Coping with shortages caused by disruptive events in automobile supply chains

Yi Jiang * Jia Shu † and Miao Song ‡

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Abstract

Unpredictable disruptive events significantly increase the difficulty of the management of automobile supply chains. In this paper, we propose an automobile production planning problem with component chips substitution in a finite planning horizon. The shortage of one chip can be compensated by another chip of the same type with a higher-end feature at an additional cost. Therefore, the automobile manufacturer can divert the on-hand inventory of chips to product lines that are more profitable in the event of shortages caused by supply chain disruptions. To cope with this, we propose a max-min robust optimization model that captures the uncertain supplies of chips. We show that the robust model has an MIP equivalence that can be solved by a commercial IP solver directly. We compare the max-min robust model with the corresponding deterministic and two-stage stochastic models for the same problem through extensive numerical experiments. The computational results show that the max-min robust model outperforms the other two models in terms of the average and worst-case profits.

Keywords: Automobile Supply Chain; Two-Stage Model; Component Substitution

*Department of Mathematics, Sichuan Normal University. Email: yijiang103@163.com

†Corresponding author. School of Management and Economics, University of Electronic Science and Technology of China. Email: jshu@seu.edu.cn

‡Department of Logistics and Maritime Studies, Faculty of Business, The Hong Kong Polytechnic University. Email: miao.song@polyu.edu.hk

1 Introduction

The last three decades have witnessed the development of automobile supply chains that are more complex and more geographically dispersed than ever before (cf. Manuj and Mentzer, 2008; Grant, 2014; JLT, 2019). The drivers for such changes include: (i) the global nature of the major vehicle manufacturers; (ii) the progressive breakdown of barriers to free trade around the world; (iii) the increasingly competitive nature of the vehicle manufacturing business; and (iv) the increasing complexity and variety of the vehicles.

These developments have led to an increase in lead times. Simultaneously, cost cutting pressures have resulted in diminished attention to alternate sourcing so that in many situations the number of suppliers capable to provide a given commodity has decreased. As a result, it is now not uncommon even for crucial classes of commodities to be manufactured by a single supplier at a limited number of facilities. This has led to some well publicized disruptions, such as the 1996 strike at a brake supplier factory that resulted in the idling of twenty-six General Motors (GM) plants, i.e., a shutdown of nearly all of the GM's North American production (cf. WSJ, 2007). After the terrorist attack on September 11, 2001, Ford Motor and Toyota Motor suffered severe oversea supply transportation disruptions (cf. Sheffi and Rice, 2005). In December 2001, Land Rover had to suspend the Discovery vehicles production due to the bankruptcy of UPF-Thompson, the only supplier for chassis frames of Discovery (cf. Sheffi and Rice, 2005). As also evidenced by Wagner et al. (2009), it may not be helpful to mitigate the adverse consequences of supply disruptions by enriching the supplier portfolio because these suppliers are not independent to each other. Indeed, disruptions of this type have been experienced in industries other than automobile (e.g., electronics, aerospace), as the respective supply chains have undergone developments similar to those described above for the automobile industry. For example, the great East Japan earthquake on March 11, 2011 and then the catastrophic flooding in Thailand in October 2011, both events led to a series of disruptions on electronics supply chains that caused huge losses of many companies (cf. Zhang, 2011; Tokui et al., 2017).

The detailed description of the part logistics steps and the decision problems involved in the automobile industry can be found in Boysen et al. (2015). Extensive literature exists about supply chain disruptions, see Kleindorfer and Saad (2005), Sodhi et al. (2012), and the references therein for qualitative studies, and Snyder et al. (2016) and the references therein for quantitative studies. Despite the vast volume of the studies, most of them adopt a conceptual framework with empirical findings or a production/inventory optimization model that helps companies prepare for and mitigate the adverse consequences caused by supply, demand, and cost disruptions, especially in the area of automobile supply chain disruptions (cf. Tomlin, 2006; Wu et al., 2007; Thun and Hoenig, 2011; MacKenzie et al., 2014; Matsuo, 2015; Simchi-Levi et al., 2015). A number of authors have recently stressed the need for companies to pay more attention to the resiliency of their supply chains; see, e.g., Christopher and Peck (2004), Sheffi and Rice (2005), Tang (2006), and Ponomarov and Holcomb (2009). The resiliency may have impact on the competitiveness of a firm and could influence its overall profit, especially when it faces disruptive events. Also at a number of firms, efforts are under way to detect as early as possible events that may result in supply chain disruptions (cf. Blackhurst et al., 2005; Qi et al., 2010). The cost of supply chain disruptions could be tremendously high, which, if handled inadequately, can result in the inability to serve the customer demand in time and the increase of substitution costs. Disruptions are, however, often inherent and inevitable in a global supply chain context, especially for the automobile industry which commonly pursues Just-in-Time (JIT) production enthusiastically. In this paper, we consider the issue of optimally managing the consequences of a component commodity shortage once a disruption has taken place in automobile supply chains.

The situation we model is the one where the shortage affects a class of commodities that are used on several vehicle lines manufactured at different automobile assembly plants. For instance, the shortage could be due to a major accident at a supplier production facility, e.g., a seat manufacturer. Another possibility could be a situation where a large batch of work-in-process materials used to produce a whole class of components (e.g., silicone wafers used in the production of engine control chips) are found to be defective: this could easily result in supply shortages if production involves long batch processes.

When a shortage due to supply chain disruptions occurs, it is possible to at least divert the existing inventory to the product lines that are more profitable for the vehicle manufacturer. In situations involving a whole class of commodities such as silicone chips, it may also be possible for certain types of chips to be used for functions for which they were not originally intended. This typically entails an added cost because chips with higher-end features can be used for low end applications and not vice versa.

The parts experiencing shortages may not be very deeply embedded in the final product. If this is the case, the vehicle manufacturer may consider continuing vehicle production and holding the vehicles until the missing parts become available and can be installed. This option, however, can become too costly especially in the case of a prolonged shortage. This is because of the cost and the logistics difficulties of storing a large number of finished vehicles. Furthermore, the negative effect on customer demand of first withholding and later releasing a large number of vehicles of the same type has to be kept into account. Finally, it should be noted that while the operation of installing the missing parts may have been easy to perform in the assembly plant, retrofitting may be an altogether different matter (cf. Boysen et al., 2015). Therefore, here we consider a situation where the only decision to be made is to choose the plants (or the lines of products) where production will be continued and the shortages are substituted by other alternatives that will not influence the function of the vehicle.

Motivated by these issues, we are going to study a production planning problem with component parts substitution (PPCPS). In the PPCPS, we are given a fixed planning horizon which is represented by a finite set of discrete time periods, a finite set of vehicle model lines, and a finite set of component part types, each of which contains a finite set of component parts in which a higher-end chip can be used to substitute a lesser one. In each time period, the automobile manufacturer targets a minimum market demand and has a production capacity for each vehicle model. The manufacturer obtains a unit revenue by producing each unit of a vehicle model in each time period. The contracted suppliers can supply the manufacturer with an amount of each chip of each type at the beginning of each time period. This amount is possibly subject to uncertainty due to disruptions. If necessary, the manufacturer can also receive the supply of each chip of each type from the emergency supply channel in the market and the supply amount is assumed to be always enough in each time period. In addition, there is an initial inventory for each chip of each type at the beginning of the planning horizon. The production of each unit of a vehicle model consumes a certain amount of each chip of each type. Four types of costs are considered in this problem:

- Substitution costs. When a chip with a higher-end feature is used to replace a lesser chip of the same type, a substitution cost incurs for such a replacement.
- Inventory holding costs. Each chip of each type, which is not used in the current time period

and will be carried over to the next time period, incurs an inventory holding cost at a constant rate.

- Acquisition costs. The manufacturer incurs a per unit acquisition cost for each chip of each type provided by the contracted suppliers.
- Emergency supply costs. The manufacturer incurs a per unit emergency supply cost for each chip of each type provided by the emergency suppliers.

The problem is to determine the optimal production plan for each vehicle model, and sourcing and substitution of each chip of each type in each time period so as to maximize the total revenues minus the total substitution, inventory holding, acquisition, and emergency supply costs over the predetermined planning horizon.

In the PPCPS, supply chain disruptions could induce the uncertain supplies of automobile component parts. Both stochastic programming and robust optimization are commonly adopted to deal with mathematical optimization problems that involve input uncertainties. We refer to the references, e.g., Birge and Louveaux (1997), Shapiro et al. (2009), on stochastic programming. For the important developments on robust optimization, and its applications in inventory and supply chain management, we refer to Ben-Tal et al. (2005), Ben-Tal et al. (2009), and the review article of Bertsimas et al. (2011), and the references therein. In a more general perspective, the PPCPS is related to process flexibility models by adding flexibility into resource provision, i.e., the availability of capacity or materials that can be used for more than one downstream process/product. We refer to Jordan and Graves (1995) and subsequent papers for the stream of research on process flexibility.

The rest of the paper is organized as follows. In Section 2, we first propose a deterministic mixed-integer programming (MIP) model for the production planning problem with component parts substitution. Based on the deterministic model, we construct a max-min robust model for the PPCPS that captures uncertainties in the contracted supply of chips. We also show how to derive the MIP equivalence of the robust model, which can be solved by a commercial MIP solver directly. In Section 3, we conduct a set of numerical experiments to demonstrate the advantage of the max-min robust model by comparing with the associated deterministic and two-stage stochastic models. Finally, we conclude the paper in Section 4.

2 Problem Description and Model Formulation

We start this section with the problem description and the development of the deterministic model over a predetermined planning horizon. Built upon the deterministic model, we then proceed to the max-min robust model that captures component parts supply uncertainties.

2.1 Deterministic Model

The production planning problem with component parts substitution (PPCPS) is optimized over a pre-specified planning horizon. We assume that the planning horizon consists of T disjoint time periods, which is denoted by $\mathcal{T} = \{1, 2, \dots, T\}$. Without loss of generality, these time periods are of equal length, each of which can represent a week. The production planning consists of a set of vehicle models and a set of component chip types, which are denoted by $\mathcal{M} = \{1, 2, \dots, M\}$ and $\mathcal{N} = \{1, 2, \dots, N\}$, respectively. Each component chip type $n \in \mathcal{N}$ consists of a set of chips, which is denoted by $\mathcal{P}_n = \{1, 2, \dots, P_n\}$. The assembly of one vehicle model j requires a_{ij}^n units of chip i of type n (denoted by (n, i)), i.e., the bill of material (BOM) ratio is a_{ij}^n for each $i \in \mathcal{P}_n$ and $n \in \mathcal{N}$. Each chip (n, i) with a higher-end feature can be used to replace chip (n, i') with a relatively low-end feature at a unit cost of $c_{ii't}^n$ in each period $t \in \mathcal{T}$ for each component chip type $n \in \mathcal{N}$ and $i, i' \in \mathcal{P}_n$. We assume that all the chips in $\mathcal{P}_n$ are ordered in such a way that chip (n, i) can substitute chip (n, i') if and only if $i > i'$, $\forall i, i' \in \mathcal{P}_n$, for each $n \in \mathcal{N}$. We note that the PPCPS can be easily generalized to accommodate any partial ordering of substitutability. We focus here on the more common case of downward substitution, which allows for simpler notation. At the beginning of the planning horizon, the initial inventory for chip (n, i) is y_{i0}^n. At the beginning of each time period t, the manufacturer receives an amount of supply of each chip (n, i) from its contracted suppliers and this amount is denoted by s_{it}^n. The unit acquisition cost of each chip (n, i) amounts to τ_{it}^n in each period t. The manufacturer can also source each chip (n, i) from an emergency supply in the market at the cost of δ_{it}^n per unit in each period t. We assume that there is adequate emergency supply of each chip (n, i) in each period t. There is a constant inventory holding cost rate h_{it}^n for each chip (n, i) at each time period t. We note that such rates at the last period of the planning horizon, i.e., h_{iT}^n, can also be viewed as the unit salvage value of chip (n, i), for all $i \in \mathcal{P}_n$ and $n \in \mathcal{N}$. We assume that the manufacturer has a minimum market demand forecast of $\underline{D}_{jt}$, i.e., these units have already been committed to customers, and a production capacity of $\overline{D}_{jt}$ for each vehicle model j in each period t. Because normally the production capacity is set to be larger than the forecasted minimum possible market demand, we can safely assume that the production capacity is always bigger than the minimum market demand, i.e., $\underline{D}_{jt} < \overline{D}_{jt}, \ \forall j \in \mathcal{M}, t \in \mathcal{T}$. In addition, we assume that the demand of each vehicle model not satisfied is lost and the market demand of each vehicle model exists for as many vehicles as the manufacturer can possibly produce. The manufacturer can accumulate r_{jt} revenue from the assembly of each unit of vehicle model j in period t. In each period, the manufacturer first receives the committed chip supplies from the contracted suppliers, then operates the emergency chip supplies and chip substitutions, and organizes the production based on the minimum demand forecast and the production capacity. The problem is to simultaneously determine the optimal number of each vehicle model to assemble in each period (which, in turn, determines the number of each chip of each type required in each period), the optimal inventory level of each chip of each type at the end of each period, the optimal amount of each chip of each type obtained from the emergency supply, and the optimal chip substitution plan in each period, for which we define the following decision variables:

- x_{jt}: integer variable representing the number of vehicle model j assembled in period t;
- d^n_{it}: number of chip (n, i) required for assembly in period t;
- y^n_{it}: inventory of chip (n, i) at the end of period t;
- e^n_{it}: number of chip (n, i) received from the emergency supply in period t;
- $z^n_{ii't}$: number of chip (n, i) used to substitute chip (n, i') in period t for each pair (i, i') such that $i > i', \ i, i' \in \mathcal{P}_n$.

The objective is to maximize the total revenues minus the total substitution, emergency supply, inventory holding, and acquisition costs over the predetermined planning horizon, each of which can be, respectively, formulated as follows:

- Total revenues: $\sum_{t=1}^{T} \sum_{j=1}^{M} r_{jt} x_{jt}$;
- Substitution costs: $\sum_{t=1}^{T} \sum_{n=1}^{N} \sum_{i=2}^{P_n} \sum_{i':i>i'} c^n_{ii't} z^n_{ii't}$;

- Emergency supply costs: $\sum_{t=1}^{T}\sum_{n=1}^{N}\sum_{i=1}^{P_n}\delta_{it}^{n}e_{it}^{n}$;
- Inventory holding costs: $\sum_{t=1}^{T}\sum_{n=1}^{N}\sum_{i=1}^{P_n}h_{it}^{n}y_{it}^{n}$;
- Acquisition costs: $\sum_{t=1}^{T}\sum_{n=1}^{N}\sum_{i=1}^{P_n}\tau_{it}^{n}s_{it}^{n}$.

We note that the contracted supply s_{it}^n is a constant in the deterministic model, for each $i \in \mathcal{P}_n$, $n \in \mathcal{N}$, and $t \in \mathcal{T}$. Therefore, the acquisition cost $\sum_{t=1}^{T}\sum_{n=1}^{N}\sum_{i=1}^{P_n}\tau_{it}^{n}s_{it}^{n}$ is a constant, which is exogenous in our model development. Thus, we can ignore this cost component in the model development. With these notations, the objective function can be formulated as follows:

$$\sum_{t=1}^{T}\sum_{j=1}^{M}r_{jt}x_{jt}-\sum_{t=1}^{T}\sum_{n=1}^{N}\sum_{i=2}^{P_n}\sum_{i':i>i'}c_{ii't}^{n}z_{ii't}^{n}-\sum_{t=1}^{T}\sum_{n=1}^{N}\sum_{i=1}^{P_n}\delta_{it}^{n}e_{it}^{n}-\sum_{t=1}^{T}\sum_{n=1}^{N}\sum_{i=1}^{P_n}h_{it}^{n}y_{it}^{n}.$$

In each period, we should have the number of each chip of each type required for assembly across all the vehicle models balance constraint, i.e.,

$$\sum_{j=1}^{M}a_{ij}^{n}x_{jt}=d_{it}^{n},\quad \forall i\in\mathcal{P}_n,\ n\in\mathcal{N},\ t\in\mathcal{T}. \tag{1}$$

The number of each vehicle model assembled should be within the range of the minimum market demand forecast and the production capacity of it in each period, i.e.,

$$\underline{D}_{jt}\le x_{jt}\le\overline{D}_{jt},\quad \forall j\in\mathcal{M},\ t\in\mathcal{T}. \tag{2}$$

For each chip of each type in each time period, we have the flow conservation constraint for the chips, i.e.,

$$y_{it}^{n}-y_{it-1}^{n}+d_{it}^{n}+\sum_{i>i'}z_{ii't}^{n}-\sum_{i'>i}z_{i'it}^{n}=s_{it}^{n}+e_{it}^{n},\quad \forall i\in\mathcal{P}_n,\ n\in\mathcal{N},\ t\in\mathcal{T}. \tag{3}$$

Finally, we have the non-negativity and integrality requirements for all the decision variables, i.e.,

$$x_{jt},d_{it}^{n},e_{it}^{n},y_{it}^{n},z_{ii't}^{n}\in\mathbb{Z}^{+},\quad \forall i>i',\ i,i'\in\mathcal{P}_n,\ n\in\mathcal{N},\ j\in\mathcal{M},\ t\in\mathcal{T}, \tag{4}$$

where $\mathbb{Z}^+$ represents the set of non-negative integers. To this end, the deterministic model of the PPCPS becomes:

$$\text{Maximize}\quad \sum_{t=1}^{T}\sum_{j=1}^{M}r_{jt}x_{jt}-\sum_{t=1}^{T}\sum_{n=1}^{N}\sum_{i=2}^{P_n}\sum_{i':i>i'}c_{ii't}^{n}z_{ii't}^{n}-\sum_{t=1}^{T}\sum_{n=1}^{N}\sum_{i=1}^{P_n}\delta_{it}^{n}e_{it}^{n}-\sum_{t=1}^{T}\sum_{n=1}^{N}\sum_{i=1}^{P_n}h_{it}^{n}y_{it}^{n} \tag{5}$$

$$\text{subject to}\quad (1),(2),(3),(4).$$

We next show that d_{it}^n, e_{it}^n, y_{it}^n, and $z_{ii't}^n$ in (4) can be relaxed to be continuous variables, i.e.,

$$x_{jt} \in \mathbb{Z}^+,\ d_{it}^n, e_{it}^n, y_{it}^n, z_{ii't}^n \geq 0, \quad \forall i > i',\ i, i' \in \mathcal{P}_n,\ n \in \mathcal{N},\ j \in \mathcal{M},\ t \in \mathcal{T}, \tag{6}$$

and we can obtain the following equivalent formulation (7).

$$\begin{aligned} \text{Maximize} \quad & \sum_{t=1}^{T}\sum_{j=1}^{M} r_{jt}x_{jt} - \sum_{t=1}^{T}\sum_{n=1}^{N}\sum_{i=2}^{P_n}\sum_{i':i>i'} c_{ii't}^n z_{ii't}^n - \sum_{t=1}^{T}\sum_{n=1}^{N}\sum_{i=1}^{P_n} \delta_{it}^n e_{it}^n - \sum_{t=1}^{T}\sum_{n=1}^{N}\sum_{i=1}^{P_n} h_{it}^n y_{it}^n \\ \text{subject to} \quad & (1), (2), (3), (6). \end{aligned} \tag{7}$$

Theorem 1 *Formulation (5) is equivalent to formulation (7).*

The proof of Theorem 1 is presented in Appendix A.1. Although formulation (7) significantly reduces the number of general non-negative integer variables, it is still a very challenging problem. As a matter of fact, it is NP-hard even when it is simplified to the single-period single-chip of single-type problem without considering the inventory holding cost. The proof of it is presented in Appendix A.2.

Theorem 2 *The deterministic model, i.e., formulation (7), is NP-hard even if $T = 1$, $N = 1$, $P_n = 1$, and $h_{11}^1 = 0$.*

2.2 Two-Stage Stochastic Model

In practice, it is hard to follow the deterministic model that assumes all the inputs are known a priori. Because disruptive events are usually unpredictable, these can result in the random supply of each chip of each type. In this part, we define a two-stage stochastic programming model to capture this uncertainty based on the deterministic model.

The two-stage stochastic programming model is a two-stage model, for which the decisions in the deterministic model is separated into two stages. The first stage concerns with the number of each vehicle model j assembled in each planning period, i.e., x_{jt}, for each $j \in \mathcal{M}$ and $t \in \mathcal{T}$. All the input parameters associated with the first stage are assumed to be known as in the deterministic model, i.e., r_{jt}, $\underline{D}_{jt}$, and $\overline{D}_{jt}$ for each $j \in \mathcal{M}$ and $t \in \mathcal{T}$. The second stage includes the rest of the decisions and parameters in the deterministic model, in which we consider the following input uncertainty:

- $\tilde{s}_{it}^n$: Random supply of chip i of type n in period t, for all $i \in \mathcal{P}_n$, $n \in \mathcal{N}$, and $t \in \mathcal{T}$.

The exact value of $\tilde{s}_{it}^n$ will not be realized until the second stage. For ease of exposition, we use the uncertain vector $\tilde{s}^n$ to represent $\tilde{s}_{it}^n$ for all $i \in \mathcal{P}_n$ and $t \in \mathcal{T}$.

The first stage decision variables represent the decisions that are implemented before the realization of the uncertain parameters in the second stage. Furthermore, constraint (2) and the subset of constraint (4) that only involves the first stage decision variables correspond to the constraints in the first stage. All of the parameters that associate with the first stage decision variables and the first stage constraints are deterministic.

The remaining decision variables in the deterministic model, which exclude the first stage decision variables, are referred to as the second stage decision variables. Similarly, the remaining constraints in the deterministic model, which exclude the first stage constraints, are the second stage constraints. We note that the uncertain parameters $\tilde{s}^n$ only appear in the right-hand-side of the second stage constraints. All the other parameters are independent of the uncertain parameters $\tilde{s}^n$. All of the first stage decision variables have nonzero coefficients in the second stage constraints. They connect the first and second stages, which reflect how the first stage decisions directly affect the second stage decisions.

