Signal, Image and Video Processing

Performing Fractional Delay via Fractional Singular Spectrum Analysis

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***Abstract*—This paper proposes a fractional singular spectrum analysis (SSA) based method for performing the fractional delay. First, the input sequence is divided into two overlapping sequences with the first sequence being the input sequence without its last point and the second sequence being the input sequence without its first point. Then, the singular value decompositions (SVD) are performed on the trajectory matrices constructed based on these two sequences. Next, the designs of both the right unitary matrix and the left unitary matrix for generating the new trajectory matrix are formulated as the quadratically constrained quadratic programing problems. The analytical solutions of these quadratically constrained quadratic programing problems are derived via the SVD approach. Finally, the fractional SSA components are obtained by performing the diagonal averaging operation and the fractional delay sequence is obtained by summing up all the fractional SSA components together. Since the fractional SSA operations are nonlinear and adaptive, our proposed method is a kind of nonlinear and adaptive approach for performing the fractional delay. Besides, by discarding some fractional SSA components, the joint fractional delay operation and the denoising operation can be performed simultaneously.**

***Keywords*—Fractional singular spectrum analysis, fractional delay, quadratically constrained quadratic programing.**

1. Introduction

For discrete time signals, their time indices are integer valued. Hence, the values of the signals between two consecutive time indices are unknown. On the other hand, the fractional delay of a signal is its delay with a non-integer sample. For the ideal case, the group delay of the signal is a constant. Hence, these intermediate values can be estimated [1]. It is an important topic because it finds a wide range of applications in the digital signal processing and the digital communications such as the time adjustment in digital receivers, speech coding, time delay estimation and modeling of musical instruments [2]-[7].

Conventional fractional delay systems are allpass linear phase finite impulse response filters [8]. As they are linear and non-adaptive [9], the choice of the predefined filters has a great effect on the acquired samples. Moreover, as the filters are allpass, these filters are with the low noise immunities. Nevertheless, the signals in many practical systems such as in the communication systems are corrupted by the noises [10]. Hence, the fractional delay systems yield very noisy samples for these practical applications.

Besides, the rate changers [11] are also used to acquire the fractional delay samples. First, the input sequence is upsampled. Then, the simple lowpass filtering is applied to the upsampled sequence. Finally, the downsampling is applied to the filtered sequence [12]. It is a remarkable fact that the rate changer method is the generalization of the existing linear interpolation approaches [13]. This is because using different filters in the rate changers correspond to different linear interpolations. Although the filter in the rate changer can suppress the noise, this approach is also linear and non-adaptive. Hence, the choice of the predefined filters still affects the acquired samples.

In the recent decades, the SSA [14] is widely studied in the signal processing community. The input sequence is divided into overlapping blocks and there is only one sample difference between two consecutive blocks. These blocks are used to form the columns of a matrix called the trajectory matrix. Then, the SVD is applied on the trajectory matrix. The product of an eigenvalue, the corresponding column in the left unitary trajectory and the transpose of the corresponding column in the right unitary matrix forms the two dimensional SSA component. By performing the diagonal averaging operation, the one dimensional SSA components are obtained [15]. It is a remarkable fact that the original input sequence can be expressed as the sum of these one dimensional SSA components [14]. Since different components have different characteristics, denoising can be performed by discarding some

SSA components [19]-[22]. As this is a kind of nonlinear and adaptive signal representation methods, the SSA technique finds in many engineering and science applications [16]-[18].

However, as the elements in the trajectory matrix are obtained directly from the input sequence, the conventional SSA based methods do not help for acquiring the samples between two consecutive time indices. To solve this problem, this paper proposes to divide the input sequences into two overlapping sequences and there is only one sample difference between these two sequences. By constructing two trajectory matrices corresponding to these two sequences and performing the SVD on these two trajectory matrices, a new trajectory matrix corresponding to the fractional delay sequence is obtained.

The outline of this paper is organized as follows. Section 2 presents our proposed fractional SSA based method for performing the fractional delay. In Section 3, two examples are shown to illustrate the effectiveness of our proposed method. At the end, a conclusion is summarized in Section 4.

