Joint Pricing and Inventory Control with Fixed and Convex/Concave Variable Production Costs

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This study considers a periodic-review joint pricing and inventory control problem for a single product, where production incurs a fixed cost plus a convex or concave variable cost. Our objective is to maximize the expected discounted profit over the entire planning horizon. We fully characterize the optimal policy for the single-period problem. As the optimal policy for the multi-period problem is too complicated to be implemented in practice, we develop well-structured heuristic policies, and establish worst-case performance bounds on the profit gap between the heuristic policies and the optimal policies. Numerical studies show that our heuristic policies perform extremely well. To further reveal the structural properties of the optimal policies, we also introduce two new concepts named κ-convexity and sym-κ-convexity, provide the associated preservation results, and then characterize the optimal policies.

1. Introduction

1.1. Motivation

As demand is usually price sensitive, incorporating the dynamic pricing mechanism increases flexibility to inventory management, allowing us to match demand with supply more effectively. For this reason, joint pricing and inventory control has been adopted by many industries. In the past decade, joint pricing and inventory control problems have also received considerable academic attention. Chen and Simchi-Levi (2012) provided an up-to-date survey of the progress made in this area. Most studies of joint pricing and inventory control problems in the literature assume that the variable cost is a linear function of the production/ordering quantity. However, many real applications has either a convex or concave variable cost. As illustrated in Lu and Song (2014), a piecewise linear convex variable cost structure may arise in scenarios such as multiple sourcing (each supplier has a different cost and capacity) in the retail sector or multiple labor costs (owing to overtime pay) in the manufacturing sector. Lu and Song (2014) provided detailed discussions of this cost structure. A piecewise linear concave variable cost structure may arise when an economy of scale exists in production or ordering, see e.g., Porteus (1971, 1972) and Fox et al. (2006). A fixed cost may represent the transportation cost and machine set-up cost.

In this study, we consider a periodic-review joint pricing and inventory control problem for a single product over a finite planning horizon with a fixed and a piecewise linear variable product cost, where the latter could be either convex or concave. At the beginning of each period, the production quantity and selling price are determined simultaneously. Demand in each period is stochastic and dependent on the selling price. Moreover, any leftover inventory is carried over to the next period, and unsatisfied demand is backlogged. The objective is to determine the production quantity and selling price in each period to maximize the expected total discounted profit over the planning horizon. Our purpose is to understand the structure of the optimal inventory control and pricing policies, so that we can offer managers practically implementable and efficient heuristic policies for solving these problems.

For convex variable cost, we first study the case in which the fixed cost is zero, and find that this problem enjoys a well-structured optimal policy. There is a threshold on the initial inventory level below which a firm should produce and above which a firm should not produce. The optimal produce-up-to level is an increasing function of the initial inventory level. Moreover, it is either a straight line with slope 1 (produce the same constant) or a flat line (produce up to the same level) over each region. The optimal pricing policy is a multi-list-price policy in which the optimal price always decreases with the produce-up-to level. When the initial inventory decreases, the firm can adopt two strategies. The first is to increase the price to decrease demand, and the second is to pay a higher marginal cost (e.g., order from a supplier with a higher unit cost or ask employees to work overtime) to push up the inventory level. One important finding is that it is never optimal for a firm to adopt both strategies at the same time. In other words, when the initial inventory level decreases, the price should remain at the same level until the capacity of the current source is exhausted, and should then increase only if the current source is at full capacity. When there is a positive fixed cost, the optimal inventory and pricing policy for the single-period problem is the same, except that there is a jump in the optimal produce-up-to level. However, the optimal policy for the multi-period problem with a fixed cost can be more complicated. Therefore, we develop a heuristic policy based on the structure of the optimal policy for the single period problem. Over extensive numerical studies, the heuristic policy achieves 99.992% of the optimal profit on average and 98.951% in the worst case. Moreover, we are able to establish a worst-case performance bound on the profit gap between the heuristic policy and optimal policy.

For concave variable cost, the structure of the optimal policy is complex even without the fixed cost. Hence, we first study the optimal policy of the single period problem, and show that it follows a generalized (s, S, p) policy. Similar to the convex variable cost problem, we develop a heuristic policy that is based on the structure of the optimal policy of the single period problem. Extensive numerical studies show that this heuristic policy achieves 99.98% of the optimal profit on average and 95.55% in the worst case. A worst-case performance bound on the profit gap between the heuristic policy and optimal policy is also established.

To understand why our heuristic policies perform well, we try to characterize the optimal policies of the general multi-period problems by introducing concepts named κ-convexity and sym-κ-convexity, which are generalizations of the sym-K-convexity introduced in Chen and Simchi-Levi (2004) and the strong $(K, \mathbf{c}, \mathbf{q})$-convexity introduced in Lu and Song (2014), respectively. After careful characterizations, we find several structural properties of the optimal policies that are consistent with the structures of our heuristic policies, which provides theoretical support for the strong performances of the heuristic policies.

1.2. Literature Review

In the subsection, we review the literature related to our model. Our study belongs to the stream of research on inventory control that started with Scarf (1960), the first paper to discuss and analyze fixed production costs and the associated optimal policy. By introducing a concept called K-convexity, Scarf (1960) showed that the (s, S) policy is optimal for the linear variable production cost case. Porteus (1971, 1972) further generalized the model of Scarf (1960) to include piecewise linear concave variable costs. Under some conditions on demand uncertainties (e.g., positive Pólya or uniform densities), the author managed to prove the optimality of a generalized (s, S) policy for this system. Fox et al. (2006) studied a two-supplier problem with log-concave demand uncertainties and showed that the optimal policy is well-structured. Zhang et al. (2012b) extended the model of Fox et al. (2006) by adding a capacity constraint on the supplier with the lower unit ordering cost, and characterized the optimal inventory control policy under this scenario. Chen (2015) tried to characterize the optimal policy of the inventory control problem with a general concave variable cost.

Convex variable production cost functions have been studied since Karlin (1960), whose focus was the influence of demand densities on the base-stock level. Henig et al. (1997) investigated the inventory policy under a given supply contract that leads to a two-linear-piece convex variable production cost without fixed cost, where the optimal policy can be characterized by two critical levels. It should be noted that the inventory model with a fixed cost plus a linear production variable cost and limited production capacity are also related to our models. Shaoxiang and Lambrecht (1996) found that the optimal policy becomes more complicated. They proved that the modified base-stock policy is not necessarily optimal, and the optimal policy exhibits the X-Y band structure. Later, Shaoxiang (2004) further characterized the policy based on a concept called (C, K)-convexity. Gallego and Scheller-Wolf (2000) introduced a related concept called CK-convexity and reported a more complete characterization of the optimal policy. Lu and Song (2014) characterized the optimal inventory control policy for a periodic-review inventory control system in which the production incurs both a fixed cost and convex variable cost, and developed a suitable heuristic policy. Chao and Zipkin (2008) discussed a production cost function with a fixed cost that is incurred once the production quantity exceeds a given threshold, and then characterized the optimal policy based on K-convexity. Caliskan-Demirag et al. (2012) considered a case in which the production cost is a step-function of the production quantity. Their characterization is based on several convexity-like concepts, including CK-convexity, (C, K)-convexity, and C-(K_1, K_2)-convexity. Other studies discussing a similar pattern include Li et al. (2009). All of these studies assume that price is exogenously given rather than constituting a decision, whereas price is a decision in this paper.

This study also contributes to the stream of research on joint pricing and inventory models. Our model has settings similar to the model in Federgruen and Heching (1999), who focused on linear ordering/production cost functions. Federgruen and Heching (1999) proved that the base-stock list-price policy is optimal, and showed the benefit of integrating pricing and inventory control decisions via numerical examples. Li and Zheng (2006) studied the same model with random yields, and showed that an extended base-stock list-price policy is optimal. When a fixed production cost is involved, however, the base-stock list-price policy is not optimal in general. Studies along this line focused primarily on the optimality of (s, S, p) policy and its extensions. For the linear variable production cost case, if demand uncertainty follows the additive model, Thomas (1974) adopted the concept of K-convexity introduced by Scarf (1960) and proved that the (s, S, p) policy is optimal. However, Chen and Simchi-Levi (2004) provided a counterexample showing that such a policy could be suboptimal if the demand model involves a multiplicative uncertainty term. They introduced a concept called sym-K-convexity and proved the optimality of a so-called (s, S, A, p) policy, which can be seen as an extension of the (s, S, p) policy. Chen et al. (2010) studied the joint pricing and inventory control problem with a concave production cost, and proved that a generalized (s, S, p) policy is optimal if demand follows an additive model and the random noise is Pólya or uniform. When production quantity is capacitated and demand uncertainty follows the additive model, both Chao et al. (2012) and Zhang et al. (2012a) used the CK-convexity introduced by Gallego and Scheller-Wolf (2000) to show that the optimal policy is of an (s, S, p)-like structure. Compared with these studies, our setting is more general in terms of cost structure, demand model, and demand distribution. Therefore, the results of this study have wider applications for various real scenarios.

The reminder of this paper is organized as follows. We provide the basic model settings in Section 2. In Section 3, we focus on the case of a convex variable cost plus a fixed cost, and fully characterize the optimal policy of the problem in two special cases. We then move to the general problem and develop an easy-to-implement heuristic policy and analyze its performance bound. We also test its performance in extensive numerical studies. Section 4 is parallel to Section 3, but our focus is on the case of a concave variable cost plus a fixed cost. Furthermore, Section 5 propose two new convexity-like concepts, namely, κ-convexity and sym-κ-convexity, together with corresponding preservation results, and then characterizes the optimal policies of the general multi-period problems for both convex and concave cases. Conclusions are drawn in Section 6. To streamline the discussion, all of the proofs of our results are presented in the Appendix.

2. Model Setting

We consider a firm that makes joint inventory and pricing decisions to satisfy a sequence of demands for a single product over a T-period planning horizon. In each period t, the firm observes the initial inventory level x at the beginning of the period, then selects a selling price p from a bounded interval $\mathcal{P}_t$ and decides a production quantity $z \geq 0$ simultaneously. Producing quantity $z > 0$ incurs a cost

$$c(z) = \sum\nolimits_{i=1}^{n} (K_i + c_i z) \mathbf{1}_{\{q_{i-1} < z \leq q_i\}}, \tag{1}$$

where $c_i \geq 0$ for all i, $0 = q_0 < q_1 < \cdots < q_{n-1} < q_n = +\infty$, $K_1 \geq 0$, $K_{i+1} = K_i - (c_{i+1} - c_i) q_i$ for all $1 \leq i < n$, and $\mathbf{1}_{\{\cdot\}}$ is the indicator function. We define $c(0) = K_1$. Observe that $c(z)$ is increasingly continuous, and consists of n linear pieces. Producing $z > 0$ incurs a fixed cost $c(0) = K_1 \geq 0$. Thus, the production cost can be expressed by $c(z)\mathbf{1}_{\{z>0\}}$ for any $z \geq 0$. In this paper, we are interested in the following two cases(see Figure 1 for an illustration):

(i) $c(z)$ is convex, implying that $c_1 < c_2 < \cdots < c_n$ and $K_1 > K_2 > \cdots > K_n$; and

(ii) $c(z)$ is concave, implying that $c_1 > c_2 > \cdots > c_n$ and $K_1 < K_2 < \cdots < K_n$.

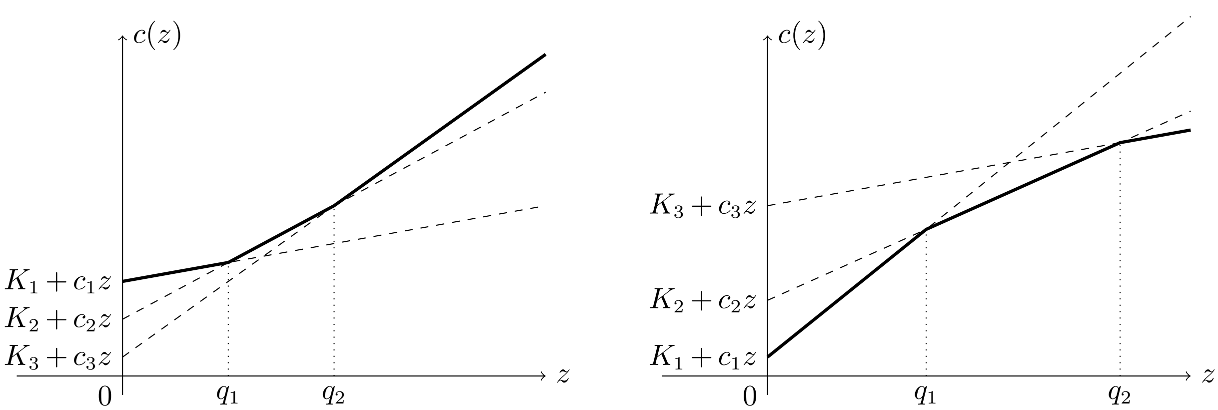

Figure 1 Cost function $c(z)$: convex case (left) and concave case (right)

After production in period t, demand D_t for this period is realized and satisfied with the on-hand inventory. We assume that it follows a general model $D_t = \xi_t d_t(p) + \varepsilon_t$, where ξ_t and ε_t are random variables with $\xi_t > 0, \mathbb{E}\xi_t = 1$ and $\mathbb{E}\varepsilon_t = 0$. Moreover, $d = d_t(p)$ denotes the expected demand associated with selling price $p \in \mathcal{P}_t$. Similar to Chen and Simchi-Levi (2004), we consider its inverse function $p = p_t(d)$ over $\mathcal{D}_t = \{d_t(p) : p \in \mathcal{P}_t\}$ and express demand in term of d as

$$D_t = \xi_t d + \varepsilon_t, \quad \forall d \in \mathcal{D}_t. \tag{2}$$

In addition, we assume that random vectors (ξ_t, ε_t) are independent across time period t, demand D_t is non-negative with probability 1, price $p_t(d)$ is continuous, and expected revenue $dp_t(d)$ is concave in $d \in \mathcal{D}_t$. Note that the demand model is called *multiplicative* when $\varepsilon_t \equiv 0$, and *additive* when $\xi_t \equiv 1$ for all $1 \le t \le T$.

After satisfying realized demand D_t in period t with the on-hand inventory, any leftover inventory is carried over to the next period, and any unsatisfied demand is backlogged. This incurs an associated inventory holding and shortage cost $h_t(I)$ in terms of inventory level I at the end of period t. We assume that

$$h_t(I) = -h_t^-(0 \wedge I) + h_t^+(0 \vee I),$$

where coefficients h_t^- and h_t^+ are non-negative for all $1 \le t \le T$, and $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$ for any real numbers a and b.

Let $\gamma \in [0, 1]$ be the discount factor. The firm's objective is to find an inventory and pricing policy to maximize the total expected discounted profit over the entire planning horizon. For each period $t = 1, \cdots, T$, given the initial inventory level x in this period, the profit-to-go function $v_t(x)$ satisfies the following dynamic programming:

$$v_t(x) = \max_{z \ge 0} \left\{ u_t(x+z) - c(z)\mathbf{1}_{\{z>0\}} \right\}, \tag{3a}$$

$$u_t(y) = \max_{d \in \mathcal{D}_t} \left\{ dp_t(d) - \mathbb{E}h_t(y - \xi_t d - \varepsilon_t) + \gamma \mathbb{E} v_{t+1}(y - \xi_t d - \varepsilon_t) \right\}, \tag{3b}$$

where $u_t(y)$ can be interpreted as the maximal expected profit-to-go after raising the inventory level to y in period t. In addition, for notational convenience, we suppose that there is no terminal value at the end of the planning horizon, i.e., $v_{T+1}(x) = 0$. In the following, we denote $z_t^*(x)$ and $d_t^*(y)$ as the optimal solutions to problems (3a) and (3b), respectively. Note that the optimal selling price can be expressed as $p_t^*(x) = p_t(d_t^*(z_t^*(x) + x))$.

To characterize the optimal policy, we define function $v_t^0(x)$ as below to represent the profit-to-go function in period t if the fixed cost $c(0) = K_1$ always incurs even when nothing is produced (i.e., $z = 0$), i.e.,

$$v_t^0(x) = \max_{z \ge 0} \left\{ u_t(x+z) - c(z) \right\}. \tag{4}$$

We denote $z_t^0(x)$ as an optimal solution to problem (4). Moreover, by $K_1 \ge 0$,

$$v_t(x) = v_t^0(x) \vee u_t(x). \tag{5}$$

Similar to an assumption made in Federgruen and Heching (1999), we assume that $\lim_{|y|\to\infty} [u_t(y) - c_i y] = -\infty$ for each $1 \le i \le n$ and hence $S_t(c_i)$ and $P_t(c_i)$ are finite, where for any a, we define

$$S_t(a) = \min \arg\max_y \{u_t(y) - ay\} \text{ and } P_t(a) = p_t(d_t^*(S_t(a))). \tag{6}$$

As $u_t(y) - ay$ is submodular in (a, y), by Theorem 2.8.2 in Topkis (1998), $S_t(a)$ is decreasing in a. This implies that $S_t(c_1) \ge \cdots \ge S_t(c_n)$ when $c(z)$ is convex, and $S_t(c_1) \le \cdots \le S_t(c_n)$ when $c(z)$ is concave. Furthermore, we define $\mathcal{O}_t$ as the set of initial inventory levels at which it is optimal to produce in period t. From the definitions of $z_t^*(x)$ and $v_t^0(x)$, it is straightforward to see that

$$\mathcal{O}_t = \{x : z_t^*(x) > 0\} = \{x : v_t^0(x) > u_t(x)\}.$$

In addition, let $\mathcal{O}_t^c$ be the complement of set $\mathcal{O}_t$. Observe that $\mathcal{O}_\sqcup \subseteq \{x < R_t\}$ with R_t given by

$$R_t = \min\{x : x \in \mathcal{O}_t^c\} = \min\left\{x : v_t^0(x) \le u_t(x)\right\}. \tag{7}$$

3. Convex Variable Cost

When the variable cost function $c(z)$ is convex, we first characterize the optimal policy of model (3) for two special cases. Subsection 3.1 studies the case without a fixed production cost (i.e., $K_1 = 0$). This is a special case of the general problem with a well-structured optimal policy and interesting insights. Moreover, its result can help us to establish the results in many other cases. Subsection 3.2 studies the single-period problem. The optimal joint pricing and inventory control policy can be fully characterized and is well-structured. More importantly, such characterization helps us to better understand the structure of the optimal policy for the general problem, and motivates us to develop a practically implementable and efficient heuristic policy in Subsection 3.3. Numerical studies testing the policy's performance are presented in Subsection 3.4.

3.1. No Fixed Cost Problem

Consider the multi-period problem with zero fixed production cost, i.e., $K_1 = 0$. In this case, by $c(0) = K_1 = 0$, we have $c(z)\mathbf{1}_{\{z>0\}} = c(z)$ for all $z \ge 0$, implying that $v_t(x) = v_t^0(x)$ and $z_t^*(x) = z_t^0(x)$ in each period t. Furthermore, because $dp_t(d)$ is concave in d and $h_t(y)$ is convex, by the convexity of $c(z)$, one can inductively verify that both problem (3b) and problem (5) are concave maximization problems, implying that $u_t(y)$ are concave for $t = T, \cdots, 1$. In this case, we have the following result for problem (5) and its optimal solution $z_t^0(x)$.

Proposition 1. *When $u_t(y)$ is concave and $c(z)$ is convex, $R_t \le S_t(c_1)$, and $v_t^0(x) > u_t(x)$ if and only if $x < R_t$. Moreover, an optimal solution to problem (4) is*

$$z_t^0(x) = \begin{cases} 0 & \text{if } x \ge S_t(c_1), \\ S_t(c_i) - x & \text{if } S_t(c_i) - q_i \le x < S_t(c_i) - q_{i-1} \text{ for } 1 \le i \le n, \\ q_i & \text{if } S_t(c_{i+1}) - q_i \le x < S_t(c_i) - q_i \text{ for } 1 \le i < n. \end{cases} \tag{8}$$

Proposition 1 completely characterizes the production policy when $u_t(y)$ is concave. Specifically, it states that a firm should produce if and only if the initial inventory level is below the threshold $R_t \le S_t(c_1)$. Moreover, it also provides a closed form of the optimal solution $z_t^0(x)$ that solves problem (4). Notice that $z_t^0(x)$ is decreasing and piecewise linear in x, which is equal to 0 or belongs to the set $\{S_t(c_i)-x, q_i : 1 \le i \le n\}$.

By Proposition 1, we are able to characterize the optimal policy in each period as below.

THEOREM 1. *When $K_1 = 0$ and $c(z)$ is convex, $\mathcal{O}_t = \{x < S_t(c_1)\}$, $z_t^*(x) = z_t^0(x)$ with $z_t^0(x)$ given in* (8)*, and $p_t^*(x) = P_t(c_i)$ if $z_t^*(x) = S_t(c_i) - x$. Furthermore, $z_t^*(x)$ and $p_t^*(x)$ are decreasing in x.*

Figure 2 illustrates the optimal policy specified in Theorem 1. In each period t, the state space can be partitioned into at most $2n$ regions. Over each region, either $z_t^*(x)$ or $p_t^*(x)$ must be a constant, whereas the other is decreasing in x, implying that the firm should either reduce the production quantity or charge a lower price in response to a higher initial inventory level. However, these two strategies should be applied alternatively, not simultaneously. In particular, when $z_t^*(x)$ is decreasing over some region, it has the specific expression $z_t^*(x) = S_t(c_i) - x$, i.e., the firm should produce up to a constant level $S_t(c_i)$ and charge a constant price $P_t(c_i)$. When $z_t^*(x) = q_i$, which means when the produce up to level $z_t^*(x) + x$ increases, the optimal price should decrease as shown in Figure 2. This is an indication of a multi-list-price policy.

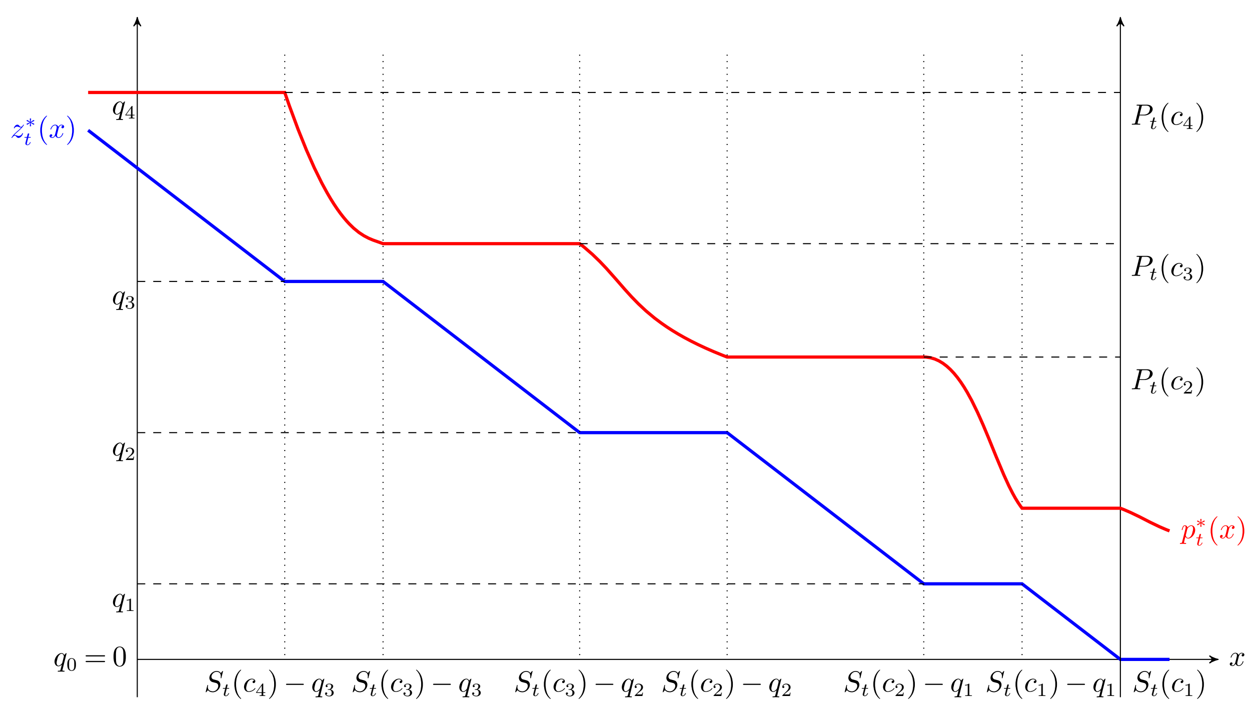

Figure 2 Optimal production quantity $z_t^*(x)$ and price $p_t^*(x)$ when $c(z)$ is convex and $K_1 = 0$

3.2. Single-Period Problem

Now we consider the optimal policy of the single-period problem with $K_1 > 0$. The single-period model corresponds to a perishable product whose inventory cannot be carried over to the next period. The optimal policy for the single-period problem can also help us to understand the optimal policy of the multi-period problem.

