the number of periods until expiration. We use Δ to denote the state without reward points: the consumer reaches the state Δ if she just used all of her points for a free product, or if she has lost all of her points due to expiration. To separate out the effect of the transaction utility and that of the expiration term, we do not include the transaction utility in this section, i.e., $k = 0$.

Let $J(\cdot,\cdot)$ denote the value function. The optimality equations can be written as

$$J(x,t) = \begin{cases} \lambda q \max\Big\{v - p + \delta J(x+1,T), \delta J(x,t-1)\Big\} + \lambda(1-q)\max\Big\{v - p + \delta J(x+1,T), \bar{v} + \delta J(x,t-1)\Big\} \\ \quad +(1-\lambda)\delta J(x,t-1), \qquad\qquad \text{if } 1 \le x \le X-1,\;\; 1 \le t \le T, \\ \lambda q \max\Big\{v + \delta J(\Delta), \delta J(X,t-1)\Big\} + \lambda(1-q)\max\Big\{v + \delta J(\Delta), \bar{v} + \delta J(X,t-1)\Big\} \\ \quad +(1-\lambda)\delta J(X,t-1), \qquad\qquad \text{if } x = X,\;\; 1 \le t \le T, \end{cases} \tag{5}$$

$$J(\Delta) = \lambda q \max\Big\{v - p + \delta J(1,T), \delta J(\Delta)\Big\} + \lambda(1-q)\max\Big\{v - p + \delta J(1,T), \bar{v} + \delta J(\Delta)\Big\} + (1-\lambda)\delta J(\Delta). \tag{6}$$

To facilitate presentation, the equations above use the convention that the states $(x, 0)$ and $(0, t)$, for all x and t, are the same as state Δ.

We first analyze the structural properties of the value function and obtain the following result.

PROPOSITION 5. *(a) For any fixed t where $1 \le t \le T$,*

$$J(x,T) - J(x-1,t-1) \le J(x+1,T) - J(x,t-1), \qquad \forall 1 \le x \le X-1. \tag{7}$$

That is, the consumer gains more utility from making a purchase in state (x, t) than in state $(x-1, t)$.

(b) For any fixed x where $0 \le x \le X-1$,

$$J(x+1,T) - J(x,t) \le J(x+1,T) - J(x,t-1), \qquad \forall t = 1,\dots,T-1. \tag{8}$$

That is, the consumer gains more utility from making a purchase in state (x, t) than in state $(x, t+1)$.

The left and right sides in (7) capture the value function difference between a purchase and no purchase in states $(x-1, t)$ and (x, t), respectively. The inequality implies that as the consumer's point inventory approaches the redemption threshold, she gains more utility from making a purchase. Similarly, the left and right sides in (8) capture the value function difference between a purchase and no purchase in states $(x, t+1)$ and (x, t), respectively. The inequality indicates that as the point inventory approaches expiration, the consumer also gains more utility from making a purchase. The intuition is that when the risk of losing reward points looms larger, a purchase can mitigate such a risk by renewing the expiration term, thereby bringing more utility to the consumer. This is consistent with the evidence documented in the empirical literature (e.g., Kivetz et al. 2006, Kopalle et al. 2012). To summarize, both the redemption threshold and the expiration term create additional motivation for the consumer to make a purchase. To the best of our knowledge, we are the first to analytically show such joint effects in a perfectly rational framework.

The following two corollaries are the immediate results of Proposition 5.

Corollary 3. *For any fixed price p,*

(a) if $v-p+\delta J(x,T)\geq\delta J(x-1,t-1)$, then $v-p+\delta J(x+1,T)\geq\delta J(x,t-1)$ for any $1\leq x\leq X-1$ and $1\leq t\leq T$; that is, when $v_0=0$, if the consumer makes a purchase in state $(x-1,t)$, then she will also make a purchase in state (x,t);

(b) if $v-p+\delta J(x,T)\geq\bar{v}+\delta J(x-1,t-1)$, then $v-p+\delta J(x+1,T)\geq\bar{v}+\delta J(x,t-1)$ for any $1\leq x\leq X-1$ and $1\leq t\leq T$; that is, when $v_0=\bar{v}$, if the consumer makes a purchase in state $(x-1,t)$, then she will also make a purchase in state (x,t).

Similar to Corollary 1 in the base model, Corollary 3(a) indicates that, conditional on a low outside option, state Δ is the state where the consumer is least willing to make a purchase. If she makes a purchase even with no points in hand, then she will keep making purchases with more points accumulated. The underlying reasons are twofold. On the one hand, when the consumer is closer to the redemption threshold, she has a higher incentive to make a purchase because she would like to earn and consume the free product earlier. The earlier she consumes the free product, the less the utility loss is due to discounting. This has already been illustrated in the base model. On the other hand, there is a new reason for the consumer to make a purchase: a purchase would renew the expiration date for her point inventory, thereby mitigating the consumer's risk of losing her points. If the consumer is willing to make a purchase in order to renew the expiration date for fewer points in hand, there is no reason for the consumer to make no purchase and thus face the risk of losing more points. Therefore, compared to the same situation in the base model, the consumer has a higher incentive to make a purchase in the presence of the expiration term. The same logic applies to the case with a high-value outside option. Corollary 3(b) argues that, conditional on a high outside option, for any fixed expiration date t, once the consumer makes a purchase with a certain number of points in hand, she will keep making purchases as she approaches the redemption threshold.

Corollary 4. *The consumer's willingness to make a purchase for the product increases with her point inventory. Specifically, for any fixed price p,*

$$v - p + \delta J(x,T) - \big(\bar{v} + \delta J(x-1,t-1)\big) \le v - p + \delta J(x+1,T) - \big(\bar{v} + \delta J(x,t-1)\big)$$

for any $1 \le x \le X-1$ and $1 \le t \le T$.

Corollary 4 shows the key driving force for the consumer's purchase acceleration behavior. The left-hand side of the inequality compares the value of a purchase with no purchase when the consumer is in state $(x-1,t)$, while the right-hand side does the same thing for state (x,t). The inequality implies that, conditional on a high outside option, the purchase option is increasingly attractive as the consumer approaches the redemption threshold. Even if the consumer does not make a purchase in state (x,t), the purchase option becomes more attractive because the potential loss due to point expiration is greater when she is closer to the redemption threshold.

Due to the complexity of the optimality equations in the presence of the expiration date, we are not able to explicitly characterize the value function. Therefore, we do not provide the price ranges under which the consumer accelerates her purchase when she approaches the redemption threshold. However, we are still able to analytically show the existence of purchase acceleration, as summarized in the following proposition.

Proposition 6. *There exists a range of prices p such that*

(a) when $v_0 = 0$, the consumer always makes a purchase/redeems;

(b) when $v_0 = \bar{v}$, for any fixed t, there exists a threshold τ such that the consumer does not make a purchase in states $\{(x,t) : 0 \le x \le \tau - 1\}$, but makes a purchase in states $\{(x,t) : \tau \le x \le X-1\}$.

Proposition 6 confirms the existence of point pressure due to the redemption threshold in the presence of the expiration date. For any fixed t, when the outside option is high, a consumer does not make a purchase when she has few points in hand. As the consumer's point inventory increases, the purchase option becomes increasingly attractive. Once the value of the purchase exceeds that of no purchase, the consumer makes a purchase and continues to do so with more points in hand. Therefore, the point pressure arises. In general, the consumer makes a purchase for two reasons when the point inventory is high enough. On the one hand, when the consumer is closer to the redemption threshold, a purchase enables her to earn the free product earlier. On the other hand, the consumer has a higher incentive to make a purchase in order to renew the expiration term, given that the potential loss due to expiration grows as her point inventory increases.

6.1 "Buy One, Get One Free" (BOGO) Program with a Finite Expiration Term

In this subsection, we analyze "Buy One, Get One Free" (BOGO) programs, a special case of BXGO programs with a finite expiration term T by taking $X = 1$. Our main objectives are twofold. First, we want to show that BOGO programs with an exogenous expiration term T can increase the firm's profit, which immediately implies that BXGO programs with finite expiration terms can increase the firm's profit. Second, we also want to leverage the analytical tractability of BOGO programs to shed insights on why BXGO programs can be profitable even without a positive transaction utility.

In a BOGO program, it suffices to keep track of the expiration term of the reward point. We use 0 to denote the state without a reward point. The optimality equations are simplified as follows:

$$\begin{aligned} J(0) &= \lambda q \max\{v - p + \delta J(T), \delta J(0)\} + \lambda(1-q)\max\{v - p + \delta J(T), \bar{v} + \delta J(0)\} \\ &\quad + (1-\lambda)\delta J(0), \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (9) \\ J(t) &= \lambda q \max\{v + \delta J(0), \delta J(t-1)\} + \lambda(1-q)\max\{v + \delta J(0), \bar{v} + \delta J(t-1)\} \\ &\quad + (1-\lambda)\delta J(t-1), \qquad\qquad\qquad\qquad\qquad\qquad \forall 1 \le t \le T. \qquad (10) \end{aligned}$$

Note that for a general BOGO program, it is still very challenging to solve the dynamic programming equations (9) and (10) explicitly. The difficulty lies in characterizing the consumer's behavior when the outside option is high. The consumer may redeem in each state $i \in \{1, 2, \ldots, T\}$, or there may exist a threshold τ such that the consumer redeems only when the point will expire within τ periods. Recall that our objective is to show that a BOGO program could benefit the firm. Hence, we focus on an analytically tractable case where consumers do not make a purchase but always redeem when the outside option is high, which provides a lower bound of the optimal profit of the BOGO program. We have the following result regarding consumer behavior.

Proposition 7.

If $\dfrac{1-\delta+2\lambda\delta-\lambda\delta[(1-\lambda)\delta]^T}{1-(1-\lambda)\delta}(v-\bar{v}) < p \le$

$$\min\left\{v + \frac{\lambda\delta\left\{1-[(1-\lambda)\delta]^T\right\}\left\{v-(1-q)\bar{v}\right\}}{1-(1-\lambda)\delta}, \frac{1-\delta+2\lambda\delta q+\lambda\delta[(1-\lambda)\delta]^{T-1}\left\{1-q-\delta q(1-\lambda)\right\}}{\lambda q\delta\left\{1-[(1-\lambda)\delta]^{T-1}\right\}}(v-\bar{v})\right\},$$

then

(a) when $v_0 = 0$*, the consumer always makes a purchase/redeems;*

(b) when $v_0 = \bar{v}$*, the consumer does not make a purchase but redeems whenever she has a point in hand.*

We use π^{BO} to denote the optimal profit when the consumer behaves as in Proposition 7. Solving this optimization problem shows that[11] if v is not too large, compared to $v-\bar{v}$, then

$$\pi^{BO}=\lambda qv\frac{1+\lambda q\delta\frac{1-[(1-\lambda)\delta]^T}{1-(1-\lambda)\delta}}{1+q[1-(1-\lambda)^T]}+\lambda(1-q)(v-\bar{v})\frac{\lambda q\delta\frac{1-[(1-\lambda)\delta]^T}{1-(1-\lambda)\delta}}{1+q[1-(1-\lambda)^T]}. \tag{11}$$

The following proposition compares π^{BO} with the profit without a reward program, which is $\max\big\{\lambda qv, \lambda(v-\bar{v})\big\}$.