2. Fractional SSA based method for performing fractional delay

2.1. Notations

Denote an input sequence as $x(n)$. Let N be its length. Divide $x(n)$ into two overlapped sequences with the first sequence being the first $N-1$ points of $x(n)$ and the second sequence being the last $N-1$ points of $x(n)$. Let the vectors of these two sequences be

$$\mathbf{x}_A = \begin{bmatrix} x(0) & \cdots & x(N-2) \end{bmatrix}^T \in \Re^{N-1} \tag{1}$$

and

$$\mathbf{x}_B = \begin{bmatrix} x(1) & \cdots & x(N-1) \end{bmatrix}^T \in \Re^{N-1}, \tag{2}$$

respectively. Denote the trajectory matrices corresponding to $\mathbf{x}_A$ and $\mathbf{x}_B$ as

$$\mathbf{X}_A = \begin{bmatrix} x(0) & x(1) & \cdots & x(N-L-1) \\ \vdots & \vdots & \ddots & \vdots \\ x(L-1) & x(L) & \cdots & x(N-2) \end{bmatrix} \in \Re^{L\times(N-L)} \tag{3}$$

and

$$\mathbf{X}_B = \begin{bmatrix} x(1) & x(2) & \cdots & x(N-L) \\ \vdots & \vdots & \ddots & \vdots \\ x(L) & x(L+1) & \cdots & x(N-1) \end{bmatrix} \in \Re^{L\times(N-L)}, \tag{4}$$

respectively. Here, $\Re^{a\times b}$ represents the set of real valued matrices and the dimensions of these matrices are *a* by *b*.

After applying the SVD on $\mathbf{X}_A$, the left unitary matrix $\mathbf{U}_A \in \Re^{L\times L}$, the right unitary matrix $\mathbf{V}_A \in \Re^{(N-L)\times(N-L)}$ and the diagonal matrix $\mathbf{\Lambda}_A \in \Re^{L\times L}$ are acquired simultaneously. That is,

$$\mathbf{X}_A = \mathbf{U}_A \begin{bmatrix} \mathbf{\Lambda}_A & \mathbf{0}_{L\times(N-2L)} \end{bmatrix} \mathbf{V}_A^H . \tag{5}$$

Here, $\mathbf{0}_{a\times b}$ represents the zero matrix with the size $a\times b$ and the conjugate transposition operator is represented by the superscript "H". Likewise, after applying the SVD on $\mathbf{X}_B$, the left unitary matrix $\mathbf{U}_B \in \Re^{L\times L}$, the right unitary matrix $\mathbf{V}_B \in \Re^{(N-L)\times(N-L)}$ and the diagonal matrix $\mathbf{\Lambda}_B \in \Re^{L\times L}$ are acquired simultaneously. That is,

$$\mathbf{X}_B = \mathbf{U}_B \begin{bmatrix} \mathbf{\Lambda}_B & \mathbf{0}_{L\times(N-2L)} \end{bmatrix} \mathbf{V}_B^H . \tag{6}$$

2.2. Formulation of the new trajectory matrix corresponding to the fractional delay sequence

Define $\tilde{\mathbf{U}}$ and $\tilde{\mathbf{V}}$ as two unitary matrices. Denote the columns of $\tilde{\mathbf{U}}$ and the columns of $\tilde{\mathbf{V}}$ as $\tilde{\mathbf{u}}_i \in \Re^L$ for $i=0,\cdots,L-1$ and $\tilde{\mathbf{v}}_i \in \Re^{N-L}$ for $i=0,\cdots,N-L$, respectively. That is,

$$\tilde{\mathbf{U}} = \begin{bmatrix} \tilde{\mathbf{u}}_0 & \cdots & \tilde{\mathbf{u}}_{L-1} \end{bmatrix} \tag{7}$$

and

$$\tilde{\mathbf{V}} = \begin{bmatrix} \tilde{\mathbf{v}}_0 & \cdots & \tilde{\mathbf{v}}_{N-L-1} \end{bmatrix}. \tag{8}$$

$\forall \gamma \in [0,1]$, our objective is to find both $\tilde{\mathbf{U}}$ and $\tilde{\mathbf{V}}$ such that $\tilde{\mathbf{U}}$ and $\tilde{\mathbf{V}}$ are closed to $\gamma\mathbf{U}_A + (1-\gamma)\mathbf{U}_B$ and $\gamma\mathbf{V}_A + (1-\gamma)\mathbf{V}_B$, respectively. Here, $\tilde{\mathbf{U}}$ can be understood as the fraction between $\mathbf{U}_A$ and $\mathbf{U}_B$. Likewise, $\tilde{\mathbf{V}}$ can be understood as the fraction between $\mathbf{V}_A$ and $\mathbf{V}_B$. Let $trace(\mathbf{Z})$ be the trace of $\mathbf{Z}$ and $\mathbf{I}_L$ be the $L\times L$ identity matrix. Then, the design of $\tilde{\mathbf{U}}$ can be formulated as the following optimization problem:

Problem (P_U)

$$\min_{\tilde{\mathbf{U}}} \quad J_U(\tilde{\mathbf{U}}) = trace\left(\left(\gamma\mathbf{U}_A + (1-\gamma)\mathbf{U}_B - \tilde{\mathbf{U}}\right)^T \left(\gamma\mathbf{U}_A + (1-\gamma)\mathbf{U}_B - \tilde{\mathbf{U}}\right) \right), \tag{9a}$$

subject to $\tilde{\mathbf{U}}^T\tilde{\mathbf{U}} = \mathbf{I}_L$. (9b)

It is important to note that the set of unitary matrices are nonconvex. Hence, Problem (P_U) is a nonconvex optimization problem [23]. Hence, it is very hard to obtain its global optimal solution. To address this difficulty, the SVD approach [24] is employed. Let

$$\begin{bmatrix} \tilde{\mathbf{u}}_0^\bullet & \cdots & \tilde{\mathbf{u}}_{L-1}^\bullet \end{bmatrix} = \gamma\mathbf{U}_A + (1-\gamma)\mathbf{U}_B . \tag{10}$$

Let the matrix containing the Lagrange multipliers be

$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda_{0,0} & \cdots & \lambda_{0,L-1} \\ \vdots & \ddots & \vdots \\ \lambda_{L-1,0} & \cdots & \lambda_{L-1,L-1} \end{bmatrix}. \tag{11}$$

Let

$$\delta(z) = \begin{cases} 1 & z = 0 \\ 0 & z \neq 0 \end{cases} \tag{12}$$

be the discrete time delta function. Then, the unconstrained optimization problem corresponding to Problem (P_U) is:

$$\min_{(\tilde{\mathbf{U}},\boldsymbol{\lambda})} \quad L_U(\tilde{\mathbf{U}},\boldsymbol{\lambda}) = trace\left(\left(\gamma\mathbf{U}_A + (1-\gamma)\mathbf{U}_B - \tilde{\mathbf{U}}\right)^T \left(\gamma\mathbf{U}_A + (1-\gamma)\mathbf{U}_B - \tilde{\mathbf{U}}\right) \right) - \sum_{p=0}^{L-1}\sum_{q=0}^{L-1} \lambda_{p,q}\left(\tilde{\mathbf{u}}_p^T\tilde{\mathbf{u}}_q - \delta(p-q)\right) . \tag{13}$$

Since

$$L_U\left(\tilde{\mathbf{U}},\boldsymbol{\lambda}\right)=\sum_{l=0}^{L-1}\left(\tilde{\mathbf{u}}_l^T\tilde{\mathbf{u}}_l-2\tilde{\mathbf{u}}_l^{\bullet T}\tilde{\mathbf{u}}_l+\tilde{\mathbf{u}}_l^{\bullet T}\tilde{\mathbf{u}}_l^{\bullet}\right) \\ -\sum_{p=0}^{L-1}\sum_{q=0}^{L-1}\lambda_{p,q}\left(\tilde{\mathbf{u}}_p^T\tilde{\mathbf{u}}_q-\delta\left(p-q\right)\right), \tag{14}$$

we have:

$$\frac{\partial}{\partial\tilde{\mathbf{u}}_l}L_U\left(\tilde{\mathbf{U}},\boldsymbol{\lambda}\right)=2\tilde{\mathbf{u}}_l-2\tilde{\mathbf{u}}_l^{\bullet}-\sum_{p=0}^{L-1}\left(\lambda_{p,l}+\lambda_{l,p}\right)\tilde{\mathbf{u}}_p \tag{15}$$

for $l=0,\cdots,N-1$. This implies that

$$\tilde{\mathbf{u}}_l-\frac{1}{2}\sum_{p=0}^{L-1}\left(\lambda_{p,l}+\lambda_{l,p}\right)\tilde{\mathbf{u}}_p=\tilde{\mathbf{u}}_l^{\bullet} \tag{16}$$

for $l=0,\cdots,N-1$. Hence, we have:

$$\tilde{\mathbf{U}}-\frac{1}{2}\tilde{\mathbf{U}}\left(\boldsymbol{\lambda}+\boldsymbol{\lambda}^T\right)=\gamma\mathbf{U}_A+\left(1-\gamma\right)\mathbf{U}_B. \tag{17}$$

In other words, we have

$$\tilde{\mathbf{U}}\left(\mathbf{I}_L-\frac{1}{2}\left(\boldsymbol{\lambda}+\boldsymbol{\lambda}^T\right)\right)=\gamma\mathbf{U}_A+\left(1-\gamma\right)\mathbf{U}_B. \tag{18}$$