We only need to consider the last period $t = T$. In this case, observe that $u_t(y)$ is obviously concave by the convexity of $h_t(y)$. Recall that $R_t = \min\{x : v_t^0(x) \le u_t(x)\}$. Thus, Proposition 1 remains valid, which implies that $R_t \le S_t(c_1)$ and $v_t(x) = v_t^0(x) > u_t(x)$ if and only if $x < R_t$, i.e., $\mathcal{O}_t = \{x < R_t\}$. Furthermore, the optimal policy can be characterized as below.

THEOREM 2. *When $t = T$ and $c(z)$ is convex, $R_t \le S_t(c_1)$, $\mathcal{O}_t = \{x < R_t\}$, $z_t^*(x) = z_t^0(x)\mathbf{1}_{\{x<R_t\}}$ with $z_t^0(x)$ given in (8), and $p_t^*(x) = P_t(c_i)$ when $z_t^*(x) = S_t(c_i) - x$. Furthermore, $z_t^*(x)$ is decreasing in x, and $p_t^*(x)$ is decreasing when $x < R_t$, increasing at $x = R_t$, and decreasing when $x > R_t$.*

Figure 3 illustrates Theorem 2. It shows that for the single-period problem, production is executed if and only if the initial inventory level x falls below a threshold $R_t \le S_t(c_1)$. When $x < R_t$, the optimal production quantity $z_t^*(x)$ and optimal price $p_t^*(x)$ are the same as those for the case studied in the previous subsection. At $x = R_t$, $z_t^*(x)$ jumps down to 0, and $p_t^*(x)$ takes an upward jump. When $x > R_t$, the firm should produce nothing and charge a lower price when the initial inventory level x increases.

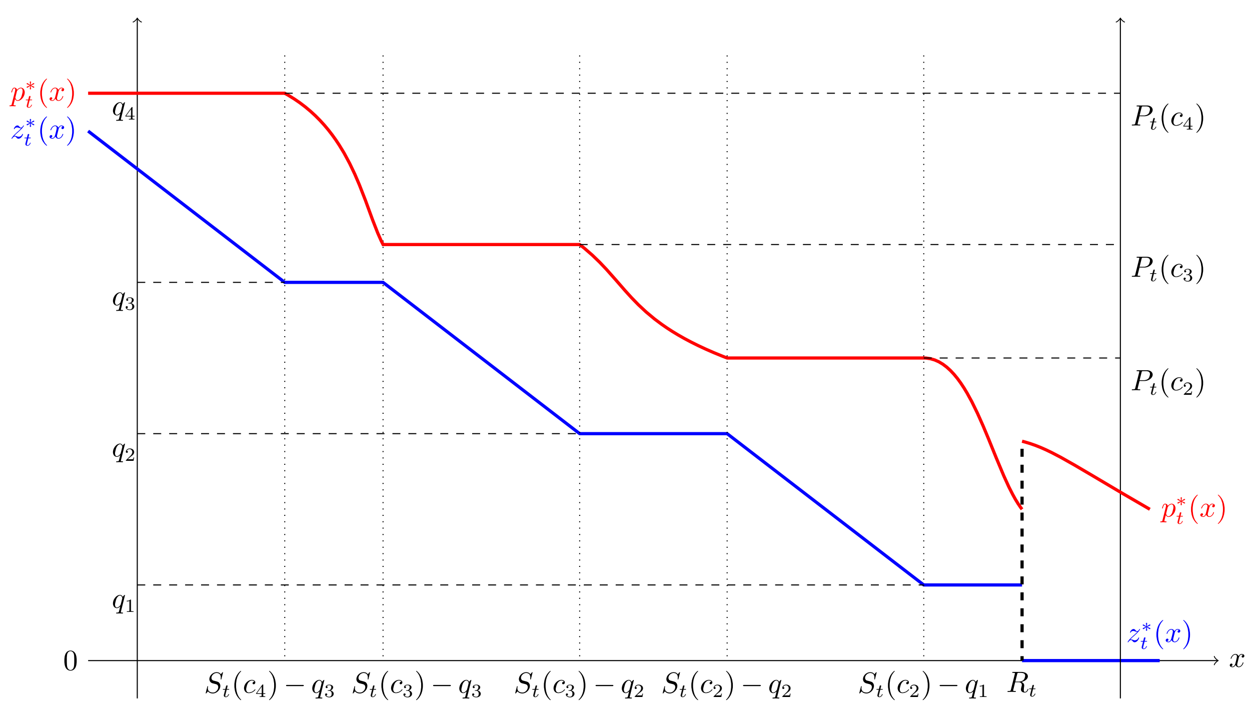

Figure 3 Optimal production quantity $z_t^*(x)$ and price $p_t^*(x)$ in the last period $t = T$ when $c(z)$ is convex

3.3. Heuristic for Multi-Period Problem

We now move to the general multi-period problem with $K_1 > 0$. Unlike the special cases studied in Subsections 3.1 and 3.2, the profit-to-go function is not concave in this case. Due to the lack of concavity, it is not surprising that both the analysis and the structure of the optimal policy become much more complicated. Thus, in this subsection, we develop easy-to-implement heuristic policies and show their worst-case performance bounds.

To circumvent the challenge brought by the lack of concavity, we need a concept called a *lower convex envelope*. Specifically, the lower convex envelope of a function $f(x)$, denoted by $f^e(x)$, follows the definition below and denotes the largest convex function such that $f^e(x) \le f(x)$ for all x:

$$f^e(x) = \inf\left\{(1-\lambda)f(x_0) + \lambda f(x_1) : x = (1-\lambda)x_0 + \lambda x_1, \lambda \in [0,1]\right\}.$$

Proposition 2 below provides an interesting and useful result for the function and its lower convex envelope, which plays an important role in estimating the performance of Algorithm 1.

PROPOSITION 2. *Suppose $f(x)$ is continuous and $f^e(x)$ is the lower convex envelope of $f(x)$. If $\liminf_{x\to+\infty} [x^{-1}f(x)] > 0$ (or $\limsup_{x\to-\infty} [x^{-1}f(x)] < 0$) and $f(x)$ has a greatest (or least) minimizer x^*, then x^* is also the greatest (or least) minimizer of $f^e(x)$ and satisfies $f(x^*) = f^e(x^*)$.*

The following heuristic policy has the same structure as the optimal policy for the single-period problem studied in Subsection 3.2. It shows how to compute the heuristic inventory and pricing policy, denoted by $\bar{z}_t(x)$ and $\bar{p}_t(x)$, backwards from period T to period 1.

ALGORITHM 1. *Initialize $\bar{v}_{T+1}(x) = v_{T+1}(x)$. Consider any $t = T, \cdots, 1$.*

1. *Compute $\bar{u}_t(y)$ as below and let $\bar{d}_t(y)$ be the corresponding optimal solution:*

$$\bar{u}_t(y) = \max_{d\in\mathcal{D}_t}\left\{dp_t(d) - \mathbb{E}h_t(y - \xi_t d - \varepsilon_t) + \gamma\mathbb{E}\bar{v}_{t+1}(y - \xi_t d - \varepsilon_t)\right\}.$$

2. *For each $1 \le i \le n$, compute $\bar{S}_t(c_i) = \min\arg\max\{\bar{u}_t(y) - c_i y\}$. Define*

$$\bar{z}^0_t(x) = \begin{cases} 0 & \text{if } x \ge \bar{S}_t(c_1), \\ \bar{S}_t(c_i) - x & \text{if } \bar{S}_t(c_i) - q_i \le x < \bar{S}_t(c_i) - q_{i-1} \text{ for } 1 \le i \le n, \\ q_i & \text{if } \bar{S}_t(c_{i+1}) - q_i \le x < \bar{S}_t(c_i) - q_i \text{ for } 1 \le i < n. \end{cases}$$

   and $\bar{v}^0_t(x) = \bar{u}_t(x + \bar{z}^0_t(x)) - c(\bar{z}^0_t(x))$.

3. *Compute $\bar{R}_t = \sup\{x : \bar{v}^0_t(x) > \bar{u}_t(x)\}$ and*

$$\bar{z}_t(x) = \begin{cases} \bar{z}^0_t(x), & \text{if } x < \bar{R}_t, \\ 0, & \text{if } x \ge \bar{R}_t. \end{cases}$$

4. *Compute* $\bar{y}_t(x) = x + \bar{z}_t(x)$*,* $\bar{p}_t(x) = p_t(\bar{d}_t(\bar{y}_t(x)))$ *and* $\bar{v}_t(x) = \begin{cases} \bar{v}_t^0(x), & x < \bar{R}_t, \\ \bar{u}_t(x), & x \geq \bar{R}_t. \end{cases}$

In Algorithm 1, given the profit-to-go function $\bar{v}_{t+1}(x)$ under the heuristic policy, Step 1 obtains $\bar{u}_t(y)$ and $\bar{d}_t(y)$ as counterparts of $u_t(y)$ and $d_t^*(y)$, which are associated with problem (3b). In other words, given the after-production inventory level y in period t, $\bar{u}_t(y)$ is the profit generated by the heuristic policy and $\bar{d}_t(y)$ is the expected demand chosen by the heuristic policy. Step 2 computes $\bar{S}_t(a)$ for any $a \in \{c_1, \cdots, c_n\}$, which is the heuristic counterpart of $S_t(a)$ in (6). Moreover, $\bar{z}_t^0(x)$ is the heuristic counterpart of production quantity $z_t^0(x)$, and $\bar{v}_t^0(x)$ represents the profit-to-go associated with $\bar{z}_t^0(x)$. Note that $\bar{z}_t^0(x) > 0$ if and only if $x < \bar{S}_t(c_1)$.

Step 3 selects a threshold point $\bar{R}_t$ as the maximum of the initial inventory level above which producing nothing is always no worse than producing a positive amount in the heuristic. In other words, if $x > \bar{R}_t$, then the firm would be better off producing nothing rather than $\bar{z}_t^0(x)$ in the heuristic. By $K_1 = c(0) > 0$, if $x \geq \bar{S}_t(c_1)$, then $\bar{v}_t^0(x) = \bar{u}_t(x) - c(0) < \bar{u}_t(x)$ and hence $\bar{R}_t < \bar{S}_t(c_1)$. Thus, $\bar{z}_t^0(x) > 0$ when $x < \bar{R}_t$, implying that $\bar{R}_t$ is the threshold below which we produce a positive amount in the heuristic. Consequently, the heuristic inventory policy $\bar{z}_t(x)$ is defined as $\bar{z}_t^0(x)$ if $x < \bar{R}_t$ and zero otherwise, which has the same structure as the optimal policy for the single-period problem (see Theorem 2). Finally, Step 4 generates the after-production inventory level $\bar{y}_t(x)$ under the heuristic inventory policy and the heuristic pricing policy $\bar{p}_t(x)$ by applying the expected demand $\bar{d}_t(y)$ of the heuristic policy computed in Step 1 and the profit $\bar{v}_t(x)$ obtained by the heuristic policy from period t to the end of the planning horizon. Notice that functions $\bar{u}_t$, $\bar{v}_t^0$, and $\bar{v}_t$ obtained in the algorithm may not be concave in general when $t < T$.

With Proposition 2, we are able to show the performance of Algorithm 1 as below.

THEOREM 3. *In any period* $t = 1, \cdots, T$*,* $\bar{v}_t(x)$ *obtained by Algorithm 1 satisfies*

$$0 \leq v_t(x) - \bar{v}_t(x) \leq \sum_{i=0}^{T-t} [(2i+1)K_1]\gamma^i - K_1\gamma^{T-t}. \tag{9}$$

Moreover, the heuristic policy is optimal, i.e., $\bar{v}_t(x) = v_t(x)$*, if any of the following conditions holds:*

(a) it is a single period problem, i.e., $t = T$*;*

(b) the fix cost $K_1 = 0$*; or*

(c) an (s, S) *policy is optimal to problem* (3a)*, e.g., demand uncertainty follows the additive model and the fixed cost* $K_1 > \left(\sum_{i=t}^{T} \gamma^{i-t} h_i^- - c_n\right) q_{n-1} - \sum_{i=1}^{n-1} (c_i - c_{i+1}) q_i$ *for any* $1 \leq t \leq T$*.*

Theorem 3 gives the performance bound of Algorithm 1 in (9), which only depends on the fixed cost K_1, the number of periods T, and the discounted factor γ. Furthermore, it also lists three important cases in which the heuristic policy is optimal. In particular, part (a) and part (b) correspond to the single-period case and the case without the fixed cost K_1, which are consistent with Theorem 1 and Theorem 2, respectively. Moreover, if the fixed cost K_1 is sufficiently large and demand uncertainty is additive, then part (c) states that the heuristic policy is also optimal and is reduced to an (s,S) policy.

When $\gamma = 1$, note that the performance bound of Algorithm 1 is quadratic in the number of periods T. We now provide another heuristic policy with a worst-case performance bound that is linear in T. It has the same structure as the optimal policy for the case without a fixed cost, which is illustrated in Figure 2.

ALGORITHM 2. *Let* $\hat{v}^0_{T+1}(x) = v_{T+1}(x)$ *and* $\hat{v}_{T+1}(x) = v_{T+1}(x)$ *for any* x. *Consider any* $t = T, \cdots, 1$.

1. *For each* y, *compute* $\hat{u}_t(y)$ *as below and let* $\hat{d}_t(y)$ *be the corresponding optimal solution:*

$$\hat{u}_t(y) = \max_{d \in \mathcal{D}_t} \left\{ dp_t(d) - \mathbb{E}h_t(y - \xi_t d - \varepsilon_t) + \gamma \mathbb{E}\hat{v}^0_{t+1}(y - \xi_t d - \varepsilon_t) \right\}.$$

2. *For any* $a \in \{c_1, \cdots, c_n\}$, *compute* $\hat{S}_t(a) = \min \arg\max_y \{\hat{u}_t(y) - ay\}$. *For any* x, *define*

$$\hat{z}_t(x) = \begin{cases} 0 & \text{if } x \geq \hat{S}_t(c_1), \\ \hat{S}_t(c_i) - x & \text{if } \hat{S}_t(c_i) - q_i \leq x < \hat{S}_t(c_i) - q_{i-1} \text{ for } 1 \leq i \leq n, \\ q_i & \text{if } \hat{S}_t(c_{i+1}) - q_i \leq x < \hat{S}_t(c_i) - q_i \text{ for } 1 \leq i < n. \end{cases}$$

   and $\hat{v}^0_t(x) = \hat{u}_t(x + \hat{z}_t(x)) - c(\hat{z}_t(x))$.

3. *For each* x, *compute* $\hat{y}_t(x) = x + \hat{z}_t(x)$, $\hat{p}_t(x) = p_t(\hat{d}_t(\hat{y}_t(x)))$, *and*

$$\begin{aligned} \hat{v}_t(x) = {} & \hat{d}_t(\hat{y}_t(x))\hat{p}_t(x) - \mathbb{E}h_t\big(\hat{y}_t(x) - \xi_t \hat{d}_t(\hat{y}_t(x)) - \varepsilon_t\big) + \gamma \mathbb{E}\hat{v}_{t+1}\big(\hat{y}_t(x) - \xi_t \hat{d}_t(\hat{y}_t(x)) - \varepsilon_t\big) \\ & - c(\hat{z}_t(x))\mathbf{1}_{\{\hat{z}_t(x) > 0\}}. \end{aligned}$$

Steps 1 and 2 of Algorithm 2 are very similar to those of Algorithm 1. The only difference is that $\hat{u}_t(y)$ and $\hat{d}_t(y)$, which are the heuristic counterparts of $u_t(y)$ and $d^*_t(y)$, are computed using $\hat{v}^0_t(x)$, which is the heuristic counterpart of $v^0_t(x)$. The basic idea of Algorithm 2 is to assume that the fixed cost K_1 is always charged in each period no matter whether the firm produces or not. This means that the value of K_1 does not affect the heuristic policy. Hence, functions $\hat{u}_t$, $\hat{v}^0_t$ and $\hat{v}_t$ obtained in the algorithm are concave. Moreover, the structure of the heuristic policy, i.e., $\hat{z}_t(x)$ and $\hat{p}_t(x)$, is the same as that of the case without a fixed cost, which is shown in Theorem 1 of Subsection 3.1. As the actual production cost in a period is zero if nothing is produced, the actual profit of implementing the heuristic policy, which is $\hat{v}_t(x)$ computed in Step 3, is larger than $\hat{v}^0_t(x)$. The performance of heuristic Algorithm 2 is given by the following theorem.

THEOREM 4. *For any period* t, $0 \leq v_t(x) - \hat{v}_t(x) \leq \sum_{i=0}^{T-t} \gamma^i K_1$.

Theorem 4 shows that the performance bound of Algorithm 2 is linear in T when $\gamma = 1$, which is better than that of Algorithm 1. Nevertheless, we need to point out that the latter has the advantage of a threshold-type structure in its Step 3, which makes Algorithm 1 optimal in the single period problem according to Theorem 3(a) and also optimal when $\xi_t \equiv 1$ and K_1 is sufficiently large according to Theorem 3(c). Through extensive numerical studies, we find that Algorithm 1 performs as well as, if not better than, Algorithm 2 when K_1 is small, and outperforms Algorithm 2 when K_1 is large, which is shown in the next subsection. In summary, we recommend Algorithm 1 to be implemented in practice. The purpose of presenting Algorithm 2 is to support Algorithm 1 by numerically comparing its performance with an algorithm whose worst-case performance bound is $O(TK_1)$.

3.4. Numerical Analysis

This subsection uses a set of numerical experiments to demonstrate that the heuristic policy according to Algorithm 1 is very close to optimal and outperforms that of Algorithm 2, especially for a large fixed cost K_1. The experiments are designed as follows. For each $n \in \{2,3\}$, 100 instances with 12 periods, i.e., $T = 12$, are generated independently and considered for all $K_1 \in \{20,40,60,80\}$.

In addition to the production cost specified by K_1, c_i for all $1 \le i \le n$, and q_i for all $1 \le i < n$, we also impose a production capacity denoted by q_n for each period. For all instances, c_n is fixed to 1, c_i for all $1 \le i < n$ are set to the order statistics of $n-1$ uniform random numbers in $[0.6,1]$, and q_i for all $1 \le i \le n$ are set to the order statistics of n uniform random numbers in $[200,1200]$. For any $1 \le t \le 12$, the inventory holding and shortage penalty cost is defined as $h_t(I) = aI^+ + bI^-$, where a and b are uniformly generated in $[0.02,0.2]$. The salvage value at the end of the planning horizon is $v_{T+1}(x) = a_{T+1}x^+ - b_{T+1}x^-$, where a_{T+1} and b_{T+1} are uniformly generated in $[0,0.4]$ and $[1.4,2.2]$, respectively. Furthermore, the discount factor γ is fixed to 0.95.

For the demand model, we let $\mathcal{D}_t = [200,500]$ for any $1 \le t \le 12$. The price as a function of demand is set to $p_t(d) = \alpha - \beta d$, where α and β are uniformly generated in $[5,6]$ and $[0.005,0.0075]$, respectively. We assume that ξ_t and ε_t are independent and have stationary distributions over time. The distribution of ξ_t is randomly selected from the following:

- a uniform distribution on the support $\{0.6,0.8,1,1.2,1.4\}$; or
- a discretized normal distribution such that $P(\xi_t = 0.6) = \Phi(0.7,1,\sigma_\xi)$, $P(\xi_t = \xi) = \Phi(\xi + 0.1,1,\sigma_\xi) - \Phi(\xi - 0.1,1,\sigma_\xi)$ for all $\xi \in \{0.8,1,1.2\}$, and $P(\xi_t = 1.4) = 1 - \Phi(1.3,1,\sigma_\xi)$, where σ_ξ is uniformly generated in $[0.1,0.3]$ and $\Phi(\cdot,\mu,\sigma)$ denotes the cumulative distribution function of a normal distribution with mean μ and standard deviation σ.

Similarly, the distribution of ε_t is randomly selected from the following:

- a uniform distribution on the support $\{-100,-60,-20,20,60,100\}$; or
- a discretized normal distribution such that $P(\varepsilon_t = -100) = \Phi(-80,0,\sigma_\varepsilon)$, $P(\varepsilon_t = \varepsilon) = \Phi(\varepsilon + 20,0,\sigma_\varepsilon) - \Phi(\varepsilon - 20,0,\sigma_\varepsilon)$ for all $\xi \in \{-60,-20,20,60\}$, and $P(\varepsilon_t = 100) = 1 - \Phi(80,0,\sigma_\varepsilon)$, where σ_ε is uniformly generated in $[30,60]$.

Table 1 Performance of Algorithms 1 and 2 (%)

$n=2$			$t=1$	$t=2$	$t=3$	$t=4$	$t=5$	$t=6$	$t=7$	$t=8$	$t=9$	$t=10$	$t=11$
$K_1=20$	Alg 1	Avg	99.998	99.998	99.997	99.997	99.997	99.996	99.996	99.996	99.995	99.994	100
		Min	99.926	99.908	99.920	99.898	99.867	99.895	99.853	99.911	99.862	99.747	99.996
	Alg 2	Avg	99.623	99.610	99.594	99.573	99.547	99.513	99.476	99.409	99.320	99.183	98.924
		Min	98.592	98.598	98.609	98.616	98.634	98.644	98.671	98.674	98.704	98.582	98.349
$K_1=40$	Alg 1	Avg	99.995	99.994	99.994	99.995	99.994	99.994	99.993	99.992	99.991	99.987	99.990
		Min	99.801	99.802	99.803	99.808	99.818	99.824	99.814	99.801	99.823	99.782	99.501
	Alg 2	Avg	98.925	98.902	98.875	98.834	98.790	98.733	98.671	98.552	98.416	98.185	97.803
		Min	97.036	97.049	97.066	97.064	96.847	96.897	96.998	96.879	96.962	96.557	96.601
$K_1=60$	Alg 1	Avg	99.997	99.996	99.996	99.995	99.994	99.993	99.991	99.989	99.979	99.965	99.934
		Min	99.927	99.926	99.903	99.908	99.873	99.844	99.840	99.813	99.659	99.561	98.951
	Alg 2	Avg	98.043	98.011	97.975	97.915	97.854	97.771	97.691	97.522	97.343	97.020	96.634
		Min	95.032	95.020	95.005	94.995	94.528	94.641	94.750	94.831	94.518	94.392	94.758
$K_1=80$	Alg 1	Avg	99.996	99.995	99.995	99.993	99.992	99.992	99.986	99.985	99.969	99.957	99.927
		Min	99.935	99.930	99.911	99.883	99.865	99.808	99.800	99.736	99.505	99.204	99.279
	Alg 2	Avg	97.035	96.994	96.950	96.870	96.794	96.686	96.590	96.368	96.150	95.729	95.391
		Min	92.648	92.607	92.627	92.635	92.065	92.238	92.368	92.399	91.783	92.183	92.693
$n=3$			$t=1$	$t=2$	$t=3$	$t=4$	$t=5$	$t=6$	$t=7$	$t=8$	$t=9$	$t=10$	$t=11$
$K_1=20$	Alg 1	Avg	100	100	100	100	100	100	100	100	100	100	100
		Min	100	99.999	99.999	99.999	99.999	99.999	99.999	99.999	99.994	99.963	99.999
	Alg 2	Avg	99.616	99.602	99.588	99.568	99.544	99.516	99.478	99.423	99.346	99.214	98.949
		Min	98.558	98.558	98.557	98.558	98.557	98.557	98.557	98.552	98.563	98.613	98.450
$K_1=40$	Alg 1	Avg	100	100	100	99.999	99.999	99.999	99.999	99.998	99.997	99.994	99.990
		Min	99.992	99.991	99.988	99.985	99.982	99.976	99.969	99.952	99.917	99.818	99.550
	Alg 2	Avg	98.943	98.918	98.893	98.856	98.812	98.764	98.700	98.604	98.484	98.254	97.853
		Min	96.724	96.713	96.728	96.741	96.713	96.767	96.765	96.712	96.967	96.702	96.744
$K_1=60$	Alg 1	Avg	99.996	99.996	99.996	99.996	99.997	99.996	99.995	99.993	99.991	99.985	99.966
		Min	99.782	99.745	99.743	99.806	99.908	99.875	99.817	99.835	99.845	99.847	99.407
	Alg 2	Avg	98.056	98.019	97.986	97.935	97.874	97.811	97.722	97.592	97.443	97.113	96.707
		Min	94.669	94.662	94.723	94.730	94.716	94.861	94.782	94.823	95.245	94.689	94.864
$K_1=80$	Alg 1	Avg	99.998	99.997	99.997	99.996	99.996	99.995	99.993	99.992	99.988	99.981	99.957
		Min	99.933	99.921	99.900	99.885	99.925	99.896	99.847	99.865	99.867	99.634	99.475
	Alg 2	Avg	97.009	96.963	96.924	96.859	96.785	96.708	96.600	96.442	96.269	95.831	95.507
		Min	92.519	92.534	92.638	92.619	92.660	92.861	92.696	92.879	93.386	92.632	92.820

For each instance and $K_1 \in \{20, 40, 60, 80\}$, the optimal dynamic programming recursion in model (3) computes the optimal policy $\{z_t^*(x), p_t^*(x)\}$ and the optimal profit $v_t(x)$, whereas Algorithms 1 and 2 obtain the heuristic policies $\{\bar{z}_t(x), \bar{p}_t(x)\}$ and $\{\hat{z}_t(x), \hat{p}_t(x)\}$ and the corresponding profits $\bar{v}_t(x)$ and $\hat{v}_t(x)$, respectively. The performances of Algorithms 1 and 2 are measured by

$$\inf_{\substack{x\in[-800,800]:\\ v_t(x)>0}} \frac{\bar{v}_t(x)}{v_t(x)} \times 100\% \qquad \text{and} \qquad \inf_{\substack{x\in[-800,800]:\\ v_t(x)>0}} \frac{\hat{v}_t(x)}{v_t(x)} \times 100\%,$$

which correspond to the percentage of the optimal profit that can be achieved by Algorithms 1 and 2, respectively. In other words, if the performance measure of a policy is $a\%$, then the profit of the policy is $a\%$ of the maximal profit of the optimal policy. Also note that the performance measure is evaluated for initial inventory levels falling into the interval $[-800, 800]$, where 800 is the maximum possible demand one can observe in any period.