The second stage decisions are made after implementing the first stage decisions and observing the realized values of the uncertain parameters in the second stage. In other words, the values of the second stage decision variables depend on the values of the first stage decision variables as well as the realized values of the second stage uncertain parameters. To this end, the two-stage stochastic model that maximizes the first stage revenue minus the expectation of the second stage cost among all realizations of $\tilde{s}^n$ for the PPCPS can be formulated as follows:

$$\begin{aligned} \max \quad & \sum_{t=1}^{T}\sum_{j=1}^{M} r_{jt}x_{jt} - \sum_{n=1}^{N} \mathbb{E}_{\tilde{s}^n}[\mathcal{Q}_n(x, \tilde{s}^n)] \\ \text{s.t.} \quad & \underline{D}_{jt} \le x_{jt} \le \overline{D}_{jt}, && \forall j \in \mathcal{M},\ t \in \mathcal{T}, \\ & x_{jt} \in \mathbb{Z}^+, && \forall j \in \mathcal{M},\ t \in \mathcal{T}, \end{aligned} \tag{8}$$

where

$$
\begin{aligned}
\mathcal{Q}_n(x,\tilde{s}^n) \quad = \quad \min \quad & \sum_{t=1}^{T}\sum_{i=2}^{P_n}\sum_{i':i>i'} c^n_{ii't} z^n_{ii't} + \sum_{t=1}^{T}\sum_{i=1}^{P_n} \delta^n_{it} e^n_{it} + \sum_{t=1}^{T}\sum_{i=1}^{P_n} h^n_{it} y^n_{it} \\
\text{s.t.} \quad & \sum_{j=1}^{M} a^n_{ij} x_{jt} = d^n_{it}, \quad \forall i \in \mathcal{P}_n,\ t \in \mathcal{T}, \\
& y^n_{it} - y^n_{it-1} + d^n_{it} + \sum_{i>i'} z^n_{ii't} - \sum_{i'>i} z^n_{i'it} = \tilde{s}^n_{it} + e^n_{it}, \quad \forall i \in \mathcal{P}_n,\ t \in \mathcal{T}, \\
& d^n_{it}, e^n_{it}, y^n_{it}, z^n_{ii't} \geq 0, \quad \forall i > i',\ i, i' \in \mathcal{P}_n,\ t \in \mathcal{T}.
\end{aligned} \tag{9}
$$

For ease of exposition, we rewrite the two-stage stochastic model (8-9) in a compact form as follows:

$$
\begin{aligned}
\max \quad & c_1^T x - \sum_{n=1}^{N} \mathbb{E}_{\tilde{b}_2^n}[\mathcal{Q}_n(x, \tilde{b}_2^n)] \\
\text{s.t.} \quad & A_1 x \leq b_1, \\
& x \in \mathbb{Z}^+,
\end{aligned} \tag{10}
$$

where

$$
\begin{aligned}
\mathcal{Q}_n(x,\tilde{b}_2^n) \quad = \quad \min \quad & c_2^{nT} y^n \\
\text{s.t.} \quad & A_2^n x + B^n y^n = \tilde{b}_2^n, \\
& y^n \geq 0,
\end{aligned} \tag{11}
$$

where the vector of the first stage decision variables and the cost coefficient vector corresponding to the first stage decision variables are denoted by x and c_1, respectively. The constraints that only involve the first stage decision variables are represented by $A_1 x \leq b_1$. All the first stage parameters are deterministic. The second stage decision variables and the associated cost coefficients are denoted by vectors y^n and c_2^n, respectively, for all $n \in \mathcal{N}$. The second stage constraints are represented by $A_2^n x + B^n y^n = \tilde{b}_2^n$ for each $n \in \mathcal{N}$. The uncertain parameters $\tilde{b}_2^n$ only appear in the right-hand-side of the second stage constraints, whereas A_2^n, B^n, and c_2^n are deterministic. The specific distribution of $\tilde{b}_2^n$ is required so as to compute the expectation. In most cases, a finite number of scenarios, e.g., $b_2^{n1}, b_2^{n2}, \ldots, b_2^{nK_n}$, are considered, and each of them happens with the probability $p_k^n, \forall k = 1, 2, \ldots, K_n$. Then model (10-11) can be rewritten as

$$
\begin{aligned}
\max \quad & c_1^T x - \sum_{n=1}^{N} \sum_{k=1}^{K_n} p_k^n c_2^{nT} y^{nk} \\
\text{s.t.} \quad & A_1 x \le b_1, \\
& x \in \mathbb{Z}^+, \\
& A_2^n x + B^n y^{nk} = b_2^{nk}, \qquad \forall n \in \mathcal{N},\ k = 1, 2, \ldots, K_n, \\
& y^{nk} \ge 0, \qquad \forall n \in \mathcal{N},\ k = 1, 2, \ldots, K_n.
\end{aligned} \tag{12}
$$

Although (12) has been widely applied in various problems, the main difficulty to implement this scenario-based approach is how to select the scenarios and calculate the probability for each scenario.

2.3 Max-Min Robust Model

The max-min robust model that we will define in this part is developed in the same way as the two-stage stochastic model except that we consider that the uncertain vector $\tilde{s}^n$ can take any value in an uncertainty set $\mathcal{U}_{\int^n}$ and the max-min robust model maximizes the first stage revenue minus the worst-case second stage cost among all realizations of $\tilde{s}^n \in \mathcal{U}_{\int^n}$ for each $n \in \mathcal{N}$. Thus, the max-min robust model for the PPCPS can be formulated as follows:

$$
\begin{aligned}
\max \quad & \sum_{t=1}^{T} \sum_{j=1}^{M} r_{jt} x_{jt} - \sum_{n=1}^{N} \max_{\tilde{s}^n \in \mathcal{U}_{\int^n}} \left[\mathcal{Q}_n(x, \tilde{s}^n) \right] \\
\text{s.t.} \quad & \underline{D}_{jt} \le x_{jt} \le \overline{D}_{jt}, \qquad \forall j \in \mathcal{M},\ t \in \mathcal{T}, \\
& x_{jt} \in \mathbb{Z}^+, \qquad \forall j \in \mathcal{M},\ t \in \mathcal{T},
\end{aligned} \tag{13}
$$

where

$$
\begin{aligned}
\mathcal{Q}_n(x, \tilde{s}^n) \quad = \quad \min \quad & \sum_{t=1}^{T} \sum_{i=2}^{P_n} \sum_{i':i>i'} c_{ii't}^n z_{ii't}^n + \sum_{t=1}^{T} \sum_{i=1}^{P_n} \delta_{it}^n e_{it}^n + \sum_{t=1}^{T} \sum_{i=1}^{P_n} h_{it}^n y_{it}^n \\
\text{s.t.} \quad & \sum_{j=1}^{M} a_{ij}^n x_{jt} = d_{it}^n, \quad \forall i \in \mathcal{P}_n,\ t \in \mathcal{T}, \\
& y_{it}^n - y_{it-1}^n + d_{it}^n + \sum_{i>i'} z_{ii't}^n - \sum_{i'>i} z_{i'it}^n = \tilde{s}_{it}^n + e_{it}^n, \quad \forall i \in \mathcal{P}_n,\ t \in \mathcal{T}, \\
& d_{it}^n, e_{it}^n, y_{it}^n, z_{ii't}^n \ge 0, \quad \forall i > i',\ i, i' \in \mathcal{P}_n,\ t \in \mathcal{T}.
\end{aligned} \tag{14}
$$

Suppose that we know the lower bound s_{it}^{nL}, the upper bound s_{it}^{nU}, and the most likely value s_{it}^{nM} for each uncertain supply $\tilde{s}_{it}^n$, $\forall i \in \mathcal{P}_n,\ n \in \mathcal{N},\ t \in \mathcal{T}$. Without loss of generality, we suppose

$s_{it}^{nL} < s_{it}^{nM} < s_{it}^{nU}$ for each $i \in \mathcal{P}_n,\ n \in \mathcal{N},\ t \in \mathcal{T}$. For each $\tilde{s}_{it}^n$, we measure its deviation from the most likely value s_{it}^{nM} by $\eta_{it}^n \in [0,1]$ which is defined as follows:

$$\eta_{it}^n = \begin{cases} (s_{it}^{nM} - \tilde{s}_{it}^n)/(s_{it}^{nM} - s_{it}^{nL}), \text{ if } \tilde{s}_{it}^n \le s_{it}^{nM} \\ (\tilde{s}_{it}^n - s_{it}^{nM})/(s_{it}^{nU} - s_{it}^{nM}), \text{ if } \tilde{s}_{it}^n > s_{it}^{nM} \end{cases} \quad \forall i \in \mathcal{P}_n,\ n \in \mathcal{N},\ t \in \mathcal{T}. \tag{15}$$

Based on the deviation measurement $\eta^n = [\eta_{11}^n, \eta_{12}^n, \ldots, \eta_{P_nT}^n]^T$ defined in (15), we define the uncertainty set $\mathcal{U}_{f^n}$ for each $n \in \mathcal{N}$ as follows:

$$\mathcal{U}_{f^n} = \left\{ \tilde{s}^n \in \mathbb{R}^{P_n \times T} \;\middle|\; \begin{array}{l} \eta_{it}^n = \begin{cases} (s_{it}^{nM} - \tilde{s}_{it}^n)/(s_{it}^{nM} - s_{it}^{nL}), \text{ if } \tilde{s}_{it}^n \le s_{it}^{nM} \\ (\tilde{s}_{it}^n - s_{it}^{nM})/(s_{it}^{nU} - s_{it}^{nM}), \text{ if } \tilde{s}_{it}^n > s_{it}^{nM} \end{cases} \forall i \in \mathcal{P}_n,\ t \in \mathcal{T}, \\ \tilde{s}_{it}^n \in [s_{it}^{nL}, s_{it}^{nU}]\ \forall i \in \mathcal{P}_n,\ t \in \mathcal{T},\ \sum_{i \in \mathcal{P}_n} \sum_{t \in \mathcal{T}} \eta_{it}^n \le 1 \end{array} \right\}. \tag{16}$$

Proposition 1 shows that the uncertainty set $\mathcal{U}_{f^n}$ is a bounded polytope and completely characterizes the extreme points of it. The proof of Proposition 1 is presented in Appendix A.3.

Proposition 1 *Define* $s^{nuvw} = [s_{11}^{nuvw}, s_{12}^{nuvw}, \ldots, s_{PT}^{nuvw}]^T, u = 1,2,\ldots,P_n,\ v = 1,2,\ldots,T,\ w = 1,2$*, such that*

$$s_{it}^{nuvw} = \begin{cases} s_{it}^{nL}, & \text{if } i = u, t = v, \\ s_{it}^{nM}, & \text{otherwise}, \end{cases} \quad \forall i \in \mathcal{P}_n,\ t \in \mathcal{T},\ \text{if } w = 1,$$

and

$$s_{it}^{nuvw} = \begin{cases} s_{it}^{nU}, & \text{if } i = u, t = v, \\ s_{it}^{nM}, & \text{otherwise}, \end{cases} \quad \forall i \in \mathcal{P}_n,\ t \in \mathcal{T},\ \text{if } w = 2.$$

The extreme points of the uncertainty set $\mathcal{U}_{f^n}$ *are* $\{s^{nuvw} : u = 1,2,\ldots,P_n,\ v = 1,2,\ldots,T,\ w = 1,2\}$.

For ease of exposition, we rewrite the max-min robust model (13-14) in a compact form as follows:

$$\begin{aligned} \max \quad & c_1^T x - \sum_{n=1}^{N} \max_{\tilde{s}^n \in \mathcal{U}_{f^n}} [\mathcal{Q}_n(x, \tilde{s}^n)] \\ \text{s.t.} \quad & A_1 x \le b_1, \\ & x \in \mathbb{Z}^+, \end{aligned} \tag{17}$$

where

$$\begin{aligned} \mathcal{Q}_n(x, \tilde{s}^n) &= \min \quad c_2^{nT} y^n \\ &\quad \text{s.t.} \quad A_2^n x + B^n y^n = \tilde{s}^n, \\ &\qquad\quad y^n \geq 0. \end{aligned} \tag{18}$$

where x, y^n, c_1, c_2^n, A_1, A_2^n, B^n, and b_1 are those that are defined in the development of the two-stage stochastic model (10-11).

Given the values of x and $\tilde{s}^n$, by taking the dual of (18), we have

$$\begin{aligned} \mathcal{Q}_n(x, \tilde{s}^n) &= \max \quad (\tilde{s}^n - A_2^n x)^T \lambda^n \\ &\quad \text{s.t.} \quad B^{nT} \lambda^n \leq c_2^n. \end{aligned}$$

Therefore, $\max_{\tilde{s}^n \in \mathcal{U}_{\int^n}} [\mathcal{Q}_n(x, \tilde{s}^n)] = \max\{(\tilde{s}^n - A_2^n x)^T \lambda^n | \tilde{s}^n \in \mathcal{U}_{\int^n}, B^{nT} \lambda^n \leq c_2^n\}$. As a result, (17) can be rewritten as:

$$\begin{aligned} \max \quad & c_1^T x - \sum_{n=1}^{N} \max\{(\tilde{s}^n - A_2^n x)^T \lambda^n | \tilde{s}^n \in \mathcal{U}_{\int^n}, B^{nT} \lambda^n \leq c_2^n\} \\ \text{s.t.} \quad & A_1 x \leq b_1, \\ & x \in \mathbb{Z}^+. \end{aligned} \tag{19}$$

Theorem 3 shows that Model (19) has a linear MIP equivalence. The proof of it is presented in Appendix A.4.