After applying the SVD on $\gamma\mathbf{U}_A+\left(1-\gamma\right)\mathbf{U}_B$, the left unitary matrix $\mathbf{U}_\gamma^{\bullet}\in\Re^{L\times L}$, the right unitary matrix $\mathbf{V}_\gamma^{\bullet}\in\Re^{L\times L}$ and the diagonal matrix $\boldsymbol{\Lambda}_\gamma^{\bullet}\in\Re^{L\times L}$ are acquired simultaneously. On the other hand, it is well known that after applying the SVD on a symmetric matrix, the right unitary matrix and the left unitary matrix are the same. As $\mathbf{I}_L-\frac{1}{2}\left(\boldsymbol{\lambda}+\boldsymbol{\lambda}^T\right)$ is a symmetric matrix, the right unitary matrix and the left unitary matrix acquired via applying the SVD on $\mathbf{I}_L-\frac{1}{2}\left(\boldsymbol{\lambda}+\boldsymbol{\lambda}^T\right)$ are the same. Let this unitary matrix be $\mathbf{U}_\lambda\in\Re^{L\times L}$. Also, the diagonal matrix $\boldsymbol{\Lambda}_\lambda\in\Re^{L\times L}$ is simultaneously obtained after applying the SVD on $\mathbf{I}_L-\frac{1}{2}\left(\boldsymbol{\lambda}+\boldsymbol{\lambda}^T\right)$. Hence, we have

$$\tilde{\mathbf{U}}\mathbf{U}_\lambda\boldsymbol{\Lambda}_\lambda\mathbf{U}_\lambda^T=\mathbf{U}_\gamma^{\bullet}\boldsymbol{\Lambda}_\gamma^{\bullet}\mathbf{V}_\gamma^{\bullet T}. \tag{19}$$

This implies that

$$\mathbf{U}_\lambda=\mathbf{V}_\gamma^{\bullet}, \tag{20}$$

$$\boldsymbol{\Lambda}_\lambda=\boldsymbol{\Lambda}_\gamma^{\bullet} \tag{21}$$

and

$$\tilde{\mathbf{U}}\mathbf{U}_\lambda=\mathbf{U}_\gamma^{\bullet}. \tag{22}$$

This further implies that

$$\tilde{\mathbf{U}}=\mathbf{U}_\gamma^{\bullet}\mathbf{U}_\lambda^T=\mathbf{U}_\gamma^{\bullet}\mathbf{V}_\gamma^{\bullet T}. \tag{23}$$

Likewise, $\tilde{\mathbf{V}}$ can be obtained in a similar way. Define

$$\tilde{\boldsymbol{\Lambda}}=\gamma\boldsymbol{\Lambda}_A+\left(1-\gamma\right)\boldsymbol{\Lambda}_B. \tag{24}$$

Here, $\tilde{\boldsymbol{\Lambda}}$ can be understood as the fraction between $\boldsymbol{\Lambda}_A$ and $\boldsymbol{\Lambda}_B$. Then, the trajectory matrix of the fractional SSA is defined as

$$\tilde{\mathbf{X}}=\tilde{\mathbf{U}}\left[\tilde{\boldsymbol{\Lambda}}\quad\mathbf{0}_{L\times(N-2L)}\right]\tilde{\mathbf{V}}^H. \tag{25}$$

Define

$$\tilde{\mathbf{u}}_i=\left[\tilde{u}_{i,0}\quad\cdots\quad\tilde{u}_{i,L-1}\right]^T \tag{26}$$

for $i=0,\cdots,L-1$ and

$$\tilde{\mathbf{v}}_i=\left[\tilde{v}_{i,0}\quad\cdots\quad\tilde{v}_{i,N-L-1}\right]^T \tag{27}$$

for $i=0,\cdots,N-L-1$. Also, denote the diagonal elements of $\tilde{\boldsymbol{\Lambda}}$ as $\tilde{\lambda}_i$ for $i=0,\cdots,L-1$. Define

$$\tilde{\mathbf{X}}_i=\tilde{\lambda}_i\tilde{\mathbf{u}}_i\tilde{\mathbf{v}}_i^H=\tilde{\lambda}_i\begin{bmatrix}\tilde{u}_{i,0}\\ \vdots\\ \tilde{u}_{i,L-1}\end{bmatrix}\left[\tilde{v}_{i,0}\quad\cdots\quad\tilde{v}_{i,N-L-1}\right]^H \\ =\begin{bmatrix}\tilde{\lambda}_i\tilde{u}_{i,0}\tilde{v}_{i,0}^* & \cdots & \tilde{\lambda}_i\tilde{u}_{i,0}\tilde{v}_{i,N-L-1}^*\\ \vdots & \ddots & \vdots\\ \tilde{\lambda}_i\tilde{u}_{i,L-1}\tilde{v}_{i,0}^* & \cdots & \tilde{\lambda}_i\tilde{u}_{i,L-1}\tilde{v}_{i,N-L-1}^*\end{bmatrix}\in\Re^{L\times(N-L)} \tag{28}$$