Proposition 8. *If*

$$\frac{v-\bar{v}}{q}\le v\le\Bigg\{\frac{(1-\delta+\lambda\delta)\Big\{1-\delta+2\lambda\delta q+\lambda\delta[(1-\lambda)\delta]^{T-1}[1-q-\delta q(1-\lambda)]\Big\}}{\lambda\delta q\big\{1-[(1-\lambda)\delta]^{T-1}\big\}\Big\{1-(1-\lambda)\delta+\lambda\delta q\big\{1-[(1-\lambda)\delta]^{T}\big\}\Big\}}$$
$$-\frac{\lambda^2\delta^2q(1-q)\big\{1-[(1-\lambda)\delta]^{T}\big\}\big\{1-[(1-\lambda)\delta]^{T-1}\big\}}{\lambda\delta q\big\{1-[(1-\lambda)\delta]^{T-1}\big\}\Big\{1-(1-\lambda)\delta+\lambda\delta q\big\{1-[(1-\lambda)\delta]^{T}\big\}\Big\}}\Bigg\}(v-\bar{v})$$

and δ is sufficiently close to 1, then $\pi^{BO}\ge\max\{\lambda qv,\lambda(v-\bar{v})\}$.

In order to understand why BOGO programs can increase profits, it is insightful to examine the structure of the optimal profit π^{BO}. As shown in (11), the first term captures the profit contribution from the consumer who purchases in state 0 and redeems in states $\{1,2,\dots,T\}$ when the value of the outside option is low, and the second term captures the profit contribution from the consumer who does not purchase in state 0 but redeems in states $\{1,2,\dots,T\}$ when the value of the outside option is high. When δ is sufficiently close to 1, the profit converges to[12]

$$\lambda qv+\lambda(1-q)(v-\bar{v})\frac{q[1-(1-\lambda)^T]}{1+q[1-(1-\lambda)^T]}.$$

Note that when $v_0=\bar{v}$, although the consumer does not make a purchase, the additional utility from redemption enables the firm to charge a high price. In fact, through BOGO programs, the firm charges an effective price v when the outside option is low (thus, the profit contribution is λqv) and a price $v-\bar{v}$ when the outside option is high (where $\frac{q[1-(1-\lambda)^T]}{1+q[1-(1-\lambda)^T]}$ is the probability that the consumer redeems). That is, BOGO programs enable the firm to implement intra-consumer price discrimination, resulting in profit improvement.

[11] The detailed problem formulation and solution are relegated to the proof of Proposition 8 in the Appendix.

[12] In the base model without an expiration term, when δ is sufficiently close to 1, the consumer may never redeem the point when $v_0=\bar{v}$ (unless the price is sufficiently low such that the consumer also makes a purchase and redeems when the outside option is high, which yields no price discrimination), given that she could always wait to redeem when $v_0=0$. In this case, the profit converges to λqv; that is, the firm could only collect revenue when $v_0=0$. Thus, the BOGO program without an expiration term fails to facilitate price discrimination within the same consumer.

6.2 Numerical Studies for BXGO Programs with Finite Expiration Terms

This subsection first illustrates consumers' purchase acceleration behavior under BXGO programs with a finite expiration term, and then investigates the interdependence between the redemption threshold and the expiration term. Throughout this section, we use a representative numerical example where $(\lambda, v, \bar{v}, q, \delta) = (0.5, 0.7, 0.28, 0.6, 0.99)$, and the firm sets $(p, X, T) = (0.76, 5, 4)$, unless specified otherwise.

Table 3 Consumer's optimal decisions

v_0	0					$\bar{v}$				
x \ t	0	1	2	3	4	0	1	2	3	4
0	1	–	–	–	–	0	–	–	–	–
1	–	1	1	1	1	–	0	0	0	0
2	–	1	1	1	1	–	0	0	0	0
3	–	1	1	1	1	–	1	0	0	0
4	–	1	1	1	1	–	1	1	0	0
5	–	1	1	1	1	–	1	1	0	0

6.2.1 Purchase acceleration Table 3 summarizes consumers' optimal decisions, where the rows correspond to the values of x (from 0 to 5), and the columns correspond to the values of t (from 0 to 4). An element 0 means that the consumer does not make a purchase or redeem, while 1 means that she makes a purchase or redeems. When $v_0 = 0$, the elements are all 1's, indicating that consumers always make a purchase or redeem. The case when $v_0 = \bar{v}$ is more interesting. When $t = 1$ (2), consumers only make a purchase when they have at least 3 (4) points. Intuitively, as a consumer's point inventory approaches the redemption threshold, she is more likely to earn a free product before the points expire; hence, she has a higher incentive to make a purchase in order to increase her chance of successful redemption. When $x = 4$, the consumer does not make a purchase or redeem in states (4, 3) and (4, 4), but purchases in states (4, 1) and (4, 2), implying that she accelerates her purchase when her points are sufficiently close to expiration. Intuitively, when there is still a long time for the points to expire (i.e., t is large), the consumer is not eager to make a purchase (at the cost of giving up the attractive outside option) because the risk of point expiration is low. However, if the points are about to expire (e.g., $t = 1$), she would make a purchase to renew the expiration term of her points even when the outside option is attractive.

The case when $v_0 = \bar{v}$ in Table 3 shows two types of purchase acceleration, demonstrating that consumers' purchase/redeem decisions depend on both the point inventory in hand and the remaining expiration periods. Moreover, the two types of purchase acceleration *interact*. A comparison between $x = 3$ and $x = 4$ shows that the consumer leans more toward making a purchase when she has more points in hand. Intuitively, with more points in hand, the potential loss due to expiration is greater; thus, the consumer prefers to make a purchase earlier in order to mitigate the risk of losing her points.

We verify that the BXGO program in this example is profitable for the firm. The profit is 0.2149, a 2.33% increase over the profit without a BXGO program $\max\{\lambda qv, \lambda(v - \bar{v})\} = 0.21$.

6.2.2 BXGO programs (nX, nT) under different n Redemption thresholds and expiration terms are both redemption hurdles. They can substitute for each other – the firm can reduce the redemption rate by either increasing the redemption threshold or decreasing the expiration term. We investigate the interdependence between these two hurdles by considering BXGO programs (nX, nT) for different n.

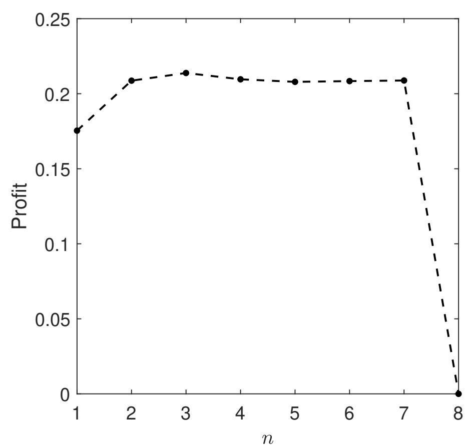


Figure 6 Profit of BXGO programs with (nX, nT) under different n.

How does the firm's profit vary with n? We explore this question by taking $(X, T) = (1, 1)$ under an exogenous price $p = 0.76$. Figure 6 depicts the profits when n varies from 1 to 8. Observe that the profit becomes 0 when $n = 8$, given that consumers do not participate in the program anymore. For $1 \leq n \leq 7$, the profits under different values of n are not equal to each other. To see why this is the case, it suffices to investigate consumers' purchase probability. We examine $n = 2$ and $n = 4$ in some detail. Tables 4 and 5 show consumers' purchase decisions when $v_0 = \bar{v}$ for $n = 2$ and 4, respectively. We underline the states where the consumer pays for a product in both tables. Clearly, the purchase probabilities in these two cases are different, leading to different profits. Together, these results suggest that interaction between the redemption thresholds and expiration terms is not straightforward and warrants careful investigation.

Table 4 The consumer's decision under $(nX, nT) = (2, 2)$ when $v_0 = \bar{v}$

x \ t	0	1	2
0	0	–	–
1	–	$\underline{1}$	0
2	–	1	1

Table 5 The consumer's decision under $(nX, nT) = (4, 4)$ when $v_0 = \bar{v}$

x \ t	0	1	2	3	4
0	0	–	–	–	–
1	–	0	0	0	0
2	–	$\underline{1}$	0	0	0
3	–	$\underline{1}$	$\underline{1}$	0	0
4	–	1	1	0	0

7. Summary and Future Directions

This research focuses on understanding the role of redemption hurdles in shaping consumer behavior and reward program profitability. The controversial nature and divergent practices of using redemption hurdles have led us to investigate several questions that are interesting to both academicians and practitioners: how do redemption hurdles drive the redemption and purchase decisions of consumers? Should firms abandon expiration terms, i.e., never allow the points to expire? How should firms balance the purchase likelihood and effective price paid by consumers? How do firms coordinate different types of redemption hurdles? These managerially important questions have drawn significant attention (Breugelmans et al. 2015) but remain unanswered to date. The current paper bridges this knowledge gap by examining two common redemption hurdles: a redemption threshold and a finite expiration term in an integrated analytical framework.

Our research makes two main contributions to the literature. First, we build a novel consumer model that endogenizes a consumer's purchase and redemption behavior. We show that redemption hurdles give rise to interesting dynamics in consumers' purchase and redemption decisions over time. To the best of our knowledge, we are the first to analytically demonstrate the point pressure phenomenon with respect to the redemption threshold and reward expiration, both of which are well documented in the empirical literature. Importantly, we reveal how (in our modeling framework) the point pressure phenomenon arises due to the intertemporal trade-offs facing strategic consumers.

Second, we show the underlying mechanisms through which redemption hurdles can increase a firm's profitability. In our base model, the firm sets a redemption threshold with no expiration term. We demonstrate that a reward program with only a redemption threshold cannot improve the firm's profit in the absence of transaction utility because consumers discount their future utility, and the firm needs to lower the price sufficiently in order to retain these consumers. However, if consumers derive transaction utility for free products, a BXGO program can be profitable even when the reward points never expire, which explains why many BXGO programs choose not to stipulate an expiration date. While this finding is not surprising, it does point to a plausible reason for why BXGO programs are popular in practice. Furthermore, we find that a redemption threshold, coupled with a finite reward expiration term, can strictly increase the firm's profit, even without any transaction utility. This happens because the expiration term creates an additional incentive for consumers to make purchases, despite the discounting effect. Specifically, BXGO programs have the potential to price discriminate the same consumer with a stochastic outside option, leading to profit improvement in the BXGO program. This result implies that consumer heterogeneity (commonly required in the revenue management literature) is not essential for the profitability of such programs. This result should be contrasted with Sun and Zhang (2019), where the key result is driven by inter-personal price discrimination between frequent and infrequent consumers.