Theorem 3 *Model (19) with the uncertainty set $\mathcal{U}_{\int^n}$ defined in (16) has an equivalent linear MIP formulation as follows.*

$$\begin{aligned} \max \quad & c_1^T x - \textstyle\sum_{n=1}^{N} z^n \\ s.t. \quad & A_1 x \leq b_1, \\ & x \in \mathbb{Z}^+, \\ & z^n \geq c_2^{nT} y^{nuvw}, && \forall u \in \mathcal{P}_n,\ n \in \mathcal{N},\ v \in \mathcal{T},\ w = 1, 2, \\ & A_2^n x + B^n y^{nuvw} = s^{nuvw}, && \forall u \in \mathcal{P}_n,\ n \in \mathcal{N},\ v \in \mathcal{T},\ w = 1, 2, \\ & y^{nuvw} \geq 0, && \forall u \in \mathcal{P}_n,\ n \in \mathcal{N},\ v \in \mathcal{T},\ w = 1, 2. \end{aligned} \tag{20}$$

For any $n \in \mathcal{N}$, the uncertainty set $\mathcal{U}_{\int^n}$ in (16) considers the case that the total deviation of the supplies $\tilde{s}_{it}^n$ for all i and t does not exceed 1. As a generalization, we can consider the case that the total deviation is bounded by a given parameter Γ, which leads to the following uncertainty set:

$$\mathcal{U}_{\int^n} = \left\{ \tilde{s}^n \in \mathbb{R}^{P_n \times T} \;\middle|\; \begin{array}{l} \eta_{it}^n = \begin{cases} (s_{it}^{nM} - \tilde{s}_{it}^n)/(s_{it}^{nM} - s_{it}^{nL}), \text{ if } \tilde{s}_{it}^n \leq s_{it}^{nM} \\ (\tilde{s}_{it}^n - s_{it}^{nM})/(s_{it}^{nU} - s_{it}^{nM}), \text{ if } \tilde{s}_{it}^n > s_{it}^{nM} \end{cases} \forall i \in \mathcal{P}_n,\ t \in \mathcal{T}, \\ \tilde{s}_{it}^n \in [s_{it}^{nL}, s_{it}^{nU}]\ \forall i \in \mathcal{P}_n,\ t \in \mathcal{T},\ \sum_{i \in \mathcal{P}_n} \sum_{t \in \mathcal{T}} \eta_{it}^n \leq \Gamma \end{array} \right\}. \tag{21}$$

$\mathcal{U}_{\int^n}$ in (21) remains a polytope, but the number of its extreme points grows exponentially in Γ. Therefore, the robust model (19) cannot be equivalently reformulated as a compact MIP. In response, we propose the following cutting plane algorithm to solve model (19) with the uncertainty set $\mathcal{U}_{\int^n}$ defined in (21).

Step 0. For any $n \in \mathcal{N}$, choose a finite set $\mathcal{U}'_{\int^n}$ such that $\mathcal{U}'_{\int^n} \subseteq \mathcal{U}_{\int^n}$.

Step 1. Solve the following master problem

$$\begin{aligned} \max \quad & c_1^T x - \textstyle\sum_{n=1}^N z^n \\ \text{s.t.} \quad & A_1 x \leq b_1, \\ & x \in \mathbb{Z}^+, \\ & z^n \geq c_2^{nT} y^n(s^n), && \forall n \in \mathcal{N},\ s^n \in \mathcal{U}'_{\int^n}, \\ & A_2^n x + B^n y^n(s^n) = s^n, && \forall n \in \mathcal{N},\ s^n \in \mathcal{U}'_{\int^n}, \\ & y^n(s^n) \geq 0, && \forall n \in \mathcal{N},\ s^n \in \mathcal{U}'_{\int^n} \end{aligned}$$

to obtain an optimal solution $(\bar{x}, \bar{z}^n, \bar{y}^n(s^n))$ for all $n \in \mathcal{N}$ and $s^n \in \mathcal{U}'_{\int^n}$.

Step 2. For any $n \in \mathcal{N}$, solve the following separation problem

$$\hat{z}^n = \max_{\tilde{s}^n \in \mathcal{U}_{\int^n}} \mathcal{Q}_n(\bar{x}, \tilde{s}^n)$$

to obtain the optimal value $\hat{z}^n$ and an optimal solution $\hat{s}^n$. If $\hat{z}^n > \bar{z}^n$, add $\hat{s}^n$ to the set $\mathcal{U}'_{\int^n}$.

Step 3. If $\hat{z}^n \leq \bar{z}^n$ for all $n \in \mathcal{N}$, $\bar{x}$ is an optimal first-stage solution to the robust model (19); otherwise, go to Step 2.

Note that the robust model (19) is equivalent to

$$\begin{aligned} \max \quad & c_1^T x - \textstyle\sum_{n=1}^N z^n \\ \text{s.t.} \quad & A_1 x \leq b_1, \\ & x \in \mathbb{Z}^+, \\ & z^n \geq \mathcal{Q}_n(x, \tilde{s}^n), \quad \forall n \in \mathcal{N},\ \tilde{s}^n \in \mathcal{U}_{\int^n}, \end{aligned}$$

while the master problem in Step 1 is equivalent to

$$\begin{aligned} \max \quad & c_1^T x - \textstyle\sum_{n=1}^N z^n \\ \text{s.t.} \quad & A_1 x \le b_1, \\ & x \in \mathbb{Z}^+, \\ & z^n \ge \mathcal{Q}_n(x, \tilde{s}^n), \quad \forall n \in \mathcal{N},\ \tilde{s}^n \in \mathcal{U}'_{fn}, \end{aligned}$$

which is a relaxation of the original robust model. The solution obtained in Step 1 is feasible to the original model if and only if

$$\bar{z}^n \ge \max_{\tilde{s}^n \in \mathcal{U}_{fn}} \mathcal{Q}_n(\bar{x}, \tilde{s}^n)$$

for all $n \in \mathcal{N}$. Therefore, we need to solve the separation problem in Step 2. Note that $\mathcal{Q}_n(\bar{x}, \tilde{s}^n)$ is defined as a minimization problem in (18). Thus, the separation problem itself is a max-min problem. If we consider the dual of (18), then the separation problem can be written as

$$\begin{aligned} \max \quad & (\tilde{s}^n - A_2^n x)^T \lambda^n \\ \text{s.t.} \quad & B^{nT} \lambda^n \le c_2^n, \\ & \tilde{s}^n \in \mathcal{U}_{fn}, \end{aligned}$$

which has a non-linear objective function. The following theorem shows that the separation problem in Step 2 has an equivalent linear MIP formulation. For notational simplicity, we use K^n to denote the number of columns in the matrix B^n. Then y^n is a $K^n \times 1$ vector of decision variables. Let B_k^n be the kth column of the matrix B^n. The proof of it is given in Appendix A.5.

Theorem 4 *For any $n \in \mathcal{N}$ and any feasible first-stage solution x,*

$$\begin{aligned} \max\nolimits_{\tilde{s}^n \in \mathcal{U}_{fn}} \mathcal{Q}_n(\bar{x}, \tilde{s}^n) = \max \quad & c_2^{nT} y^n \\ s.t. \quad & A_2^n x + B^n y^n = \tilde{s}^n, \\ & 0 \le c_2^n - B_k^{nT} \lambda^n \le M \zeta_k^n, && \forall k \in \{1, 2, \dots, K^n\}, \\ & 0 \le y_k^n \le M(1 - \zeta_k^n), && \forall k \in \{1, 2, \dots, K^n\}, \\ & \zeta_k^n \in \{0, 1\}, && \forall k \in \{1, 2, \dots, K^n\}, \\ & \eta_{it}^n \ge (s_{it}^{nM} - \tilde{s}_{it}^n)/(s_{it}^{nM} - s_{it}^{nL}), && \forall i \in \mathcal{P}_n,\ t \in \mathcal{T}, \\ & \eta_{it}^n \ge (\tilde{s}_{it}^n - s_{it}^{nM})/(s_{it}^{nU} - s_{it}^{nM}), && \forall i \in \mathcal{P}_n,\ t \in \mathcal{T}, \\ & \tilde{s}_{it}^n \in [s_{it}^{nL}, s_{it}^{nU}], && \forall i \in \mathcal{P}_n,\ t \in \mathcal{T}, \\ & \textstyle\sum_{i \in \mathcal{P}_n} \sum_{t \in \mathcal{T}} \eta_{it}^n \le \Gamma, \end{aligned}$$

where M is a sufficiently large number.

3 Computational Results

The purpose of this section is to compare the performance of the max-min robust model with those of the deterministic and two-stage stochastic models proposed in Section 2. We first describe how we generate the test instances and the implementation procedure of the experiments. We then demonstrate that the max-min robust model outperforms the other two models in both optimality and robustness through extensive numerical experiments. All the randomly generated MIP instances are solved by the CPLEX 12.4 MIP solver. All the test instances are implemented on a Dell workstation with 3.20 GHz Intel i7 CPU and 16G memory. The maximum CPU time needed among all the instances solved does not exceed 3,600 seconds. Because the models are proposed to make the strategic production decision for each future period, one hour solution time will not discount the potential practical usefulness of the proposed models.

3.1 Instance Generation

In this part, we describe how we generate the inputs in each numerical experiment. We use $U(a, b)$ and $U(\Delta)$ to denote a uniform distribution in the closed interval $[a, b]$ and a discrete uniform distribution that takes values in the finite set Δ, respectively. In addition, we use $N(\mu, \sigma)$ to represent a normal distribution with mean μ and standard deviation σ. Then, we use $N(\mu, \sigma, a, b)$ to represent a truncated normal distribution that is based on $N(\mu, \sigma)$, in which $[a, b]$ specifies the truncation interval. Furthermore, we use $T(a, b, c)$ to denote a triangular distribution with lower bound a, most likely value b, and upper bound c.

3.1.1 Planning Horizon, Time Periods, Type of Vehicles, Capacities, and Demands

We assume that the automobile manufacturer has a fixed planning horizon. In each week of the planning horizon, the top management of the automobile manufacturer determines the minimum target market demand and the assembly line capacity for each vehicle type. Therefore, the length of one period is set to be one week. We generate the number of time periods, the number of vehicle types, and the minimum target demand and the production capacity of each vehicle type of each time period as follows.

- Number of time periods: We set the number of time periods $T \in \{2, 4, 6, 8, 10, 12, 16, 20, 24\}$;

- Vehicle types: We generate the number of vehicle types $M \sim U(\{3, 5, 10\})$;
- Minimum demand: We generate the minimum target market demand of each vehicle type at each time period as $\underline{D}_{jt} \sim U(\{0, 1, \dots, 1000\}),\ \forall j \in \mathcal{M},\ t \in \mathcal{T}$;
- Production capacity: We generate the production capacity of each vehicle type at each time period as $\overline{D}_{jt} = 5000(1 + \phi_{jt})$ where $\phi_{jt} \sim U(-0.2, 0.2)$ for each $j \in \mathcal{M},\ t \in \mathcal{T}$.

3.1.2 Type of Chips, Number of Chips of Each Type, BOM Ratio, and Initial Inventory

We generate the number of chip types, the number of chips of each type, the BOM ratio of each chip of each type for each vehicle type, and the initial inventory level of each chip of each type as follows.