Table 1 presents the profits of Algorithms 1 and 2 as percentages of the optimum for any n, K_1, and t. As 100 instances are available for each combination, we report the average and minimum profits for the 100 instances in the rows headed "Avg" and "Min", respectively. Note that t corresponds to a $(T-t+1)$-period problem. As $T=12$, we omit the results of $t=12$ because our heuristic policy is always optimal for the single period problem.

Table 1 demonstrates that Algorithm 1 is very close to optimal. For all of the instances generated, it achieves 99.992% of the optimal profit on average and obtains 98.951% of the optimal profit in the worst case. When n increases from 2 to 3, the average and worst-case profits change from 99.989% and 98.951% to 99.995% and 99.407%, respectively, which indicates that there is no significant difference when n varies. Moreover, the performance tends to improve as t decreases. Because the profit for period t corresponds to that of a $(T-t+1)$-period problem, this observation implies that the performance of Algorithm 1 (in terms of percentage) will not deteriorate when the planning horizon expands. We also observe that the performance of Algorithm 1 is not sensitive to the fixed cost K_1, unlike Algorithm 2. The performance of Algorithm 2 is also satisfactory, achieving 98.055% of the optimal profit on average. However, its performance is shadowed by that of Algorithm 1, especially for a large K_1. For example, when $K_1 = 80$, the average and minimum profits of Algorithm 2 are 96.521% and 91.783%, respectively, which are 3.465% and 7.421% smaller than those of Algorithm 1, respectively.

4. Concave Variable Cost

In this section, we focus on the case where the variable cost function $c(z)$ is concave. Specifically, we first characterize the optimal joint pricing and inventory policy for the single-period problem in Subsection 4.1, which is well-structured. Based on this structure, a practically implementable and efficient heuristic policy is developed for the multi-period problem in Subsection 4.2. The performance of the heuristic is explored through extensive numerical studies in Subsection 4.3. This section is parallel to Section 3, which considers the convex variable cost, but here we do not study the special case of zero fixed cost because it does not lead to a significantly simpler optimal policy for the multi-period problem when the variable cost is concave.

4.1. Single-Period Problem

In this subsection, we study the optimal policy of the single-period problem. Hence, we only need to focus on the last period $t=T$. As function $u_t(y)$ in problem (3a) is concave, it is known (see, e.g., Chapter 9 in Porteus 2002) that a *generalized* (s,S) *policy* is optimal for problem (3a). That is, there are $m \le n$ and

$$s_m \le s_{m-1} \le \cdots \le s_1 \le S_1 \le \cdots \le S_{m-1} \le S_m, \tag{10}$$

such that it is optimal to raise the inventory level to S_m if $x < s_m$, to S_i if $s_{i+1} \le x < s_i$ for $1 \le i < m$, and to x if $x \ge s_1$. Furthermore, Lemma 9.13 in Porteus (2002) shows how to calculate $\{(s_j, S_j) : 1 \le j \le m\}$. To develop the heuristic policy for the multi-period problem, we provide an alternative method for determining the optimal generalized (s,S) policy for problem (3a), which is shown in the following algorithm.

ALGORITHM 3. 1. *For each* $1 \le i < n$, *compute*

$$r_i = \max_{i<j\le n} \sup \{x : [u_t(S_t(c_i)) - c(S_t(c_i) - x)] < [u_t(S_t(c_j)) - c(S_t(c_j) - x)]\}.$$

2. *Let* $\mathcal{I} = \{1 \le i < n : r_i < S_t(c_i)\} \cup \{n\}$ *and denote by* $\mathcal{I} = \{i_1 < i_2 < \cdots < i_k = n\}$. *Observe that* $\mathcal{I} = \{n\}$ *if it has only one element.*
3. *Initialize* $\mathcal{J} = \{i_1\}$. *For each* $1 < l < k$, *sequentially add index* $i_l \in \mathcal{I}$ *into* $\mathcal{J}$ *when* $r_{i_l} < \min\{r_{i_{l'}} : 1 \le l' < l\}$. *Finally, add index* $i_k = n$ *into* $\mathcal{J}$ *and denote by* $\mathcal{J} = \{j_1 < \cdots < j_{m-1} < j_m = n\}$. *Observe that* $\mathcal{J} = \{n\}$ *if it has only one element.*
4. *Compute sequence* $\{(s_l, S_l) : 1 \le l \le m\}$ *by letting* $s_1 = S_1 = S_t(c_{j_1})$, $s_l = r_{j_{l-1}}$ *and* $S_l = S_t(c_{j_l})$ *for each* $l = 2, \cdots, m$.

In Algorithm 3, r_i in Step 1 is well-defined because the concavity of $c(z)$ and $S_t(c_i) \le S_t(c_j)$ imply that $[u_t(S_t(c_i)) - c(S_t(c_i) - x)] - [u_t(S_t(c_j)) - c(S_t(c_j) - x)]$ is increasing in x. This monotonicity implies that

$$u_t(S_t(c_i)) - c(S_t(c_i) - x) \ge \max_{j:i<j\le n} [u_t(S_t(c_j)) - c(S_t(c_j) - x)]$$

if $x > r_t$ and

$$u_t(S_t(c_i)) - c(S_t(c_i) - x) < \max_{j:i<j\le n} [u_t(S_t(c_j)) - c(S_t(c_j) - x)]$$

if $x < r_t$. That is, given the initial inventory level x, if $x > r_i$, then raising the inventory level to $S_t(c_i)$ is more beneficial than raising it to $S_t(c_j)$ for any $j > i$; otherwise, it is less beneficial than raising the inventory level to $S_t(c_j)$ for some $j > i$.

Step 2 generates the index set $\mathcal{I}$ by collecting all indices i such that either $r_i < S_t(c_i)$ or $i = n$. Step 3 further generates an index set $\mathcal{J} \subseteq \mathcal{I}$ such that $r_i > r_j$ for any $i, j \in \mathcal{J}$ with $i < j$. Note that s_l and S_l obtained in Step 4 for $1 \le l \le m$ satisfy inequality (10), $s_1 = S_1$, and $S_m = S_t(c_n)$.

The following proposition helps us to characterize the optimal solution to problem (3a).

Proposition 3. *If a generalized* (s,S) *policy is optimal to problem* (3a)*, then* $R_t \le s_1$ *and* $v_t^0(x) > u_t(x)$ *if and only if* $x < R_t$*, and for any* $x < R_t$*,*

$$z_t^0(x) = \begin{cases} S_m - x, & \text{if } x < s_m, \\ S_l - x, & \text{if } s_{l+1} \le x < s_l \text{ and } 1 \le l < m, \end{cases} \tag{11}$$

where the sequence $\{(s_l, S_l) : 1 \le l \le m\}$ *is computed by Algorithm 3.*

Proposition 3 states that there is a threshold $R_t \le s_1$ below which we produce and above which we do not produce. Please note that this study uses R_t instead of s_1 to denote the reproduce point. Moreover, Proposition 3 provides a closed form of an optimal solution $z_t^0(x)$ that solves problem (4) when $x < R_t$, which only depends on $S_t(c_i)$ and $u_t(S_t(c_i))$ for all $1 \le i \le n$.

We are ready to characterize the optimal policy in the last period $t = T$ as follows. Recall that $R_t = \min\{x : v_t^0(x) \le u_t(x)\}$.

Theorem 5. *When* $t = T$ *and* $c(z)$ *is concave,* $\mathcal{O}_t = \{x < R_t\}$*,* $R_t \le s_1$*,* $z_t^*(x) = z_t^0(x)\mathbf{1}_{\{x<R_t\}}$ *with* $z_t^0(x)$ *given in* (11)*, and* $p_t^*(x) = P_t(c_{j_l})$ *when* $z_t^*(x) = S_l - x$*. Furthermore,* $z_t^*(x)$ *is decreasing in* x*, and* $p_t^*(x)$ *is increasing when* $x \le R_t$ *and decreasing when* $x \ge R_t$*.*

Figure 4 illustrates Theorem 5. It shows that for the single-period problem, production is executed if and only if the initial inventory level x falls below a threshold R_t. The state space on the left side of R_t can be partitioned into m regions with $m \le n$. Over each region, we have $z_t^*(x) = S_t(c_j) - x$ and $p_t = P_t(c_j)$ for some $1 \le j \le n$, i.e., the firm should produce up to a constant level $S_t(c_j)$ and charge a constant price $P_t(c_j)$. As the initial inventory level x increases, the optimal production quantity $z_t^*(x)$ decreases when $x < R_t$, jumps down to 0 at $x = R_t$ and then remains 0 for all $x > R_t$. Moreover, the optimal price $p_t^*(x)$ increases when $x \le R_t$ and then decreases when $x \ge R_t$. This suggests that if production is executed, then the firm should produce less and charge more in response to a higher initial inventory level. However, if production is not executed, then the firm should offer a deeper price discount for a higher initial inventory level.

4.2. Heuristic for the Multi-Period Problem

Similar to Section 3.3, in this subsection we provide a heuristic policy and show its worst-case performance for a concave $c(z)$. The following heuristic policy has the same structure as the optimal policy for the single-period problem illustrated in Figure 4 in Subsection 4.1. It shows how to compute the heuristic inventory and pricing policy, denoted by $\bar{z}_t(x)$ and $\bar{p}_t(x)$.

Algorithm 4. *Initialize* $\bar{v}_{T+1}(x) = v_{T+1}(x)$*. Consider any* $t = T, \cdots, 1$*.*

1. *Compute* $\bar{u}_t(y)$ *as below and let* $\bar{d}_t(y)$ *be the corresponding optimal solution:*

$$\bar{u}_t(y) = \max_{d \in \mathcal{D}_t} \left\{ d p_t(d) - \mathbb{E} h_t(y - \xi_t d - \varepsilon_t) + \gamma \mathbb{E} \bar{v}_{t+1}(y - \xi_t d - \varepsilon_t) \right\}.$$

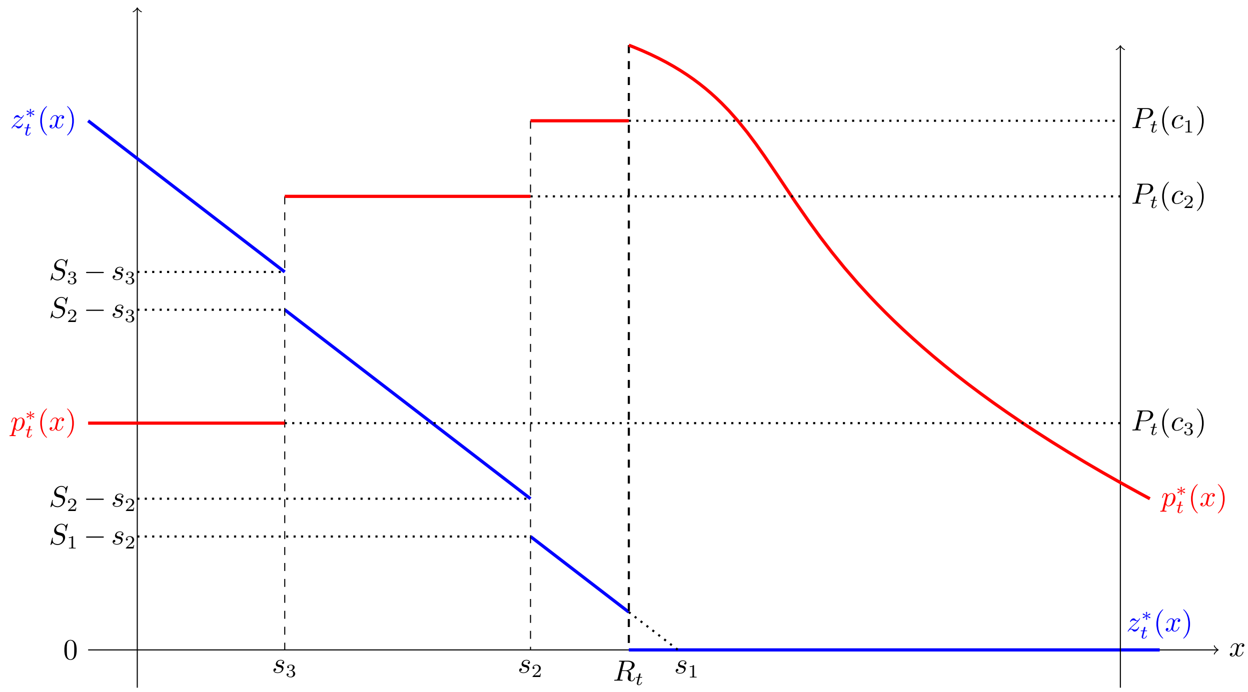

Figure 4 **Optimal production quantity $z_t^*(x)$ and price $p_t^*(x)$ in the last period $t = T$ when $c(z)$ is concave**

2. *For each $1 \le i \le n$, compute $\bar{S}_t(c_i) = \min\arg\max\{\bar{u}_t(y) - c_i y\}$ and $\bar{u}_t(\bar{S}_t(c_i))$. Apply Algorithm 3 to obtain index set $\mathcal{J}$ and sequence $\{(s_l, S_l) : 1 \le l \le m\}$ with $S_t(c_i)$ and $u_t(S_t(c_i))$ replaced by $\bar{S}_t(c_i)$ and $\bar{u}_t(\bar{S}_t(c_i))$ for all $1 \le i \le n$, respectively. Define*

$$\bar{z}_t^0(x) = \begin{cases} S_m - x, & \text{if } x < s_m, \\ S_l - x, & \text{if } s_{l+1} \le x < s_l \text{ and } 1 \le l < m, \end{cases}$$

and $\bar{v}_t^0(x) = \bar{u}_t(x + \bar{z}_t^0(x)) - c(\bar{z}_t^0(x))$.

3. *Compute $\bar{R}_t = \inf\{x < s_1 : \bar{v}_t^0(x) \le \bar{u}_t(x)\}$ and*

$$\bar{z}_t(x) = \begin{cases} \bar{z}_t^0(x), & \text{if } x < \bar{R}_t, \\ 0, & \text{if } x \ge \bar{R}_t. \end{cases}$$

4. *Compute $\bar{y}_t(x) = x + \bar{z}_t(x)$, $\bar{p}_t(x) = p_t(\bar{d}_t(\bar{y}_t(x)))$ and* $\bar{v}_t(x) = \begin{cases} \bar{v}_t^0(x), & x < \bar{R}_t, \\ \bar{u}_t(x), & x \ge \bar{R}_t. \end{cases}$

Notice that functions $\hat{u}_t$, $\hat{v}_t^0$ and $\hat{v}_t$ obtained in the algorithm are not necessarily concave. Similar to Theorem 3, we can prove the following results for the performance of Algorithm 4.

THEOREM 6. *In any period $t = 1, \cdots, T$, $\bar{v}_t(x)$ obtained by Algorithm 4 satisfies*

$$0 \le v_t(x) - \bar{v}_t(x) \le \sum_{i=1}^{T-t} i K_n \gamma^i. \tag{12}$$

Moreover, the heuristic algorithm is optimal, i.e., $\bar{v}_t(x) = v_t(x)$, if a generalized (s, S) policy is optimal to problem (3a)*, e.g.,* *(a) the single period problem, i.e, $t=T$;*

(b) demand uncertainty follows the additive model and ε_t is a positive Pólya or positive uniform random variable; and

(c) demand uncertainty follows the additive model and $K_1 > (\sum_{i=t}^{T}\gamma^{i-t}h_i^- - c_{n-1})q_{n-1}$ for any $1\le t\le T$.

Theorem 6 provides a performance bound of Algorithm 4 that depends on K_n, the number of periods T, and the discounted factor γ. Furthermore, it shows that this heuristic algorithm is optimal in three interesting cases, where part (a) and part (c) are consistent with Theorem 3, and part (b) is implied by Theorem 3 in Chen et al. (2010).

As the bound given by (12) is quadratic in the number of periods T, we provide another heuristic policy whose performance bound is linear in T. The basic idea is to replace the cost function $c(z)$ with $(K_n+c_nz)\mathbf{1}_{\{z>0\}}$ in each period. Hence, the profit-to-go function in this heuristic policy is not necessarily concave but symmetric-K_n concave. Therefore, the structure of this policy is the same as that of the optimal policy in Chen and Simchi-Levi (2004).

Algorithm 5. *Let $\hat{v}^0_{T+1}(x)=v_{T+1}(x)$ and $\hat{v}_{T+1}(x)=v_{T+1}(x)$ for any x. Consider any $t=T,\cdots,1$.*

1. *For each y, compute $\hat{u}_t(y)$ as below and let $\hat{d}_t(y)$ be the corresponding optimal solution:*

$$\hat{u}_t(y)=\max_{d\in\mathcal{D}_t}\left\{dp_t(d)-\mathbb{E}h_t(y-\xi_t d-\varepsilon_t)+\gamma\mathbb{E}\hat{v}^0_{t+1}(y-\xi_t d-\varepsilon_t)\right\}.$$

2. *For each x, compute $\hat{v}^0_t(x)$ as below and let $\hat{z}_t(x)$ be the corresponding optimal solution*

$$\hat{v}^0_t(x)=\max_{z\ge 0}\left\{\hat{u}_t(x+z)-(K_n+c_nz)\mathbf{1}_{\{z>0\}}\right\},$$

3. *For each x, compute $\hat{y}_t(x)=x+\hat{z}_t(x)$, $\hat{p}_t(x)=p_t(\hat{d}_t(\hat{y}_t(x)))$, and*

$$\begin{aligned}\hat{v}_t(x)=\;&\hat{d}_t(\hat{y}_t(x))\hat{p}_t(x)-\mathbb{E}h_t\big(\hat{y}_t(x)-\xi_t\hat{d}_t(\hat{y}_t(x))-\varepsilon_t\big)+\gamma\mathbb{E}\hat{v}_{t+1}\big(\hat{y}_t(x)-\xi_t\hat{d}_t(\hat{y}_t(x))-\varepsilon_t\big)\\&-c(\hat{z}_t(x))\mathbf{1}_{\{\hat{z}_t(x)>0\}}.\end{aligned}$$

In this algorithm, $\hat{v}^0_t(x)$ is used to compute the heuristic policy $\hat{z}_t(x)$ and $\hat{p}_t(x)$, which, when implemented, leads to the actual profit $\hat{v}_t(x)$. The performance of the heuristic policy obtained by Algorithm 5 is given below.

Theorem 7. *For any period t, $0\le v_t(x)-\hat{v}_t(x)\le\sum_{i=0}^{T-t}\gamma^i(K_n-K_1)$.*

Just as Algorithm 2 supports Algorithm 1, the purpose of presenting Algorithm 5 is to support Algorithm 4 by numerically comparing its performance with an algorithm whose worst-case performance bound is $O(TK_n)$. In the next subsection, our extensive numerical studies show that Algorithm 4 outperforms Algorithm 5, despite the fact that its worst-case performance bound is not as good as that of Algorithm 5.

4.3. Numerical Analysis

Table 2 Performance of Algorithms 4 and 5 (%)

$n=2$			$t=1$	$t=2$	$t=3$	$t=4$	$t=5$	$t=6$	$t=7$	$t=8$	$t=9$	$t=10$	$t=11$
$K_1=20$	Alg 4	Avg	99.989	99.988	99.988	99.988	99.985	99.979	99.975	99.971	99.958	99.923	99.827
		Min	99.518	99.454	99.465	99.499	99.481	99.310	99.325	99.208	99.137	98.448	96.487
	Alg 5	Avg	99.848	99.841	99.832	99.813	99.793	99.773	99.710	99.643	99.533	99.059	98.099
		Min	96.717	96.747	96.770	96.581	96.609	96.775	96.269	96.260	96.707	95.255	92.363
$K_1=40$	Alg 4	Avg	99.999	99.999	99.999	99.999	99.999	99.998	99.997	99.995	99.993	99.985	99.953
		Min	99.981	99.978	99.973	99.968	99.955	99.919	99.882	99.859	99.768	99.541	98.173
	Alg 5	Avg	99.919	99.916	99.902	99.891	99.880	99.854	99.815	99.758	99.641	99.197	98.328
		Min	98.192	98.172	98.090	98.090	98.028	97.889	97.826	97.670	96.705	96.110	92.481
$K_1=60$	Alg 4	Avg	100	100	100	100	100	100	100	100	99.999	99.998	99.991
		Min	99.996	99.995	99.994	99.992	99.988	99.972	99.959	99.951	99.917	99.839	99.098
	Alg 5	Avg	99.955	99.950	99.942	99.934	99.925	99.903	99.878	99.840	99.697	99.284	98.491
		Min	99.101	99.044	98.871	99.011	98.896	98.487	98.747	98.527	97.469	96.782	92.635
$K_1=80$	Alg 4	Avg	100	100	100	100	100	100	100	100	100	99.999	99.994
		Min	99.998	99.998	99.997	99.996	99.994	99.985	99.978	99.974	99.956	99.913	99.427
	Alg 5	Avg	99.975	99.970	99.967	99.961	99.953	99.938	99.916	99.885	99.730	99.353	98.593
		Min	99.437	99.346	99.388	99.308	99.376	99.053	98.963	99.031	98.208	96.719	92.742

$n=3$			$t=1$	$t=2$	$t=3$	$t=4$	$t=5$	$t=6$	$t=7$	$t=8$	$t=9$	$t=10$	$t=11$
$K_1=20$	Alg 4	Avg	99.976	99.971	99.971	99.969	99.964	99.963	99.960	99.951	99.931	99.890	99.804
		Min	98.176	97.837	97.892	97.857	97.735	97.686	97.852	97.924	97.519	97.194	95.550
	Alg 5	Avg	99.885	99.874	99.860	99.840	99.815	99.775	99.732	99.650	99.465	98.961	97.763
		Min	97.553	97.437	97.300	97.315	97.349	97.149	96.714	96.096	95.909	93.880	88.254
$K_1=40$	Alg 4	Avg	99.986	99.986	99.987	99.985	99.983	99.983	99.982	99.976	99.965	99.943	99.890
		Min	99.082	99.135	99.179	99.099	98.927	98.889	99.047	99.101	98.673	97.747	97.268
	Alg 5	Avg	99.947	99.938	99.927	99.911	99.895	99.862	99.834	99.763	99.596	99.091	97.973
		Min	98.906	98.767	98.699	98.661	98.763	98.475	98.005	97.530	97.076	93.964	88.312
$K_1=60$	Alg 4	Avg	99.998	99.998	99.998	99.997	99.996	99.995	99.993	99.990	99.983	99.966	99.935
		Min	99.868	99.854	99.847	99.803	99.734	99.632	99.529	99.524	99.239	98.575	97.856
	Alg 5	Avg	99.972	99.966	99.957	99.945	99.927	99.904	99.891	99.814	99.646	99.179	98.096
		Min	99.672	99.587	99.475	99.284	99.312	99.094	98.752	97.863	97.787	94.020	88.160
$K_1=80$	Alg 4	Avg	100	99.999	99.999	99.999	99.999	99.999	99.998	99.996	99.993	99.986	99.947
		Min	99.971	99.967	99.962	99.951	99.933	99.901	99.874	99.840	99.740	99.502	98.067
	Alg 5	Avg	99.981	99.977	99.970	99.962	99.948	99.929	99.917	99.854	99.698	99.245	98.196
		Min	99.760	99.694	99.606	99.511	99.386	99.333	98.845	98.120	98.169	94.107	88.233

The numerical experiments in this subsection are designed in the same fashion as those in Subsection 3.4 except for the following two changes, which ensure the concavity of $c(x)$. First, we let $c_1 = 1$ and c_i for all $1 < i \leq n$ be the reverse order statistics of $n-1$ uniform random numbers in $[0.6, 1]$. Second, the capacity q_n is set to $+\infty$ and q_i for all $1 \leq i < n$ are set to the order statistics of $n-1$ uniform random numbers in $[200, 1200]$. As in Subsection 3.4, 100 instances with $T = 12$

are generated independently for each $n \in \{2,3\}$. For each instance and $K_1 \in \{20,40,60,80\}$, the optimal dynamic programming recursion in model (3), Algorithm 4, and Algorithm 5 are applied to obtain the policies $\{z_t^*(x), p_t^*(x)\}$, $\{\bar{z}_t(x), \bar{p}_t(x)\}$, and $\{\hat{z}_t(x), \hat{p}_t(x)\}$ and the corresponding profits $v_t(x)$, $\bar{v}_t(x)$, and $\hat{v}_t(x)$, respectively. For any given n, K_1 and t, Table 2 summarizes the profits of Algorithms 4 and 5 as percentages of the optimum for the average and worst case of 100 instances.