for $i=0,\cdots,L-1$ as the two dimensional fractional SSA components. In our work, the conjugate operator is represented as the superscript "*". It is no hard to see that

$$\tilde{\mathbf{X}}=\sum_{i=0}^{L-1}\tilde{\mathbf{X}}_i. \tag{29}$$

Let the j^{th} off-diagonal of $\tilde{\mathbf{X}}_i$ be $\mathbf{y}_{i,j}$ for $i=0,\cdots,L-1$ and for $j=0,\cdots,N-2$. Define $\mu_{i,j}$ for $i=0,\cdots,L-1$ and for $j=0,\cdots,N-2$ as the average value across all the elements in $\mathbf{y}_{i,j}$ based on the de-Hankelization approach. That is, to perform the conventional diagonal averaging. Here, the elements of the one dimensional fractional SSA components are defined as:

$$\mu_{i,k}=\begin{cases}\dfrac{\sum_{j=0}^{k}\tilde{\lambda}_i\tilde{u}_{i,j}\tilde{v}_{i,k-j}^*}{k+1} & 0\le k\le L-1\\ \dfrac{\sum_{j=0}^{L-1}\tilde{\lambda}_i\tilde{u}_{i,j}\tilde{v}_{i,k-j}^*}{L} & L\le k\le N-L\\ \dfrac{\sum_{j=k-N+L}^{L-1}\tilde{\lambda}_i\tilde{u}_{i,j}\tilde{v}_{i,k-j}^*}{N-k} & N-L+1\le k\le N-2\end{cases} \tag{30}$$

for $i=0,\cdots,L-1$ and for $0\le k\le N-2$. Denote the i^{th} one dimensional fractional SSA component as

$$\boldsymbol{\mu}_i=\left[\mu_{i,0}\quad\cdots\quad\mu_{i,N-2}\right]^T\in\Re^{N-1} \tag{31}$$

for $i=0,\cdots,L-1$.

3. Computer numerical simulation results

Since the SSA is a time frequency analysis technique, a time frequency analysis based method should be compared. Nevertheless, there is no similar nonlinear adaptive time frequency analysis based interpolation method. Hence, a linear non-adaptive time frequency analysis based interpolation method is compared instead. As our approach only interpolates one point between two consecutive points in the signal, the same case is performed in the conventional linear interpolation approach. For the simplicity reason, the weighted sum between two consecutive points of the signal is employed as the interpolation point in the linear interpolation approach. In order to have a fair comparison, the weight is selected as the value of γ.

For our proposed fractional SSA based method, a small value of L, say $L=10$, is chosen. This is because all the fractional SSA components can be demonstrated easily. In this session, two examples are used for an illustration. For the first example, the original input sequence is chosen as the sinusoidal waveform. That is,

$$x(n)=\sin(\omega_0 n). \quad (32)$$

This input sequence is chosen because the sinusoidal waveform is widely used in the digital communication system and the delay is always happened in the communication channel. Here, $N=1001$ is chosen. This is because the infinite length sequence can be approximated by the finite length sequence if $N>>L$, say $N>100L$. Besides, ω_0 is denoted as the angular velocity of $x(n)$. In particular,

$$\omega_0=\frac{20\pi}{N-1} \quad (33)$$

is chosen because 10 cycles can be demonstrated graphically.

Let $\hat{x}(n)$ be the acquired sequence. Also, let $\sigma(n)$ be the zero mean unit variance identically and independently Gaussian distributed noise. It is assumed that $\hat{x}(n)$ is additively corrupted by the above type of noises with the noise power denoted as P. That is:

$$\hat{x}(n)=x(n)+\sqrt{P}\sigma(n) \text{ for } n=0,\cdots,N-1. \quad (34)$$

Suppose that only some of the fractional SSA components are summed together to reconstruct the original input sequence. Let $\tilde{x}(n)$ be the reconstructed sequence. Let $\tilde{X}(\omega)$, $\hat{X}(\omega)$ and $X(\omega)$ be the discrete time Fourier transforms of $\tilde{x}(n)$, $\hat{x}(n)$ and $x(n)$, respectively.

Define Q as the ratio of the energy of the original input sequence to the energy of the difference between the magnitude of the reconstructed sequence and the magnitude of the original input sequence. That is:

$$Q=\frac{\int_{-\pi}^{\pi}|X(\omega)|^2 d\omega}{\int_{-\pi}^{\pi}\left\||\tilde{X}(\omega)|-|X(\omega)|\right\|^2 d\omega}. \quad (35)$$

Since an ideal fractional delay system only modifies the phase response of the input sequence, the magnitude response of the input sequence is not affected. Hence, Q is employed as the performance metric to evaluate the performance of various methods.

