https://doi.org/10.1287/ijoc.2019.0933

Submitted to *INFORMS Journal on Computing*
manuscript JOC-2018-03-OA-029

Multistage Stochastic Power Generation Scheduling Co-Optimizing Energy and Ancillary Services

Jianqiu Huang†
Department of Industrial and Systems Engineering, University of Florida, Gainesville, FL 32611, jianqiuhuang@ufl.edu

Kai Pan†*
Department of Logistics and Maritime Studies, Faculty of Business, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong, kai.pan@polyu.edu.hk

Yongpei Guan
Department of Industrial and Systems Engineering, University of Florida, Gainesville, FL 32611, guan@ise.ufl.edu

†Co-first authors in alphabetical order

With the increasing penetration of intermittent renewable energy and fluctuating electricity loads, power system operators are facing significant challenges in maintaining system load balance and reliability. In addition to traditional energy markets that are designed to balance power generation and load, ancillary service markets have been recently introduced to help manage the considerable uncertainty by reserving certain generation capacities against unexpected events. In this paper, we develop a multistage stochastic optimization model for system operators to efficiently schedule power generation assets to co-optimize power generation and regulation reserve service (a critical ancillary service product) under uncertainty. In addition, to improve the computational efficiency of the proposed multistage stochastic integer program, we explore its polyhedral structure by investigating physical characteristics of individual generators, the system-wide requirements that couple all of the generators, and the scenario tree structure for our proposed multistage model. We start with the single-generator polytope and provide convex hull descriptions for the two-period case under different parameter settings. We then provide several families of multi-period strong valid inequalities linking different scenarios and covering decision variables that represent both power generation and regulation reserve amounts. We further extend our study by exploring the multi-generator polytope and derive strong valid inequalities linking different generators and covering multiple periods. To enhance computational performance, polynomial-time separation algorithms are developed for the exponential number of inequalities. Finally, we verify the effectiveness of our proposed strong valid inequalities by applying them as user cuts under the branch-and-cut scheme to solve multistage stochastic network-constrained power generation scheduling problems.

Key words: ancillary services; power generation scheduling; stochastic optimization; strong valid inequalities; convex hull

* Corresponding author

1. Introduction

With renewable energy's continuous penetration and the rapid development of distributed energy resources, the modern power system is evolving quickly and significantly. Renewable energy complicates the power system substantially by introducing considerable uncertainty due to its intermittent nature. For instance, the electricity generation from wind and solar energy depends greatly on the weather, which is uncertain, making scheduling other generators using traditional fuels like coal and natural gas a huge challenge.

The energy market is already complex because 1) electricity cannot be stored at a very large scale compared to normal commercial products, as in general, it needs to be consumed immediately after being generated, 2) the power system is large-scale and geographically distributed, and 3) the physical characteristics of all of the components in the system are highly complex and coupled. It is very difficult to maintain an exact match between electricity generation and the load at all times under uncertainty. To ensure the efficient and reliable operation of the energy market, system operators are responsible for deciding the power generation schedule to meet the electricity load and maintain cost-effectiveness. In daily operations, system operators are required to solve a large-scale security-constrained unit commitment problem to obtain the corresponding power generation schedule. With the penetration of renewable energy, power system operators are further challenged to schedule power generation assets and to manage uncertainties.

To meet this challenge, ancillary service markets were recently introduced by different Independent System Operators (ISOs) in the wholesale electricity markets to protect the power system against unexpected fluctuations on both the generation (i.e., supply) and load (i.e., demand) sides. Ancillary services are mainly provided by the generation side with reserved generator capacities, although energy storage and demand response can also serve as ancillary services. In current power systems, there are usually three types of ancillary service products: regulation reserve, spinning reserve, and supplemental reserve. Regulation reserve is the most commonly used type. It reserves a certain amount of generation capacity to handle potential future uncertainties like momentary changes on both the generation and load sides and a sudden loss of a generator or transmission line. It is maintained for a quick response to automatic generation control signals to balance fluctuations on both the generation and load sides (Zhou et al. 2016). Although ancillary services can help hedge against uncertainties, there still exist significant difficulties in coordinating traditional energy markets and new ancillary service markets.

Traditional practices sequentially clear the energy and ancillary service markets, considering them separately. However, sequential clearing cannot guarantee a globally optimal power generation schedule coupled with ancillary services (Cheung 2008, Carlson et al. 2012, Zhou et al. 2016).