Table 2 shows that Algorithm 4 is close to optimal as it achieves 99.98% and 95.55% of the optimal profit for the average and worst case, respectively. Moreover, the performance improves slightly as t decreases. Therefore, its excellent performance could be preserved for problems with long planning horizon. The performance also gets better for a larger K_1. This observation can be explained by part (c) of Theorem 6, which suggests that Algorithm 4 may perform very well for a sufficiently large K_1.

On average, Algorithm 5 also achieves 99.641% of the optimal profit, which is very satisfactory. However, this is still 0.339% smaller than the average performance of Algorithm 4. The worst-case performance of Algorithm 5 is 88.16% of the optimal profit, which is 7.39% smaller than that of Algorithm 4. Therefore, we conclude that Algorithm 4 performs better than Algorithm 5.

5. Characterization of Optimal Policy

In this section, we try to characterize the optimal policies of the general multi-period problems. As shown by counter examples in Lu and Song (2014) and Chen (2015), we know that even for pure inventory control problems without a pricing decision, the optimal policy can be very complicated such that a full characterization is not possible or meaningful. However, this does not mean that the optimal policies do not have any structural properties. The purpose of this section is to show that the structures of the optimal policies have some common features with those of the heuristic policies developed in this study.

A commonly used method in the literature is to construct a convex-like concept, show its preservation in the dynamic programming problem, and then characterize the optimal policy on the basis of such a convex-like concept. We follow this idea and introduce the following concepts.

Definition 1. Given a non-negative and increasing function $\kappa(x)$ defined on $\Re_+$, a function $f(x)$ is *κ-convex* if the following inequality holds for any $a, b \geq 0$ and $x_0 + a \leq x_1 - b$:

$$b[f(x_0+a) - f(x_0)] + a[f(x_1-b) - f(x_1)] \leq a\kappa(b). \tag{13}$$

It is *sym-κ-convex* if the following inequality holds for any $a, b \geq 0$ and $x_0 + a \leq x_1 - b$:

$$b[f(x_0+a) - f(x_0)] + a[f(x_1-b) - f(x_1)] \leq [a\kappa(b)] \vee [b\kappa(a)]. \tag{14}$$

Moreover, $f(x)$ is *κ-concave* (or *sym-κ-concave*) if $-f(x)$ is κ-convex (or sym-κ-convex).

To better understand this new concept intuitively, we rewrite (13) as

$$\frac{f(x_0+a)-f(x_0)}{a} \le \frac{f(x_1)-f(x_1-b)}{b} + \frac{\kappa(b)}{b}.$$

Please note that $x_0 \le x_0+a \le x_1-b \le x_1$. Similarly, (14) holds if and only if

$$\frac{f(x_0+a)-f(x_0)}{a} \le \frac{f(x_1)-f(x_1-b)}{b} + \frac{\kappa(a)}{a} \vee \frac{\kappa(b)}{b}.$$

Hence, the above two inequalities basically say that the slope of a κ-convex or sym-κ-convex function $f(x)$ does not decrease by some proper adjustment of $\kappa(x)$.

Definition 1 is closely related to K-convexity in Scarf (1960), sym-K-convexity in Chen and Simchi-Levi (2004), strong $(K,[c_1,\cdots,c_n],[q_1,\cdots,q_n])$-convexity in Lu and Song (2014), and c-convexity in Chen (2015). For convenience, their definitions are provided below.

Definition 2. Given $K \ge 0$, a function $f(x)$ is

(a) *K-convex* if $f((1-\lambda)x_0+\lambda x_1) \le (1-\lambda)f(x_0)+\lambda[f(x_1)+K]$ for any $0\le\lambda\le 1$ and $x_0 \le x_1$;

(b) *sym-K-convex* if the following inequality holds for any $0 \le \lambda = 1-\mu \le 1$ and x_0, x_1:

$$f(\mu x_0+\lambda x_1) \le \mu f(x_0)+\lambda f(x_1)+(\lambda\vee\mu)K; \tag{15}$$

(c) *strong $(K,[c_1,\cdots,c_n],[q_1,\cdots,q_n])$-convex* for $0 \le c_1 < \cdots < c_n$, and $0 < q_1 < \cdots < q_n = +\infty$ if $b[f(x_0+a)-f(x_0)]+a[f(x_1-b)-f(x_1)] \le a\kappa(b)$ for any $a \vee b \le a+b \le x_1-x_0$, where

$$\kappa(x) = K_1\mathbf{1}_{\{0\le x\le q_1\}} + \sum_{i=2}^{n}[K_i+(c_i-c_1)x]\mathbf{1}_{\{q_{i-1}<x\le q_i\}}, \tag{16}$$

where $K_1 = K$ and $K_{i+1} = K_i-(c_{i+1}-c_i)q_i$ for $i = 1,\cdots,n-1$; and

(d) *c-convex* for some non-negative, increasing, and concave function $c(x)$ if $f(\mu x_0+\lambda x_1) \le \mu f(x_0)+\lambda[f(x_1)+c(\mu(x_1-x_0))]$ for any $0\le\lambda = 1-\mu \le 1$ and $x_0 \le x_1$.

It is easy to see that a strong $(K,[c_1,\cdots,c_n],[q_1,\cdots,q_n])$-convexity is equivalent to the κ-convexity with $\kappa(x)$ given by (16). Moreover, the following proposition shows how the (sym-)κ-convexity is related to the other convexities.

Proposition 4. *(a) K-convexity is equivalent to κ-convexity with $\kappa(x)=K$.*

(b) Sym-K-convexity is implied by sym-κ-convexity with $\kappa(x)=K$.

(c) Given a non-negative, increasing, and concave function $c(x)$, c-convexity is implied by κ-convexity with $\kappa(x)=c(x)$.

We now provide two propositions on the preservation of κ-convexity or sym-κ-convexity in a class of parametric optimization problems associated with our applications.

Proposition 5. *Given random variables ε and $\xi \in [L,U]$ with $0 < L \le U$, a convex function h defined on $\mathcal{Z}$, and any non-negative and increasing function $\kappa(x)$ defined on $\Re_+$, suppose*

$$f(x) = \min_{z\in\mathcal{Z}}\left\{\mathbb{E}g(x-\xi z-\varepsilon)+h(z)\right\}.$$

If $g(x)$ is κ-convex and $U = L$ (i.e., ξ is deterministic), then $f(x)$ is also κ-convex; if $g(x)$ is sym-κ-convex and either $\kappa(x)$ is constant or $U \le 2L$, then $f(x)$ is also sym-κ-convex.

PROPOSITION 6. *For the cost function $c(z)$ given in (1), consider*

$$f(x) = \min_{z \ge 0}\{g(x+z) + c(z)\mathbf{1}_{\{z>0\}}\}.$$

When $c(z)$ is convex, the following statements hold:

(a) *if $g(x)$ is κ-convex with $\kappa(x) = c(x) - c_1 x$, then so is $f(x)$;*

(b) *if $g(x)$ is sym-κ-convex with $\kappa(x) = c(x) - c_1 x$, then so is $f(x)$; and*

(c) *if $K_n \ge 0$ and $g(x)$ is sym-κ-convex with $\kappa(x) = K_1$, then so is $f(x)$.*

Moreover, when $c(z)$ is concave, the following statements hold:

(d) *if $g(x)$ is κ-convex with $\kappa(x) = c(x) - c_n x$, then so is $f(x)$; and*

(e) *if $g(x)$ is sym-κ-convex with $\kappa(x) = c(x) - c_n x$, then so is $f(x)$.*

Notice that the preservation results presented in the above two propositions extend the corresponding results of previous studies such as Scarf (1960), Chen and Simchi-Levi (2004), Lu and Song (2014), and Chen (2015). They could be potentially useful to many similar applications. In particular, we are ready to characterize the optimal policy for the multi-period problem in our application for each period t. Recall that $\mathcal{O}_t$ is the set where it is optimal to produce. Hence, its complementary set $\mathcal{O}_t^c$ is the set where it is optimal to produce nothing. In the demand model, the multiplicative term has a random variable $\xi \in [L, U]$ with $0 < L \le U$.

THEOREM 8. *When $c(z)$ is convex, $z_t^*(x) + x$ increases with $x \in \mathcal{O}_t$, and it is equal to $S_t(c_i)$ if $S_t(c_i) - q_i < x \le S_t(c_i) - q_{i-1}$ for $1 \le i \le n$. Furthermore, $\{x < R_t\} \subseteq \mathcal{O}_t$ and the following statements hold:*

(a) *if $U = L$ or $K_n \ge 0$, then $\{x \ge S_t(c_1)\} \subseteq \mathcal{O}_t^c$; and*

(b) *if $U \le 2L$, then $\{x \ge S_t(c_1 - c_n)\} \subseteq \mathcal{O}_t^c$.*

When $c(z)$ is concave, $z_t^*(x) + x$ decreases with $x \in (-\infty, R_t)$, and it is equal to some $S_t(c_i)$ for any $x < R_t$. Furthermore, $\{x < R_t\} \subseteq \mathcal{O}_t$ and the following statements hold:

(a) *if $U = L$, then $\{x \ge S_t(c_n)\} \subseteq \mathcal{O}_t^c$; and*

(b) *if $U \le 2L$, then $\{x \ge S_t(0)\} \subseteq \mathcal{O}_t^c$.*

Please note that when $c(z)$ is convex, $S_t(c_1) < S_t(c_1 - c_n)$. Hence, the result of (a) is better than the result of (b). When $U = L$, the demand model is additive. It is expected that we can get a better result in this case because it is a special case of $U \le 2L$. $K_n \ge 0$ implies that average production cost $c(z)/z$ decreases with the production quantity z. This nice property helps us to establish the result without any assumption about the support of the random variable ξ. Theorem 8 shows that the structures of optimal policies share some common features with those of the heuristic policies developed in this study.

6. Conclusion

In this paper, we study the joint pricing and inventory control problem with a fixed cost and a convex or concave variable cost. We fully characterize the optimal policies for the single-period problems. The optimal policies are well-structured, which motivates us to develop practically implementable heuristic policies for the general multi-period problems. The heuristic policies have worst-case performance bounds, and their close-to-optimal performances are shown in our extensive numerical studies. In our characterizations of the optimal policies in Section 5, we propose new variations of convexity, namely, κ-convexity and sym-κ-convexity. We expect that these concepts will have applications in other problems with a similar cost structure.

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Appendix

Proof of Proposition 1

We first prove that $v_t^0(x) - u_t(x)$ is decreasing in x and negative at $x = S_t(c_1)$. To see it, consider any $x < \tilde{x}$. By the definition of $v_t^0(x)$ given in (4),

$$\begin{aligned} v_t^0(\tilde{x}) - u_t(\tilde{x}) &= \max_{z \geq 0} \{u_t(\tilde{x} + z) - c(z) - u_t(\tilde{x})\} \\ &\leq \max_{z \geq 0} \{u_t(x + z) - u_t(x) - c(z)\} = v_t^0(x) - u_t(x), \end{aligned}$$

where the inequality holds by the concavity of $u_t(y)$ and $z \geq 0$. Furthermore, if $x = S_t(c_1)$, then

$$\begin{aligned} v_t^0(x) - u_t(x) &= \max_{y \geq x} \{[u_t(y) - c_1 y] + [c_1 y - c(y - x)]\} - u_t(x) \\ &\leq [u_t(x) - c_1 x] + \max_{y \geq x} \{[c_1 y - c(y - x)]\} - u_t(x) \\ &\leq [-c_1 x] + [c_1 x - c(0)] = -c(0) \leq 0, \end{aligned}$$

where the first inequality holds because $x = S_t(c_1)$ maximizes the concave function $u_t(y) - c_1 y$, and the second inequality holds because $c_1 y - c(y-x)$ is decreasing in y by $c(z) = \max\{K_i + c_i z : 1 \le i \le n\} \ge c_1 z$ for any $z \ge 0$. Thus, $v_t^0(x) > u_t(x)$ if and only if $x < R_t$ and $R_t < S_t(c_1)$.

To see $z_t^0(x)$ given in (8) solves problem (4), notice that problem (4) is a concave maximization problem. It is well-known in convex analysis that as a sufficient condition for the optimality of $z_t^0(x)$, we only need to verify the following inequality for $z = z_t^0(x)$.

$$-\partial^- u_t(x+z) + \partial^- c(z) \le 0 \le -\partial^+ u_t(x+z) + \partial^+ c(z), \tag{A.1}$$

where $\partial^+ f(x)$ and $\partial^- f(x)$ denote the right-derivative and left-derivative of function $f(x)$, respectively, and we specify $\partial^- c(0) = -\infty$. Three cases are considered as below.

(a) If $S_t(c_i) - q_i \le x < S_t(c_i) - q_{i-1}$ for some $1 \le i \le n$, then obviously $z_t^0(x) = S_t(c_i) - x$ satisfies $q_{i-1} < z_t^0(x) \le q_i$. By the definition of $c(z)$, inequality (A.1) reduces to

$$\begin{cases} -\partial^- u_t(S_t(c_i)) + c_i \le 0 \le -\partial^+ u_t(S_t(c_i)) + c_i, & \text{if } z_t^0(x) < q_i, \\ -\partial^- u_t(S_t(c_i)) + c_i \le 0 \le -\partial^+ u_t(S_t(c_i)) + c_{i+1}, & \text{if } z_t^0(x) = q_i. \end{cases}$$

Both inequalities are satisfied because $S_t(c_i)$ is a minimizer of the convex function $-u_t(y) + c_i y$ and $c_i < c_{i+1}$ by the convexity of $c(z)$.

(b) When $S_t(c_{i+1}) - q_i \le x < S_t(c_i) - q_i$ for some $1 \le i < n$, by $z_t^0(x) = q_i$, inequality (A.1) becomes

$$-\partial^- u_t(x+q_i) + c_i \le 0 \le -\partial^+ u_t(x+q_i) + c_{i+1}, \ \forall S_t(c_{i+1}) - q_i \le x < S_t(c_i) - q_i.$$

Because both $-\partial^- u_t(y)$ and $-\partial^+ u_t(y)$ are increasing in y by the convexity of $-u_t(y)$, a sufficient condition to the above inequality is

$$-\partial^- u_t\left(S_t(c_i)\right) + c_i \le 0 \le -\partial^+ u_t\left(S_t(c_{i+1})\right) + c_{i+1},$$

which is satisfied since that $S_t(c_i)$ is a minimizer of the convex function $-u_t(y) + c_i y$.

(c) When $x \ge S_t(c_1)$, by the definition of $c(z)$, inequality (A.1) reduces to $0 \le -\partial^+ u_t(x) + c_1$. It holds because $-\partial^+ u_t(x)$ is increasing in x by the convexity of $-u_t(x)$, and $-\partial^+ u_t(S_t(c_1)) + c_1 \ge 0$ since that $S_t(c_1)$ is a minimizer of the convex function $-u_t(y) + c_1 y$. □

Proof of Theorem 1

Since that $z_t^*(x)$ has been characterized in Proposition 1, we only need to consider the optimal price $p_t^*(x) = p_t(d_t^*(y_t^*(x)))$, where by (8), the inventory level after producing $y_t^*(x) = x + z_t^0(x)$ is

$$y_t^*(x) = \begin{cases} x, & \text{if } x \ge S_t(c_1), \\ S_t(c_i), & \text{if } S_t(c_i) - q_i \le x < S_t(c_i) - q_{i-1} \text{ for } 1 \le i \le n, \\ x + q_i, & \text{if } S_t(c_{i+1}) - q_i \le x < S_t(c_i) - q_i \text{ for } 1 \le i < n, \end{cases}$$

By the definition of $P_t(a)$, it is straightforward to see $p_t^*(x) = P_t(c_i)$ when $z_t^*(x) = z_t^0(x) = S_t(c_i) - x$. To see the monotonicity of $p_t^*(x)$, because $p_t(d)$ is decreasing in d, it suffices to show $d_t^*(y)$ is increasing in y and $y_t^*(x)$ is increasing in x. The monotonicity of $y_t^*(x)$ can be directly verified from its expression. Moreover, since that the objective function of problem (3b) is supermodular in (y, d) by $\xi_t \geq 0$ and the concavity of $-h_t(x) + \gamma v_{t+1}(x)$, we know from Theorem 2.8.2 in Topkis (1998) that $d_t^*(y)$ is increasing in y. □

Proof of Theorem 2

We only need to show the monotonicity of $z_t^*(x)$ and $p_t^*(x)$. Other statements are either straightforward or can be verified by an argument similar to the proof of Theorem 1.

Because $z_t^0(x)$ is non-negative and decreasing in x, obviously $z_t^*(x) = z_t^0(x)\mathbf{1}_{\{x<R_t\}}$ is also decreasing in x. In addition, when $x < R_t$, because $z_t^*(x) = z_t^0(x)$, similar to the proof of Theorem 1, $p_t^*(x)$ is also decreasing in x. When $x \geq R_t$, we have $p_t^*(x) = p_t(d_t^*(x))$ because the inventory level after production is x. $d_t^*(y)$ is increasing in y since that the objective function of problem (3b) is supermodular in (y, d) by the concavity of u_t and $\xi_t \geq 0$. Thus, $p_t^*(x)$ is decreasing in $x \geq R_t$. Furthermore, because the inventory level after production is $x + z_t^*(x) > x$ when $x < R_t$ and it is equal to x when $x = R_t$, we know that $p_t^*(x)$ takes an upward jump at $x = R_t$. □

Proof of Proposition 2

Because x^* is the least minimizer of $f(x)$ if and only if $-x^*$ is the greatest minimizer of $f(-x)$, and $f^e(x)$ is the lower convex envelope of $f(x)$ if and only if $f^e(-x)$ is the lower convex envelope of $f(-x)$, it suffices to focus on the greatest minimizer part, where the least minimizer part immediately follows by considering function $f(-x)$ instead.

Given the greatest minimizer x^* of $f(x)$, it is also a minimizer of $f^e(x)$ because by the definition of the lower convex envelope, the following inequality is satisfied for any x:

$$\begin{aligned} f^e(x^*) &\leq f(x^*) = \inf\{(1-\lambda)f(x^*) + \lambda f(x^*) : \lambda \in [0,1]\} \\ &\leq \inf\{(1-\lambda)f(x_0) + \lambda f(x_1) : x = (1-\lambda)x_0 + \lambda x_1, \lambda \in [0,1]\} = f^e(x). \end{aligned}$$

Moreover, letting $x = x^*$ in the above inequality yields $f^e(x^*) \leq f(x^*) \leq f^e(x^*)$, i.e., $f(x^*) = f^e(x^*)$. To further ensure that x^* is the greatest minimizer of $f^e(x)$, by $f(x^*) \geq f^e(x^*)$, we only need to verify $f^e(x) > f(x^*)$ for any $x > x^*$. Notice from the definition of $f^e(x)$ that

$$\begin{aligned} f^e(x) &= f(x) \wedge \inf\{(1-\lambda)f(x_0) + \lambda f(x_1) : x = (1-\lambda)x_0 + \lambda x_1, 0 < \lambda < 1\} \\ &= f(x) \wedge \inf\{[bf(x-a) + af(x+b)]/(a+b) : a > 0 \text{ and } b > 0\}. \end{aligned}$$

For any fixed $x > x^*$, because x^* is the greatest minimizer, we have $f(x) > f(x^*)$ and hence as a sufficient condition to $f^e(x) > f(x^*)$, it remains to verify

$$0 < \inf_{a>0,b>0} \left\{ \frac{b}{a+b}[f(x-a) - f(x^*)] + \frac{a}{a+b}[f(x+b) - f(x^*)] \right\}. \tag{A.2}$$

First, we derive a lower bound of $f(x+b)$. On the one hand, because $\liminf_{x\to+\infty} [x^{-1}f(x)] > 0$, there exist $\delta_0 > 0$ and $B_0 > 0$ such that for any $b \geq B_0$, $(x+b)^{-1}f(x+b) \geq 2\delta_0$ and hence

$$\frac{f(x+b) - f(x^*)}{x+b-x^*} \geq \frac{2\delta_0(x+b) - f(x^*)}{x+b-x^*} = 2\delta_0 + \frac{2\delta_0 x^* - f(x^*)}{x+b-x^*}.$$

Because its right side converges to $2\delta_0 > 0$ as b goes to infinity, there exists $B \geq B_0$ such that

$$\forall\, b \geq B: \; f(x+b) - f(x^*) \geq \delta_0(x+b-x^*).$$

On the other hand, when $0 < b \leq B$, because $x^* < x+b \leq x+B$, x^* is the greatest minimizer, and function $f(y)$ is continuous on the compact set $[x, x+B]$, it follows that

$$\inf_{0<b\leq B} \frac{f(x+b) - f(x^*)}{x+b-x^*} \geq \inf_{0\leq b\leq B} \frac{f(x+b) - f(x^*)}{x+B-x^*} = \min_{0\leq b\leq B} \frac{f(x+b) - f(x^*)}{x+B-x^*} > 0.$$

The above inequality suggests that there exists positive δ_1 such that

$$\forall\, 0 < b \leq B: \; f(x+b) - f(x^*) \geq \delta_1(x+b-x^*).$$

In summary, we have $f(x+b) \geq f(x^*) + \delta(x+b-x^*)$ with $\delta = \delta_0 \wedge \delta_1 > 0$ for all $b > 0$.

By the obtained inequality on $f(x+b)$, to see inequality (A.2) holds, we only need to verify

$$0 < \inf_{a>0,b>0} \left\{ \frac{b}{a+b}[f(x-a) - f(x^*)] + \frac{a}{a+b}\delta(x+b-x^*) \right\}. \tag{A.3}$$

Denote by $\theta(a,b)$ the objective function on the right side. It is straightforward to prove that

$$\begin{aligned}\theta(a,b) &= \frac{b[f(x-a) - f(x^*) + a\delta] + a\delta(x-x^*)}{a+b} \\ &= [f(x-a) - f(x^*) + a\delta] + \frac{-a[f(x-a) - f(x^*) + a\delta] + a\delta(x-x^*)}{a+b},\end{aligned}$$

where term b appears only in the denominator of the second term on the right side. Thus, $\theta(a,b)$ is either increasing or decreasing in b, implying that for any $b > 0$,

$$\theta(a,b) \geq \theta(a,0) \wedge \lim_{b\to+\infty} \theta(a,b) = [\delta(x-x^*)] \wedge [f(x-a) - f(x^*) + a\delta].$$

By substituting this into the desired inequality (A.3), it remains to prove

$$0 < \inf_{a>0} \left\{ [\delta(x-x^*)] \wedge [f(x-a) - f(x^*) + \delta a] \right\} = \inf_{y<x} \left\{ [\delta(x-x^*)] \wedge [f(y) - f(x^*) + \delta(x-y)] \right\}. \tag{A.4}$$

Let $\bar{x} = (x^* + x)/2$, where $x^* < \bar{x} < x$ by $x^* < x$. On the one hand, when $y \leq \bar{x}$, obviously

$$f(y) - f(x^*) + \delta(x-y) \geq \delta(x-y) \geq \delta(x-\bar{x}) = \tfrac{1}{2}\delta(x-x^*).$$

On the other hand, when $\bar{x} < y < x$, $f(y) - f(x^*) > 0$ uniformly on $[\bar{x}, x]$ because x^* is the largest minimizer and function $f(y)$ is continuous. Thus, there exists $\delta' > 0$ such that

$$\min_{\bar{x} \le y \le x} \frac{f(y) - f(x^*)}{x - x^*} = \delta',$$

implying that $f(y) - f(x^*) + \delta(x - y) \ge f(y) - f(x^*) \ge \delta'(x - x^*)$. In summary, we know that

$$\inf_{y < x} [f(y) - f(x^*) + \delta(x - y)] \ge \left[(\tfrac{1}{2}\delta) \wedge \delta' \right] (x - x^*),$$

which yields the desirable inequality (A.4). This completes the proof, i.e., x^* is indeed the greatest minimizer of $f^e(x)$. $\square$

Proof of Theorem 3

Because Algorithm 1 generates a feasible solution $[\bar{z}_t(x), \bar{d}_t(x)]$ to problem (3), $v_t(x) \ge \bar{v}_t(x)$ in all periods t. We now focus on the upper bound of $v_t(x) - \bar{v}_t(x)$. To see it, consider for each t the lower convex envelopes of $-\bar{v}_t(x)$ and $-\bar{u}_t(x)$, denoted by $-\bar{v}_t^e(x)$ and $-\bar{u}_t^e(x)$, respectively. Define constants A_t and B_t for each $1 \le t \le T+1$ as below:

$$A_t = \sup_x [\bar{v}_t^e(x) - \bar{v}_t(x)] \quad \text{and} \quad B_t = \sup_x [v_t(x) - \bar{v}_t(x)].$$

Observe that $A_{T+1} = B_{T+1} = 0$ by $v_{T+1}(x) = \bar{v}_{T+1}(x) = 0$ in Algorithm 1. Moreover, since the heuristic policy has the same structure as the optimal policy for the single-period problem (see the discussion below Algorithm 1), by Theorem 2, we also have $B_T = 0$.