In order to determine which fractional SSA components are chosen for our proposed fractional SSA based method, first find a fractional SSA component such that the value of Q is maximized. In this case, this fractional SSA component is selected. Then, find a new component and add this new component to the selected component such that the new value of Q is maximized. If the value of Q is increased, then this new component is also selected and the above procedures are repeated. Otherwise, this new selected component is discarded and the algorithm is terminated. It is found that only the first fractional SSA component is selected. To account for this phenomenon, Figure 1 shows the fractional SSA components obtained by our proposed fractional SSA based method when $\gamma=0.3$ and $P=0.05$. It can be seen from Figure 1 that the first fractional SSA component retains the shape of the waveform of the original input sequence. On the other hand, the rest fractional SSA components contain a lot of noises. Hence, only the first fractional SSA component is selected.

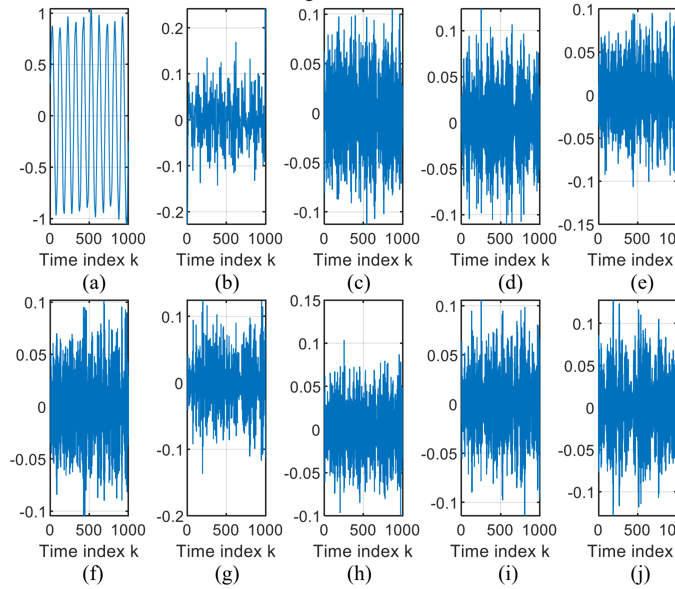


Figure 1. (a) The first fractional SSA component, (b) the second fractional SSA component, (c) the third fractional SSA component, (d) the fourth fractional SSA component, (e) the fifth fractional SSA component, (f) the sixth fractional SSA component, (g) the seventh fractional SSA component, (h) the eighth fractional SSA component, (i) the ninth fractional SSA component and (j) the tenth fractional SSA component when $\gamma=0.3$ and $P=0.05$.

Besides, Figure 2 shows the sequences reconstructed by both the linear interpolation based method and our proposed fractional SSA based method. Although the sequence reconstructed by the linear interpolation based method preserves the shape of the waveform of the original input sequence, the reconstructed sequence is severely corrupted by the noise. On the other hand, the sequence reconstructed by our proposed fractional SSA based method preserves the shape of the waveform of the original input sequence without severely corrupted by the noise. This demonstrates that our proposed fractional SSA based method has a higher noise immunity ability than the linear interpolation based method.

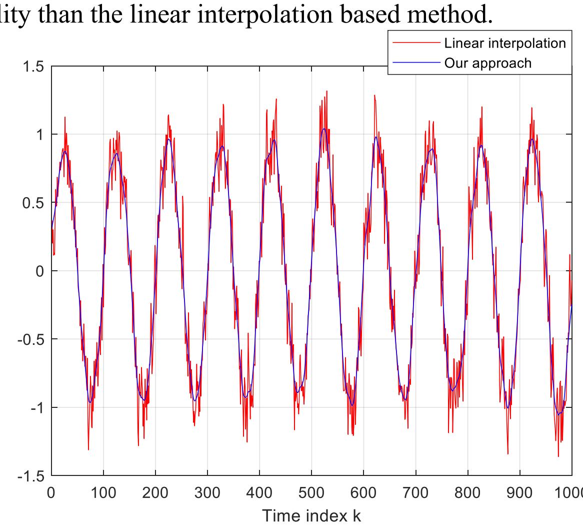


Figure 2. The sequences reconstructed by various methods when $\gamma=0.3$ and $P=0.05$.