Our results have clear managerial implications. First, firms should recognize the time-varying nature of consumers' price sensitivity (willingness to pay for the product). Such variations are likely to arise due to exogenous factors such as competition; yet, according to our analysis, they can be effectively managed with reward programs. Firms can use reward programs to retain consumers with free products when their willingness to pay is low, but can also seize the opportunity to charge consumers a high price when their willingness to pay is high. Therefore, we unveil an interesting and hitherto unstudied mechanism of within-consumer price discrimination. Second, our analysis of transaction utility illustrates the importance of choosing rewards that are relevant and valuable for consumers, and the effectiveness of a reward program hinges on the "right" choice of reward (e.g., Kivetz and Simonson 2002). Reward programs with low transaction utility should recognize the importance of setting a finite expiration term, which is a necessary condition for the reward program to be profitable. Third, every firm should try to facilitate consumers' reward redemption and boost their transaction utility. In recent years, many firms have converted the punch cards to digital ones – a move that is likely to increase the redemption utility for consumers. Finally, we show that the optimal design of redemption hurdles is not straightforward: the interdependence between the two types of redemption hurdles and the price are nontrivial, and acting purely based on intuition is risky. For example, increasing the redemption threshold and expiration term in the same proportion does not yield the same profit. Therefore, striking a balance between the two requires a careful analysis.

There are several possible extensions to our research. First, we assume that the purchased product and the rewarded product are the same. However, they do not have to be. Indeed, rewards may take many different forms in practice. We analytically model one type of reward (i.e., free products). It would be informative to compare BXGO programs with other reward programs. Second, we do not consider certain practical reward program features. One such feature is customer tiers. Instead of granting a free product after reaching a predetermined redemption threshold, a firm can offer a status reward with additional benefits (e.g., Kopalle et al. 2012, Chun and Ovchinnikov 2019). However, the BXGO programs we consider usually do not have a customer tier component; therefore, customer tiers are not captured in our model. We believe that a tiered design is an important topic for future research. Third, our theoretical perspective can benefit from complementary empirical research in the future. For example, it would be interesting to examine how transaction utility varies across different product/service categories. As our analytical results demonstrate, such knowledge would be helpful in terms of informing the design of reward programs. Fourth, while our rationality-based framework follows recent literature (e.g., Lewis 2004, Chun and Ovchinnikov 2019), there are psychological considerations such as consumer procrastination (Shu and Gneezy 2010), the salience effect of goal pursuit (Bonezzi et al. 2011), and "redemption momentum" (Dorotic et al. 2014), which deserve consideration and can be meaningfully incorporated into future research. Finally, while our model assumes that consumers form correct expectations in their decision-making, it is worthwhile to empirically investigate whether there is systematic deviation from rational expectations.

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APPENDIX

An Analysis of "Buy X, Get One Free" Reward Programs

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Proof of Proposition 1.

For any $0 \le x \le X-2$, accordingly to (1), we have

$$J(x+1) = \lambda q \max\Big\{v-p+\delta J(x+2), \delta J(x+1)\Big\} + \lambda(1-q)\max\Big\{v-p+\delta J(x+2), \bar{v}+\delta J(x+1)\Big\} + (1-\lambda)\delta J(x+1), \tag{12}$$

$$J(x) = \lambda q \max\Big\{v-p+\delta J(x+1), \delta J(x)\Big\} + \lambda(1-q)\max\Big\{v-p+\delta J(x+1), \bar{v}+\delta J(x)\Big\} + (1-\lambda)\delta J(x). \tag{13}$$

We consider four cases.

Case 1: Suppose

$$v-p+\delta J(x+2) \ge \delta J(x+1), \quad v-p+\delta J(x+1) \ge \delta J(x).$$

Then, (12) and (13) reduce to

$$J(x+1) = \lambda q\big(v-p+\delta J(x+2)\big) + \lambda(1-q)\max\Big\{v-p+\delta J(x+2), \bar{v}+\delta J(x+1)\Big\} + (1-\lambda)\delta J(x+1),$$
$$J(x) = \lambda q\big(v-p+\delta J(x+1)\big) + \lambda(1-q)\max\Big\{v-p+\delta J(x+1), \bar{v}+\delta J(x)\Big\} + (1-\lambda)\delta J(x).$$

Therefore,

$$\begin{aligned}
&J(x+1)-J(x)\\
=&\lambda q\delta\big(J(x+2)-J(x+1)\big) + (1-\lambda)\delta\big(J(x+1)-J(x)\big)\\
&+\lambda(1-q)\Big(\max\Big\{v-p+\delta J(x+2), \bar{v}+\delta J(x+1)\Big\} - \max\Big\{v-p+\delta J(x+1), \bar{v}+\delta J(x)\Big\}\Big).
\end{aligned}$$

Subcase 1: Suppose $v-p+\delta J(x+2) \ge \bar{v}+\delta J(x+1)$. Then

$$\begin{aligned}
J(x+1)-J(x) =&\lambda q\delta\big(J(x+2)-J(x+1)\big) + (1-\lambda)\delta\big(J(x+1)-J(x)\big)\\
&+\lambda(1-q)\Big(v-p+\delta J(x+2) - \max\Big\{v-p+\delta J(x+1), \bar{v}+\delta J(x)\Big\}\Big)\\
\le&\lambda q\delta\big(J(x+2)-J(x+1)\big) + (1-\lambda)\delta\big(J(x+1)-J(x)\big)\\
&+\lambda(1-q)\Big(v-p+\delta J(x+2) - \big(v-p+\delta J(x+1)\big)\Big)\\
=&\lambda\delta\big(J(x+2)-J(x+1)\big) + (1-\lambda)\delta\big(J(x+1)-J(x)\big).
\end{aligned}$$

So,

$$J(x+1)-J(x)\le\frac{\lambda\delta}{1-(1-\lambda)\delta}\big(J(x+2)-J(x+1)\big)<J(x+2)-J(x+1).$$

Subcase 2: Suppose $v-p+\delta J(x+2)<\bar{v}+\delta J(x+1)$. Then

$$\begin{aligned}J(x+1)-J(x)=&\lambda q\delta\big(J(x+2)-J(x+1)\big)+(1-\lambda)\delta\big(J(x+1)-J(x)\big)\\&+\lambda(1-q)\Big(\bar{v}+\delta J(x+1)-\max\Big\{v-p+\delta J(x+1),\bar{v}+\delta J(x)\Big\}\Big)\\\le&\lambda q\delta\big(J(x+2)-J(x+1)\big)+(1-\lambda)\delta\big(J(x+1)-J(x)\big)\\&+\lambda(1-q)\Big(\bar{v}+\delta J(x+1)-\big(\bar{v}+\delta J(x)\big)\Big)\\=&\lambda q\delta\big(J(x+2)-J(x+1)\big)+(1-\lambda q)\delta\big(J(x+1)-J(x)\big).\end{aligned}$$

So,

$$J(x+1)-J(x)\le\frac{\lambda q\delta}{1-(1-\lambda q)\delta}\big(J(x+2)-J(x+1)\big)<J(x+2)-J(x+1).$$

Case 2: Suppose

$$v-p+\delta J(x+2)\ge\delta J(x+1),\quad v-p+\delta J(x+1)<\delta J(x).$$

It follows immediately that

$$J(x+2)-J(x+1)\ge\frac{p-v}{\delta},\quad J(x+1)-J(x)<\frac{p-v}{\delta}.$$

Therefore,

$$J(x+1)-J(x)<J(x+2)-J(x+1).$$

Case 3: Suppose

$$v-p+\delta J(x+2)<\delta J(x+1),\quad v-p+\delta J(x+1)\ge\delta J(x).$$

We show this case is impossible by arriving a contradiction. If $v\ge p$, then it is impossible that $v-p+\delta J(x+2)<\delta J(x+1)$ holds because of the monotonicity of $J(\cdot)$. Next, we focus on the case $v<p$.

Since $v-p+\delta J(x+2)<\delta J(x+1)$, (12) reduces to

$$J(x+1)=\lambda q\delta J(x+1)+\lambda(1-q)\big(\bar{v}+\delta J(x+1)\big)+(1-\lambda)\delta J(x+1)=\delta J(x+1)+\lambda(1-q)\bar{v},$$

and thus, $J(x+1)=\frac{\lambda(1-q)}{1-\delta}\bar{v}$. Putting $v<p$ into $v-p+\delta J(x+1)\ge\delta J(x)$ yields that

$$J(x)<J(x+1)=\frac{\lambda(1-q)}{1-\delta}\bar{v}.\tag{14}$$

On the other hand, according to (13),

$$J(x) \geq \lambda q \delta J(x) + \lambda(1-q)\big(\bar{v} + \delta J(x)\big) + (1-\lambda)\delta J(x) = \delta J(x) + \lambda(1-q)\bar{v},$$

and thus

$$J(x) \geq \frac{\lambda(1-q)}{1-\delta}\bar{v},$$

which contradicts (14). Hence, this case is impossible.

<u>Case 4:</u> Suppose

$$v - p + \delta J(x+2) < \delta J(x+1), \quad v - p + \delta J(x+1) < \delta J(x).$$

Then (12) reduces to

$$J(x+1) = \lambda q \delta J(x+1) + \lambda(1-q)\big(\bar{v} + \delta J(x+1)\big) + (1-\lambda)\delta J(x+1) = \delta J(x+1) + \lambda(1-q)\bar{v},$$

and thus

$$J(x+1) = \frac{\lambda(1-q)}{1-\delta}\bar{v}.$$

Likewise, one can check

$$J(x) = \frac{\lambda(1-q)}{1-\delta}\bar{v}.$$

Therefore,

$$J(x+1) - J(x) = 0.$$

The monotonicity of $J(\cdot)$ indicates that

$$J(x+2) - J(x+1) \geq 0.$$

Hence,

$$J(x+2) - J(x+1) \geq J(x+1) - J(x).$$

This completes the proof. ■

Proof of Proposition 2.

Before the proof of Proposition 2, we first introduce two lemmas: Lemmas 4 and 5.