- Chip types: We generate the number of chip types $N \sim U(\{3, 5, 7\})$;
- Number of chips of each type: We generate the number of chips of each type $P_n \sim U(\{3, 4, 5\})$, $\forall n \in \mathcal{N}$;
- BOM ratio: We generate the BOM ratio $a_{ij}^n \sim U(\{1, 2, 3, 4\})$, $\forall i \in \mathcal{P}_n,\ n \in \mathcal{N},\ j \in \mathcal{M}$;
- Initial inventory: We implement with three sets of initial inventory levels, i.e., high, medium, and low. We let $\xi_{i0}^n \sim U(-0.2, 0.2)$, $\forall i \in \mathcal{P}_n,\ n \in \mathcal{N}$. The generation of high, medium, and low initial inventory levels corresponds to $y_{i0}^n \sim 5000(1 + \xi_{i0}^n)$, $y_{i0}^n \sim 2500(1 + \xi_{i0}^n)$, and $y_{i0}^n \sim 100(1 + \xi_{i0}^n)$, respectively, $\forall i \in \mathcal{P}_n,\ n \in \mathcal{N}$.

3.1.3 Supplies of Chips

The numerical experiments adopt two approaches to generate the supplies of chips, i.e., an autoregressive chip supply value approach and a healthy level of the chip supply chain approach.

The first approach is based on the AR(1) model. Suppose that $\{\tilde{s}_{it}^n, \forall i \in \mathcal{P}_n, t \in \mathcal{T}\}$ are independent across n. The following notations are employed to define the joint distribution for $\{\tilde{s}_{it}^n, \forall i \in \mathcal{P}_n, t \in \mathcal{T}\}$ used in our numerical experiments. First, for any $i \in \mathcal{P}_n$, we generate μ_i^n and σ_i^n uniformly in the intervals $[5000, 15000]$ and $[1000, 3000]$, respectively. These two parameters denote the mean and standard deviation of $\tilde{s}^n_{it}$ for all $t \in \mathcal{T}$, respectively. For any $i, j \in \mathcal{P}_n$, let α^n_{ij} be a number uniformly generated in $[0, 1]$. Let ϵ^n_{jt} for all $j \in \mathcal{P}_n$ and $t \in \mathcal{T}$ denote independent standard normal random variables. In addition, let ρ^n be a number generated uniformly in $[0.2, 0.8]$. The joint distribution for $\{\tilde{s}^n_{it}, \forall i \in \mathcal{P}_n, t \in \mathcal{T}\}$ is defined as follows.

- Consider $t = 1$. For any $i \in \mathcal{P}_n$, let

$$\tilde{s}^n_{i1} = \mu^n_i + \sigma^n_i \frac{\sum_{j \in \mathcal{P}_n} \alpha^n_{ij} \epsilon^n_{j1}}{\sqrt{\sum_{j \in \mathcal{P}_n} (\alpha^n_{ij})^2}}. \tag{22}$$

  It is straightforward to see that $\{\tilde{s}^n_{i1}, \forall i \in \mathcal{P}_n\}$ follows a multivariate normal distribution where $E[\tilde{s}^n_{i1}] = \mu^n_i$ for all $i \in \mathcal{P}_n$, $Var(\tilde{s}^n_{i1}) = (\sigma^n_i)^2$ for all $i \in \mathcal{P}_n$, and

$$\begin{aligned} Cov(\tilde{s}^n_{i1}, \tilde{s}^n_{i'1}) &= E\left[\left(\sigma^n_i \frac{\sum_{j \in \mathcal{P}_n} \alpha^n_{ij} \epsilon^n_{j1}}{\sqrt{\sum_{j \in \mathcal{P}_n} (\alpha^n_{ij})^2}}\right)\left(\sigma^n_{i'} \frac{\sum_{j \in \mathcal{P}_n} \alpha^n_{i'j} \epsilon^n_{j1}}{\sqrt{\sum_{j \in \mathcal{P}_n} (\alpha^n_{i'j})^2}}\right)\right] \\ &= \sigma^n_i \sigma^n_{i'} \frac{\sum_{j \in \mathcal{P}_n} \sum_{j' \in \mathcal{P}_n} E[\alpha^n_{ij} \epsilon^n_{j1} \alpha^n_{i'j'} \epsilon^n_{j'1}]}{\sqrt{\sum_{j \in \mathcal{P}_n} (\alpha^n_{ij})^2} \sqrt{\sum_{j \in \mathcal{P}_n} (\alpha^n_{i'j})^2}} = \sigma^n_i \sigma^n_{i'} \frac{\sum_{j \in \mathcal{P}_n} \alpha^n_{ij} \alpha^n_{i'j}}{\sqrt{\sum_{j \in \mathcal{P}_n} (\alpha^n_{ij})^2} \sqrt{\sum_{j \in \mathcal{P}_n} (\alpha^n_{i'j})^2}} \end{aligned}$$

  for any $i, i' \in \mathcal{P}_n$.

- Consider $t = 2, 3, \ldots, T$. For any $i \in \mathcal{P}_n$, let

$$\tilde{s}^n_{it} = \rho^n \tilde{s}^n_{i,t-1} + (1 - \rho^n)\mu^n_i + \sqrt{1 - (\rho^n)^2} \cdot \sigma^n_i \frac{\sum_{j \in \mathcal{P}_n} \alpha^n_{ij} \epsilon^n_{jt}}{\sqrt{\sum_{j \in \mathcal{P}_n} (\alpha^n_{ij})^2}}. \tag{23}$$

  We can see that $\tilde{s}^n_{it}$ follows an AR(1)-process. Furthermore, $\{\tilde{s}^n_{it}, \forall i \in \mathcal{P}_n\}$ follows a multivariate normal distribution where $E[\tilde{s}^n_{it}] = \mu^n_i$ for all $i \in \mathcal{P}_n$, $Var(\tilde{s}^n_{it}) = (\sigma^n_i)^2$ for all $i \in \mathcal{P}_n$, and

$$\begin{aligned} Cov(\tilde{s}^n_{it}, \tilde{s}^n_{i't}) = E\Bigg[&\left(\rho^n(\tilde{s}^n_{i,t-1} - \mu^n_i) + \sqrt{1 - (\rho^n)^2} \cdot \sigma^n_i \frac{\sum_{j \in \mathcal{P}_n} \alpha^n_{ij} \epsilon^n_{jt}}{\sqrt{\sum_{j \in \mathcal{P}_n} (\alpha^n_{ij})^2}}\right) \\ &\times \left(\rho^n(\tilde{s}^n_{i',t-1} - \mu^n_{i'}) + \sqrt{1 - (\rho^n)^2} \cdot \sigma^n_{i'} \frac{\sum_{j \in \mathcal{P}_n} \alpha^n_{i'j} \epsilon^n_{jt}}{\sqrt{\sum_{j \in \mathcal{P}_n} (\alpha^n_{i'j})^2}}\right)\Bigg] \\ = (\rho^n)^2 \rho^n_{i'} Cov(\tilde{s}^n_{i,t-1}, \tilde{s}^n_{i',t-1}) &+ (1 - (\rho^n)^2) Cov(\tilde{s}^n_{i1}, \tilde{s}^n_{i'1}) = Cov(\tilde{s}^n_{i1}, \tilde{s}^n_{i'1}) \end{aligned}$$

  for any $i, i' \in \mathcal{P}_n$.

Based on the joint distribution for $\{\tilde{s}_{it}^n, \forall i \in \mathcal{P}_n, t \in \mathcal{T}\}$ defined above, we generate the forecasted supply and the realized supply as follow.

- Forecasted supply: For each test instance, we generate 100 random samples according to (22) and (23) for each $n \in \mathcal{N}$. Let s_{it}^{nL}, s_{it}^{nM}, and s_{it}^{nU} be the minimum, average, and maximum of the 100 samples, respectively, for each $i \in \mathcal{P}_n$ and $t \in \mathcal{T}$. The three models are implemented by (i) The deterministic model: We replace s_{it}^n with s_{it}^{nM}. (ii) The stochastic model: For each $n \in \mathcal{N}$, we consider 100 scenarios, i.e., $K_n = 100$, $\forall n \in \mathcal{N}$. Each scenario is sampled by $T(s_{it}^{nL}, s_{it}^{nM}, s_{it}^{nU})$ for all $i \in \mathcal{P}_n$ and $t \in \mathcal{T}$. We assume $p_k^n = 0.01$, $\forall k \in \{1, 2, \ldots, 100\}$. (iii) The robust model: For each $n \in \mathcal{N}$, the uncertainty set $\mathcal{U}_{\int^n}$ is defined using $(s_{it}^{nL}, s_{it}^{nM}, s_{it}^{nU})$ for all $i \in \mathcal{P}_n$ and $t \in \mathcal{T}$.
- Realized supply: We generate 1,000 realizations of $\tilde{s}_{it}^n$ using (22) and (23).

The generated supply parameters are then truncated to $[0, \infty)$ to ensure that the supplies are always nonnegative.

In the second approach, we introduce a parameter $\tilde{H}$ that represents the healthy level of the chip supply chain. $\tilde{H}$ is assumed to be uniformly distributed in $[0, 1]$. $\tilde{s}_{it}^n$ is set to

$$\tilde{s}_{it}^n = a_i^n \tilde{H} + \tilde{b}_{it}^n, \tag{24}$$

where $a_i^n > 0$ is a given parameters and $\tilde{b}_{it}^n$ is a random variable independent across t and (n, i). Thus, conditional on the supply chain health $\tilde{H}$, the chip supplies are independent random variables. For example, the outbreak of COVID-19 can be viewed as a scenario where the health level $\tilde{H}$ is very low. This would generally reduce the supplies of all chips during the planning horizon, i.e., $\tilde{s}_{it}^n$ for all t and (n, i) are very likely to take small values. As $a_i^n > 0$, (24) obviously leads to such a result. In this study, a_i^n is generated uniformly in $[0.2, 2]$. $\tilde{b}_{it}^n$ is normally distributed with mean $\mu_n^{b,i}$ and standard deviation $\sigma_n^{b,i}$ drawn uniformly from the intervals $[0, 1]$ and $[1, 10]$, respectively. As in the AR(1)-based first approach, we can generate the forecasted and realized supplies according to (24). These generated supplies are also truncated to $[0, \infty)$.

3.1.4 Revenues and Cost Parameters

The revenue and cost related parameters are generated as follows.

- Unit revenue: We generate $r_{jt} = (1 + \vartheta_{jt})j \times 10^5$, where $\vartheta_{jt} \sim U(-0.1, 0.1)$, $\forall j \in \mathcal{M},\ t \in \mathcal{T}$;

- Unit substitution cost: We generate $c^n_{ii't} = (i - i')^{(1+\theta^n_{ii't})} \times 10^3$, where $\theta^n_{ii't} \sim U(-0.1, 0.1)$, $\forall i > i',\ i, i' \in \mathcal{P}_n,\ n \in \mathcal{N},\ t \in \mathcal{T}$;

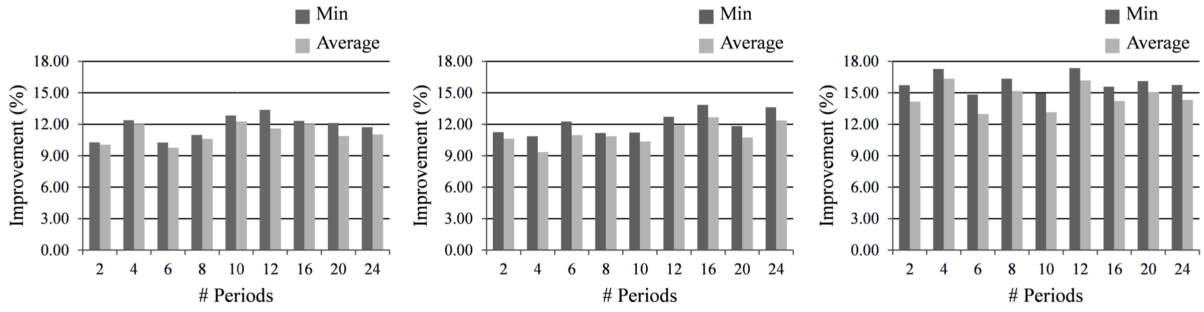
- Emergency chip supply cost rate: We generate $\delta^n_{it} \sim 2(1+\epsilon^n_{it})i \times 10^3$, where $\epsilon^n_{it} \sim U(-0.2, 0.2)$, $\forall i \in \mathcal{P}_n,\ n \in \mathcal{N},\ t \in \mathcal{T}$;

- Inventory holding cost rate: We generate $h^n_{it} \sim (1 + \varepsilon^n_{it})i \times 10^2$, where $\varepsilon^n_{it} \sim U(-0.2, 0.2)$, $\forall i \in \mathcal{P}_n,\ n \in \mathcal{N},\ t \in \mathcal{T}$.