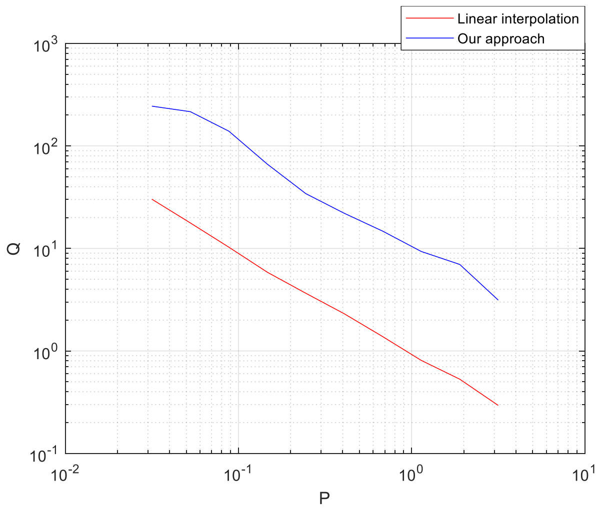
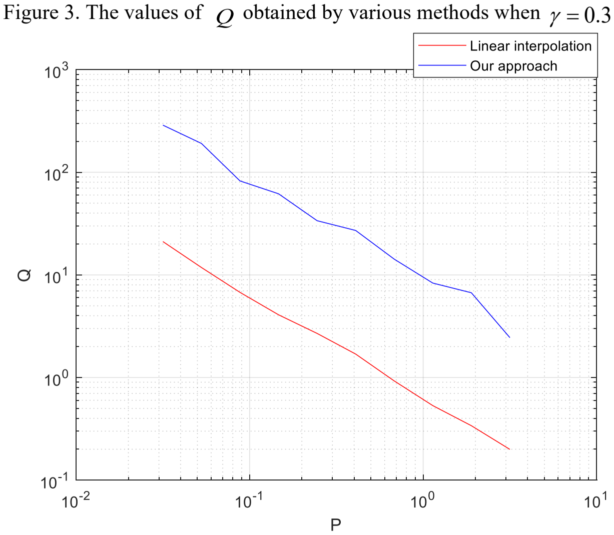
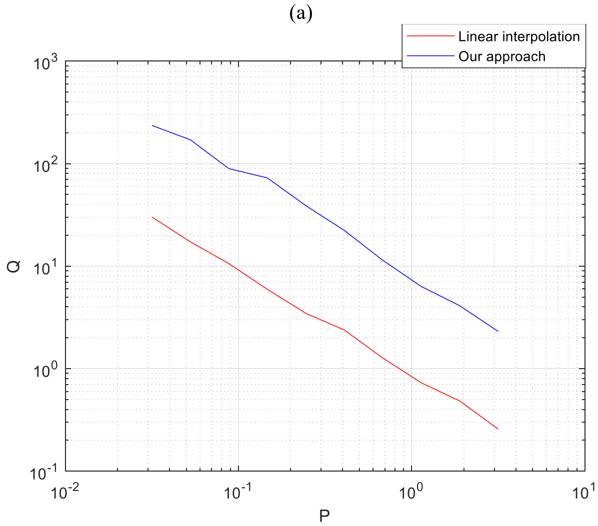


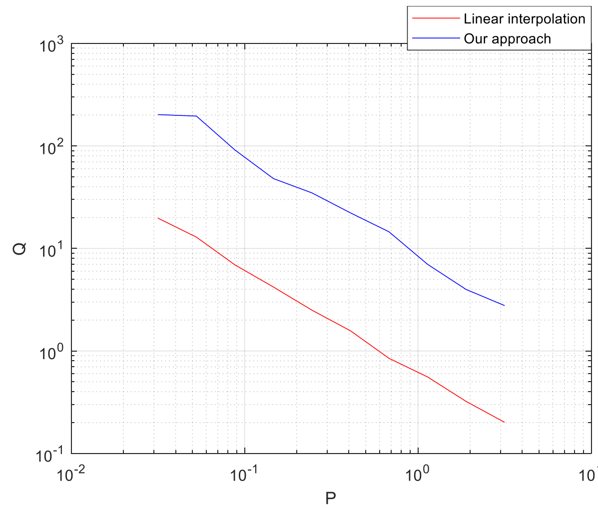
Figure 3. The values of Q obtained by various methods when $\gamma = 0.3$.



(a)



(b)



(c)

Figure 4. The effects of P on the values of Q at various va1lues of γ. (a) $\gamma = 0.1$. (b) $\gamma = 0.7$. (c) $\gamma = 0.9$.

In order to evaluate the performances of our proposed fractional SSA based method at various noise levels, Figure 3 shows the values of Q at different values of P when $\gamma = 0.3$. From Figure 3, it is not hard to see that our proposed fractional SSA based method can always achieve higher values of Q than the linear interpolation based method. In fact, our proposed fractional SSA based method achieves an average of 10.3694dB improvement compared to the linear interpolation based method. This demonstrates the outperformance of our proposed fractional SSA based method.