LEMMA 4. *We have $v + k + \delta J(0) \geq \delta J(X)$. That is, the consumer always redeems when the outside option is low.*

Proof of Lemma 4. Suppose for a contradiction that $v+k+\delta J(0)<\delta J(X)$, then (2) reduces to

$$J(X)=\lambda q\delta J(X)+\lambda(1-q)\big(\bar{v}+\delta J(X)\big)+(1-\lambda)\delta J(X)=\delta J(X)+\lambda(1-q)\bar{v},$$

and thus

$$J(X)=\frac{\lambda(1-q)}{1-\delta}\bar{v}. \tag{15}$$

Note also that

$$J(0)\geq\lambda qJ(0)+\lambda(1-q)\big(\bar{v}+\delta J(0)\big)+(1-\lambda)\delta J(0)=\delta J(0)+\lambda(1-q)\bar{v},$$

and thus

$$J(0)\geq\frac{\lambda(1-q)}{1-\delta}\bar{v}. \tag{16}$$

Combining (15) and (16) gives that $v+k+\delta J(0)\geq\delta J(X)$, which contradicts our supposition. This completes the proof. ■

Lemma 5. *For any fixed price p, if $v-p+\delta J(X)\geq\bar{v}+\delta J(X-1)$, then $v+k+\delta J(0)\geq\bar{v}+\delta J(X)$.*

Proof of Lemma 5. According to (1),

$$\begin{aligned}J(X-1)&=\lambda q\big(v-p+\delta J(X)\big)+\lambda(1-q)\big(v-p+\delta J(X)\big)+(1-\lambda)\delta J(X-1)\\&=\lambda\big(v-p+\delta J(X)\big)+(1-\lambda)\delta J(X-1),\end{aligned}$$

and thus

$$J(X-1)=\frac{\lambda\big(v-p+\delta J(X)\big)}{1-(1-\lambda)\delta}.$$

Note that

$$\begin{aligned}&v-p+\delta J(X)-\bar{v}-\delta J(X-1)=v-p+\delta J(X)-\bar{v}-\delta\frac{\lambda\big(v-p+\delta J(X)\big)}{1-(1-\lambda)\delta}\\&=\frac{1}{1-(1-\lambda)\delta}\Big\{(1-\delta)v-\big(1-(1-\lambda)\delta\big)\bar{v}-(1-\delta)\big(p-\delta J(X)\big)\Big\}\geq 0,\end{aligned}$$

so

$$(1-\delta)v-\big(1-(1-\lambda)\delta\big)\bar{v}\geq(1-\delta)\big(p-\delta J(X)\big). \tag{17}$$

Moreover,

$$J(X)\geq J(X-1)=\frac{\lambda\big(v-p+\delta J(X)\big)}{1-(1-\lambda)\delta},$$

and thus

$$J(X) \geq \frac{\lambda(v-p)}{1-\delta}, \tag{18}$$

from which we obtain

$$p \geq v - \frac{1-\delta}{\lambda} J(X). \tag{19}$$

Suppose for a contradiction that $v + k + \delta J(0) < \bar{v} + \delta J(X)$, then

$$\begin{aligned} J(X) &= \lambda q\big(v + k + \delta J(0)\big) + \lambda(1-q)\big(\bar{v} + \delta J(X)\big) + (1-\lambda)\delta J(X) \\ &= \lambda q\big(v + k + \delta J(0)\big) + \lambda(1-q)\bar{v} + (1-\lambda q)\delta J(X). \end{aligned} \tag{20}$$

Hence,

$$J(X) = \frac{\lambda q\big(v + k + \delta J(0)\big) + \lambda(1-q)\bar{v}}{1-(1-\lambda q)\delta}. \tag{21}$$

We have

$$\begin{aligned} & v + k + \delta J(0) - \bar{v} - \delta J(X) \\ =& v + k + \delta J(0) - \bar{v} - \delta \frac{\lambda q\big(v + k + \delta J(0)\big) + \lambda(1-q)\bar{v}}{1-(1-\lambda q)\delta} && \text{[by (21)]} \\ =& \frac{1}{1-(1-\lambda q)\delta}\Big\{(1-\delta)(v+k) - (1-\delta+\lambda\delta)\bar{v} + (1-\delta)\delta J(0)\Big\} \\ \geq& \frac{1}{1-(1-\lambda q)\delta}\Big\{(1-\delta)\big(p - \delta J(X)\big) + (1-\delta)\delta J(0) + (1-\delta)k\Big\} && \text{[by (17)]} \\ =& \frac{1-\delta}{1-(1-\lambda q)\delta}\Big\{p - \delta J(X) + \delta J(0) + k\Big\}. \end{aligned} \tag{22}$$

If $p \geq v - \bar{v}$, then by (22),

$$\begin{aligned} v + k + \delta J(0) - \bar{v} - \delta J(X) &\geq \frac{1-\delta}{1-(1-\lambda q)\delta}\Big\{v - \bar{v} - \delta J(X) + \delta J(0) + k\Big\} \\ &= \frac{1-\delta}{1-(1-\lambda q)\delta}\Big\{v + k + \delta J(0) - \bar{v} - \delta J(X)\Big\}. \end{aligned}$$

Hence,

$$\left(1 - \frac{1-\delta}{1-\delta+\lambda q\delta}\right)\Big\{v + k + \delta J(0) - \bar{v} - \delta J(X)\Big\} \geq 0,$$

so, $v + k + \delta J(0) \geq \bar{v} + \delta J(X)$, which contradicts our supposition.

Now, we consider the case $p < v - \bar{v}$. In this case, (18) indicates that

$$J(X) \geq \frac{\lambda\big\{v - (v - \bar{v})\big\}}{1-\delta} = \frac{\lambda \bar{v}}{1-\delta}. \tag{23}$$

According to (22),

$$\begin{aligned}
&v+k+\delta J(0)-\bar{v}-\delta J(X)\\
\geq&\frac{1-\delta}{1-(1-\lambda q)\delta}\Big\{p-\delta J(X)+\delta J(0)+k\Big\}\\
\geq&\frac{1-\delta}{1-(1-\lambda q)\delta}\Big\{v-\frac{1-\delta}{\lambda}J(X)-\delta J(X)+\delta J(0)+k\Big\} && \text{[by (19)]}\\
=&\frac{1-\delta}{1-(1-\lambda q)\delta}\Big\{\frac{\big(1-(1-\lambda q)\delta\big)J(X)-\lambda(1-q)\bar{v}}{\lambda q}-\frac{1-\delta+\lambda\delta}{\lambda}J(X)\Big\} && \text{[by (20)]}\\
=&\frac{1-\delta}{1-(1-\lambda q)\delta}\frac{1-q}{\lambda q}\Big\{(1-\delta)J(X)-\lambda\bar{v}\Big\}\\
\geq&0, && \text{[by (23)]}
\end{aligned}$$

contradicting our supposition. This completes the proof. ■

Now, it is ready to prove Proposition 2. To simplify the presentation, we split Proposition 2 into four lemmas, Lemmas 6-9, corresponding to Parts (a)-(d) of Proposition 2.

LEMMA 6. *If $p \leq p^D(0)$, then the consumer always purchases/redeems when she arrives the market, independent of the realization of the outside option.*

Proof of Lemma 6. Suppose

$$v-p+\delta J(1)\geq\bar{v}+\delta J(0). \tag{24}$$

Corollary 1(b) and Lemma 5 imply that the consumer always purchases/redeems when the outside option is high, which immediately implies that she also purchases/redeems in each state when the outside option is low. Given the consumer's behavior, (1) and (2) reduce to

$$\begin{aligned}
J(x)&=\lambda\big(v-p+\delta J(x+1)\big)+(1-\lambda)\delta J(x), && \text{for } 0\leq x\leq X-1,\\
J(X)&=\lambda\big(v+k+\delta J(0)\big)+(1-\lambda)\delta J(X).
\end{aligned}$$

Solving the above set of equations yields that

$$\begin{aligned}
J(0)&=\frac{\lambda v}{1-\delta}-\frac{\lambda p}{1-\delta}\frac{1-A_1^X}{1-A_1^{X+1}}+\frac{\lambda k}{1-\delta}\frac{A_1^X(1-A_1)}{1-A_1^{X+1}},\\
\delta J(1)&=\frac{1-(1-\lambda)\delta}{\lambda}J(0)-v+p.
\end{aligned}$$

Putting $J(0)$ and $\delta J(1)$ back to (24) yields

$$p\leq\frac{(1-A_1^{X+1})(v-\bar{v})+A_1^X(1-A_1)k}{1-A_1^X}=p^D(0).$$

This completes the proof. ■

Lemma 7. *If $p^D(\tau-1) < p \le \min\{p^B(\tau), p^D(\tau)\}$ for any threshold $\tau \in \{1,\ldots,X-1\}$, then*

(a) when $v_0 = 0$, the consumer purchases/redeems in each state;

(b) when $v_0 = \bar{v}$, the consumer does not purchase in states $\{0,1,\ldots,\tau-1\}$, purchases in states $\{\tau,\tau+1,\ldots,X-1\}$, and redeems in state X.

Proof of Lemma 7. Suppose

$$v - p + \delta J(1) \ge \delta J(0), \tag{25}$$

$$v - p + \delta J(\tau+1) \ge \bar{v} + \delta J(\tau), \tag{26}$$

$$v - p + \delta J(\tau) < \bar{v} + \delta J(\tau-1), \tag{27}$$

for any fixed $1 \le \tau \le X-1$.

Suppose $v_0 = 0$. Inequality (25) and Corollary 1(a) imply that the consumer purchases in states $\{0,1,\ldots,X-1\}$. Lemma 4 shows that she redeems in state X.

Suppose $v_0 = \bar{v}$. Inequalities (26), (27), and Corollary 2 imply that the consumer does not purchase in states $\{0,1,\ldots,\tau-1\}$, and purchases in states $\{\tau,\ldots,X-1\}$. Furthermore, inequality (26) and Lemma 5 imply that the consumer redeems in state X.

Given the consumer's behavior, (1) and (2) reduce to

$$\begin{aligned}
J(x) &= \lambda q\big(v - p + \delta J(x+1)\big) + \lambda(1-q)\big(\bar{v} + \delta J(x)\big) + (1-\lambda)\delta J(x), && \text{for } 0 \le x \le \tau-1,\\
J(x) &= \lambda\big(v - p + \delta J(x+1)\big) + (1-\lambda)\delta J(x), && \text{for } \tau \le x \le X-1,\\
J(X) &= \lambda\big(v + k + \delta J(0)\big) + (1-\lambda)\delta J(X).
\end{aligned}$$

Solving the above set of equations yields that

$$\begin{aligned}
J(0) = \frac{1}{(1-\delta)(1-A^\tau A_1^{X-\tau+1})}\Big\{&\lambda\big(q(v-p) + (1-q)\bar{v}\big)(1-A^\tau) + A^\tau(1-A_1^{X-\tau+1})\lambda v\\
&- A^\tau(1-A_1^{X-\tau})\lambda p + A^\tau A_1^{X-\tau}(1-A_1)\lambda k\Big\}.
\end{aligned}$$

Putting back to (25), (26), and (27) gives that

$$\begin{aligned}
p &\le \frac{\big(A^\tau(1-A_1^{X-\tau+1}) + q(1-A^\tau)\big)v - A^\tau(1-A_1^{X-\tau+1})(1-q)\bar{v} + A^\tau A_1^{X-\tau}(1-A_1)k}{A^\tau(1-A_1^{X-\tau}) + q(1-A^\tau)} = p^B(\tau),\\
p &\le \frac{\big(1 - A_1^{X-\tau+1} + q(1-A^\tau)A_1^{X-\tau+1}\big)(v-\bar{v}) + A_1^{X-\tau}(1-A_1)k}{1 - A_1^{X-\tau} + q(1-A^\tau)A_1^{X-\tau+1}} = p^D(\tau),\\
p &> \frac{\big(1 - A_1^{X-\tau+2} + q(1-A^{\tau-1})A_1^{X-\tau+2}\big)(v-\bar{v}) + A_1^{X-\tau+1}(1-A_1)k}{1 - A_1^{X-\tau+1} + q(1-A^{\tau-1})A_1^{X-\tau+2}} = p^D(\tau-1),
\end{aligned}$$

respectively. This completes the proof. ■

Lemma 8. *Suppose $p^D(X-1) < p \le \min\big\{p^B(X), p^D(X)\big\}$. Then,* *(a) when $v_0 = 0$, the consumer purchases/redeems in each state;*

(b) when $v_0 = \bar{v}$, the consumer does not purchase in states $\{0, 1, \ldots, X-1\}$, but redeems in state X.