3.2 Numerical Experiments

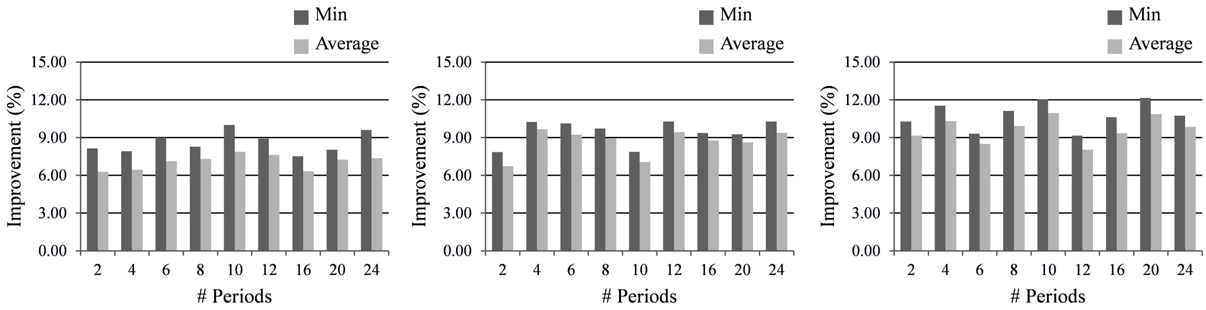
In the numerical experiment, we generate 100 random instances for each time period length T that correspond to the high, medium, and low initial inventory levels, respectively. For each instance, each value of T, and each type of the initial inventory level, we compare the deterministic model (7), the two-stage stochastic model (12), and the max-min robust model (20). We solve all the instances to optimality and obtain the optimal first stage solution, i.e., the number of vehicle model j assembled in period t for each $j \in \mathcal{M},\ t \in \mathcal{T}$, for each of the three models. For each of the first stage decision variable solutions obtained in the implementation of the deterministic, two-stage stochastic, and max-min robust models and each of the realizations, we solve the corresponding second stage problem (9) and compute the cost of both the first and second stages for this realization.

For each model of each value of T and each type of the initial inventory level implemented, we calculate the average and the minimum among the profits corresponding to the 1,000 realizations, which are denoted by P^{Ave}_D and P^{Min}_D for the deterministic model, P^{Ave}_S and P^{Min}_S for the two-stage stochastic model, P^{Ave}_R and P^{Min}_R for the max-min robust model, respectively. We evaluate the performances of the deterministic, two-stage stochastic, and max-min robust models based on the comparison of the average and minimum profits because they approximate the expectation and the worst-case profits associated with implementing the corresponding first stage solution under the true distribution of the uncertain supplies of chips. The improvements of the max-min robust model with respect to the deterministic and two-stage stochastic models in terms of the average profit are calculated by $P^{Ave}_R/P^{Ave}_D - 1$ and $P^{Ave}_R/P^{Ave}_S - 1$, respectively. Similarly, the improvements of the max-min robust model with respect to the deterministic and two-stage stochastic models in terms of the minimum profit are calculated by $P_R^{Min}/P_D^{Min} - 1$ and $P_R^{Min}/P_S^{Min} - 1$, respectively.



(a) H: Robust vs. Deterministic (b) M: Robust vs. Deterministic (c) L: Robust vs. Deterministic

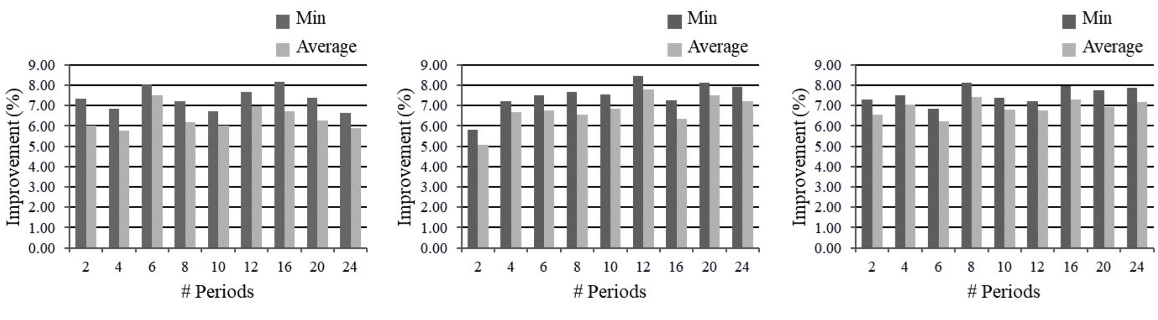
Figure 1: Improvement comparison: robust model with $\Gamma = 1$ vs. deterministic model (approach I)



(a) H: Robust vs. Stochastic (b) M: Robust vs. Stochastic (c) L: Robust vs. Stochastic

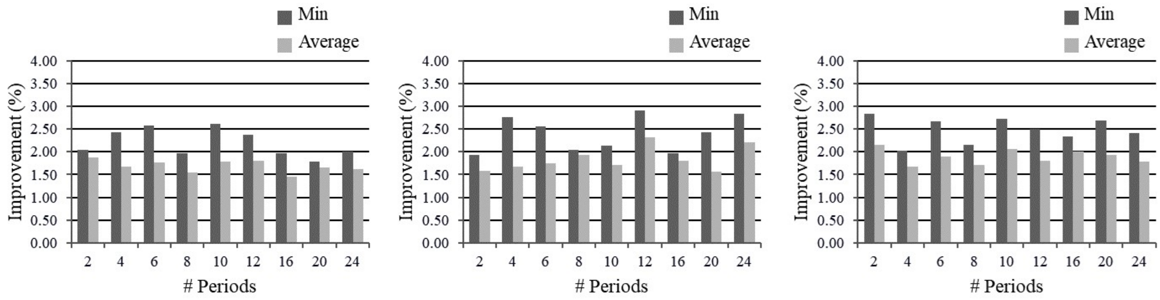
Figure 2: Improvement comparison: robust model with $\Gamma = 1$ vs. stochastic model (approach I)

Figures 1 & 2 illustrate the average improvements of the 100 instances in terms of the average profit and the worst-case profit for different types of the initial inventory level using the first approach to generate the forecasted supply. Figures 1 & 2 show that the max-min robust model ($\Gamma = 1$) outperforms both the deterministic and two-stage stochastic models. In particular, the worst-case and average profit improvements of the max-min robust model ($\Gamma = 1$) are 8.60% and 7.06% higher than those of the two-stage stochastic model, respectively, for the high initial inventory level. These two improvements are 9.45% and 8.65%, respectively, for the medium initial inventory level. These two improvements increase to 10.77% and 9.66%, respectively, for the low initial inventory level. When we compare the max-min robust model ($\Gamma = 1$) with the deterministic model, these two improvements reach to 11.80% and 11.14%, respectively, for the high initial inventory level. Similarly, these two improvements increase to 12.08% and 11.09%, respectively, for the medium initial inventory level. For the low initial inventory level, the two improvements further increase to 15.99% and 14.62%, respectively. The computational results show that the improvements of the max-min robust model ($\Gamma = 1$) comparing with the deterministic model are higher than those of the max-min robust model ($\Gamma = 1$) comparing with the two-stage stochastic model. This translates directly into the fact that the two-stage stochastic model performs better than the deterministic model. This is because the two-stage stochastic model considers the input uncertainties, whereas the deterministic model completely ignores them. In addition, the max-min robust model ($\Gamma = 1$) has a better improvement in terms of the worst-case profit than it in terms of the average profit. This is due to the fact that the max-min robust model ($\Gamma = 1$) takes into account the worst-case profit among all possible realizations within the uncertainty set. Moreover, the max-min robust model ($\Gamma = 1$) performs better in terms of both the average and worst-case profits when the initial inventory level decreases. Similar observations can be obtained from Figures 3 & 4 by comparing the robust model ($\Gamma = P_nT$) with the deterministic and the two-stage stochastic models, respectively. Figures 5, 6, 7, & 8 exhibit similar results when we use the second approach to generate the forecasted supply.



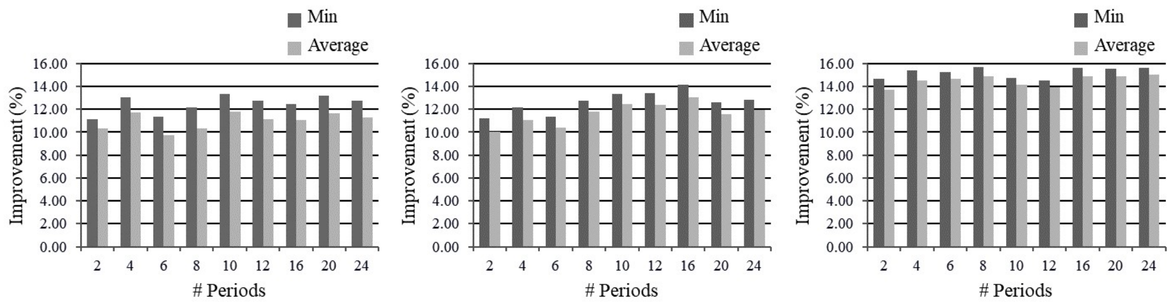
(a) H: Robust vs. Deterministic (b) M: Robust vs. Deterministic (c) L: Robust vs. Deterministic

Figure 3: Improvement comparison: robust model with $\Gamma = P_nT$ vs. deterministic model (approach I)



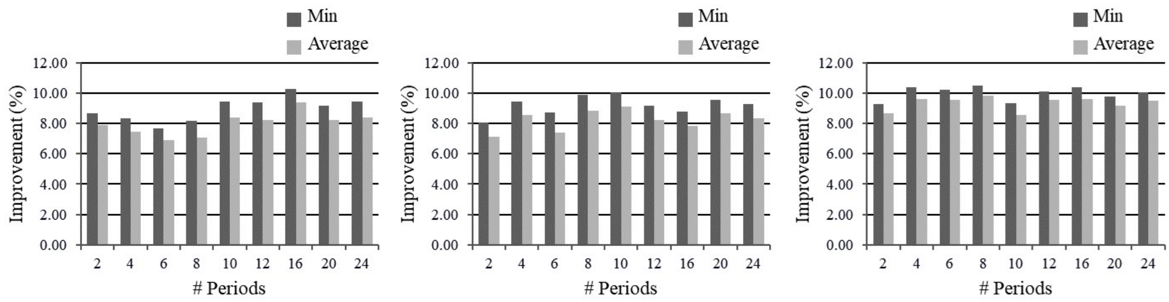
(a) H: Robust vs. Stochastic (b) M: Robust vs. Stochastic (c) L: Robust vs. Stochastic

Figure 4: Improvement comparison: robust model with $\Gamma = P_n T$ vs. stochastic model (approach I)