In order to evaluate the performances of our proposed fractional SSA based method at various values of γ, Figure 4 presents the effects of P on the values of Q at various values of γ. From Figure 4, it is not hard to see that the effects of P on the values of Q are similar for different values of γ. This demonstrates the robustness of our proposed fractional SSA based method.

Now, consider an example of an electrocardiogram. It is worth noting that the electrocardiogram has a sudden jump in the QS duration. This part of signal consists of a wide frequency range. In practice, the electrocardiogram is corrupted by the noise generated by the acquisition system and the background environment. However, the noise characteristic is unknown. To model the noise, first the empirical mode decomposition is applied to the noisy electrocardiogram. For this realization of the electrocardiogram, it is found that there are eight intrinsic mode functions. The last five intrinsic mode functions are summed together to obtain the denoised electrocardiogram. Then, the zero mean unit variance identically and independently Gaussian distributed noise is added to denoised electrocardiogram.

In order to demonstrate the robustness of our proposed method, the same set of the parameters employed in the above example is used here. That is, $L = 10$, $N = 1001$, $\gamma = 0.3$ and $P = 0.05$. Figure 5 shows the sequences reconstructed by both the linear interpolation based method and our proposed fractional SSA based method. Likewise, the sequence reconstructed by the linear interpolation approach is severely corrupted by the noise. On the other hand, the sequence reconstructed by our proposed fractional SSA based method preserves the shape of the waveform of the original input sequence without severely corrupted by the noise.

Figure 6 shows the values of Q at different values of P when $\gamma = 0.3$. From Figure 6, it is not hard to see that our proposed fractional SSA based method can always achieve higher values of Q than the linear interpolation based method. This demonstrates the effectiveness of our fractional SSA based method.

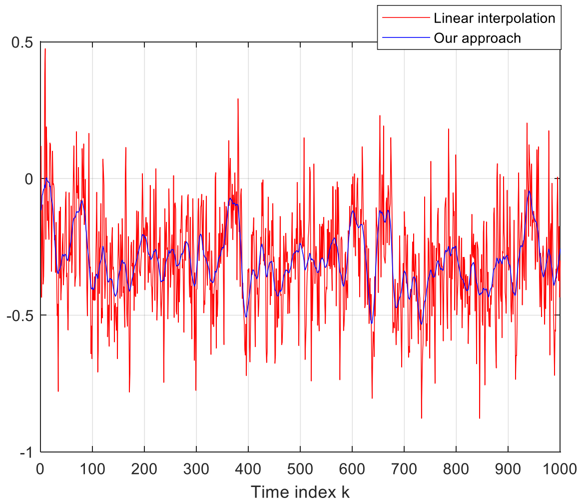


Figure 5. The sequences reconstructed by various methods when $\gamma = 0.3$ and $P = 0.05$.

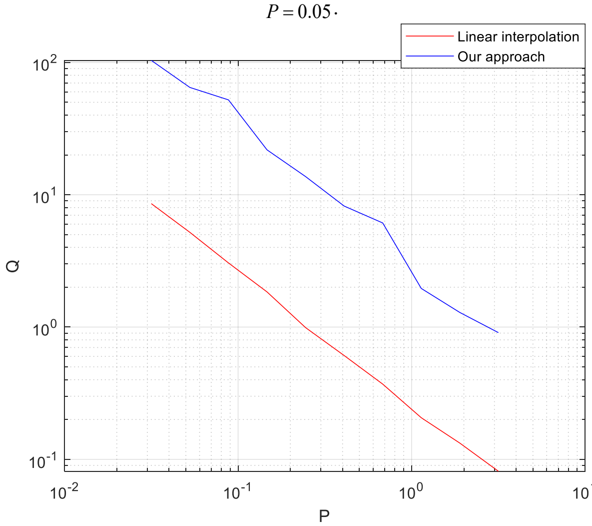


Figure 6. The values of Q obtained by various methods when $\gamma = 0.3$.

4. Conclusions

In this paper, a fractional SSA based method for performing the fractional delay is proposed. First, two overlapping sequences are extracted from the input sequence. Then, both the right unitary matrix and the left unitary matrix are designed for generating the new trajectory matrix. These design problems are formulated as the quadratically constrained quadratic programing problems and the analytical solutions are derived via the SSA approach. Since our proposed fractional SSA based method does not involve the linear time invariant filters, the proposed method is nonlinear and adaptive. Hence, the acquired sequence does not dependent on any parameter except the total number of the rows of the trajectory matrix and the fractional delay number. The computer numerical simulation results illustrate the effectiveness of our fractional SSA based method.

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