In any period $t \le T$, because $0 \le v_{t+1}(x) - \bar{v}_{t+1}(x) \le B_{t+1}$, and $\bar{u}_t(x)$ given in Step 1 is a counterpart of $u_t(x)$ given in (3b) with $v_{t+1}(x)$ replaced by $\bar{v}_{t+1}(x)$, we have that

$$0 \le u_t(y) - \bar{u}_t(y) \le \gamma B_{t+1}. \tag{A.5}$$

Define $\hat{u}_t(y)$ below as a counterpart of $u_t(x)$ in (3b) with $v_{t+1}(x)$ replaced by $\bar{v}_{t+1}^e(x)$:

$$\hat{u}_t(y) = \max_{d \in \mathcal{D}_t} \left\{ dp_t(d) - \mathbb{E}h_t(y - \xi_t d - \varepsilon_t) + \gamma \mathbb{E}\bar{v}_{t+1}^e(y - \xi_t d - \varepsilon_t) \right\}.$$

Because $-\bar{v}_{t+1}^e(x) \le -\bar{v}_{t+1}(x) \le A_{t+1} - \bar{v}_{t+1}^e(x)$ by the definition of A_{t+1}, we know that

$$-\hat{u}_t(y) \le -\bar{u}_t(y) \le -\hat{u}_t(y) + \gamma A_{t+1}, \tag{A.6}$$

where $\hat{u}_t(y)$ is obviously concave by the convexity of $h_t(x)$ and the concavity of $\bar{v}_{t+1}^e(x)$. Because $-\hat{u}_t(y)$ is a convex function no more than $-\bar{u}_t(y)$ by the first inequality in (A.6), and $-\bar{u}_t^e(y)$ is the lower convex envelope of $-\bar{u}_t(y)$, we know that $-\hat{u}_t(y) \le -\bar{u}_t^e(y)$. Furthermore, by the second inequality in (A.6) and $-\bar{u}_t^e(y) \le -\bar{u}_t(y)$, we have that $-\bar{u}_t^e(y) \le -\bar{u}_t(y) \le -\bar{u}_t^e(y) + \gamma A_{t+1}$, i.e.,

$$0 \le \bar{u}_t^e(y) - \bar{u}_t(y) \le \gamma A_{t+1}. \tag{A.7}$$

This together with (A.5) ensures that

$$u_t(y) - \bar{u}_t^e(y) \le \gamma B_{t+1} - [\bar{u}_t^e(y) - \bar{u}_t(y)] \le \gamma B_{t+1}. \tag{A.8}$$

To see the relation among B_t, B_{t+1} and A_{t+1}, define functions

$$\hat{v}_t(x) = \max_{z\ge 0} \left\{\bar{u}_t^e(x+z) - c(z)\mathbf{1}_{\{z>0\}}\right\}, \tag{A.9a}$$

$$\hat{v}_t^0(x) = \max_{z\ge 0} \left\{\bar{u}_t^e(x+z) - c(z)\right\}, \tag{A.9b}$$

which are counterparts of $v_t(x)$ in (3a) and $v_t^0(x)$ in (4) with $u_t(y)$ replaced by $\bar{u}_t^e(y)$, respectively. By inequality (A.8) and $0 \le c(z) - c(z)\mathbf{1}_{\{z>0\}} \le K_1$, we have that

$$v_t(x) - \hat{v}_t(x) \le \gamma B_{t+1} \quad \text{and} \quad 0 \le \hat{v}_t(x) - \hat{v}_t^0(x) \le K_1. \tag{A.10}$$

Since that $\bar{u}_t^e(y)$ in (A.9) is concave, similar to the results for the single-period problem, there is some threshold $\hat{R}_t$ such that

$$\hat{v}_t(x) = \begin{cases} \hat{v}_t^0(x) \ge \bar{u}_t^e(x), & \text{if } x < \hat{R}_t, \\ \bar{u}_t^e(x) \ge \hat{v}_t^0(x), & \text{if } x \ge \hat{R}_t. \end{cases}$$

We next prove that $\bar{z}_t^0(x)$ in Step 2 solves problem (A.9b), i.e.,

$$\hat{v}_t^0(x) = \bar{u}_t^e(x + \bar{z}_t^0(x)) - c(\bar{z}_t^0(x)). \tag{A.11}$$

The basic idea is to apply Proposition 2. We make three observations below:

(i) The least minimizer of $c_i y - \bar{u}_t(y)$ defined in Step 2, is finite, because $|u_t(y) - \bar{u}_t(y)| \le \gamma B_{t+1}$ by (A.5) and $\lim_{|y|\to\infty} [u_t(y) - c_i y] = -\infty$ as assumed in Section 2. Moreover, we can show $\lim_{y\to-\infty} y^{-1}[\bar{u}_t(y) - c_i y] > 0$ by applying the following lemma, whose proof is presented after the proof of Theorem 3.

Lemma 1. *Consider $c(z)$ defined in (1). For any $1 \le t \le T$, there exist $\dot{x}_t, \dot{y}_t > -\infty$ such that*

(a) $z_t^*(x) = S_t(c_n) - x$ and $v_t(x) = v_t(\dot{x}_t) - c_n(\dot{x}_t - x)$ for any $x \le \dot{x}_t$;

(b) $u_t(y) = u_t(\dot{y}_t) - (h_t^- + \gamma \mathbf{1}_{\{t<T\}} c_n)(\dot{y}_t - y)$ for any $y \le \dot{y}_t$.

Lemma 1 (b) implies

$$\lim_{y\to-\infty} y^{-1}[u_t(y) - c_i y] = h_t^- + \gamma \mathbf{1}_{\{t<T\}} c_n - c_i.$$

As $\lim_{|y|\to\infty} [u_t(y) - c_i y] = -\infty$, Lemma 1 (b) also shows that $h_t^- + \gamma \mathbf{1}_{\{t<T\}} c_n - c_i > 0$. Thus, $\lim_{y\to-\infty} y^{-1}[u_t(y) - c_i y] > 0$. $|u_t(y) - \bar{u}_t(y)| \le \gamma B_{t+1}$ then yields $\lim_{y\to-\infty} y^{-1}[\bar{u}_t(y) - c_i y] > 0$.

(ii) $\bar{u}_t(y)$ is continuous. In fact, note that $\bar{v}_{T+1}(x)=0$ is obviously continuous. Suppose $\bar{v}_{t+1}(x)$ is continuous in some period $1 \le t \le T$. In period t, because functions $dp_t(d)$, $h_t(y)$ and $\bar{v}_{t+1}(y)$ are continuous, we know that function $\bar{u}_t(y)$ in Step 1 is also continuous. Moreover, $\bar{v}_t^0(x)$ in Step 2 is continuous because $\bar{z}_t^0(x)$ is continuous. According to the continuity of $\bar{u}_t(y)$ and $\bar{v}_t^0(x)$, the definition of $\bar{R}_t$ in Step 3 implies $\bar{u}_t(\bar{R}_t)=\bar{v}_t^0(\bar{R}_t)$ as long as $\bar{R}_t$ is finite. Therefore, $\bar{v}_t(x)$ is also continuous.

(iii) $c_i y - \bar{u}_t^e(y)$ is the lower convex envelope of $c_i y - \bar{u}_t(y)$ for any $1 \le i \le n$ because

$$\begin{aligned} c_i y - \bar{u}_t^e(y) &= c_i y + \inf\{-(1-\lambda)\bar{u}_t(y_0) - \lambda \bar{u}_t(y_1) : y = (1-\lambda)y_0 + \lambda y_1, \lambda \in [0,1]\} \\ &= \inf\{(1-\lambda)[c_i y_0 - \bar{u}_t(y_0)] + \lambda[c_i y_1 - \bar{u}_t(y_1)] : y = (1-\lambda)y_0 + \lambda y_1, \lambda \in [0,1]\}. \end{aligned}$$

where the first equation holds since $-\bar{u}_t^e(y)$ is the lower convex envelope of $-\bar{u}_t(y)$.

With the above observations, we know from Proposition 2 that $\bar{S}_t(c_i)$ is also the least minimizer of $c_i x - \bar{u}_t^e(x)$. Because $\bar{z}_t^0(x)$ specified in Step 2 only depends on the values of $\bar{S}_t(c_i)$ for $1 \le i \le n$, by the concavity of $\bar{u}_t^e(y)$, similar to the proof of Proposition 1, we conclude that $\bar{z}_t^0(x)$ indeed solves problem (A.9b).

By (A.11) and $\bar{v}_t^0(x) = \bar{u}_t(x + \bar{z}_t^0(x)) - c(\bar{z}_t^0(x))$ in Step 2, we next prove

$$0 \le \hat{v}_t(x) - \bar{v}_t(x) \le K_1 + \gamma A_{t+1}. \tag{A.12}$$

Recall from their definitions that

$$\hat{v}_t(x) = \begin{cases} \hat{v}_t^0(x), & \text{if } x < \hat{R}_t \\ \bar{u}_t^e(x), & \text{if } x \ge \hat{R}_t \end{cases}, \qquad \bar{v}_t(x) = \begin{cases} \bar{v}_t^0(x), & \text{if } x < \bar{R}_t \\ \bar{u}_t(x), & \text{if } x \ge \bar{R}_t \end{cases}.$$

Four cases are distinguished as below, where we let $\bar{z} = \bar{z}_t^0(x)$ for notational simplicity.

(i) When $x < \bar{R}_t \wedge \hat{R}_t$, by (A.11) and the definition of $\bar{v}_t^0(x)$, we can express

$$\hat{v}_t^0(x) - \bar{v}_t^0(x) = [\bar{u}_t^e(x+\bar{z}) - c(\bar{z})] - [\bar{u}_t(x+\bar{z}) - c(\bar{z})].$$

Thus, inequality (A.12) is immediately yielded by (A.7).

(ii) When $\bar{R}_t \le x < \hat{R}_t$, we know from (A.9b) and $\bar{R}_t = \sup\{x : \bar{v}_t^0(x) > \bar{u}_t(x)\}$ that

$$\begin{aligned} \hat{v}_t^0(x) - \bar{u}_t(x) &\ge \bar{u}_t^e(x) - \bar{u}_t(x), \\ \hat{v}_t^0(x) - \bar{u}_t(x) &\le \hat{v}_t^0(x) - \bar{v}_t^0(x) = \bar{u}_t^e(x+\bar{z}) - \bar{u}_t(x+\bar{z}). \end{aligned}$$

Thus, inequality (A.12) is yielded by (A.7).

(iii) When $\hat{R}_t \le x < \bar{R}_t$, because $\bar{u}_t^e(x) = \hat{v}_t(x) \ge \hat{v}_t^0(x)$, by the definitions of $\hat{v}_t^0(x)$ and $\bar{v}_t^0(x)$,

$$\bar{u}_t^e(x) - \bar{v}_t^0(x) \ge \hat{v}_t^0(x) - \bar{v}_t^0(x) = \bar{u}_t^e(x+\bar{z}) - \bar{u}_t(x+\bar{z}).$$

Thus, $\bar{u}_t^e(x) - \bar{v}_t^0(x) \geq 0$ by (A.7). Furthermore, by inequalities (A.10) and (A.7),

$$\begin{aligned}\bar{u}_t^e(x) - \bar{v}_t^0(x) &\leq \left[\hat{v}_t^0(x) + K_1\right] - [\bar{u}_t(x+\bar{z}) - c(\bar{z})] \\ &\leq \left[\hat{v}_t^0(x) + K_1\right] - [\bar{u}_t^e(x+\bar{z}) - c(\bar{z})] + \gamma A_{t+1}.\end{aligned}$$

Thus, $\bar{u}_t^e(x) - \bar{v}_t^0(x) \leq K_1 + \gamma A_{t+1}$ by (A.11), implying inequality (A.12) holds.

(iv) When $x \geq \bar{R}_t \vee \hat{R}_t$, inequality (A.12) immediately follows from (A.7).

By the definition of B_t, as well as inequalities (A.10) and (A.12), we conclude that

$$B_t = \sup_x \{[v_t(x) - \hat{v}_t(x)] + [\hat{v}_t(x) - \bar{v}_t(x)] \leq \gamma B_{t+1} + (K_1 + \gamma A_{t+1}). \tag{A.13}$$

To see the relation between A_t and A_{t+1}, observe from (A.10) and (A.12) that

$$-K_1 \leq \hat{v}_t^0(x) - \bar{v}_t(x) \leq \hat{v}_t(x) - \bar{v}_t(x) \leq K_1 + \gamma A_{t+1}.$$

By properly arranging terms in the above inequality, it leads to

$$-[K_1 + \hat{v}_t^0(x)] \leq -\bar{v}_t(x) \leq K_1 + \gamma A_{t+1} - \hat{v}_t^0(x). \tag{A.14}$$

By the convexity of $c(z)$, $\hat{v}_t^0(x)$ given in (A.9b) is concave. Because $-\bar{v}_t^e(x)$ is the lower convex envelope of $-\bar{v}_t(x)$, by (A.14), $-[K_1 + \hat{v}_t^0(x)] \leq -\bar{v}_t^e(x)$ and hence $-\bar{v}_t(x) \leq 2K_1 + \gamma A_{t+1} - \bar{v}_t^e(x)$, implying

$$A_t = \sup_x [\bar{v}_t^e(x) - \bar{v}_t(x)] \leq 2K_1 + \gamma A_{t+1}.$$

In summary, we conclude that $B_T = 0$, $A_T = 2K_1$ and for any $1 \leq t < T$,

$$A_t \leq 2K_1 + \gamma A_{t+1} \quad \text{and} \quad B_t \leq K_1 + \gamma(A_{t+1} + B_{t+1}).$$

By some basic algebra, it can be verified that for each $t < T$,

$$A_t \leq \sum\nolimits_{i=0}^{T-t} (2K_1)\gamma^i \quad \text{and} \quad B_t \leq \sum\nolimits_{i=0}^{T-t} [(2i+1)K_1]\gamma^i - K_1\gamma^{T-t}.$$

Thus, by the definition of B_t, we obtain the upper bound of $v_t(x) - \bar{v}_t(x)$ in Theorem 3.

To see these sufficient conditions for $\bar{v}_t(x) = v_t(x)$, notice that condition (a) is ensured by $B_T = 0$ as proved. Moreover, condition (b) can be derived from Theorem 1. Thus, it suffices to focus on condition (c). Suppose that $B_{t+1} = 0$, i.e., $\bar{v}_{t+1}(x) = v_{t+1}(x)$, for some $1 \leq t \leq T$. Then $\bar{u}_t(x) = u_t(x)$ and $\bar{S}_t(c_n) = S_t(c_n)$. In addition, by the definition of $c(z)$, it can be verified that

$$K_n = K_1 + (c_1 - c_2)q_1 + \cdots + (c_n - c_{n-1})q_{n-1},$$

where K_n is the intercept of the last linear piece of $c(z)$. Thus, the given assumption on K_1 can be expressed by $K_n > (H_t - c_n)q_{n-1}$ for $H_t = \sum_{i=t}^{T} \gamma^{i-t} h_i^-$. In addition, because demand uncertainty is additive, we are able to prove the following result on $z_t^*(x)$:

LEMMA 2. *Consider $c(z)$ defined in (1). Suppose that $K_n > (H_t - c_n)q_{n-1}$ for any $1 \le t \le T$ and $c(z) \ge K_n + c_n z$ for any $z \ge 0$. Then, for any $1 \le t \le T$, $z_t^*(x) = S_t(c_n) - x$ if $x < R_t^n$ and $z_t^*(x) = 0$ otherwise, where $S_t(c_n) - R_t^n > q_{n-1}$ and*

$$R_t^n = \sup\{x \le S_t(c_n) : [u_t(S_t(c_n)) - c_n S_t(c_n)] > [u_t(x) - c_n x] + K_n\}. \tag{A.15}$$

The proof of Lemma 2 is moved to the end of this subsection to streamline the discussion. Lemma 2 states that it is optimal to produce if and only if x is below the threshold R_t^n; moreover, if $x \le R_t^n$, then it is optimal to raise the inventory level to $S_t(c_n)$ by producing at least q_{n-1} units. That is, an (s, S) policy is optimal to problem (3a).

It suffices to show $R_t^n = \bar{R}_t$ for the threshold $\bar{R}_t$ obtained in Step 3 because this ensures $\bar{z}_t(x)$ obtained in Step 3 is equal to $z_t^*(x)$ and $\bar{v}_t(x)$ in Step 4 is equal to $v_t(x)$. There are two cases.

(i) To see $R_t^n \le \bar{R}_t$, by $S_t(c_n) - R_t^n > q_{n-1}$, we have $\bar{v}_t^0(x) = u_t(S_t(c_n)) - c_n(S_t(c_n) - x) - K_n$ for any $x \le R_t^n$. Also note that $\bar{u}_t(x) = u_t(x)$ for all x. The definitions of R_t^n and $\bar{R}_t$ immediately yield $R_t^n \le \bar{R}_t$.

(ii) To see $R_t^n \ge \bar{R}_t$, for any $x > R_t^n$, because it is optimal not to produce by Lemma 2,

$$u_t(x) \ge \max_{z \ge 0} \{u_t(x+z) - c(z)\} \ge u_t(x + \bar{z}_t^0(x)) - c(\bar{z}_t^0(x)).$$

Thus, $x > \bar{R}_t$ by the definition of $\bar{R}_t$, implying that $R_t^n \ge \bar{R}_t$. □

Proof of Lemma 1: Let $\dot{x}_{T+1} = 0$. $v_{T+1}(x) = 0$ for any x implies $v_{T+1}(x) = v_{T+1}(\dot{x}_{T+1})$ for any $x \le \dot{x}_{T+1}$. Consequently, for any given $1 \le t \le T$, we can assume for induction that there exists $\dot{x}_{t+1} > -\infty$ such that $v_{t+1}(x) = v_{t+1}(\dot{x}_{t+1}) - \mathbf{1}_{\{t<T\}} c_n(\dot{x}_{t+1} - x)$ for any $x \le \dot{x}_{t+1}$. Let $\dot{y}_t = 0 \wedge \dot{x}_{t+1}$. Recall that $\xi_t d + \epsilon_t \ge 0$ with probability 1 for any $d \in \mathcal{D}_t$. As $h_t(I) = -h_t^-(0 \wedge I) + h_t^+(0 \vee I)$, we have

$$\begin{aligned} u_t(y) &= \max_{d \in \mathcal{D}_t} \left\{ d p_t(d) + h_t^- \mathbb{E}(y - \xi_t d - \varepsilon_t) + \gamma v_{t+1}(\dot{x}_{t+1}) - \gamma \mathbf{1}_{\{t<T\}} c_n \mathbb{E}(\dot{x}_{t+1} - (y - \xi_t d - \varepsilon_t)) \right\} \\ &= (h_t^- + \gamma \mathbf{1}_{\{t<T\}} c_n) y + \gamma v_{t+1}(\dot{x}_{t+1}) + \max_{d \in \mathcal{D}_t} \left\{ d p_t(d) - h_t^- d - \gamma \mathbf{1}_{\{t<T\}} c_n (\dot{x}_{t+1} + d) \right\} \end{aligned}$$

for any $y \le \dot{y}_t$, i.e., $u_t(y) = u_t(\dot{y}_t) - (h_t^- + \gamma \mathbf{1}_{\{t<T\}} c_n)(\dot{y}_t - y)$ for any $y \le \dot{y}_t$. Also note that Section 2 assumes $\lim_{|y| \to \infty} [u_t(y) - c_i y] = -\infty$ for all $1 \le i \le n$. It is straightforward that $h_t^- + \gamma \mathbf{1}_{\{t<T\}} c_n - c_i > 0$ and $S_t(c_i) \ge \dot{y}_t$ for all $1 \le i \le n$.

Let

$$\dot{x}_t = \dot{y}_t - q_{n-1} \vee \left(\frac{K_n}{h_t^- + \gamma \mathbf{1}_{\{t<T\}} c_n - c_n} + 1 \right) > -\infty.$$

Consider any $x \le \dot{x}_t$. For any $0 \le z \le q_{n-1}$, we have $x + z \le \dot{y}_t$. As $h_t^- + \gamma \mathbf{1}_{\{t<T\}} c_n - c_i > 0$ for all $1 \le i \le n$, the definition of $c(z)$ implies that $u_t(x+z) - c(z)$ is increasing in $0 \le z \le q_{n-1}$ and hence

$$\max_{0 \le z \le q_{n-1}} \left\{ u_t(x+z) - c(z) \right\} = u_t(x + q_{n-1}) - c(q_{n-1}). \tag{A.16}$$

Also note that $x+q_{n-1} \le \dot{x}_t + q_{n-1} \le \dot{y}_t \le S_t(c_n)$. Therefore,

$$\max_{z\ge q_{n-1}} \{u_t(x+z)-c(z)\} = \max_{y\ge x+q_{n-1}} \{u_t(y)-c(y-x)\} = \max_{y\ge x+q_{n-1}} \{u_t(y)-K_n-c_n(y-x)\}$$
$$= u_t(S_t(c_n)) - K_n - c_n(S_t(c_n)-x) \ge u_t(\dot{y}_t) - K_n - c_n(\dot{y}_t - x), \quad \text{(A.17)}$$

where the inequality follows from the definition of $S_t(c_n)$. (A.16) and (A.17) yield

$$v_t^0(x) = \max_{z\ge 0} \{u_t(x+z)-c(z)\} = u_t(S_t(c_n)) - K_n - c_n(S_t(c_n)-x) \ge u_t(\dot{y}_t) - K_n - c_n(\dot{y}_t - x).$$

Recall that $x \le \dot{x}_t \le \dot{y}_t$. We obtain $u_t(x) = u_t(\dot{y}_t) - (h_t^- + \gamma \mathbf{1}_{\{t<T\}} c_n)(\dot{y}_t - x)$ and so

$$\begin{aligned} u_t(\dot{y}_t) - K_n - c_n(\dot{y}_t - x) &= u_t(x) + (h_t^- + \gamma \mathbf{1}_{\{t<T\}} c_n - c_n)(\dot{y}_t - x) - K_n \\ &\ge u_t(x) + (h_t^- + \gamma \mathbf{1}_{\{t<T\}} c_n - c_n)(\dot{y}_t - \dot{x}_t) - K_n \\ &\ge u_t(x) + (h_t^- + \gamma \mathbf{1}_{\{t<T\}} c_n - c_n)\left(\frac{K_n}{h_t^- + \gamma \mathbf{1}_{\{t<T\}} c_n - c_n} + 1\right) - K_n > u_t(x), \end{aligned}$$

where the inequalities are yielded by $h_t^- + \gamma \mathbf{1}_{\{t<T\}} c_n - c_n > 0$, $x \le \dot{x}_t$, and the definition of $\dot{x}_t$. As a result,

$$v_t(x) = v_t^0(x) \vee u_t(x) = v_t^0(x) = u_t(S_t(c_n)) - K_n - c_n(S_t(c_n)-x) = v_t(\dot{x}_t) - c_n(\dot{x}_t - x),$$

and $z_t(x) = S_t(c_n) - x$ for all $x \le \dot{x}_t$. □

Proof of Lemma 2: Notice that if let $H_{T+1} = 0$, then we can inductively define $H_t = h_t^- + \gamma H_{t+1}$ for $t = T, \cdots, 1$. We divide the proof into two parts.