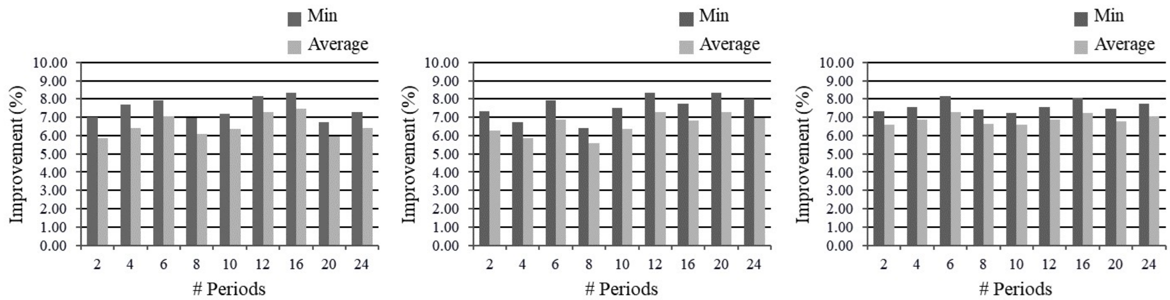
(a) H: Robust vs. Deterministic (b) M: Robust vs. Deterministic (c) L: Robust vs. Deterministic

Figure 5: Improvement comparison: robust model with $\Gamma = 1$ vs. deterministic model (approach II)



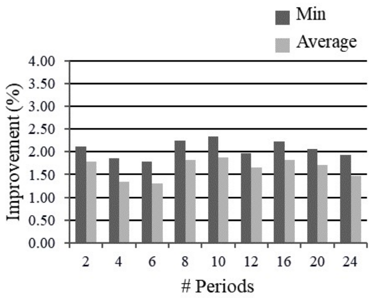
(a) H: Robust vs. Stochastic (b) M: Robust vs. Stochastic (c) L: Robust vs. Stochastic

Figure 6: Improvement comparison: robust model with $\Gamma = 1$ vs. stochastic model (approach II)

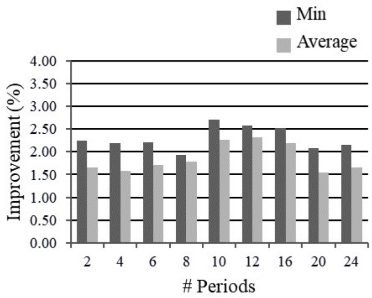


(a) H: Robust vs. Deterministic (b) M: Robust vs. Deterministic (c) L: Robust vs. Deterministic

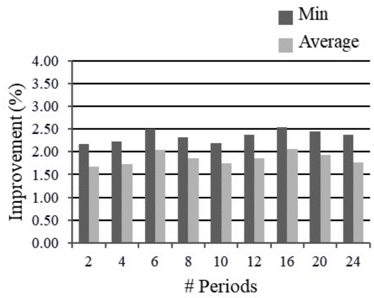
Figure 7: Improvement comparison: robust model with $\Gamma = P_nT$ vs. deterministic model (approach II)



(a) H: Robust vs. Stochastic



(b) M: Robust vs. Stochastic



(c) L: Robust vs. Stochastic

Figure 8: Improvement comparison: robust model with $\Gamma = P_n T$ vs. stochastic model (approach II)

4 Conclusions

In this paper, we propose a max-min robust optimization approach for the production planning problem with component parts substitution that arises from automobile supply chain disruptions. The PPCPS is defined on a fixed planning horizon with multiple periods. The robust model of the PPCPS is a two-stage profit maximization model that optimizes the decision on the number of each vehicle model to assemble before the realization of the uncertain supplies of chips of each type in each planning period. The decisions on the chip substitution, emergency sourcing, consumption, and inventory flow are made after implementing the decision of the number of each vehicle model to assemble and observing the realized values of the uncertain supplies of chips of each type in each planning period. The uncertainty set for the possible realizations of the uncertain supply of each chip of each type is a bounded polytope that is defined based on the lower bound, the most likely value, and the upper bound of this uncertain parameter. The proposed robust model protects against any disturbance of the uncertain parameter that falls into the uncertainty set. We analyze the structural properties of the robust model and give its computationally tractable MIP equivalence. We evaluate the performance of the proposed robust model by comparing it with the corresponding deterministic and two-stage stochastic models for the same problem via extensive numerical experiments. The computational results show that the average improvements of the robust model in comparison with the other two models are non-neglectable.

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We would like to thank the DE, the AE, and the referees for their constructive comments that led to this improved version. This research was supported by the National Natural Science Foundation of China (Grants 72091213, 71922901, 71831004, and 71525001), the Jiangsu Provincial Six Talent Peaks Project (Grant TD-RJFW-001), the Jiangsu Province "333" Project (Grant BRA2018068), and Hong Kong Research Grants Council General Research Fund (Grant PolyU 15240816E).

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Appendix A Proofs

A.1 Proof of Theorem 1

Proof: Firstly, because x_{jt} is integral, d^n_{it} must be integral by constraint (1). As shown in Figure 1, constraint (3) is a standard network flow constraint that satisfies the totally unimodular property (cf. Nemhauser and Wolsey, 1988). Thus, e^n_{it}, y^n_{it}, and $z^n_{ii't}$ correspond to integral flows in the network. □

A.2 Proof of Theorem 2

Proof: Consider the case with $T = 1$, $N = 1$, and $P = 1$. We let $\delta^1_{11} := \max_{j\in\mathcal{M}}\left\{\dfrac{r_{j1}}{a^1_{1j}}\right\} + 1$ so that the optimal emergency supply ${e^1_{11}}^* = 0$. Then, the objective of formulation (7) becomes $\sum_{j\in\mathcal{M}} r_{j1}x_{j1} - h^1_{11}y^1_{11}$. In addition, constraint (3) becomes $y^1_{11} + d^1_{11} = s^1_{11}$ by further imposing $y^1_{10} = 0$. This implies that $d^1_{11} = s^1_{11} - y^1_{11} \le s^1_{11}$ by $y^1_{11} \ge 0$. Due to this, constraint (1) becomes $\sum_{j=1}^{M} a^1_{1j}x_{j1} = d^1_{11} \le s^1_{11}$ in which s^1_{11} is a constant. Together with $h^1_{11} = 0$, formulation (7) is reduced to the knapsack problem by further imposing $\underline{D}_{j1} = 0$ and $\overline{D}_{j1} = 1$ for all $j \in \mathcal{M}$, which proves the NP-hardness of the deterministic model. □

A.3 Proof of Proposition 1

Proof: For any $\tilde{s}^n \in \mathcal{U}_{f^n}$, consider $\eta^n = [\eta^n_{11}, \eta^n_{12}, \ldots, \eta^n_{P_nT}]^T \in [0,1]^{P_nT}$. Let us define $\alpha^n_{uvw},\ u = 1, 2, \ldots, P_n,\ v = 1, 2, \ldots, T,\ w = 1, 2$, such that

$$\alpha^n_{uv1} = \begin{cases} \eta^n_{uv} + \frac{1-\|\eta^n\|_1}{P_nT}\frac{s^{nU}_{uv}-s^{nM}_{uv}}{s^{nU}_{uv}-s^{nL}_{uv}}, & \text{if } \tilde{s}^n_{uv} \le s^{nM}_{uv}, \\ \frac{1-\|\eta^n\|_1}{P_nT}\frac{s^{nU}_{uv}-s^{nM}_{uv}}{s^{nU}_{uv}-s^{nL}_{uv}}, & \text{if } \tilde{s}^n_{uv} > s^{nM}_{uv}, \end{cases}$$

and

$$\alpha^n_{uv2} = \begin{cases} \frac{1-\|\eta^n\|_1}{P_nT}\frac{s^{nM}_{uv}-s^{nL}_{uv}}{s^{nU}_{uv}-s^{nL}_{uv}}, & \text{if } \tilde{s}^n_{uv} \le s^{nM}_{uv}, \\ \eta^n_{uv} + \frac{1-\|\eta^n\|_1}{P_nT}\frac{s^{nM}_{uv}-s^{nL}_{uv}}{s^{nU}_{uv}-s^{nL}_{uv}}, & \text{if } \tilde{s}^n_{uv} > s^{nM}_{uv}. \end{cases}$$

Because $\|\eta^n\|_1 = \sum_{i=1}^{P_n}\sum_{t=1}^{T}\eta^n_{it} \le 1$ by the definition in (15), we have $\alpha^n_{uvw} \ge 0,\ \forall u =$

$1,2,\ldots,P_n,\ v=1,2,\ldots,T,\ w=1,2$. Furthermore,

$$\begin{aligned}\sum_{u=1}^{P_n}\sum_{v=1}^{T}\sum_{w=1}^{2}\alpha_{uvw}^n &= \sum_{u=1}^{P_n}\sum_{v=1}^{T}(\alpha_{uv1}^n+\alpha_{uv2}^n)\\ &= \sum_{u=1}^{P_n}\sum_{v=1}^{T}\left(\eta_{uv}^n+\frac{1-\|\eta^n\|_1}{P_nT}\left(\frac{s_{uv}^{nU}-s_{uv}^{nM}}{s_{uv}^{nU}-s_{uv}^{nL}}+\frac{s_{uv}^{nM}-s_{uv}^{nL}}{s_{uv}^{nU}-s_{uv}^{nL}}\right)\right)\\ &= \sum_{u=1}^{P_n}\sum_{v=1}^{T}\left(\eta_{uv}^n+\frac{1-\|\eta^n\|_1}{P_nT}\right)=1\end{aligned}$$

For any $i\in\mathcal{P}_n,\ t\in\mathcal{T}$, by the definition of s^{nuvw}, we have $s_{it}^{nit1}=s_{it}^{nL}, s_{it}^{nit2}=s_{it}^{nU}$, and $s_{it}^{nuvw}=s_{it}^{nM}$, for any $u\neq i,\ v\neq t$. Therefore,

$$\begin{aligned}\sum_{u=1}^{P_n}\sum_{v=1}^{T}\sum_{w=1}^{2}\alpha_{uvw}^n s_{it}^{nuvw} &= \alpha_{it1}^n s_{it}^{nL}+\alpha_{it2}^n s_{it}^{nU}+(1-\alpha_{it1}^n-\alpha_{it2}^n)s_{it}^{nM}\\ &= \alpha_{it1}^n s_{it}^{nL}+\alpha_{it2}^n s_{it}^{nU}+(1-\eta_{it}^n-\tfrac{1-\|\eta^n\|_1}{P_nT})s_{it}^{nM}.\end{aligned}$$

If $\tilde{s}_{it}^n\le s_{it}^{nM}$, adopting $\eta_{it}^n=(\tilde{s}_{it}^n-s_{it}^{nM})/(s_{it}^{nL}-s_{it}^{nM})$, we have

$$\begin{aligned}&\sum_{w=1}^{2}\sum_{u=1}^{P_n}\sum_{v=1}^{T}\alpha_{uvw}^n s_{it}^{nuvw}\\ =\ &\left(\eta_{it}^n+\tfrac{1-\|\eta^n\|_1}{P_nT}\tfrac{s_{it}^{nU}-s_{it}^{nM}}{s_{it}^{nU}-s_{it}^{nL}}\right)s_{it}^{nL}+\tfrac{1-\|\eta^n\|_1}{P_nT}\tfrac{s_{it}^{nM}-s_{it}^{nL}}{s_{it}^{nU}-s_{it}^{nL}}s_{it}^{nU}+\left(1-\eta_{it}^n-\tfrac{1-\|\eta^n\|_1}{P_nT}\right)s_{it}^{nM}\\ =\ &\eta_{it}^n s_{it}^{nL}+\tfrac{1-\|\eta^n\|_1}{P_nT}\left(\tfrac{s_{it}^{nU}-s_{it}^{nM}}{s_{it}^{nU}-s_{it}^{nL}}s_{it}^{nL}+\tfrac{s_{it}^{nM}-s_{it}^{nL}}{s_{it}^{nU}-s_{it}^{nL}}s_{it}^{nU}\right)+\left(1-\eta_{it}^n-\tfrac{1-\|\eta^n\|_1}{P_nT}\right)s_{it}^{nM}\\ =\ &\eta_{it}^n s_{it}^{nL}+\tfrac{1-\|\eta^n\|_1}{P_nT}s_{it}^{nM}+\left(1-\eta_{it}^n-\tfrac{1-\|\eta^n\|_1}{P_nT}\right)s_{it}^{nM}\\ =\ &\eta_{it}^n s_{it}^{nL}+(1-\eta_{it}^n)s_{it}^{nM}\\ =\ &\tfrac{\tilde{s}_{it}^n-s_{it}^{nM}}{s_{it}^{nL}-s_{it}^{nM}}s_{it}^{nL}+\tfrac{s_{it}^{nL}-\tilde{s}_{it}^n}{s_{it}^{nL}-s_{it}^{nM}}s_{it}^{nM}=\tilde{s}_{it}^n.\end{aligned}$$