Proof of Lemma 8. Suppose

$$v - p + \delta J(1) \geq \delta J(0), \tag{28}$$

$$v - p + \delta J(X) < \bar{v} + \delta J(X-1), \tag{29}$$

$$v + k + \delta J(0) \geq \bar{v} + \delta J(X). \tag{30}$$

Suppose $v_0 = 0$. Inequality (28) and Corollary 1(a) imply that the consumer purchases in states $\{0, 1, \ldots, X-1\}$, while Lemma 4 shows that she redeems in state X.

Suppose $v_0 = \bar{v}$. Inequality (29) and Corollary 2 imply that the consumer does not purchase in states $\{0, 1, \ldots, X-1\}$, while (30) shows that she redeems in state X.

Given the consumer's behavior, (1) and (2) reduce to

$$\begin{aligned}
J(x) &= \lambda q\big(v - p + \delta J(x+1)\big) + \lambda(1-q)\big(\bar{v} + \delta J(x)\big) + (1-\lambda)\delta J(x), \qquad \text{for } 0 \leq x \leq X-1, \\
J(X) &= \lambda\big(v + k + \delta J(0)\big) + (1-\lambda)\delta J(X).
\end{aligned}$$

Solving the above set of equations yields that

$$J(0) = \frac{1}{1 - A^X A_1}\Big\{\lambda\big(q(v-p) + (1-q)\bar{v}\big)\frac{1 - A^X}{1-\delta} + A^X \frac{\lambda(v+k)}{1-(1-\lambda)\delta}\Big\}.$$

Putting back to (28), (29), and (30) gives that

$$\begin{aligned}
p &\leq \frac{1 - A_1 A^X}{1 - A^X} v + \frac{1-q}{q}\frac{A^X(1-A_1)}{1-A^X}(v - \bar{v}) + \frac{A^X(1-A_1)}{q(1-A^X)} k = p^B(X), \\
p &> \frac{(1 - A + A_1 - A_1 A^X)(v - \bar{v}) + (1-A)A_1 k}{1 - A + A_1 A - A_1 A^X} = p^D(X-1), \\
p &\leq \frac{(1 - A^{X+1})(v - \bar{v}) + (1-A)k}{A(1-A^X)} = p^D(X),
\end{aligned}$$

respectively. This completes the proof. ■

LEMMA 9. *Suppose $p^D(X) < p \leq \frac{(1-A^{X+1})v + A^X(1-A)k}{1-A^X}$. Then,*

(a) when $v_0 = 0$, the consumer purchases/redeems in each state;

(b) when $v_0 = \bar{v}$, the consumer neither purchase nor redeem in any state.

Proof of Lemma 9. Suppose

$$v - p + \delta J(1) \geq \delta J(0), \tag{31}$$

$$v + k + \delta J(0) < \bar{v} + \delta J(X). \tag{32}$$

Suppose $v_0 = 0$. Inequality (31) and Corollary 1(a) imply that the consumer purchases in states $\{0, 1, \ldots, X-1\}$, while Lemma 4 shows that she redeems in state X.

Suppose $v_0 = \bar{v}$. Inequality (32) and Lemma 5 imply that the consumer does not purchase in state $X-1$. Thus, according to Corollary 2, the consumer does not purchase in states $\{0, 1, \ldots, X-1\}$. Inequality (32) directly tells us that the consumer does not redeem in state X.

Given the consumer's behavior, (1) and (2) reduce to

$$J(x) = \lambda q\big(v - p + \delta J(x+1)\big) + \lambda(1-q)\big(\bar{v} + \delta J(x)\big) + (1-\lambda)\delta J(x), \qquad \text{for } 0 \leq x \leq X-1,$$

$$J(X) = \lambda q\big(v + k + \delta J(0)\big) + \lambda(1-q)\big(\bar{v} + \delta J(X)\big) + (1-\lambda)\delta J(X).$$

Solving the above set of equations yields that

$$J(0) = \frac{1}{(1-A^{X+1})(1-\delta)}\Big\{\lambda\big(qv + (1-q)\bar{v}\big)(1-A^{X+1}) - \lambda qp(1-A^X) + \lambda qkA^X(1-A)\Big\}.$$

Putting back to (31) and (32) gives that

$$p \leq \frac{(1-A^{X+1})v + A^X(1-A)k}{1-A^X},$$

$$p > \frac{(1-A^{X+1})(v-\bar{v}) + (1-A)k}{A(1-A^X)} = p^D(X),$$

respectively. This completes the proof. ■

Proof of Lemma 1.

The balance equations can be written as follows,

$$\begin{aligned}
\lambda q q_0 &= \lambda q_X, & \\
\lambda q q_i &= \lambda q q_{i-1}, & \text{for } i = 1, \ldots, \tau-1, \\
\lambda q_\tau &= \lambda q q_{\tau-1}, & \\
\lambda q_i &= \lambda q_{i-1}, & \text{for } i = \tau+1, \ldots, X, \\
\sum_{i=0}^{X} q_i &= 1. &
\end{aligned}$$

Solving the above set of equations gives the stationary probabilities. ■

Proof of Lemma 2.

Let $\pi_b(\tau, X)$ denote the firm's profit in Case (b) for a given (τ, X), where $1 \le \tau \le X-1$. Thus,

$$\pi_b^* = \max_{\tau, X} \pi_b(\tau, X).$$

We comment here that τ is not a decision variable, rather, it is jointly decided by the model parameters and firm's decision (p, X). For each pair of (p, X), there exists a unique pair of (τ, X).

Define $\pi_b^{ub}(\tau, X) = \frac{\lambda q X}{\tau + q(X-\tau+1)} p^D(\tau)$, which is an upperbound of $\pi_b(\tau, X)$. In order to show $\pi_b^* \le \max\{\lambda q v, \lambda(v-\bar{v})\}$, it suffices to show

$$\max_{\tau, X} \pi_b^{ub}(\tau, X) \le \max\{\lambda q v, \lambda(v-\bar{v})\}. \tag{33}$$

Suppose $\tau = X - i$ for any fixed i, where $1 \le i \le X-1$. Then,

$$\max_{\tau, X} \pi_b^{ub}(\tau, X) = \max_{1 \le i \le X-1, X} \pi_b^{ub}(X-i, X).$$

Here, we keep the distance between τ and X fixed, and examine the property of $\pi_b^{ub}(X-i, X)$ for each fixed $1 \le i \le X-1$.

Part (a): Suppose $q \le \frac{1}{2}$.

One can check

$$p^D(\tau) = p^D(X-i) = \frac{1 - A_1^{i+1} + q(1-A^{X-i})A_1^{i+1}}{1 - A_1^i + q(1-A^{X-i})A_1^{i+1}} (v - \bar{v})$$

is decreasing in X, and the purchase probability

$$\frac{\lambda q X}{\tau + q(X-\tau+1)} = \frac{\lambda q X}{X - (1-q)i + q}$$

is also decreasing in X because $q \le \frac{1}{2}$, so $\pi_b^{ub}(X-i, X)$ decreases in X. Hence, for each fixed i where $1 \le i \le X-1$, we have

$$\pi_b^{ub}(X-i, X) \le \pi_b^{ub}(1, i+1)$$

because $X \ge i+1$. Thus,

$$\max_{1 \le i \le X-1, X} \pi_b^{ub}(X-i, X) = \max_{1 \le i} \pi_b^{ub}(1, i+1) = \max_{X \ge 2} \pi_b^{ub}(1, X),$$

where the last equality holds because of variable substitution.

Therefore, to show (33), it suffices to show $\max_{X \ge 2} \pi_b^{ub}(1, X) \le \max\{\lambda q v, \lambda(v-\bar{v})\}$. For any $X \ge 2$, we have

$$\pi_b^{ub}(1, X) - \lambda(v - \bar{v}) = \frac{\lambda q X}{qX+1} p^D(1) - \lambda(v - \bar{v})$$

$$
\begin{aligned}
&= \frac{\lambda q X}{qX+1}\frac{1-A_1^X+q(1-A)A_1^X}{1-A_1^{X-1}+q(1-A)A_1^X}(v-\bar{v})-\lambda(v-\bar{v})\\
&= \frac{\lambda(v-\bar{v})}{(qX+1)\{1-A_1^{X-1}+q(1-A)A_1^X\}}\Big\{qXA_1^{X-1}(1-A_1)-\big(1-A_1^{X-1}+q(1-A)A_1^X\big)\Big\}.
\end{aligned}
$$

Let $L(q)=qXA_1^{X-1}(1-A_1)-\big(1-A_1^{X-1}+q(1-A)A_1^X\big)$. We expect to show $L(q)<0$ for any $0\le q\le 1$. Note that

$$
\begin{aligned}
L(0) &= -1+A_1^{X-1}<0,\\
L(1) &= XA_1^{X-1}(1-A_1)-\Big\{1-A_1^{X-1}+(1-A)A_1^X\Big\}\\
&= (1-A_1)\Big\{XA_1^{X-1}-(1+A_1+\cdots+A_1^{X-2})-A_1^X\Big\}\\
&= (1-A_1)\Big\{(A_1^{X-1}-1)+(A_1^{X-1}-A_1^X)+(X-2)A_1^{X-1}-\sum_{i=1}^{X-2}A_1^i\Big\}\\
&= (1-A_1)\Big\{(1-A_1)\Big(A_1^{X-1}-\sum_{i=0}^{X-2}A_1^i\Big)+(X-2)A_1^{X-1}-\sum_{i=1}^{X-2}A_1^i\Big\}\\
&< 0.
\end{aligned}
$$

Note also that

$$
L'(q)=XA_1^{X-1}(1-A_1)-A_1^X\left(\frac{1-\delta}{1-\delta+\lambda q\delta}\right)^2,\quad L''(q)=2A_1^X(1-\delta)^2(1-\delta+\lambda q\delta)^{-1}\lambda\delta>0,
$$

so the convexity with $L(0)<0$ and $L(1)<0$ imply that $L(q)<0$ for any $0\le q\le 1$. This completes the proof of Part (a).