(a) We first inductively prove that $v_t^H(x) = v_t(x) - H_t x$ and $u_t^H(x) = u_t(x) - H_t x$ are decreasing in x for each t. Suppose $v_{t+1}^H(x)$ is decreasing in x for some $1 \le t \le T$, where the statement is trivial at $t=T$ by $v_{T+1}^H(x) = v_{T+1}(x) = 0$. By $\xi_t = 1$ and (3b), we have that

$$u_t^H(y) = \max_{d\in\mathcal{D}_t} \{d[p_t(d) - H_t] + \mathbb{E} w_t(y-d-\varepsilon_t) + \gamma \mathbb{E} v_{t+1}^H(y-d-\varepsilon_t)\},$$

where $w_t(x) = (\gamma H_{t+1} - H_t)x - h_t(x) = -(h_t^- + h_t^+)(0 \vee x)$ is clearly decreasing in x. Thus, $u_t^H(y)$ is decreasing in y. In addition, by (3a), we can express

$$v_t^H(x) = \max_{z\ge 0} \{u_t^H(x+z) - [c(z) - H_t z]\mathbf{1}_{\{z>0\}}\},$$

which is clearly decreasing in x by monotonicity of function $u_t^H(x)$.

(b) We next inductively prove that $v_t(x)$ is K_n-concave and equal to the function below for each t:

$$v_t^n(x) = \max_{z\ge 0} \{u_t(x+z) - (K_n + c_n z)\mathbf{1}_{\{z>0\}}\}. \quad \text{(A.18)}$$

Suppose $v_{t+1}(x)$ is K_n-concave for some $1 \le t \le T$, where the statement is trivial at $t=T$ by $v_{T+1}(x) = 0$. Then function $u_t(x)$ is also K_n-concave by applying Proposition 5(a) to problem (3b). For problem (A.18), by Scarf (1960), $v_t^n(x)$ is K_n-concave, and $z_t^*(x) = [S_t(c_n) - x]\mathbf{1}_{\{x<R_t^n\}}$

given in Lemma 2 solves problem (A.18). Thus, to complete this proof, we only need to further prove that $z_t^*(x)$ also solves problem (3a) and satisfies $z_t^*(x) > q_{n-1}$ for any $x < R_t^n$.

To see it, given any $x < R_t^n$, by $K_n > (H_t - c_n)q_{n-1}$, the definition of R_t^n in (A.15), monotonicity of $u_t(x) - H_t x$ proved in part (a), and the definition of $z_t^*(x)$, we have that

$$\begin{aligned}(H_t - c_n)q_{n-1} < K_n &\leq [u_t(S_t(c_n)) - c_n S_t(c_n)] - [u_t(x) - c_n x] \\ &\leq (H_t - c_n)[S_t(c_n) - x] = (H_t - c_n)z_t^*(x).\end{aligned}$$

implying that $z_t^*(x) > q_{n-1}$ for any $x < R_t^n$. Thus, either $z_t^*(x) = 0$ or $z_t^*(x) > q_{n-1}$ for all x. By $(K_n + c_n z)\mathbf{1}_{\{z>0\}} = c(z)\mathbf{1}_{\{z>0\}}$ for either $z = 0$ or $z > q_{n-1}$, we have that

$$\begin{aligned}v_t^n(x) &= \max_z \left\{u_t(x+z) - c(z)\mathbf{1}_{\{z>0\}} : z = 0 \text{ or } z > q_{n-1}\right\} \\ &\leq \max_z \left\{u_t(x+z) - c(z)\mathbf{1}_{\{z>0\}} : z \geq 0\right\} = v_t(x).\end{aligned}$$

On the other hand, because $c(z) \geq K_n + c_n z$ for any $z \geq 0$, we also have that

$$\begin{aligned}v_t(x) &= \max_z \left\{u_t(x+z) - c(z)\mathbf{1}_{\{z>0\}} : z \geq 0\right\} \\ &\leq \max_z \left\{u_t(x+z) - (K_n + c_n z)\mathbf{1}_{\{z>0\}} : z \geq 0\right\} = v_t^n(x).\end{aligned}$$

In summary, we conclude that $v_t(x) = v_t^n(x)$ is K_n-concave and $z_t^*(x)$ solves problem (3a). □

Proof of Theorem 4

We inductively show that for $t = T+1, \cdots, 1$, $\hat{v}_t^0(x)$ is concave and

$$0 \leq v_t(x) - \hat{v}_t(x) \leq v_t(x) - \hat{v}_t^0(x) \leq B_t,$$

where $B_{T+1} = 0$ and $B_t = \sum_{i=0}^{T-t} \gamma^i K_1$. Notice that we can express $B_t = K_1 + \gamma B_{t+1}$ for $t = T, \cdots, 1$. Suppose the statement is true in period $t+1$ for some $1 \leq t \leq T$. In period t, obviously $\hat{u}_t(y)$ in Step 1 is concave and satisfies that

$$0 \leq u_t(y) - \hat{u}_t(y) \leq \gamma B_{t+1}. \tag{A.19}$$

Similar to the proof of Theorem 1, we can verify that $\hat{v}_t^0(x)$ obtained in Step 2 satisfies that

$$\hat{v}_t^0(x) = \max_{z \geq 0} \left\{\hat{u}_t(x+z) - c(z)\right\},$$

and $\hat{z}_t(x)$ generated in Step 2 solves the above problem. It is straightforward to see that $\hat{v}_t^0(x)$ is concave. By (A.19) and $K_1 \geq c(z) - [c(z)\mathbf{1}_{\{z>0\}}] \geq 0$, it follows that

$$\begin{aligned}\hat{v}_t^0(x) &\geq \max_{z \geq 0} \left\{[u_t(x+z) - \gamma B_{t+1}] - [c(z)\mathbf{1}_{\{z>0\}} + K_1]\right\} \\ &= v_t(x) - (K_1 + \gamma B_{t+1}) = v_t(x) - B_t.\end{aligned}$$

Also note that

$$
\begin{aligned}
\hat{v}_t^0(x) &= \hat{d}_t(\hat{y}_t(x))\hat{p}_t(x) - \mathbb{E}h_t\big(\hat{y}_t(x) - \xi_t\hat{d}_t(\hat{y}_t(x)) - \varepsilon_t\big) \\
&+ \gamma\mathbb{E}\hat{v}_{t+1}^0\big(\hat{y}_t(x) - \xi_t\hat{d}_t(\hat{y}_t(x)) - \varepsilon_t\big) - c(\hat{z}_t(x)) \\
&\le \hat{d}_t(\hat{y}_t(x))\hat{p}_t(x) - \mathbb{E}h_t\big(\hat{y}_t(x) - \xi_t\hat{d}_t(\hat{y}_t(x)) - \varepsilon_t\big) \\
&+ \gamma\mathbb{E}\hat{v}_{t+1}\big(\hat{y}_t(x) - \xi_t\hat{d}_t(\hat{y}_t(x)) - \varepsilon_t\big) - c(\hat{z}_t(x))\mathbf{1}_{\{\hat{z}_t(x)>0\}} \\
&= \hat{v}_t(x) \\
&\le \max_{y\ge x, d\in\mathcal{D}_t} \big\{dp_t(d) - \mathbb{E}h_t\big(y - \xi_t d - \varepsilon_t\big) + \gamma\mathbb{E}\hat{v}_{t+1}\big(y - \xi_t d - \varepsilon_t\big) - c(y-x)\mathbf{1}_{\{y>x\}}\big\} \\
&\le \max_{y\ge x, d\in\mathcal{D}_t} \big\{dp_t(d) - \mathbb{E}h_t\big(y - \xi_t d - \varepsilon_t\big) + \gamma\mathbb{E}v_{t+1}\big(y - \xi_t d - \varepsilon_t\big) - c(y-x)\mathbf{1}_{\{y>x\}}\big\} \\
&= v_t(x).
\end{aligned}
$$

We obtain $0 \le v_t(x) - \hat{v}_t(x) \le v_t(x) - \hat{v}_t^0(x) \le B_t$. □

Proof of Proposition 3

When $u_t(y)$ and $c(z)$ are concave, because $c(z)$ is piecewise linear, by Lemma 9.13 in Porteus (2002), we know that a general (s, S) policy is optimal for problem (5). Therefore, $v_t^0(x) > u_t(x)$ if and only if $x < R_t$ for some R_t. To see $R_t \le S_1$, we first prove $R_t \le S_t(c_n)$. In fact, at $x = S_t(c_n)$,

$$
\begin{aligned}
v_t^0(x) &= \max_{y\ge x} \{[u_t(y) - c_n y] + [c_n y - c(y-x)]\} \\
&\le [u_t(x) - c_n x] + \max_{y\ge x} \{[c_n y - c(y-x)]\} \\
&\le [u_t(x) - c_n x] + [c_n x - c(0)] \le u_t(x),
\end{aligned}
$$

where the first inequality holds because $S_t(c_n)$ maximizes the concave function $u_t(y) - c_n y$, and the second inequality holds because $c_n y - c(y-x)$ is decreasing in y by the definition of $c(z)$ and $c_n \le c_i$ for all $1 \le i \le n$. Because $v_t^0(x) > u_t(x)$ if and only if $x < R_t$, $R_t \le S_t(c_n)$ by the above inequality.

To further see $R_t \le S_1$, notice that S_1 obtained in Algorithm 3 is equal to $S_t(c_{j_1}) = S_t(c_{i_1})$, the least element of $\{S_t(c_i) : i \in \mathcal{I}\}$. If $R_t > S_1$, then there is some $i \in \mathcal{I}$ such that $S_t(c_i) < R_t$, where $i < n$ because $R_t \le S_t(c_n)$ as proved. Consider the optimal solution corresponding to the initial inventory level $x = S_t(c_i)$. By $x < R_t$, the general (s, S) policy states that it is optimal to raise the inventory up to some $S_t(c_j)$ for some $j > i$. However, because $x > r_i$ by $i \in \mathcal{I}$, we know from the definition of r_i that $u_t(x) \ge u_t(x) - c(0) \ge u_t(S_t(c_j)) - c(S_t(c_j) - x)$ for all $j > i$, i.e., it is not optimal to raise the inventory up to any $S_t(c_j) > x$. This is a contradiction. Thus, $R_t \le S_1$.

We now focus on the expression of $v_t^0(x)$ when $x < R_t$. Because it is optimal to raise the inventory level up to $S_t(c_i) > x$ for some $1 \le i \le n$, we can express

$$
v_t^0(x) = \max_{1\le i\le n} \{u_t(S_t(c_i)) - c(S_t(c_i) - x) : x < S_t(c_i)\}. \tag{A.20}
$$

Consider any $i \in \{1, 2, \cdots, n\} \setminus \mathcal{I}$. The definition of $\mathcal{I}$ in Step 2 implies $i < n$ and $S_t(c_i) \le r_i$. When $x \ge S_t(c_i)$, clearly i is infeasible to problem (A.20). When $x < S_t(c_i)$, we have $x < r_i$ by $S_t(c_i) \le r_i$, and by the definition of r_i in Step 1, there is some $i < j \le n$ such that $x < S_t(c_j)$ and

$$u_t\left(S_t(c_i)\right) - c\left(S_t(c_i) - x\right) < u_t(S_t(c_j)) - c\left(S_t(c_j) - x\right).$$

Thus, index i is suboptimal to problem (A.20). In summary, any index $i \notin \mathcal{I}$ is either infeasible or suboptimal to problem (A.20), implying that problem (A.20) is equivalent to

$$v_t^0(x) = \max_{i \in \mathcal{I}} \left\{u_t(S_t(c_i)) - c(S_t(c_i) - x) : x < S_t(c_i)\right\}.$$

Because $R_t \le S_1$ and S_1 is the least element in $\{S_t(c_i) : i \in \mathcal{I}\}$, the constraint $x < S_t(c_i)$ in the above problem is redundant for any $x < R_t$, implying that problem (A.20) is equivalent to

$$v_t^0(x) = \max_{i \in \mathcal{I}} \left\{u_t(S_t(c_i)) - c(S_t(c_i) - x)\right\}. \tag{A.21}$$

It remains to show that $z_t^0(x)$ given in (11) solves problem (A.21) when $x < R_t \le S_1 = s_1$, which, as $\mathcal{J} \subseteq \mathcal{I} \subseteq \{j : j_1 \le j \le n\}$, is an immediate result of the following lemma.

Lemma 3. *For any given $u_t(y)$,*

$$\max_{j: j_1 \le j \le n} \left\{u_t(S_t(c_j)) - c(S_t(c_j) - x)\right\} = \begin{cases} u_t(S_m) - c(S_m - x), & \text{if } x < s_m, \\ u_t(S_l) - c(S_l - x), & \text{if } s_{l+1} \le x < s_l \text{ and } 1 \le l < m, \end{cases}$$

where $j_1 = \min\{j : j \in \mathcal{J}\}$ and $\{(s_l, S_l) : 1 \le l \le m\}$ are computed by Algorithm 3.

The proof of the lemma is presented subsequently. □

Proof of Lemma 3: For any $1 \le i < n$, recall that the definition of r_i and the concavity of $c(z)$ implies

$$u_t\left(S_t(c_i)\right) - c\left(S_t(c_i) - x\right) \ge \max_{j: i < j \le n} \left[u_t(S_t(c_j)) - c\left(S_t(c_j) - x\right)\right] \quad \forall x > r_i$$

$$u_t\left(S_t(c_i)\right) - c\left(S_t(c_i) - x\right) < \max_{j: i < j \le n} \left[u_t(S_t(c_j)) - c\left(S_t(c_j) - x\right)\right] \quad \forall x < r_i.$$

Furthermore, according to the continuity of $c(z)$, we have

$$u_t\left(S_t(c_i)\right) - c\left(S_t(c_i) - r_i\right) = \max_{j: i < j \le n} \left[u_t(S_t(c_j)) - c\left(S_t(c_j) - r_i\right)\right]$$

for any i such that $r_i < S_t(c_i)$, i.e., $i \in \mathcal{I} \setminus \{n\}$.

Consider any $x \ge s_{l+1} = r_{j_l}$ for some $1 \le l < m$. Note that $j_l \in \mathcal{J} \setminus \{n\} \subseteq \mathcal{I} \setminus \{n\}$. Thus,

$$u_t(S_l) - c(S_l - x) = u_t(S_t(c_{j_l})) - c(S_t(c_{j_l}) - x) \ge \max_{j: j_l < j \le n} \left[u_t(S_t(c_j)) - c\left(S_t(c_j) - x\right)\right]. \tag{A.22}$$

Consider any $x < s_l = r_{j_{l-1}}$ for some $1 < l \le m$. Assume for induction that

$$\max_{j: i+1 \le j \le j_l} \left[u_t(S_t(c_j)) - c\left(S_t(c_j) - x\right)\right] \le \max_{j: j_l \le j \le n} \left[u_t(S_t(c_j)) - c\left(S_t(c_j) - x\right)\right].$$

for some $j_1 \le i < j_l$, which obviously holds when $i = j_l - 1$. We can show that $r_i \ge s_l = r_{j_{l-1}}$ by considering the following cases:

(i) Note that there exists no i such that $i \in \mathcal{J}$ and $j_{l-1} < i < j_l$, which means that $r_i \geq r_{j_{l-1}}$ for all $i \in \mathcal{I}$ and $j_{l-1} < i < j_l$.

(ii) Suppose that $i \in \mathcal{I}$ and $i \leq j_{l-1}$. As $j_{l-1} \in \mathcal{J}$, the definition of $\mathcal{J}$ implies that $r_{j_{l-1}} \leq r_i$ as $i \in \mathcal{I}$ and $i \leq j_{l-1}$.

(iii) Suppose that $i \notin \mathcal{I}$. Then we have $r_i \geq S_t(c_i) \geq S_t(c_{j_1}) = S_1 = s_1 \geq s_l$, where the second inequality follows from the monotonicity of $S_t(a)$ and $c_i \leq c_{j_1}$.

Thus, $x < s_l$ implies $x < r_i$ and hence

$$\begin{aligned} & u_t\left(S_t(c_i)\right) - c\left(S_t(c_i) - x\right) < \max_{j:i<j\leq n}\left[u_t(S_t(c_j)) - c\left(S_t(c_j) - x\right)\right] \\ = & \left\{\max_{j:i+1\leq j\leq j_l}\left[u_t(S_t(c_j)) - c\left(S_t(c_j) - x\right)\right]\right\} \vee \left\{\max_{j:j_l\leq j\leq n}\left[u_t(S_t(c_j)) - c\left(S_t(c_j) - x\right)\right]\right\} \\ \leq & \left\{\max_{j:j_l\leq j\leq n}\left[u_t(S_t(c_j)) - c\left(S_t(c_j) - x\right)\right]\right\}, \end{aligned}$$

where the second inequality follows from the induction assumption. Consequently,

$$\max_{j:i\leq j\leq j_l}\left[u_t(S_t(c_j)) - c\left(S_t(c_j) - x\right)\right] \leq \max_{j:j_l\leq j\leq n}\left[u_t(S_t(c_j)) - c\left(S_t(c_j) - x\right)\right] \qquad \forall j_1 \leq i \leq j_l. \tag{A.23}$$

Now we can complete the proof by applying (A.22) and (A.23).

(i) If $x < s_m$, letting $i = j_1$ and $l = m$ in (A.23) yields

$$\max_{j:j_1\leq j\leq n}\left[u_t(S_t(c_j)) - c\left(S_t(c_j) - x\right)\right] \leq u_t(S_t(c_{j_m})) - c\left(S_t(c_{j_m}) - x\right) = u_t(S_m) - c(S_m - x).$$

We obtain the desired result as $S_m = S_t(c_n) \in \{S_t(c_j) : j_1 \leq j \leq n\}$.

(ii) If $s_{l+1} \leq x < s_l$ for some $1 < l < m$,

$$\begin{aligned} & \max_{j:j_1\leq j\leq n}\left[u_t(S_t(c_j)) - c\left(S_t(c_j) - x\right)\right] \\ = & \left\{\max_{j:j_1\leq j\leq j_l}\left[u_t(S_t(c_j)) - c\left(S_t(c_j) - x\right)\right]\right\} \vee \left\{\max_{j:j_l\leq j\leq n}\left[u_t(S_t(c_j)) - c\left(S_t(c_j) - x\right)\right]\right\} \\ = & \left\{\max_{j:j_l\leq j\leq n}\left[u_t(S_t(c_j)) - c\left(S_t(c_j) - x\right)\right]\right\} = u_t(S_l) - c(S_l - x), \end{aligned}$$

where the second and third equalities are obtained by (A.23) and (A.22), respectively.

(iii) If $s_2 \leq x < s_1$, the desired result follows immediately from (A.22) with $l = 1$. □

Proof of Theorem 5

Since $u_t(y)$ is concave in the last period $t = T$, a general (s, S) policy is optimal to problem (3a). The properties of $z_t^*(x)$ follows immediately from Proposition 3. Moreover, when $z_t^*(x) = S_l - x = S_t(c_{j_l}) - x$ for some $1 \leq l \leq m$, $z_t^*(x) + x = S_t(c_{j_l})$ and hence $p_t^*(x) = P_t(c_{j_l})$ by the definition of $P_t(a)$ in (6).

It remains to show the monotonicity of $p_t^*(x)$. Recall that $p_t^*(x) = p_t(d_t^*(y_t^*(x)))$, where $y_t^*(x)$ denotes the inventory level after producing, and $p_t(d)$ is decreasing in d as assumed. Moreover, because the objective function of problem (3b) is supermodular in (y,d) by the concavity of $-h_t(x)+\gamma v_{t+1}(x)$ and $\xi_t \geq 0$, we know that $d_t^*(y)$ is increasing in y. Thus, $p_t^*(x)$ is increasing (or decreasing) in x if $y_t^*(x)$ is decreasing (or increasing) in x. In particular, $p_t^*(x)$ is increasing when $x < R_t$ because $y_t^*(x)$ is decreasing when $x < R_t$ by the specification of the general (s,S) policy. When $x \geq R_t$, since that $y_t^*(x) = x$ is increasing in x, the associated optimal price $p_t^*(x)$ is decreasing in x. Furthermore, because $y_t^*(x) > x$ when $x < R_t$ and $y_t^*(x) = x$ at $x = R_t$, we know that $p_t^*(x)$ takes an upward jump at $x = R_t$. In summary, $p_t^*(x)$ is increasing when $x \leq R_t$ and then decreasing when $x \geq R_t$. □

Proof of Theorem 6

Because Algorithm 4 generates a feasible solution $[\bar{z}_t(x), \bar{d}_t(x)]$ of problem (3), $v_t(x) \geq \bar{v}_t(x)$ in all periods t. To find the upper bound of $v_t(x) - \bar{v}_t(x)$, let $-\bar{v}_t^e(x)$ and $-\bar{u}_t^e(x)$ be the lower convex envelopes of $-\bar{v}_t(x)$ and $-\bar{u}_t(x)$, respectively. Moreover, define

$$A_t = \sup_x [\bar{v}_t^e(x) - \bar{v}_t(x)] \quad \text{and} \quad B_t = \sup_x [v_t(x) - \bar{v}_t(x)],$$

where $A_{T+1} = B_{T+1} = 0$ by $v_{T+1}(x) = \bar{v}_{T+1}(x) = 0$. Proposition 3 and Theorem 5 yield that $B_T = 0$. Furthermore, as shown in the proof of Theorem 3, we can obtain (A.7) and (A.8), i.e.,

$$0 \leq \bar{u}_t^e(y) - \bar{u}_t(y) \leq \gamma A_{t+1} \tag{A.24}$$

and

$$u_t(y) - \bar{u}_t^e(y) \leq \gamma B_{t+1}. \tag{A.25}$$

Similar to the proof of Theorem 3, let

$$\hat{v}_t(x) = \max_{z \geq 0} \left\{ \bar{u}_t^e(x+z) - c(z)\mathbf{1}_{\{z>0\}} \right\}, \tag{A.26}$$

which, by (A.25) implies

$$v_t(x) - \hat{v}_t(x) \leq \gamma B_{t+1}. \tag{A.27}$$

We next prove that there exists some $\hat{R}_t \leq s_1$ such that

$$\hat{v}_t(x) = \begin{cases} \bar{v}_t^0(x) > \bar{u}_t^e(x), & \text{if } x < \hat{R}_t, \\ \bar{u}_t^e(x) \geq \bar{v}_t^0(x), & \text{if } \hat{R}_t \leq x < s_1, \\ \bar{u}_t^e(x), & \text{if } x \geq s_1, \end{cases} \tag{A.28}$$

where s_1 and $\bar{z}_t^0(x)$ are computed in Step 2. Similar to the proof of Theorem 3, we make three observations below:

(i) Note that Lemma 1 holds as long as $c(z)$ is non-negative, nondecreasing, and piecewise linear continuous. Following the argument in the proof of Theorem 3, $c_i y - \bar{u}_t(y)$ has a finite least minimizer and $\lim_{y \to -\infty} y^{-1}[\bar{u}_t(y) - c_i y] > 0$.

(ii) $\bar{u}_t(y)$ is continuous. This can be shown inductively by assuming that $\bar{v}_{t+1}(x)$ for some $1 \le t \le T$ is continuous, which obviously holds for $\bar{v}_{T+1}(x) = 0$. Note that function $\bar{u}_t(y)$ in Step 1 is continuous as $dp_t(d)$, $h_t(y)$ and $\bar{v}_{t+1}(y)$ are all continuous. Furthermore, Lemma 3 shows that

$$\bar{v}_t^0(x) = \max_{j: j_1 \le j \le n} \left\{ \bar{u}_t(\bar{S}_t(c_j)) - c(\bar{S}_t(c_j) - x) \right\} \qquad \forall x < s_1,$$

which is continuous as $c(z)$ is continuous. To see the continuity of $\bar{v}_t$, note that $s_1 = S_1$. Therefore, $\lim_{x \uparrow s_1} \bar{z}_t^0(x) = 0$ and $\lim_{x \uparrow s_1} \bar{v}_t^0(x) = \bar{u}_t(s_1) - K_1 \le \bar{u}_t(s_1)$. Applying the continuity of $\bar{v}_t^0$ and $\bar{u}_t$, the definition of $\bar{v}_t$ immediately yields $\bar{v}_t(\bar{R}_t) = \lim_{x \uparrow \bar{R}_t} \bar{v}_t^0(x) = \bar{u}_t(\bar{R}_t)$ and so $\bar{v}_t$ is continuous.

(iii) As shown in the proof of Theorem 3, $c_i y - \bar{u}_t^e(y)$ is the lower convex envelope of $c_i y - \bar{u}_t(y)$ for any $1 \le i \le n$.