Similarly, we can also obtain

$$\sum_{w=1}^{2}\sum_{u=1}^{P_n}\sum_{v=1}^{T}\alpha_{uvw}^n s_{it}^{nuvw}=\tilde{s}_{it}^n,\ \text{if}\ \tilde{s}_{it}^n>s_{it}^{nM}.$$

To this end, we have shown that any $\tilde{s}^n\in\mathcal{U}_{f^n}$ can be represented as a convex combination of $\{s^{nuvw}:u=1,2,\ldots,P_n,\ v=1,2,\ldots,T,\ w=1,2\}$. We next show that any convex combination of $\{s^{nuvw}:u=1,2,\ldots,P_n,\ v=1,2,\ldots,T,\ w=1,2\}$ is in the set $\mathcal{U}_{f^n}$.

Consider $\alpha_{uvw}^n\ge 0$ for any $u=1,2,\ldots,P_n,\ v=1,2,\ldots,T,\ w=1,2$ such that $\sum_{w=1}^{2}\sum_{u=1}^{P_n}\sum_{v=1}^{T}\alpha_{uvw}^n=1$. Let $\tilde{s}^n=\sum_{w=1}^{2}\sum_{u=1}^{P_n}\sum_{v=1}^{T}\alpha_{uvw}^n s^{nuvw}$. For any $i\in\mathcal{P}_n,\ t\in\mathcal{T}$, the definition of s^{nuvw} gives rise to

$$\tilde{s}^n_{it} = \sum_{u=1}^{P_n}\sum_{v=1}^{T}\sum_{w=1}^{2} \alpha^n_{uvw} s^{nuvw}_{it} = \alpha^n_{it1} s^{nL}_{it} + \alpha^n_{it2} s^{nU}_{it} + (1-\alpha^n_{it1}-\alpha^n_{it2}) s^{nM}_{it}.$$

If $\tilde{s}^n_{it} \le s^{nM}_{it}$, (15) gives rise to

$$\begin{aligned} \eta^n_{it} &= \tfrac{\tilde{s}^n_{it}-s^{nM}_{it}}{s^{nL}_{it}-s^{nM}_{it}} = \tfrac{\alpha^n_{it1}s^{nL}_{it}+\alpha^n_{it2}s^{nU}_{it}-(\alpha^n_{it1}+\alpha^n_{it2})s^{nM}_{it}}{s^{nL}_{it}-s^{nM}_{it}} \\ &< \tfrac{\alpha^n_{it1}s^{nL}_{it}+\alpha^n_{it2}s^{nL}_{it}-(\alpha^n_{it1}+\alpha^n_{it2})s^{nM}_{it}}{s^{nL}_{it}-s^{nM}_{it}} = \alpha^n_{it1}+\alpha^n_{it2}, \end{aligned}$$

where the inequality follows from $s^{nL}_{it} < s^{nU}_{it}$ and $s^{nL}_{it} - s^{nM}_{it} < 0$. Similarly, if $\tilde{s}^n_{it} > s^{nM}_{it}$, we have

$$\begin{aligned} \eta^n_{it} &= \tfrac{\tilde{s}^n_{it}-s^{nM}_{it}}{s^{nU}_{it}-s^{nM}_{it}} = \tfrac{\alpha^n_{it1}s^{nL}_{it}+\alpha^n_{it2}s^{nU}_{it}-(\alpha^n_{it1}+\alpha^n_{it2})s^{nM}_{it}}{s^{nU}_{it}-s^{nM}_{it}} \\ &< \tfrac{\alpha^n_{it1}s^{nU}_{it}+\alpha^n_{it2}s^{nU}_{it}-(\alpha^n_{it1}+\alpha^n_{it2})s^{nM}_{it}}{s^{nU}_{it}-s^{nM}_{it}} = \alpha^n_{it1}+\alpha^n_{it2}. \end{aligned}$$

Therefore, we have $\eta^n_{it} < \alpha^n_{it1} + \alpha^n_{it2}$ for any $i \in \mathcal{P}_n,\ t \in \mathcal{T}$. Because $\eta^n_{it} \ge 0$ for any $i \in \mathcal{P}_n$ and $t \in \mathcal{T}$, it follows immediately that $\|\eta^n\|_1 = \sum_{i=1}^{P_n}\sum_{t=1}^{T}\eta^n_{it} < \sum_{i=1}^{P_n}\sum_{t=1}^{T}\alpha^n_{it1} + \alpha^n_{it2} = \sum_{i=1}^{P_n}\sum_{t=1}^{T}\sum_{w=1}^{2}\alpha^n_{itw} = 1$, which implies that $\tilde{s}^n \in \mathcal{U}_{f^n}$.

So far we have shown that the uncertainty set $\mathcal{U}_{f^n}$ is the convex hull of $\{s^{nuvw} : u = 1,2,\dots,P_n,\ v = 1,2,\dots,T,\ w = 1,2\}$. We now proceed to show that $\{s^{nuvw} : u = 1,2,\dots,P_n,\ v = 1,2,\dots,T,\ w = 1,2\}$ are the extreme points of $\mathcal{U}_{f^n}$.

We note that $s^{nL}_{it} < s^{nM}_{it} < s^{nU}_{it}$ for all $i \in \mathcal{P}_n$ and $t \in \mathcal{T}$. Any s^{nuvw} where $u = 1,2,\dots,P_n,\ v = 1,2,\dots,T,\ w = 1$ cannot be represented as a convex combination of $\{s^{nu'v'w} : u' = 1,2,\dots,P_n,\ v' = 1,2,\dots,T,\ w = 1,2,\ u' \neq u,\ v' \neq v\}$, because $s^{nuv1}_{uv} = s^{nL}_{uv} < s^{nM}_{uv} \le s^{nuvw}$ for any $u' \neq u,\ v' \neq v$. Similarly, any s^{nuvw} where $u = 1,2,\dots,P_n,\ v = 1,2,\dots,T,\ w = 2$ cannot be represented as a convex combination of $\{s^{nu'v'w} : u' = 1,2,\dots,P_n,\ v' = 1,2,\dots,T,\ w = 1,2,\ u' \neq u,\ v' \neq v\}$ either, so we complete the proof. □

A.4 Proof of Theorem 3

Proof: In (19), if the value of λ^n is given, $\max\{(\tilde{s}^n - A^n_2 x)^T \lambda^n | \tilde{s}^n \in \mathcal{U}_{f^n}, B^{nT}\lambda^n \le c^n_2\}$ becomes an LP. The optimal solution to this maximization problem can be attained at an extreme point of $\mathcal{U}_{f^n}$, which is given by Proposition 1. Therefore, (19) with the uncertainty set $\mathcal{U}_{f^n}$ is equivalent to

$$\begin{aligned} \max\quad & c_1^T x - \textstyle\sum_{n=1}^N z^n \\ \text{s.t.}\quad & A_1 x \le b_1, \\ & z^n \ge \max\left\{ (s^{nuvw} - A^n_2 x)^T \lambda^n \,\middle|\, B^{nT}\lambda^n \le c^n_2 \right\}, \quad \forall u \in \mathcal{P}_n,\ n \in \mathcal{N},\ v \in \mathcal{T},\ w = 1,2, \\ & x \in \mathbb{Z}^+. \end{aligned} \tag{25}$$

Moreover, the dual of $\max\left\{(s^{nuvw} - A_2^n x)^T \lambda^n \,\middle|\, B^{nT}\lambda^n \leq c_2^n\right\}$ is

$$\begin{aligned} \min \quad & c_2^{nT} y^{nuvw} \\ \text{s.t.} \quad & A_2^n x + B^n y^{nuvw} = s^{nuvw}, \\ & y^{nuvw} \geq 0. \end{aligned}$$

Substituting into (25), we can obtain the linear MIP equivalence of (19) with the uncertain set $\mathcal{U}_{\int^n}$ as (20).

According to the max-min approach, this model protects any change inside the uncertainty set $\mathcal{U}_{\int^n}$ for all $n \in \mathcal{N}$. That is, the total profit will never be below the optimal value of (20) as long as $\tilde{s}^n \in \mathcal{U}_{\int^n}$ for all $n \in \mathcal{N}$. □

A.5 Proof of Theorem 4

Proof: For any given $\tilde{s}^n \in \mathcal{U}_{\int^n}$, we have

$$Q_n(x, \tilde{s}^n) = c_2^{nT} y^n$$

if y^n and λ^n are feasible to the primal and dual problems, respectively, i.e.,

$$A_2^n x + B^n y^n = \tilde{s}^n, \qquad y^n \geq 0, \qquad \text{and} \qquad B^{nT}\lambda^n \leq c_2^n,$$

and satisfy the complementary slackness conditions, i.e.,

$$(c_2^n - B_k^{nT}\lambda^n) y_k^n = 0, \qquad \forall k \in \{1, 2, \ldots, K^n\}.$$

Let ζ_k^n be a binary variable such that $\zeta_k^n = 1$ if and only if $y_k^n = 0$. Then the complementary slackness conditions can be written as

$$c_2^n - B_k^{nT}\lambda^n \leq M\zeta_k^n \qquad \text{and} \qquad y_k^n \leq M(1 - \zeta_k^n), \qquad \forall k \in \{1, 2, \ldots, K^n\}.$$

Thus, $Q_n(x, \tilde{s}^n)$ is equal to $c_2^{nT} y^n$ given that

$$A_2^n x + B^n y^n = \tilde{s}^n,\ 0 \leq c_2^n - B_k^{nT}\lambda^n \leq M\zeta_k^n,\ 0 \leq y_k^n \leq M(1 - \zeta_k^n),\ \text{and}\ \zeta_k^n \in \{0, 1\}, \quad \forall k \in \{1, 2, \ldots, K^n\}.$$

Also note that $\mathcal{U}_{\int^n}$ in (21) can be equivalently defined as

$$\mathcal{U}_{\int^n} = \left\{ \tilde{s}^n \in \mathbb{R}^{P_n \times T} \;\middle|\; \begin{array}{l} \eta_{it}^n \geq (s_{it}^{nM} - \tilde{s}_{it}^n)/(s_{it}^{nM} - s_{it}^{nL}),\ \forall i \in \mathcal{P}_n,\ t \in \mathcal{T}, \\ \eta_{it}^n \geq (\tilde{s}_{it}^n - s_{it}^{nM})/(s_{it}^{nU} - s_{it}^{nM}),\ \forall i \in \mathcal{P}_n,\ t \in \mathcal{T}, \\ \tilde{s}_{it}^n \in [s_{it}^{nL}, s_{it}^{nU}]\ \forall i \in \mathcal{P}_n,\ t \in \mathcal{T},\ \sum_{i \in \mathcal{P}_n} \sum_{t \in \mathcal{T}} \eta_{it}^n \leq \Gamma \end{array} \right\}.$$

This immediately yields the desired result. □