Part (b): Suppose $q>\frac{1}{2}$ and $v-\bar{v}\le\frac{v}{1+A_1}$.

Define $\tilde{i}=\max\{i:(1-q)i<q\}$. For any fixed $i\ge\tilde{i}+1$ and X, $\frac{\lambda qX}{X-(1-q)i+q}$ decreases in X, and thus it follows Part (a) that

$$
\pi_b^{ub}(X-i,X)\le\pi_b^{ub}(1,i+1)\le\max\{\lambda qv,\lambda(v-\bar{v})\}.
$$

It remains to show for each $i\le\tilde{i}$ and X,

$$
\pi_b^{ub}(X-i,X)\le\max\{\lambda qv,\lambda(v-\bar{v})\}.
$$

Note that

$$
\begin{aligned}
p^D(X-i) &= \frac{1-A_1^{i+1}+q(1-A^{X-i})A_1^{i+1}}{1-A_1^i+q(1-A^{X-i})A_1^{i+1}}(v-\bar{v})\\
&\le \frac{1-A_1^{i+1}+q(1-A)A_1^{i+1}}{1-A_1^i+q(1-A)A_1^{i+1}}(v-\bar{v})
\end{aligned}
$$

$$\leq \frac{1-A_1^2+q(1-A)A_1^2}{1-A_1+q(1-A)A_1^2}(v-\bar{v}),$$

where the first inequality holds because $p^D(X-i)$ decreases in X and $X \geq i+1$, and the last inequality holds because $\frac{1-A_1^{i+1}+q(1-A)A_1^{i+1}}{1-A_1^{i}+q(1-A)A_1^{i+1}}$ decreases in i. Note also that $\frac{\lambda qX}{X-(1-q)i+q} < \frac{\lambda qX}{X} = \lambda q$, so for any $i \leq \tilde{i}$ and X, it follows that

$$\begin{aligned}
\pi_b^{ub}(X-i,X) &\leq \lambda q \frac{1-A_1^2+q(1-A)A_1^2}{1-A_1+q(1-A)A_1^2}(v-\bar{v}) \\
&= \lambda q \Big\{1+\frac{A_1(1-A_1)}{1-A_1+q(1-A)A_1^2}\Big\}(v-\bar{v}) \\
&\leq \lambda q(1+A_1)(v-\bar{v}) \\
&\leq \lambda qv,
\end{aligned}$$

where the last inequality holds because $v-\bar{v} \leq \frac{v}{1+A_1}$. This completes the proof of Part (b). ■

Proof of Lemma 3.

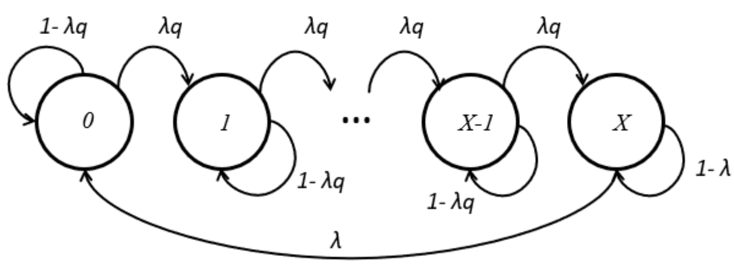

Figure 7 State transition of a consumer in Case (c)

The state transition of a consumer in Case (c) is depicted in Figure 7. The balance equations can be written as follows,

$$\begin{aligned}
&\lambda q q_0 = \lambda q_X, \\
&\lambda q q_i = \lambda q q_{i+1}, && \text{for } i=1,\dots,X-1, \\
&\lambda q_X = \lambda q q_{X-1}, \\
&\sum_{i=0}^{X} q_i = 1.
\end{aligned}$$

Solving the above set of equations yields that

$$\begin{aligned}
&q_i = \frac{1}{X+q}, && \text{for } i=0,1,\dots,X-1, \\
&q_X = \frac{q}{X+q}.
\end{aligned}$$

The consumer's purchase probability is $\lambda q \sum_{i=0}^{X-1} q_i = \frac{\lambda q X}{X+q}$. Hence, the firm faces the following optimization problem,

$$\max_{p,X} \ \frac{\lambda q X}{X+q} p$$
$$s.t.\ \frac{1-A+A_1-A_1A^X}{1-A+A_1A-A_1A^X}(v-\bar{v}) < p \leq \min\Big\{\frac{1-A_1A^X}{1-A^X}v + \frac{1-q}{q}\frac{A^X(1-A_1)}{1-A^X}(v-\bar{v}), \frac{1-A^{X+1}}{A(1-A^X)}(v-\bar{v})\Big\},$$

where the above constraint is obtained from $p^D(X-1) < p \leq \min\{p^B(X), p^D(X)\}$. Now, we show $\pi_c^* \leq \max\{\lambda q v, \lambda(v-\bar{v})\}$ under either of the conditions.

Part (a): Suppose $q \leq A$.

We have

$$\begin{aligned}
&\pi_c^* - \lambda(v-\bar{v}) \\
\leq& \frac{\lambda q X}{X+q}\frac{1-A^{X+1}}{A(1-A^X)}(v-\bar{v}) - \lambda(v-\bar{v}) \\
\leq& \frac{\lambda q X}{X+q}\frac{1-q^{X+1}}{q(1-q^X)}(v-\bar{v}) - \lambda(v-\bar{v}) \\
=& \frac{\lambda(v-\bar{v})}{(X+q)(1-q^X)}\Big\{X(1-q^{X+1}) - (X+q)(1-q^X)\Big\} \\
=& \frac{\lambda(v-\bar{v})}{(X+q)(1-q^X)}\Big\{Xq^X(1-q) - q(1-q^X)\Big\} \\
=& \frac{\lambda(v-\bar{v})q(1-q)}{(X+q)(1-q^X)}\Big\{Xq^{X-1} - (1+q+\cdots+q^{X-1})\Big\} \\
<& 0,
\end{aligned}$$

where the first inequality holds because $p \leq \frac{1-A^{X+1}}{A(1-A^X)}(v-\bar{v})$, and the second inequality holds because $\frac{1-A^{X+1}}{A(1-A^X)}$ decreases in A.

Part (b): Suppose $q > A$ and $v - \bar{v} \leq Av$.

We have

$$\begin{aligned}
&\pi_c^* - \lambda q v \\
\leq& \frac{\lambda q X}{X+q}\frac{1-A^{X+1}}{A(1-A^X)}(v-\bar{v}) - \lambda q v \\
=& \lambda q\Big\{\frac{X}{X+q}\frac{1-A^{X+1}}{A(1-A^X)}(v-\bar{v}) - v\Big\} \\
\leq& \lambda q\Big\{\frac{X}{X+q}\frac{1-A^{X+1}}{1-A^X}v - v\Big\} && [\text{by } v-\bar{v} \leq Av] \\
=& \frac{\lambda q v}{(X+q)(1-A^X)}\Big\{X(1-A^{X+1}) - (X+q)(1-A^X)\Big\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\lambda q v}{(X+q)(1-A^X)}\Big\{XA^X(1-A) - q(1-A^X)\Big\} \\
&= \frac{\lambda q v}{(X+q)(1-A^X)}(1-A)\Big\{XA^X - q(1+A+\cdots+A^{X-1})\Big\} \\
&\leq \frac{\lambda q v}{(X+q)(1-A^X)}(1-A)\Big\{XA^X - (A+A^2+\cdots+A^X)\Big\} \qquad \text{[by } q > A\text{]} \\
&\leq 0.
\end{aligned}
$$

This completes the proof. ■

Proof of Proposition 3.

Since $\pi_a^* \leq \lambda(v-\bar{v})$ and $\pi_d^* \leq \lambda q v$, it suffices to show under either conditions,

$$\pi_b^* \leq \max\big\{\lambda(v-\bar{v}), \lambda q v\big\}, \tag{34}$$

$$\pi_c^* \leq \max\big\{\lambda(v-\bar{v}), \lambda q v\big\}. \tag{35}$$

Part (a): Suppose $q \leq \frac{1}{2}$ and $\delta \geq \frac{2}{2+\lambda}$.

Lemma 2(a) immediately implies that (34) holds. To show (35), according to Lemma 3(a), it suffices to show $q \leq A$. One can check

$$
\begin{aligned}
q - A &= q - \frac{\lambda q \delta}{1-\delta+\lambda q\delta} = \frac{q}{1-\delta+\lambda q\delta}\big\{1-\delta(1+\lambda-\lambda q)\big\} \\
&\leq \frac{q}{1-\delta+\lambda q\delta}\Big\{1-\frac{2}{2+\lambda}(1+\lambda-\lambda q)\Big\} \qquad \text{[by } \delta \geq \frac{2}{2+\lambda}\text{]} \\
&\leq \frac{q}{1-\delta+\lambda q\delta}\Big\{1-\frac{2}{2+\lambda}\big(1+\lambda-\frac{\lambda}{2}\big)\Big\} \qquad \text{[by } q \leq \frac{1}{2}\text{]} \\
&= \frac{q}{1-\delta+\lambda q\delta}\Big\{1-\frac{2(1+\frac{\lambda}{2})}{2+\lambda}\Big\} \\
&= 0.
\end{aligned}
$$

This completes the proof of Part (a).

Part (b): Suppose $q > \frac{1}{2}$ and $v - \bar{v} \leq \min\{Av, \frac{v}{1+A_1}\}$.

Lemma 2(b) immediately implies that (34) holds. If $q \leq A$, (35) holds immediately by Lemma 3(a). Otherwise, since $v - \bar{v} \leq Av$, (35) holds by Lemma 3(b). This completes the proof of Part (b). ■

Proof of Proposition 4.

Before the proof of Proposition 4, we first introduce the following lemma.

LEMMA 10. *(a) If* $k \geq \frac{1-A_1^X - XA_1^X(1-A_1)}{XA_1^X(1-A_1)}(v-\bar{v})$, *then* $\pi_a^* \geq \lambda(v-\bar{v})$.

(b) If $k \geq \frac{1-A^X - XA^X(1-A)}{XA^X(1-A)}v$, *then* $\pi_d^* \geq \lambda qv$.