With the above observations, we know from Proposition 2 that $\bar{S}_t(c_i)$ is also the least minimizer of $c_i x - \bar{u}_t^e(x)$. Furthermore, $c_i \bar{S}_t(c_i) - \bar{u}_t^e(\bar{S}_t(c_i)) = c_i \bar{S}_t(c_i) - \bar{u}_t(\bar{S}_t(c_i))$, which implies $\bar{u}_t^e(\bar{S}_t(c_i)) = \bar{u}_t(\bar{S}_t(c_i))$. Because $\bar{z}_t^0(x)$ specified in Step 2 only depends on the values of $\bar{S}_t(c_i)$ and $\bar{u}_t(S_t(c_i))$ for $1 \le i \le n$, by the concavity of $\bar{u}_t^e(y)$, similar to the proof of Proposition 3, there exists some $\hat{R}_t \le s_1$ such that

$$\hat{v}_t(x) = \begin{cases} \bar{u}_t^e(x + \bar{z}_t^0) - c(\bar{z}_t^0) > \bar{u}_t^e(x), & \text{if } x < \hat{R}_t, \\ \bar{u}_t^e(x) \ge \bar{u}_t^e(x + \bar{z}_t^0) - c(\bar{z}_t^0), & \text{if } \hat{R}_t \le x < s_1, \\ \bar{u}_t^e(x), & \text{if } x \ge s_1. \end{cases}$$

Also note that $x + \bar{z}_t^0 \in \{\bar{S}_t(c_i) : 1 \le i \le n\}$ and $\bar{u}_t^e(\bar{S}_t(c_i)) = \bar{u}_t(\bar{S}_t(c_i))$. It is straightforward that

$$\bar{u}_t^e(x + \bar{z}_t^0) - c(\bar{z}_t^0) = \bar{u}_t(x + \bar{z}_t^0) - c(\bar{z}_t^0) = \bar{v}_t^0(x) \qquad \forall x < s_1,$$

which immediately yields (A.28).

As in the proof of Theorem 3, we next prove

$$0 \le \hat{v}_t(x) - \bar{v}_t(x) \le \gamma A_{t+1}. \tag{A.29}$$

Let $\bar{z} = \bar{z}_t^0(x)$ for notational simplicity. Note that (A.28) implies $\hat{R}_t = \inf\{x < s_1 : v_t^0(x) \le \bar{u}_t^e(x)\}$. Recall that $\bar{R}_t = \inf\{x < s_1 : v_t^0(x) \le \bar{u}_t(x)\}$ and $\bar{u}_t^e(x) \ge \bar{u}_t(x)$. We obtain $\hat{R}_t \le \bar{R}_t$. Thus, it is sufficient to consider the following three cases.

(i) When $x < \bar{R}_t \wedge \hat{R}_t$, by (A.28) and the definition of $\bar{v}_t(x)$, we have $\hat{v}_t(x) - \bar{v}_t(x) = 0$, which obviously satisfies (A.29).

(ii) When $\hat{R}_t \le x < \bar{R}_t \le s_1$, we know that $\hat{v}_t(x) - \bar{v}_t(x) = \bar{u}_t^e(x) - \bar{v}_t^0(x) \ge 0$. Furthermore, by the definition of $\bar{R}_t$, $x < \bar{R}_t$ implies $v_t^0(x) > \bar{u}_t(x)$ and so

$$\hat{v}_t(x) - \bar{v}_t(x) = \bar{u}_t^e(x) - \bar{v}_t^0(x) \le \bar{u}_t^e(x) - \bar{u}_t(x).$$

Inequality (A.29) immediately follows from (A.24).

(iii) When $x \ge \bar{R}_t \vee \hat{R}_t$, $\hat{v}_t(x) - \bar{v}_t(x) = \bar{u}_t^e(x) - \bar{u}_t(x)$. Inequality (A.29) immediately follows from (A.24).

By the definition of B_t, as well as inequalities (A.27) and (A.29), we conclude that

$$B_t = \sup_x \{[v_t(x) - \hat{v}_t(x)] + [\hat{v}_t(x) - \bar{v}_t(x)]\} \le \gamma B_{t+1} + \gamma A_{t+1}.$$

To see the relation between A_t and A_{t+1}, consider the following function as a counterpart of $\hat{v}_t(x)$ with $c(z)\mathbf{1}_{\{z>0\}}$ replaced by $K_n + c_n z$:

$$\tilde{v}_t^0(x) = \max_{z \ge 0} \{\bar{u}_t^e(x+z) - (K_n + c_n z)\}.$$

As $0 \le (K_n + c_n z) - c(z)\mathbf{1}_{\{z>0\}} \le K_n$ for all $z \ge 0$, we have

$$0 \le \hat{v}_t(x) - \tilde{v}_t^0(x) \le K_n, \quad \text{i.e.,} \quad -[K_n + \tilde{v}_t^0(x)] \le -\hat{v}_t(x) \le -\tilde{v}_t^0(x).$$

Also note that (A.29) yields $-\hat{v}_t(x) \le -\bar{v}_t(x) \le \gamma A_{t+1} - \hat{v}_t(x)$. Therefore,

$$-[K_n + \tilde{v}_t^0(x)] \le -\bar{v}_t(x) \le \gamma A_{t+1} - \tilde{v}_t^0(x). \tag{A.30}$$

Because $-\bar{v}_t^e(x)$ is the lower convex envelope of $-\bar{v}_t(x)$, and $-\tilde{v}_t^0(x)$ is obviously convex by its definition, we know from the first inequality in (A.30) that $-[K_n + \tilde{v}_t^0(x)] \le -\bar{v}_t^e(x)$. Furthermore, by the second inequality in (A.30), it leads to $-\bar{v}_t(x) \le K_n + \gamma A_{t+1} - \bar{v}_t^e(x)$, implying that

$$A_t = \sup_x [\bar{v}_t^e(x) - \bar{v}_t(x)] \le K_n + \gamma A_{t+1}.$$

In summary, we conclude that $B_T = 0$, $A_T = K_n$ and for any $1 \le t < T$,

$$A_t \le K_n + \gamma A_{t+1} \text{ and } B_t \le \gamma(A_{t+1} + B_{t+1}).$$

By some basic algebra, it can be verified that for each $t < T$,

$$A_t \le \sum_{i=0}^{T-t} K_n \gamma^i \text{ and } B_t \le \sum_{i=1}^{T-t} i K_n \gamma^i.$$

Thus, by the definition of B_t, we obtain the upper bound of $v_t(x) - \bar{v}_t(x)$.

To see these sufficient conditions for $\bar{v}_t(x) = v_t(x)$, notice that in the proof of Theorem 5 we indeed shows that if $u_t(y)$ is concave, then a general (s, S) policy is optimal to problem (3a), and all other discussions in the proof of Theorem 5 remains valid. Thus, conditions (a) and (b) lead to the desired result $\bar{v}_t(x) = v_t(x)$, where we refer readers to Theorem 3 in Chen et al. (2010) for the latter. To see the sufficient condition (c), if we can prove $v_t(x)$ satisfies the following alternative definition, then this proof is completed by applying the well-known result in Scarf (1960):

$$\begin{aligned} v_t(x) &= \max_{z\geq 0}\left\{u_t(x+z) - (K_n + c_n z)\mathbf{1}_{\{z>0\}}\right\} \\ &= u_t(x) \vee \max_{z\geq 0}\left\{u_t(x+z) - (K_n + c_n z)\right\}, \end{aligned} \tag{A.31}$$

where the second equality holds by $K_n \geq 0$. In fact, by $c(z) = \min\{K_i + c_i z : 1 \leq i \leq n\}$, $v_t(x)$ given in (3a) can be equivalently expressed by

$$v_t(x) = u_t(x) \vee \max_{z\geq 0}\max_{1\leq i\leq n}\left\{u_t(x+z) - (K_i + c_i z)\right\}. \tag{A.32}$$

Obviously, (A.31) holds for any x such that $\max_{z\geq 0}\max_{1\leq i<n}\left\{u_t(x+z) - K_i - c_i z\right\} \leq u_t(x)$. Therefore, it is sufficient to prove (A.31) under the condition that $\max_{z\geq 0}\max_{1\leq i<n}\left\{u_t(x+z) - K_i - c_i z\right\} > u_t(x)$, i.e., there exist $z^* \geq 0$ and $1 \leq i^* < n$ such that

$$u_t(x+z^*) - K_{i^*} - c_{i^*}z^* = \max_{z\geq 0}\max_{1\leq i<n}\left\{u_t(x+z) - K_i - c_i z\right\} > u_t(x). \tag{A.33}$$

Observe that the given assumption on K_1 can be expressed by $K_1 > (H_t - c_{n-1})q_{n-1}$ with $H_t = \sum_{i=t}^{T}\gamma^{i-t}h_i^-$; moreover, part (a) of the proof of Lemma 2 remains valid in this case, which states that $u_t(x) - H_t x$ is decreasing in x. By $(H_t - c_i)q_{n-1} \leq (H_t - c_{n-1})q_{n-1} < K_1 \leq K_i$ for any $1 \leq i < n$, (A.33) implies

$$\begin{aligned} (H_t - c_{i^*})q_{n-1} &\leq K_{i^*} < u_t(x+z^*) - u_t(x) - c_{i^*}z^* \\ &= \left[u_t(x+z^*) - H_t(x+z^*)\right] - \left[u_t(x) - H_t x\right] + (H_t - c_{i^*})z \leq (H_t - c_{i^*})z, \end{aligned}$$

where the last inequality holds since that $u_t(x) - H_t x$ is decreasing in x. As $K_{i^*} \geq 0$, the above inequality yields $H_t - c_{i^*} > 0$ and hence $z \geq q_{n-1}$. Furthermore, the concavity of $c(z)$ implies $K_i + c_i z \geq K_n + c_n z$ for any $1 \leq i < n$ and $z \geq q_{n-1}$. Therefore,

$$u_t(x+z^*) - K_{i^*} - c_{i^*}z^* \leq u_t(x+z^*) - K_n - c_n z^* \leq \max_{z\geq 0}\left\{u_t(x+z) - (K_n + c_n z)\right\}.$$

By combining it and inequality (A.33), we conclude that $v_t(x)$ given by (A.32) satisfies the alternative definition (A.31). □

Proof of Theorem 7

Let $B_{T+1} = 0$ and $B_t = \sum_{i=0}^{T-t}\gamma^i(K_n - K_1)$. We inductively show that for $t = T+1, \cdots, 1$,

$$B_t \geq v_t(x) - \hat{v}_t^0(x) \geq v_t(x) - \hat{v}_t(x) \geq 0.$$

Obviously the above inequality holds for $t = T+1$ by $v_{T+1}(x) = \hat{v}^0_{T+1}(x) = \hat{v}_{T+1}(x)$. Suppose it is true in period $t+1$ for some $1 \le t \le T$. In period t, observe that $\hat{u}_t(y)$ in Step 1 satisfies $\gamma B_{t+1} \ge u_t(y) - \hat{u}_t(y) \ge 0$ by the inductive assumption $B_{t+1} \ge v_{t+1}(x) - \hat{v}^0_{t+1}(x) \ge 0$. Note that

$$\max_{z\ge 0}\{K_n + c_n z - c(z)\} = \max_{z\ge 0}\max_{1\le i\le n}\{K_n + c_n z - K_i - c_i z\} \le \max_{1\le i\le n}\{K_n - K_i\} = K_n - K_1,$$

where the inequality follows from $c_n \le c_i$ for all $1 \le i \le n$, i.e., $K_n + c_n z \le c(z) + (K_n - K_1)$ for all $z \ge 0$. As $B_t = (K_n - K_1) + \gamma B_{t+1}$, $\hat{v}^0_t(x)$ obtained in Step 2 satisfies

$$\begin{aligned}
\hat{v}^0_t(x) &= \max_{z\ge 0}\left\{\hat{u}_t(x+z) - (K_n + c_n z)\mathbf{1}_{\{z>0\}}\right\} \\
&\ge \max_{z\ge 0}\left\{[u_t(x+z) - \gamma B_{t+1}] - c(z)\mathbf{1}_{\{z>0\}} - (K_n - K_1)]\right\} = v_t(x) - B_t.
\end{aligned}$$

Moreover, by the definitions of $\hat{v}^0_t(x)$, $\hat{z}_t(x)$, $\hat{d}_t(x)$, $\hat{y}_t(x)$, $\hat{p}_t(x)$ and $\hat{v}_t(x)$ in Steps 2 and 3,

$$\begin{aligned}
\hat{v}^0_t(x) &= \hat{d}_t(\hat{y}_t(x))\hat{p}_t(x) - \mathbb{E}h_t\big(\hat{y}_t(x) - \xi_t\hat{d}_t(\hat{y}_t(x)) - \varepsilon_t\big) + \gamma\mathbb{E}\hat{v}^0_{t+1}\big(\hat{y}_t(x)\big) - (K_n + c_n\hat{z}_t(x))\mathbf{1}_{\{\hat{z}_t(x)>0\}} \\
&\le \hat{d}_t(\hat{y}_t(x))\hat{p}_t(x) - \mathbb{E}h_t\big(\hat{y}_t(x) - \xi_t\hat{d}_t(\hat{y}_t(x)) - \varepsilon_t\big) + \gamma\mathbb{E}\hat{v}_{t+1}\big(\hat{y}_t(x)\big) - c(\hat{z}_t(x))\mathbf{1}_{\{\hat{z}_t(x)>0\}} = \hat{v}_t(x),
\end{aligned}$$

where the inequality holds by the inductive assumption $\hat{v}^0_{t+1}(x) \le \hat{v}_{t+1}(x)$ and $c(z) \le K_n + c_n z$ for any $z \ge 0$. Finally, by the definitions of $\hat{v}_t(x)$ and $v_t(x)$ and the inductive assumption $v_{t+1}(x) \ge \hat{v}_{t+1}(x)$,

$$\begin{aligned}
\hat{v}_t(x) &\le \max_{y\ge x, d\in\mathcal{D}_t}\left\{dp_t(d) - \mathbb{E}h_t\left(y - \xi_t d - \varepsilon_t\right) + \gamma\mathbb{E}\hat{v}_{t+1}\left(y - \xi_t d - \varepsilon_t\right) - c(y-x)\mathbf{1}_{\{y>x\}}\right\} \\
&\le \max_{y\ge x, d\in\mathcal{D}_t}\left\{dp_t(d) - \mathbb{E}h_t\left(y - \xi_t d - \varepsilon_t\right) + \gamma\mathbb{E}v_{t+1}\left(y - \xi_t d - \varepsilon_t\right) - c(y-x)\mathbf{1}_{\{y>x\}}\right\} = v_t(x).
\end{aligned}$$

In summary, we conclude that $B_t \ge v_t(x) - \hat{v}^0_t(x) \ge v_t(x) - \hat{v}_t(x) \ge 0$ for any $1 \le t \le T+1$. □

Proof of Proposition 4

Observe that if $x_\Delta = x_1 - x_0$, $a = \lambda x_\Delta$ and $b = \mu x_\Delta$ in Definition 1, then $f(x)$ is κ-convex if and only if the inequality below holds for any $x_0 \le x_1 = x_0 + x_\Delta$ and $0 < \lambda \le 1 - \mu < 1$:

$$\lambda f(x_1 - \mu x_\Delta) + \mu f(x_0 + \lambda x_\Delta) \le \lambda f(x_1) + \mu f(x_0) + \lambda\kappa(\mu x_\Delta).$$

Thus, part (c) is satisfied. In addition, K-convexity is implied by κ-convexity with $\kappa(x) = K$ and $\lambda = 1 - \mu$ in part (a). To see the other direction in part (a), i.e., K-convexity also implies κ-convexity with $\kappa(x) = K$, consider any $x_0 \le x_1 = x_0 + x_\Delta$ and $0 \le \lambda \le 1 - \mu \le 1$. If $f(x)$ is K-convex, then

$$\begin{aligned}
f(x_1 - \mu x_\Delta) &\le \mu f(x_0) + (1-\mu)[f(x_1) + K], \\
f(x_0 + \lambda x_\Delta) &\le (1-\lambda) f(x_0) + \lambda[f(x_1) + K].
\end{aligned}$$

By taking the sum of the two inequalities multiplied by λ and μ, respectively, it implies that

$$\lambda f(x_1 - \mu x_\Delta) + \mu f(x_0 + \lambda x_\Delta) \le \mu f(x_0) + \lambda[f(x_1) + K],$$

Thus, f is also κ-convex with $\kappa(x) = K$ by Definition 1.

It remains to show part (b). Given a sym-κ-convex function $f(x)$ with $\kappa(x) = K$, to see its sym-K-convexity, we only need to verify inequality (15) for any $0 \leq \lambda \leq 1$ and $x_0 > x_1$. Let $\bar{\lambda} = 1 - \lambda$ and $\bar{x}_i = x_{1-i}$ for $i = 0, 1$. Clearly $0 \leq \bar{\lambda} \leq 1$ and $\bar{x}_0 < \bar{x}_1$. By the sym-κ-convexity of $f(x)$,

$$\begin{aligned} f((1-\lambda)x_0 + \lambda x_1) &= f((1-\bar{\lambda})\bar{x}_0 + \bar{\lambda}\bar{x}_1) \\ &\leq [(1-\bar{\lambda})f(\bar{x}_0) + \bar{\lambda}f(\bar{x}_1)] + [(1-\bar{\lambda}) \vee \bar{\lambda}]K \\ &= [\lambda f(x_1) + (1-\lambda)f(x_0)] + [\lambda \vee (1-\lambda)]K. \end{aligned}$$

Thus, inequality (15) holds, i.e., $f(x)$ is sym-K-convex. □

Proof of Proposition 5

For any $a, b \geq 0$ and $x_0 + a \leq x_1 - b$, we need to prove the inequality below:

$$b[f(x_0 + a) - f(x_0)] + a[f(x_1 - b) - f(x_1)] \leq \theta(a, b),$$

where to unify the discussion, we introduce $\theta(a, b) = a\kappa(b)$ if $g(y)$ is κ-convex, and $\theta(a, b) = [a\kappa(b)] \vee [b\kappa(a)]$ if $g(y)$ is sym-κ-convex. Let $x_\Delta = x_1 - x_0$ and note that $x_\Delta \geq a + b \geq 0$. Since the above inequality is trivial when $x_\Delta = 0$, we assume $x_\Delta > 0$ in the following.

Suppose that $z_i \in \mathcal{Z}$ solves the problem associated with parameter x_i for each $i = 0, 1$, that is,

$$f(x_0) = \mathbb{E}g(x_0 - \xi z_0 - \varepsilon) + h(z_0), \quad f(x_1) = \mathbb{E}g(x_1 - \xi z_1 - \varepsilon) + h(z_1).$$

Let $\rho = x_\Delta^{-1}(z_1 - z_0)$ and $\lambda = 0 \vee \rho \wedge L^{-1}$. Observe that $0 \wedge (\rho x_\Delta) \leq \lambda x \leq 0 \vee (\rho x_\Delta)$ for any $0 < x < x_\Delta$. In particular, because $0 < a \leq x_\Delta - b < x_\Delta$, we have $z_0 \wedge z_1 \leq z_0 + \lambda a \leq z_0 \vee z_1$ and $z_0 \wedge z_1 \leq z_1 - \lambda b \leq z_0 \vee z_1$, implying that $z_0 + \lambda a$ and $z_1 - \lambda b$ belong to the convex set $\mathcal{Z}$. By the definition of f, we have

$$\begin{aligned} f(x_0 + a) &\leq \mathbb{E}\, g(x_0 + a - \xi(z_0 + \lambda a) - \varepsilon) + h(z_0 + \lambda a), \\ f(x_1 - b) &\leq \mathbb{E}\, g(x_1 - b - \xi(z_1 - \lambda b) - \varepsilon) + h(z_1 - \lambda b). \end{aligned}$$

By substituting above four inequalities into the desired inequality to eliminate terms related to function $f(x)$, we only need to prove

$$\mathbb{E}\mathcal{G} + [ah(z_1 - \lambda b) + bh(z_0 + \lambda a)] - [ah(z_1) + bh(z_0)] \leq \theta(a, b), \tag{A.34}$$

where $\mathcal{G}$ given below collects all terms related to function $g(y)$:

$$\begin{aligned} \mathcal{G} = {}& b[g(x_0 + a - \xi(z_0 + \lambda a) - \varepsilon) - g(x_0 - \xi z_0 - \varepsilon)] \\ &+ a[g(x_1 - b - \xi(z_1 - \lambda b) - \varepsilon) - g(x_1 - \xi z_1 - \varepsilon)]. \end{aligned}$$

Because $h(z)$ is convex and $z_0 + \lambda a, z_1 - \lambda b \in [z_0 \wedge z_1, z_0 \vee z_1]$, we know that

$$
\begin{aligned}
|z_1 - z_0| h(z_0 + \lambda a) &\le (|z_1 - z_0| - \lambda a) h(z_0) + (\lambda a) h(z_1), \\
|z_1 - z_0| h(z_1 - \lambda b) &\le (\lambda b) h(z_0) + (|z_1 - z_0| - \lambda b) h(z_1).
\end{aligned}
$$

By taking sum of the two inequalities multiplied by b and a respectively, we have

$$
|z_1 - z_0| [a h(z_1 - \lambda b) + b h(z_0 + \lambda a)] \le |z_1 - z_0| [a h(z_1) + b h(z_0)].
$$

Thus, to see inequality (A.34), we only need to verify $\mathcal{G} \le \theta(a, b)$ for any $\xi \in [L, U]$. We distinguish among three cases as below. We recall that $\lambda \xi = 0 \vee (\rho \xi) \wedge (L^{-1} \xi)$.

(i) If $\rho\xi = 1$ or $\rho\xi > 1$ and $\xi = L$, then $\lambda\xi = (\rho\xi) \wedge (L^{-1}\xi) = 1$. Clearly, $\mathcal{G} = 0$ by its definition.

(ii) If $\rho\xi < 1$, then $\rho\xi < L^{-1}\xi$ and $\lambda\xi = 0 \vee (\rho\xi) < 1$. It ensures $0 < 1 - \lambda\xi \le 1$ and hence

$$
x_0 - \xi z_0 < x_0 + a - \xi(z_0 + \lambda a) \le x_1 - b - \xi(z_1 - \lambda b) < x_1 - \xi z_1.
$$

By the definition of $\mathcal{G}$ and the (sym-)κ-convexity of g, as well as the above inequality,

$$
\begin{aligned}
(1 - \lambda\xi)\mathcal{G} &= [(1 - \lambda\xi) b][g(x_0 + a - \xi(z_0 + \lambda a) - \varepsilon) - g(x_0 - \xi z_0 - \varepsilon)] \\
&\quad + [(1 - \lambda\xi) a][g(x_1 - b - \xi(z_1 - \lambda b) - \varepsilon) - g(x_1 - \xi z_1 - \varepsilon)] \\
&\le \theta\left((1 - \lambda\xi) a, (1 - \lambda\xi) b\right).
\end{aligned}
$$

By $0 < 1 - \lambda\xi \le 1$ and the monotonicity of function $\kappa(z)$, it is straightforward to see that $\mathcal{G} \le \theta(a, b)$ for either $\theta(a, b) = a\kappa(b)$ or $\theta(a, b) = [a\kappa(b)] \vee [b\kappa(a)]$.

(iii) If $\rho\xi > 1$ and $\xi > L$, then $\lambda\xi = (\rho\xi) \wedge (L^{-1}\xi) > 1$, which implies that

$$
x_1 - \xi z_1 < x_1 - b - \xi(z_1 - \lambda b) \le x_0 + a - \xi(z_0 + \lambda a) < x_0 - \xi z_0.
$$

By the (sym-)κ-convexity of g and the definition of $\mathcal{G}$, similar to the previous case, we have that

$$
(\lambda\xi - 1)\mathcal{G} \le \theta\left((\lambda\xi - 1) b, (\lambda\xi - 1) a\right).
$$

Recall that $\xi > L$ corresponds to the setting that g is sym-κ-convex, i.e., $\theta(a, b) = [a\kappa(b)] \vee [b\kappa(a)]$. Substituting it into the above inequality, it follows that

$$
\mathcal{G} \le [a\kappa((\lambda\xi - 1) b)] \vee [b\kappa((\lambda\xi - 1) a)].
$$

If $\kappa(z)$ is constant, then clearly $\mathcal{G} \le \theta(a, b)$. If $U \le 2L$, then $\lambda\xi - 1 \le L^{-1}\xi - 1 \le L^{-1}U - 1 \le 1$ by the definition of λ, and hence $\mathcal{G} \le \theta(a, b)$ by the monotonicity of $\kappa(z)$. □

Proof of Proposition 6

Given any $a, b \geq 0$ and $x_0 + a \leq x_1 - b$, we need to verify the following inequality,

$$b[f(x_0+a) - f(x_0)] + a[f(x_1-b) - f(x_1)] \leq \theta(a,b).$$

Similar to the proof of Proposition 5, we introduce $\theta(a,b)$ as below to unify the discussion.

(a) $\theta(a,b) = a\kappa(b)$ if $c(z)$ is convex and $g(x)$ is κ-convex with $\kappa(x) = c(x) - c_1 x$;

(b) $\theta(a,b) = [a\kappa(b)] \vee [b\kappa(a)]$ if $c(z)$ is convex and $g(x)$ is sym-κ-convex with $\kappa(x) = c(x) - c_1 x$;

(c) $\theta(a,b) = [a\kappa(b)] \vee [b\kappa(a)]$ if $c(z)$ is convex, $K_n \geq 0$, and $g(x)$ is sym-κ-convex with $\kappa(x) = K_1$;

(d) $\theta(a,b) = a\kappa(b)$ if $c(z)$ is concave and $g(x)$ is κ-convex with $\kappa(x) = c(x) - c_n x$; and

(e) $\theta(a,b) = [a\kappa(b)] \vee [b\kappa(a)]$ if $c(z)$ is concave and $g(x)$ is sym-κ-convex with $\kappa(x) = c(x) - c_n x$.

We assume $a > 0$ in the following because otherwise the desired inequality holds obviously.

Let $h(z) = c(-z)\mathbf{1}_{\{z<0\}}$ and reformulate the problem as

$$f(x) = \min_{y,z} \{g(y) + h(z) : y + z = x, z \leq 0\}.$$

For each $i = 0, 1$, suppose (y_i, z_i) solves the above problem related to parameter x_i, i.e., $f(x_i) = g(y_i) + h(z_i)$, $y_i + z_i = x_i$ and $z_i \leq 0$. Observe that if $c(z)$ is convex and $z_0, z_1 < 0$, then at $x = x_0$ and $x = x_1$, we can express

$$f(x) = \max_{z\leq 0}\{g(x-z) + h(z)\} = \max_{z\leq 0}\{g(x-z) + c(-z)\}.$$

Because this is a special case of the problem studied in Proposition 5 corresponding to $\xi = 1$ and $\varepsilon = 0$, the desired inequality has been verified. Thus, in the following, we only focus on the case where either $c(z)$ is convex with $z_0 z_1 = 0$ or $c(z)$ is concave.

Let $x_\Delta = x_1 - x_0$ and $z_\Delta = z_1 - z_0$, where note that $y_1 - y_0 = x_\Delta - z_\Delta$. Furthermore, consider $\lambda = 0 \vee z_\Delta \wedge a$ and $\mu \in \{0, b\}$, where observe that $(z_0 + \lambda) \vee (z_1 - \mu) \leq z_0 \vee z_1$. Hence, by $z_i \leq 0$ and definition of $f(x)$, we know that

$$\begin{aligned} f(x_0+a) &\leq g(y_0 + a - \lambda) + h(z_0 + \lambda), \\ f(x_1 - b) &\leq g(y_1 - b + \mu) + h(z_1 - \mu). \end{aligned}$$

Because $f(x_i) = g(y_i) + h(z_i)$, we know from the above inequalities that

$$b[f(x_0+a) - f(x_0)] + a[f(x_1-b) - f(x_1)] \leq \mathcal{F}(\lambda, \mu),$$

where function $\mathcal{F}(\lambda, \mu)$ on the right side is given by

$$\begin{aligned} \mathcal{F}(\lambda,\mu) &= b[g(y_0 + a - \lambda) - g(y_0)] + b[h(z_0+\lambda) - h(z_0)] \\ &\quad + a[g(y_1 - b + \mu) - g(y_1)] + a[h(z_1 - \mu) - h(z_1)]. \end{aligned}$$

Thus, to complete this proof, it suffices to prove $\mathcal{F}(\lambda,\mu) \leq \theta(a,b)$ if either $c(z)$ is convex with $z_0 z_1 = 0$ or $c(z)$ is concave. We consider three cases as below.

(a) When $z_\Delta \leq 0$, $\lambda = 0$ and consider $\mu = 0$ only,

$$\mathcal{F}(0,0) = b[g(y_0 + a) - g(y_0)] + a[g(y_1 - b) - g(y_1)].$$

Because $g(y)$ is (sym-)κ-convex and $y_1 - y_0 = x_\Delta - z_\Delta \geq a + b$, we conclude $\mathcal{F}(0,0) \leq \theta(a,b)$.

(b) When $z_\Delta \geq a$, $\lambda = a$ and consider $\mu = b$ only,

$$\mathcal{F}(a,b) = b[h(z_0 + a) - h(z_0)] + a[h(z_1 - b) - h(z_1)].$$

(i) If $c(z)$ is convex with $z_0 z_1 = 0$, then $z_0 = -z_\Delta < 0 = z_1$. By $h(z) = c(-z)\mathbf{1}_{\{z<0\}}$,

$$\begin{aligned}\mathcal{F}(a,b) &= b[h(-z_\Delta + a) - h(-z_\Delta)] + a[h(-b) - h(0)] \\ &\leq b[c(z_\Delta - a) - c(z_\Delta)] + ac(b).\end{aligned}$$

Because $c(z-a) - c(z)$ is decreasing in z by convexity of $c(z)$, we know that

$$\mathcal{F}(a,b) \leq b[c(0) - c(a)] + ac(b) = bK_1 + ac(b) - bc(a). \tag{A.35}$$

If $g(x)$ is κ-convex or sym-κ-convex with $\kappa(z) = c(z) - c_1 z$, then $\theta(a,b) = a\kappa(b)$ or $\theta(a,b) = [a\kappa(b)] \vee [b\kappa(a)]$ by definition of $\theta(a,b)$. In each case, inequality (A.35) ensures

$$\mathcal{F}(a,b) \leq ac(b) - bc(a) = a\kappa(b) - b\kappa(a) \leq \theta(a,b).$$

If $K_n \geq 0$ and $g(x)$ is sym-κ-convex with $\kappa(z) = K_1$, then $\theta(a,b) = (a \vee b)K_1$ by definition of $\theta(a,b)$. In this case, we can verify $\mathcal{F}(a,b) \leq \theta(a,b)$ as below:

- When $a \leq b$, because $K_i \geq K_n \geq 0$ for any $1 \leq i \leq n$ by convexity of $c(z)$,

$$z^{-1}c(z) = \sum\nolimits_{i=1}^{n} (z^{-1}K_i + c_i)\mathbf{1}_{\{q_{i-1} < z \leq q_i\}}$$

is decreasing in z when $z > 0$. Thus, by inequality (A.35) and $b \geq a$, we conclude

$$\mathcal{F}(a,b) \leq bK_1 + ab\left[b^{-1}c(b) - a^{-1}c(a)\right] \leq bK_1.$$

- When $a > b$, notice that $ac(b) \leq (a-b)c(0) + bc(a) = (a-b)K_1 + bc(a)$ by convexity of $c(z)$. Hence, inequality (A.35) ensures that

$$\mathcal{F}(a,b) \leq bK_1 + [(a-b)K_1 + bc(a)] - bc(a) = aK_1.$$

(ii) When $c(z)$ is concave, recall that either $\theta(a,b) = a\kappa(b)$ or $\theta(a,b) = [a\kappa(b)] \vee [b\kappa(a)]$ with $\kappa(z) = c(z) - c_n z$. Observe that $h(z) = c(-z)\mathbf{1}_{\{z<0\}}$ is concave over $\Re_-$ by $c(0) = K_1 \geq 0$ and concavity of $c(z)$ over $\Re_+$. Moreover, $h(z) + c_n z = [c(-z) + c_n z]\mathbf{1}_{\{z<0\}}$ is decreasing over $\Re_-$ by monotonicity of $c(z) - c_n z$ over $\Re_+$.

- When $z_\Delta > b$, by concavity of $h(z)$ and monotonicity of $h(z)+c_n z$,

$$\begin{aligned}\mathcal{F}(a,b) &= b[h(z_0+a)-h(z_0)] + a[h(z_1-b)-h(z_1)]\\ &\le b(-c_n a) + a[h(-b)-h(0)]\\ &= -abc_n + a[c(b)-0] = a\kappa(b) \le \theta(a,b).\end{aligned}$$

- When $z_\Delta \le b$, by definition of $\mathcal{F}(a,b)$, concavity of $h(z)$ and $z_1 - b \le z_1 - z_\Delta = z_0$,

$$\begin{aligned}\mathcal{F}(a,b) &= [(b-z_\Delta)h(z_0+a) - (a+b-z_\Delta)h(z_0) + ah(z_1-b)]\\ &\quad + [z_\Delta h(z_0+a) - (z_\Delta - a)h(z_0) - ah(z_1)]\\ &\le z_\Delta h(z_0+a) - (z_\Delta - a)h(z_0) - ah(z_1)\\ &= (z_\Delta - a)[h(z_0+a)-h(z_0)] + a[h(z_0+a)-h(z_1)].\end{aligned}$$

Furthermore, by $a \le z_\Delta \le b$, monotonicity of $h(z)+c_n z$ and concavity of $h(z)$,

$$\begin{aligned}\mathcal{F}(a,b) &\le (z_\Delta - a)(-c_n a) + a[h(z_1 - z_\Delta + a) - h(z_1)]\\ &\le (z_\Delta - a)(-c_n a) + a[h(0 - z_\Delta + a) - h(0)]\\ &= (z_\Delta - a)(-c_n a) + ac(z_\Delta - a)\\ &= a\kappa(z_\Delta - a) \le a\kappa(b) \le \theta(a,b).\end{aligned}$$

(c) When $0 < z_\Delta < a$, $\lambda = z_\Delta$ and consider both candidates $\mu \in \{0, b\}$. Instead of showing either $\mathcal{F}(z_\Delta, 0)$ or $\mathcal{F}(z_\Delta, b)$ is no more than $\theta(a,b)$, we only need to prove that their convex combination $\mathcal{G} = (1 - z_\Delta/a)\,\mathcal{F}(z_\Delta,0) + (z_\Delta/a)\,\mathcal{F}(z_\Delta,b)$ is no more than $\theta(a,b)$. In fact, by definition of $\mathcal{F}(\lambda,\mu)$ and $z_\Delta = z_1 - z_0$, we can express $\mathcal{G}$ by

$$\begin{aligned}\mathcal{G} &= b[g(y_0+a-z_\Delta) - g(y_0)] + b[h(z_0+z_\Delta) - h(z_0)]\\ &\quad + (a-z_\Delta)[g(y_1-b)-g(y_1)] + z_\Delta[h(z_1-b)-h(z_1)]\\ &= b[g(y_0+a-z_\Delta)-g(y_0)] + (a-z_\Delta)[g(y_1-b)-g(y_1)]\\ &\quad + (b-z_\Delta)h(z_1) - bh(z_0) + z_\Delta h(z_1-b).\end{aligned}$$

By $y_1 - y_0 = x_\Delta - z_\Delta \ge (a - z_\Delta) + b$ and (sym-)κ-convexity of $g(y)$,

$$\mathcal{G} \le \theta(a-z_\Delta, b) + (b-z_\Delta)h(z_1) - bh(z_0) + z_\Delta h(z_1-b).$$

Therefore, to see $\mathcal{G} \le \theta(a,b)$, it suffices to prove

$$\theta(a-z_\Delta,b) + (b-z_\Delta)h(z_1) - bh(z_0) + z_\Delta h(z_1-b) \le \theta(a,b). \tag{A.36}$$

(i) When $c(z)$ is convex with $z_0 z_1 = 0$, we have $z_0 = -z_\Delta < 0 = z_1$ by $z_\Delta > 0$. In addition, by $h(z) = c(z)\mathbf{1}_{\{z<0\}}$, the desired inequality (A.36) is equivalent to

$$\theta(a-z_\Delta,b) + z_\Delta c(b) - bc(z_\Delta) \le \theta(a,b). \tag{A.37}$$

• If $g(x)$ is κ-convex with $\kappa(x) = c(x) - c_1 x$, then $\theta(a,b) = a\kappa(b)$ by definition of $\theta(a,b)$. In this case, inequality (A.37) is obviously.

• If $g(x)$ is sym-κ-convex with $\kappa(x) = c(x) - c_1 x$, then $\theta(a,b) = [a\kappa(b)] \vee [b\kappa(a)]$. In this case, inequality (A.37) further reduces to

$$[ac(b) - bc(z_\Delta)] \vee [bc(a - z_\Delta) + z_\Delta c(b) - bc(z_\Delta)] \le [ac(b)] \vee [bc(a)].$$

Because $[ac(b) - bc(z_\Delta)] \le ac(b)$, inequality (A.37) is equivalent to

$$b[c(a - z_\Delta) - c(z_\Delta)] + z_\Delta c(b) \le [ac(b)] \vee [bc(a)].$$

Furthermore, by $0 < z_\Delta < a$, a sufficient condition to the desired inequality (A.37) is

$$\begin{aligned} b[c(a - z_\Delta) - c(z_\Delta)] + z_\Delta c(b) &\le (a^{-1} z_\Delta)[ac(b)] + \left(1 - a^{-1} z_\Delta\right)[bc(a)] \\ &= z_\Delta c(b) + a^{-1} b\left[(a - z_\Delta)\, c(a)\right], \end{aligned}$$

i.e., $\frac{c(a-z_\Delta)-c(z_\Delta)}{a-z_\Delta} \le \frac{c(a)}{a}$, which is true because monotonicity and convexity of $c(z)$ imply that

$$\frac{c(a - z_\Delta) - c(z_\Delta)}{a - z_\Delta} \le \frac{c(a - z_\Delta) - c(0)}{(a - z_\Delta) - 0} \le \frac{c(a) - c(0)}{a - 0} = \frac{c(a) - K_1}{a} \le \frac{c(a)}{a}.$$

• If $K_n \ge 0$ and $g(x)$ is sym-κ-convex with $\kappa(x) = K_1$, then $\theta(a,b) = (a \vee b)K_1$. In this case, inequality (A.37) further reduces to

$$[(a - z_\Delta) \vee b]K_1 + [z_\Delta c(b) - bc(z_\Delta)] \le (a \vee b)K_1.$$

Note that $K_n \ge 0$ and (1) imply that $z^{-1}c(z)$ is decreasing in z when $z > 0$. Thus, when $z_\Delta \le b$, inequality (A.37) holds because

$$z_\Delta c(b) - bc(z_\Delta) = z_\Delta b\left[b^{-1}c(b) - z_\Delta^{-1}c(z_\Delta)\right] \le 0.$$

When $z_\Delta > b$, because $z^{-1}[c(z) - K_1]$ is increasing in z when $z > 0$ by convexity of $c(z)$ and $c(0) = K_1$, we can verify that

$$\begin{aligned} z_\Delta c(b) - bc(z_\Delta) &= (z_\Delta - b)K_1 + z_\Delta b\left\{b^{-1}[c(b) - K_1] - z_\Delta^{-1}[c(z_\Delta) - K_1]\right\} \\ &\le (z_\Delta - b)K_1. \end{aligned}$$

Thus, a sufficient condition to the desired inequality is

$$[(a - z_\Delta) \vee b]K_1 + (z_\Delta - b)K_1 \le (a \vee b)K_1,$$

i.e., $[(a - b) \vee z_\Delta]K_1 \le (a \vee b)K_1$, which is true because $z_\Delta < a$ in this case.

(ii) When $c(z)$ is concave, if $z_\Delta \le b$, then by concavity of $h(z)$ and $z_1 - b \le z_0 < z_1$.

$$(b - z_\Delta)h(z_1) - bh(z_0) + z_\Delta h(z_1 - b) \le 0.$$

In this case, inequality (A.36) reduces to $\theta(a-z_\Delta,b)\le\theta(a,b)$, which holds obviously for both $\theta(a,b)=a\kappa(b)$ and $\theta(a,b)=[a\kappa(b)]\vee[b\kappa(a)]$ with $\kappa(z)=c(z)-c_nz$. If $z_\Delta>b$, then $z_0<z_1-b$. Because $h(z)$ is concave and $h(z)+c_nz$ is decreasing when $z\le 0$,

$$\begin{aligned}(b-z_\Delta)h(z_1)-bh(z_0)+z_\Delta h(z_1-b) &= (z_\Delta-b)[h(z_1-b)-h(z_1)]+b[h(z_1-b)-h(z_0)]\\ &\le (z_\Delta-b)[h(-b)-h(0)]+b[-c_n(z_1-b-z_0)]\\ &= (z_\Delta-b)c(b)-bc_n(z_\Delta-b)\\ &= (z_\Delta-b)\kappa(b).\end{aligned}$$

Thus, inequality (A.36) is satisfied if for any $b<z_\Delta<a$, the following inequality holds.

$$\theta(a-z_\Delta,b)+(z_\Delta-b)\kappa(b)\le\theta(a,b).$$

When $\theta(a,b)=a\kappa(b)$, it is obviously true. When $\theta(a,b)=[a\kappa(b)]\vee[b\kappa(a)]$, it reduces to

$$[(a-b)\kappa(b)]\vee[b\kappa(a-z_\Delta)+(z_\Delta-b)\kappa(b)]\le[a\kappa(b)]\vee[b\kappa(a)].$$

By $[(a-b)\kappa(b)]\le[a\kappa(b)]$, it holds if $b\kappa(a-z_\Delta)+(z_\Delta-b)\kappa(b)\le a\kappa(b)$, i.e.,

$$bc(a-z_\Delta)\le bc(b)+(a-z_\Delta)c(b).$$

The above inequality is true because its left side is no more than $bc(b)$ if $a-z_\Delta\le b$, and no more than $(a-z_\Delta)c(b)$ if $a-z_\Delta>b$ because by the concavity of $c(z)$, $z^{-1}c(z)$ is decreasing in z when $z>0$. □

Proof of Theorem 8

First of all, observe that $\{x<R_t\}\subseteq\mathcal{O}_t$ immediately follows from the definition of R_t by (7). We now characterize the produce up to level $y_t^*(x)=z_t^*(x)+x$ for $x\in\mathcal{O}_t$. Note that $y_t^*(x)$ solves the problem

$$v_t^0(x)=\max_{y\ge x}[u_t(y)-c(y-x)].$$

(i) If $c(z)$ is convex, then its objective function is supermodular in (x,y). Since its feasible set forms a lattice, by Theorem 2.8.2 in Topkis (1998), we know that $y_t^*(x)$ is increasing in $x\in\mathcal{O}_t$. Moreover, for any $1\le i\le n$, because $c(z)\ge K_i+c_iz$ and $S_t(c_i)$ maximizes function $u_t(y)-c_iy$,

$$v_t^0(x)\le\max_{y\ge x}[u_t(y)-K_i-c_i(y-x)]\le u_t(S_t(c_i))-K_i-c_i\left[S_t(c_i)-x\right].$$

On the other hand, if $S_t(c_i)-q_i<x\le S_t(c_i)-q_{i-1}$, then by definitions of $c(z)$ and $v_t^0(x)$,

$$u_t(S_t(c_i))-K_i-c_i\left[S_t(c_i)-x\right]=u_t(S_t(c_i))-c\left(S_t(c_i)-x\right)\le v_t^0(x).$$

Therefore $z_t^*(x)=S_t(c_i)-x$ and $y_t^*(x)=S_t(c_i)$ if $\mathcal{X}\in\mathcal{O}_t$ and $S_t(c_i)-q_i<x\le S_t(c_i)-q_{i-1}$.

(ii) If $c(z)$ is concave and $x < R_t$, because a production is always executed over the region $(-\infty, R_t)$, then it leads no loss of optimality to ignore the constraint $y \geq x$. In this case, observe that $y_t^*(-x)$ solves the problem

$$v_t^0(-x) = \max_y [u_t(y) - c(y+x)].$$

Since its objective function is supermodular in (x, y), we know that $y_t^*(-x)$ is increasing in x, i.e., $y_t^*(x)$ is decreasing in x. Moreover, because $c(z) = \min_{1\leq i\leq n} [K_i + c_i z]$, we can express

$$v_t^0(x) = \max_{1\leq i\leq n} c_i x + \max_y [u_t(y) - c_i y].$$

Thus, we can choose $y_t^*(x) \in S_t(c_i)$ for some $1 \leq i \leq n$.

Next we characterize $\mathcal{O}_t^c$. Because $v_{T+1}(x) = 0$, by applying Proposition 5 to problem (3b) and Proposition 6 to problem (3a) for all $t = T, \cdots, 1$, we can inductively prove the following statements,

- When $U = L$, $v_t(x)$ and $u_t(x)$ are κ-concave with $\kappa(z) = c(z) - (c_1 \wedge c_n)z$;
- When $U \leq 2L$, $v_t(x)$ and $u_t(y)$ are sym-κ-concave with $\kappa(z) = c(z) - (c_1 \wedge c_n)z$; and
- When $c(z)$ is convex and $K_n \geq 0$, $v_t(x)$ and $u_t(y)$ are sym-κ-concave with $\kappa(z) = K_1$.

With results provided above, we are ready to show $\{x \geq S_t(c_0)\} \subseteq \mathcal{O}_t^c$ for a constant c_0 specified in each case (e.g., $c_0 = c_1$ if $c(z)$ is convex and either $U = L$ or $K_n \geq 0$). To unify the discussion, introduce $\theta(a, b) = a\kappa(b)$ if $U = L$, $\theta(a, b) = [a\kappa(b)] \vee [b\kappa(a)]$ if $U \leq 2L$, and $\theta(a, b) = (a \vee b)K_1$ if $c(z)$ is convex and $K_n \geq 0$. For any $x \geq S_t(c_0)$ and $b > 0$, denote by $a = x - S_t(c_0) \geq 0$. Because $S_t(c_0)$ is a global maximizer of $u_t(y) - c_0 y$, by the concave-like property of $u_t(y)$ and hence $u_t(y) - c_0 y$, we know that

$$\begin{aligned}(a+b)[u_t(x) - c_0 x] &\geq b\,[u_t(S_t(c_0)) - c_0 S_t(c_0)] + a\,[u_t(x+b) - c_0(x+b)] - \theta(a, b)\\ &\geq (a+b)[u_t(x+b) - c_0(x+b)] - \theta(a, b).\end{aligned}$$

Reformulate the above inequality as $u_t(x) - u_t(x+b) + \mathcal{A} \geq 0$ for $\mathcal{A} = c_0 b + \theta(a, b)/(a+b)$. If we are able to further prove $\mathcal{A} \leq c(b)$, then $u_t(x) \geq u_t(x+b) - c(b)$. That is, it is not optimal to produce at $x \geq S_t(c_0)$, i.e., $\{x \geq S_t(c_0)\} \subseteq \mathcal{O}_t^c$. Thus, we only need to verify $\mathcal{A} \leq c(b)$ in each case.

(i) When $U = L$, $c_0 = c_1$ if $c(z)$ is convex, and $c_0 = c_n$ if $c(z)$ is concave. In both cases, we can express $\theta(a, b) = a\kappa(b) = a[c(b) - c_0 b]$. Therefore,

$$\mathcal{A} = c_0 b + \frac{a[c(b) - c_0 b]}{a+b} \leq \frac{ac(b)}{a+b} \leq c(b).$$

(ii) When $U \leq 2L$ and $c(z)$ is convex, $\kappa(z) = c(z) - c_1 z$, $c_0 = c_1 - c_n$ and

$$\mathcal{A} = (c_1 - c_n)b + \frac{a[c(b) - c_1 b]}{a+b} \vee \frac{b[c(a) - c_1 a]}{a+b}.$$

By $\frac{a}{a+b} \leq 1$ and $c(b) - c_1 b \geq 0$, as well as $c(a) - c_1 a \leq c(a+b) - c_1(a+b)$,

$$\mathcal{A} \leq (c_1 - c_n)b + [c(b) - c_1 b] \vee \sup_{a\geq 0} \frac{b[c(a+b) - c_1(a+b)]}{a+b}.$$

Because $z^{-1}[c(z) - c_n z] = \max_{1\le i\le n}[z^{-1}K_i + c_i - c_n]$ is the maximum of n quasi-convex functions, and $c(z) = K_n + c_n z$ when z is sufficiently large, we know that for any $a \ge 0$,

$$\mathcal{A} \le (c_1 - c_n)b + [c(b) - c_1 b] \vee [b(c_n - c_1)] = [c(b) - c_n b] \vee 0 \le c(b).$$

(iii) When $U \le 2L$ and $c(z)$ is concave, $\kappa(z) = c(z) - c_n z$, $c_0 = 0$ and we can express

$$\mathcal{A} = (ab)\frac{[b^{-1}c(b)] \vee [a^{-1}c(a)] - c_n}{a+b}.$$

Because $z^{-1}c(z)$ is decreasing in z (see Proof of Proposition 6), if $a \ge b$, then

$$\mathcal{A} = (ab)\frac{b^{-1}c(b) - c_n}{a+b} = \frac{a[c(b) - c_n b]}{a+b} \le [c(b) - c_n b] \le c(b).$$

Moreover, if $a < b$, then by monotonicity of $z^{-1}c(z)$ again, we also have

$$\mathcal{A} = (ab)\frac{a^{-1}c(a) - c_n}{a+b} = \frac{bc(a) - c_n ab}{a+b} \le \frac{bc(a+b)}{a+b} \le c(b).$$

(iv) When $c(z)$ is convex and $K_n \ge 0$, then $\kappa(z) = c(z) - c_1 z$, $c_0 = c_1$ and obviously

$$\mathcal{A} = c_1 b + \frac{(a \vee b)K_1}{a+b} \le c_1 b + K_1 \le c(b).$$

In summary, we verified that $u_t(x) - u_t(x+b) + c(b) \ge 0$ for any $x \ge S_t(c_0)$ and $b > 0$. $\square$