Proof. In Case (a), the consumer always purchases/redeems, regardless of the value of the outside option. Hence, the stationary probability q_i for state $i \in \{0,1,\ldots,X\}$ is equal to each other; that is, $q_i = \frac{1}{X+1}$. Thus, the purchase probability is $\lambda\sum_{i=0}^{X-1} q_i = \frac{\lambda X}{X+1}$. Proposition 2(a) shows the price boundary as follows,

$$p \leq \frac{(1-A_1^{X+1})(v-\bar{v}) + A_1^X(1-A_1)k}{1-A_1^X}.$$

Hence,

$$\pi_a^* = \frac{\lambda X}{X+1}\frac{(1-A_1^{X+1})(v-\bar{v}) + A_1^X(1-A_1)k}{1-A_1^X}.$$

One can verify that if

$$k \geq \frac{1-A_1^X - XA_1^X(1-A_1)}{XA_1^X(1-A_1)}(v-\bar{v}),$$

then $\pi_a^* \geq \lambda(v-\bar{v})$. Similarly, one can show

$$\pi_d^* = \frac{\lambda qX}{X+1}\frac{(1-A^{X+1})v + A^X(1-A)k}{1-A^X},$$

and verify that if

$$k \geq \frac{1-A^X - XA^X(1-A)}{XA^X(1-A)}v,$$

then $\pi_d^* \geq \lambda qv$. This completes the proof. ■

Proof of Proposition 4. If $v - \bar{v} \geq qv$, then $\max\{\lambda(v-\bar{v}), \lambda qv\} = \lambda(v-\bar{v})$. The result follows immediately from Lemma 10(a). Otherwise, $\max\{\lambda(v-\bar{v}), \lambda qv\} = \lambda qv$. The result follows immediately from Lemma 10(b). ■

Proof of Proposition 5.

Part (b) holds immediately because of the monotonicity of $J(x,\cdot)$. Hereafter, we focus on showing Part (a). Note that $J(x,T) - J(x-1,t-1) \leq J(x+1,T) - J(x,t-1)$ is equivalent to

$$J(x+1,T) - J(x,T) \geq J(x,t-1) - J(x-1,t-1). \tag{36}$$

We first show that, in order to show (36), it suffices to show

$$J(x+1,T) - J(x,T) \geq J(x,t-2) - J(x-1,t-2). \tag{37}$$

According to (5), for any $1 \leq x \leq X-1$,

$$J(x,t-1) = \lambda q \max\Big\{v - p + \delta J(x+1,T), \delta J(x,t-2)\Big\}$$

$$+\lambda(1-q)\max\Big\{v-p+\delta J(x+1,T),\bar{v}+\delta J(x,t-2)\Big\}+(1-\lambda)\delta J(x,t-2), \quad (38)$$

$$J(x-1,t-1)=\lambda q\max\Big\{v-p+\delta J(x,T),\delta J(x-1,t-2)\Big\}$$
$$+\lambda(1-q)\max\Big\{v-p+\delta J(x,T),\bar{v}+\delta J(x-1,t-2)\Big\}+(1-\lambda)\delta J(x-1,t-2). \quad (39)$$

We consider two cases.

Case 1: Suppose $v-p+\delta J(x+1,T)\geq\delta J(x,t-2)$.

Then, (38) reduces to

$$J(x,t-1)=\lambda q\big(v-p+\delta J(x+1,T)\big)$$
$$+\lambda(1-q)\max\Big\{v-p+\delta J(x+1,T),\bar{v}+\delta J(x,t-2)\Big\}+(1-\lambda)\delta J(x,t-2). \quad (40)$$

Subcase 1.1: Suppose $v-p+\delta J(x+1,T)\geq\bar{v}+\delta J(x,t-2)$.

Then, (40) reduces to

$$J(x,t-1)=\lambda\big(v-p+\delta J(x+1,T)\big)+(1-\lambda)\delta J(x,t-2).$$

While, according to (39),

$$J(x-1,t-1)\geq\lambda q\big(v-p+\delta J(x,T)\big)+\lambda(1-q)\big(v-p+\delta J(x,T)\big)+(1-\lambda)\delta J(x-1,t-2)$$
$$=\lambda\big(v-p+\delta J(x,T)\big)+(1-\lambda)\delta J(x-1,t-2).$$

Hence,

$$J(x,t-1)-J(x-1,t-1)\leq\lambda\delta\big(J(x+1,T)-J(x,T)\big)+(1-\lambda)\delta\big(J(x,t-2)-J(x-1,t-2)\big).$$

Therefore,

$$J(x+1,T)-J(x,T)-\big(J(x,t-1)-J(x-1,t-1)\big)$$
$$\geq J(x+1,T)-J(x,T)-\Big\{\lambda\delta\big(J(x+1,T)-J(x,T)\big)+(1-\lambda)\delta\big(J(x,t-2)-J(x-1,t-2)\big)\Big\}$$
$$=(1-\lambda\delta)\big(J(x+1,T)-J(x,T)\big)-(1-\lambda)\delta\big(J(x,t-2)-J(x-1,t-2)\big)$$
$$\geq(1-\lambda\delta)\Big\{J(x+1,T)-J(x,T)-\big(J(x,t-2)-J(x-1,t-2)\big)\Big\}.$$

Clearly, if $J(x+1,T)-J(x,T)\geq J(x,t-2)-J(x-1,t-2)$, then (36) holds immediately.

Subcase 1.2: Suppose $v-p+\delta J(x+1,T)<\bar{v}+\delta J(x,t-2)$.

Then, (40) reduces to

$$J(x,t-1)=\lambda q\big(v-p+\delta J(x+1,T)\big)+\lambda(1-q)\bar{v}+(1-\lambda q)\delta J(x,t-2).$$

While, according to (39),

$$\begin{aligned}J(x-1,t-1) &\geq \lambda q\big(v-p+\delta J(x,T)\big)+\lambda(1-q)\big(\bar{v}+\delta J(x-1,t-2)\big)+(1-\lambda)\delta J(x-1,t-2)\\ &= \lambda q\big(v-p+\delta J(x,T)\big)+\lambda(1-q)\bar{v}+(1-\lambda q)\delta J(x-1,t-2).\end{aligned}$$

Hence,

$$J(x,t-1)-J(x-1,t-1)\leq \lambda q\delta\big(J(x+1,T)-J(x,T)\big)+(1-\lambda q)\delta\big(J(x,t-2)-J(x-1,t-2)\big).$$

Therefore,

$$\begin{aligned}&J(x+1,T)-J(x,T)-\big(J(x,t-1)-J(x-1,t-1)\big)\\ \geq &J(x+1,T)-J(x,T)-\Big\{\lambda q\delta\big(J(x+1,T)-J(x,T)\big)+(1-\lambda q)\delta\big(J(x,t-2)-J(x-1,t-2)\big)\Big\}\\ = &(1-\lambda q\delta)\big(J(x+1,T)-J(x,T)\big)-(1-\lambda q)\delta\big(J(x,t-2)-J(x-1,t-2)\big)\\ \geq &(1-\lambda q\delta)\Big\{J(x+1,T)-J(x,T)-\big(J(x,t-2)-J(x-1,t-2)\big)\Big\}.\end{aligned}$$

Again, if $J(x+1,T)-J(x,T)\geq J(x,t-2)-J(x-1,t-2)$, then (36) holds immediately.

Case 2: Suppose $v-p+\delta J(x+1,T)<\delta J(x,t-2)$.

Then, (38) reduces to

$$J(x,t-1)=\lambda(1-q)\bar{v}+\delta J(x,t-2).$$

While,

$$J(x-1,t-1)\geq \lambda(1-q)\bar{v}+\delta J(x-1,t-2).$$

Thus,

$$J(x,t-1)-J(x-1,t-1)\leq \delta\big(J(x,t-2)-J(x-1,t-2)\big)\leq J(x,t-2)-J(x-1,t-2).$$

Clearly, if $J(x+1,T)-J(x,T)\geq J(x,t-2)-J(x-1,t-2)$, then (36) holds immediately.

We have established that, to show $J(x+1,T)-J(x,T)\geq J(x,t-1)-J(x-1,t-1)$, it suffices to show

$$J(x+1,T)-J(x,T)\geq J(x,t-2)-J(x-1,t-2).$$

Continuing in this fashion, we finally obtain that it suffices to show

$$J(x+1,T)-J(x,T)\geq J(x,0)-J(x-1,0)=J(\Delta)-J(\Delta)=0,$$

which holds automatically because of the monotonicity of $J(\cdot,T)$. This completes the proof. ■

Proof of Proposition 6.

Before the proof of Proposition 6, we introduce two lemmas: Lemmas 11 and 12.

LEMMA 11. *For any fixed price p, if $v-p+\delta J(1,T)\geq \delta J(\Delta)$, then*

(i) $v-p+\delta J(x,T)\geq \delta J(x-1,t-1)$ for any $1\leq x\leq X$ and $1\leq t\leq T$;

(ii) $v+\delta J(\Delta)\geq \delta J(X,t-1)$ for any $1\leq t\leq T$.

Proof of Lemma 11. Since $v-p+\delta J(1,T)\geq \delta J(\Delta)$, Part (i) holds immediately by Proposition 5(a). Now, we show Part (ii). Suppose for a contradiction that $v+\delta J(\Delta)<\delta J(X,t-1)$ for any $1\leq t\leq T$, then

$$\begin{aligned}J(X,t)&=\lambda q\delta J(X,t-1)+\lambda(1-q)\big(\bar v+\delta J(X,t-1)\big)+(1-\lambda)\delta J(X,t-1)\\&=\lambda(1-q)\bar v+\delta J(X,t-1)\\&\leq \lambda(1-q)\bar v+\delta J(X,t).\end{aligned}$$

Thus,

$$J(X,t)\leq \frac{\lambda(1-q)\bar v}{1-\delta}. \tag{41}$$

On the other hand,

$$\begin{aligned}J(\Delta)&\geq \lambda q\delta J(\Delta)+\lambda(1-q)\big(\bar v+\delta J(\Delta)\big)+(1-\lambda)\delta J(\Delta)\\&=\lambda(1-q)\bar v+\delta J(\Delta),\end{aligned}$$

and thus,

$$J(\Delta)\geq \frac{\lambda(1-q)\bar v}{1-\delta}. \tag{42}$$

Hence,

$$v+\delta J(\Delta)>\delta J(X,t)\geq \delta J(X,t-1),$$

where the first inequality holds because of (41) and (42), and the last inequality holds because of the monotonicity of $J(X,\cdot)$. Therefore, we arrive a contradiction. This completes the proof. ■

LEMMA 12. *For any fixed price p, if there exists a threshold $\tau\in\{1,\dots,X-1\}$ such that*

$$\begin{aligned}v-p+\delta J(\tau,T)&<\bar v+\delta J(\tau-1,t-1),\\ v-p+\delta J(\tau+1,T)&\geq \bar v+\delta J(\tau,t-1),\end{aligned}$$

then

$$\begin{aligned}&v-p+\delta J(x,T)<\bar v+\delta J(x-1,t-1), &&\textit{for any } 1\leq x\leq \tau,\\ &v-p+\delta J(x,T)\geq \bar v+\delta J(x-1,t-1), &&\textit{for any } \tau+1\leq x\leq X.\end{aligned}$$

Proof of Lemma 12. It holds immediately because of the monotonicity of $J(x,T) - J(x-1,t-1)$ in Proposition 5(a). ■

Proof of Proposition 6. It suffices to show there exists a range of prices p and a threshold τ (where $1 \leq \tau \leq X-1$) such that

$$v - p + \delta J(1,T) \geq \delta J(\Delta), \tag{43}$$

$$v - p + \delta J(\tau,T) < \bar{v} + \delta J(\tau - 1, t-1), \tag{44}$$

$$v - p + \delta J(\tau + 1,T) \geq \bar{v} + \delta J(\tau, t-1), \tag{45}$$

where (43) implies that the consumer always purchases/redeems when $v_0 = 0$ according to Lemma 11, while (44)-(45), according to Lemma 12, imply that when $v_0 = \bar{v}$, there exists a threshold τ such that the consumer does not purchase in states $\{(x,t) : 0 \leq x \leq \tau - 1\}$ but purchases in states $\{(x,t) : \tau \leq x \leq X-1\}$.

Observe that if there exists a range of prices p such that

$$v - p + \delta J(1,T) < \bar{v} + \delta J(\Delta), \tag{46}$$

$$v - p + \delta J(X,T) \geq \bar{v} + \delta J(X-1, t-1), \tag{47}$$

then, due to the monotonicity of $J(x,T) - J(x-1,t-1)$, there must exist a threshold τ such that (44) and (45) hold for the same price p. Hereafter, we focus on showing there exists a range of prices such that (43), (46), and (47) hold simultaneously; that is, the price needs to satisfy the following constraint

$$v - \bar{v} + \delta J(1,T) - \delta J(\Delta) < p \leq \min\Big\{v + \delta J(1,T) - \delta J(\Delta), v - \bar{v} + \delta J(X,T) - \delta J(X-1,t-1)\Big\}.$$

Clearly,

$$v - \bar{v} + \delta J(1,T) - \delta J(\Delta) < v + \delta J(1,T) - \delta J(\Delta).$$

According to Proposition 5(a),

$$\begin{aligned} J(X,T) - J(X-1,T) &\geq J(X-1,t-1) - J(X-2,t-1) \\ J(X-1,T) - J(X-2,T) &\geq J(X-2,t-1) - J(X-3,t-1) \\ &\vdots \\ J(2,T) - J(1,T) &\geq J(1,t-1) - J(\Delta). \end{aligned}$$

Summing up the above inequalities yields that

$$J(X,T) - J(1,T) \geq J(X-1,t-1) - J(\Delta).$$

Therefore,

$$v - \bar{v} + \delta J(1,T) - \delta J(\Delta) < v - \bar{v} + \delta J(X,T) - \delta J(X-1,t-1).$$

This completes the proof. ■

Proof of Proposition 7.

Suppose

$$v - p + \delta J(T) \geq \delta J(0), \tag{48}$$
$$v - p + \delta J(T) < \bar{v} + \delta J(0), \tag{49}$$
$$v + \delta J(0) \geq \bar{v} + \delta J(T-1), \tag{50}$$

then the consumer always purchases/redeems when $v_0 = 0$, and does not purchase but always redeems when $v_0 = \bar{v}$. Next, we focus on deriving the conditions of the price p.

Given the consumer's behavior, the dynamic programming equations (9) and (10) reduce to

$$J(t) = \lambda\big(v + \delta J(0)\big) + (1-\lambda)\delta J(t-1),$$
$$J(0) = \lambda q\big(v - p + \delta J(T)\big) + \lambda(1-q)\bar{v} + (1-\lambda q)\delta J(0).$$

Solving the above set of equations yields that

$$J(0) = \frac{\lambda\{q(v-p) + (1-q)\bar{v}\}\{1-(1-\lambda)\delta\} + \lambda^2 q\delta\{1-\big((1-\lambda)\delta\big)^T\}v}{\{1-(1-\lambda)\delta\}\{1-(1-\lambda q)\delta\} - \lambda^2 q\delta^2 - \lambda q\delta(1-\delta)\big((1-\lambda)\delta\big)^T},$$
$$J(T) = \frac{1-\big((1-\lambda)\delta\big)^T}{1-(1-\lambda)\delta}\lambda v + \frac{\lambda\delta + (1-\delta)\big((1-\lambda)\delta\big)^T}{1-(1-\lambda)\delta}J(0),$$
$$J(T-1) = \frac{1-\big((1-\lambda)\delta\big)^{T-1}}{1-(1-\lambda)\delta}\lambda v + \frac{\lambda\delta + (1-\delta)\big((1-\lambda)\delta\big)^{T-1}}{1-(1-\lambda)\delta}J(0).$$

Putting back to (48), (49), and (50) gives that

$$p \leq v + \frac{\lambda\delta\{1-\big((1-\lambda)\delta\big)^T\}\{v-(1-q)\bar{v}\}}{1-(1-\lambda)\delta},$$
$$p > \frac{1-\delta+2\lambda\delta-\lambda\delta\big((1-\lambda)\delta\big)^T}{1-(1-\lambda)\delta}(v-\bar{v}),$$
$$p \leq \frac{1-\delta+2\lambda\delta q+\lambda\delta\big((1-\lambda)\delta\big)^{T-1}\{1-q-\delta q(1-\lambda)\}}{\lambda q\delta\{1-\big((1-\lambda)\delta\big)^{T-1}\}}(v-\bar{v}),$$

respectively. This completes the proof. ■

Proof of Proposition 8.

Before the proof of Proposition 8, we first characterize a consumer's state transition, write down and solve the balance equations, and formulate the optimization problem. Figure 8 characterizes the consumer's state transition when the consumer behaves as in Proposition 7. Let q_i denote the stationary probability of state i. The balance equations can be written as follows,

$$\lambda q q_0 = \lambda(q_1 + q_2 + \cdots + q_T) + (1-\lambda)q_1,$$

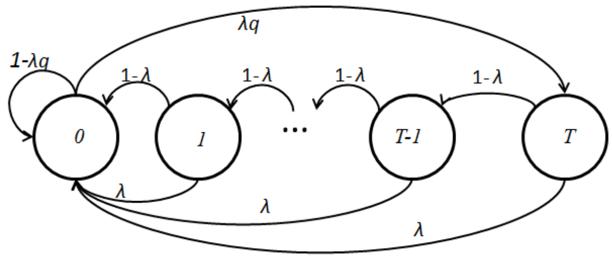

Figure 8 State transition in "Buy 1, Get 1 Free" with a general T

$$
\begin{aligned}
&q_i = (1-\lambda)q_{i+1}, && \text{for} \quad i = 1,2,\dots,T-1,\\
&q_T = \lambda q q_0,\\
&\sum_{i=0}^{T} q_i = 1.
\end{aligned}
$$

Solving the above set of equations gives the stationary probabilities:

$$
\begin{aligned}
q_0 &= \frac{1}{1+q[1-(1-\lambda)^T]},\\
q_i &= \frac{\lambda q(1-\lambda)^{T-i}}{1+q[1-(1-\lambda)^T]}, && \forall i = 1,2,\dots,T.
\end{aligned}
$$

Since the consumer makes a purchase only in state 0 when the outside option is low and does not purchase when the outside option is high, the long run average revenue collected from the consumer is $\lambda q q_0 p = \frac{\lambda q}{1+q[1-(1-\lambda)^T]}p$. Hence, the seller faces the following optimization problem,

$$
\begin{aligned}
&\max_p \frac{\lambda q}{1+q[1-(1-\lambda)^T]}p && (51)\\
s.t. \quad & p \le v + \frac{\lambda\delta\{1-[(1-\lambda)\delta]^T\}\{v-(1-q)\bar{v}\}}{1-(1-\lambda)\delta},\\
& p > \frac{1-\delta+2\lambda\delta-\lambda\delta[(1-\lambda)\delta]^T}{1-(1-\lambda)\delta}(v-\bar{v}),\\
& p \le \frac{1-\delta+2\lambda\delta q+\lambda\delta[(1-\lambda)\delta]^{T-1}\{1-q-\delta q(1-\lambda)\}}{\lambda q\delta\{1-[(1-\lambda)\delta]^{T-1}\}}(v-\bar{v}).
\end{aligned}
$$

Proof of Proposition 8. We first derive π^{BO}. According to the constraints in Problem (51), one can check if

$$
v \le \left\{ \frac{(1-\delta+\lambda\delta)\Big\{1-\delta+2\lambda\delta q+\lambda\delta[(1-\lambda)\delta]^{T-1}[1-q-\delta q(1-\lambda)]\Big\}}{\lambda\delta q\big\{1-[(1-\lambda)\delta]^{T-1}\big\}\Big\{1-(1-\lambda)\delta+\lambda\delta q\big\{1-[(1-\lambda)\delta]^T\big\}\Big\}} \right.
$$
$$
\left. - \frac{\lambda^2\delta^2 q(1-q)\big\{1-[(1-\lambda)\delta]^T\big\}\big\{1-[(1-\lambda)\delta]^{T-1}\big\}}{\lambda\delta q\big\{1-[(1-\lambda)\delta]^{T-1}\big\}\Big\{1-(1-\lambda)\delta+\lambda\delta q\big\{1-[(1-\lambda)\delta]^T\big\}\Big\}} \right\}(v-\bar{v}),
$$

then

$$\begin{aligned}
p^{BO} =& v + \frac{\lambda\delta\{1-[(1-\lambda)\delta]^T\}\{v-(1-q)\bar{v}\}}{1-(1-\lambda)\delta},\\
\pi^{BO} =& \frac{\lambda q}{1+q[1-(1-\lambda)^T]}\left\{v + \frac{\lambda\delta\{1-[(1-\lambda)\delta]^T\}\{v-(1-q)\bar{v}\}}{1-(1-\lambda)\delta}\right\}\\
=& \lambda qv \frac{1-(1-\lambda)\delta+\lambda\delta q\{1-[(1-\lambda)\delta]^T\}}{1-(1-\lambda)\delta+q\{1-(1-\lambda)\delta\}\{1-(1-\lambda)^T\}}\\
&+\lambda(1-q)(v-\bar{v})\frac{\lambda\delta q\{1-[(1-\lambda)\delta]^T\}}{1-(1-\lambda)\delta+q\{1-(1-\lambda)\delta\}\{1-(1-\lambda)^T\}}\\
=& \lambda qv \frac{1+\lambda q\delta\frac{1-[(1-\lambda)\delta]^T}{1-(1-\lambda)\delta}}{1+q\{1-(1-\lambda)^T\}} + \lambda(1-q)(v-\bar{v})\frac{\lambda q\delta\frac{1-[(1-\lambda)\delta]^T}{1-(1-\lambda)\delta}}{1+q\{1-(1-\lambda)^T\}}.
\end{aligned}$$

As $\delta \to 1$,

$$\pi^{BO} \to \lambda qv + \lambda(1-q)(v-\bar{v})\frac{q\{1-(1-\lambda)^T\}}{1+q\{1-(1-\lambda)^T\}},$$

which is, clearly, greater than λqv. Note that π^{BO} increases in δ, hence, if δ is sufficiently close to 1,

$$\pi^{BO} \geq \lambda qv = \max\{\lambda qv, \lambda(v-\bar{v})\},$$

where the last equality holds because $\frac{v-\bar{v}}{q} \leq v$. This completes the proof. ■