To overcome this coordination difficulty and thereby use ancillary services more effectively, a challenging co-optimization model that considers energy and ancillary services simultaneously needs to be solved efficiently. The co-optimization model enables a global optimum to schedule generators for both power generation and ancillary service requirements. As the regulation reserve is a critical and the most commonly used ancillary service, in this paper, we study the power generation scheduling model by incorporating both power generation and the regulation reserve for system operators. Our study leads to a unit commitment (UC) model that co-optimizes power generation and the regulation reserve. Previous studies on reserve requirements in a traditional UC model can be found in Li and Shahidehpour (2005), Ostrowski et al. (2012), Morales-España et al. (2013), and Knueven et al. (2018), among others. For instance, Knueven et al. (2018) incorporate the spinning reserve requirement by using a variable to represent the maximum power available from each generator. These studies focus on deterministic models, which have difficulty tackling renewable energy and electricity load uncertainties. Specifically, it is possible to have too much or too little reserve committed ahead of time to accommodate volatile renewable generation in the next operational time horizon (usually 24 hours).

To better handle the significant uncertainties within the power system, we propose a multistage decision making under uncertainty approach to schedule the generation units and accordingly study a multistage stochastic UC model that co-optimizes both power generation and the regulation reserve. Traditional stochastic UC (SUC) was initially proposed by Takriti et al. (1996), Carøe and Schultz (1998), and Takriti et al. (2000) to tackle electricity load uncertainty. Recently, two-stage SUC models have been studied extensively by Gröwe et al. (1995), Carøe and Schultz (1998), Cheung et al. (2015), and others to deal with various uncertainties, while several approaches, such as decomposition algorithms (Wang et al. 2008, Zheng et al. 2013, Schumacher et al. 2017) and Lagrangian relaxation (Ozturk et al. 2004, Papavasiliou and Oren 2013), have been proposed to solve them. Multistage SUC has advantages in incorporating forecasting information with varying degrees of accuracy and in utilizing realized information accumulated over time (Takriti et al. 1996, Birge and Louveaux 2011). This enables more efficient decisions based on uncertainty dynamics and allows us to model renewable generation output and load dependencies among different time periods by using a scenario tree. The related studies on multistage SUC models can be found in Wu et al. (2007), Cerisola et al. (2009), among others. However, the scenario tree-based multistage SUC model is computationally challenging because the size of the scenario tree grows significantly over time. To address this, stochastic dynamic programming (Nowak and Römisch 2000), approximate dynamic programming (Powell 2007, Zhang and Nikovski 2011), and stochastic dual dynamic integer programming (Zou et al. 2018, 2019) have been proposed to solve multistage SUC models. Furthermore, in general, because the multistage SUC model is naturally a stochastic integer program (SIP), various algorithms for SIP can be applied to solve the problem, such as advanced decomposition and Lagrangian relaxation (Carpentier et al. 1996, Nowak and Römisch 2000, Wu et al. 2007, Luedtke 2014, Liu et al. 2016), progressive hedging (Rockafellar and Wets 1991, Løkketangen and Woodruff 1996, Gade et al. 2016), cutting planes (Ahmed et al. 2004, Sen and Higle 2005, Guan et al. 2006, 2009, Luedtke et al. 2010, Zhang et al. 2014), column generation (Sen et al. 2006), and the value function approach (Huang and Ahmed 2009).

In this paper, we conduct a comprehensive polyhedral study by deriving strong valid inequalities and convex hull descriptions to improve the computational performance of the corresponding multistage SUC model that co-optimizes energy generation and the regulation reserve. There are existing studies on the polyhedral structures of deterministic UC polytopes, such as the convex hull representation of the minimum-up/-down time and logical constraints (Lee et al. 2004, Rajan and Takriti 2005), the generation upper/lower bound strengthening constraints (Morales-España et al. 2013), the convex hull representation of the polytope including generation limits, start-up and shut-down capabilities, and minimum-up/-down time constraints (Gentile et al. 2017), valid inequalities for the minimum-up/-down time constraints with multiple generators (Bendotti et al. 2018), strong valid inequalities to strengthen the ramping polytope (Ostrowski et al. 2012, Damcı-Kurt et al. 2016), perfect formulations for the ramping polytope (Knueven et al. 2018, Guan et al. 2018), and the convex hull descriptions of three-period polytopes as well as strong valid inequalities for multi-period versions of the integrated minimum-up/-down time and ramping polytope (Pan and Guan 2016a) and the corresponding SUC polytope (Pan and Guan 2016b). However, most of them either do not consider uncertainty or only optimize power generation instead of co-optimizing power generation and the regulation reserve. For instance, Pan and Guan (2016b) solve a single-generator self-scheduling problem for an independent power producer, who is the decision maker, to generate optimal bidding strategies, leading to the maximum profit. Accordingly, the physical constraints for a single generator are considered without considering regulation reserve. In contrast, in this paper, we consider a different problem with multiple generators to satisfy system-wide demand balance requirements and other physical and network constraints, where the decision maker is a power system operator who aims to minimize the total cost considering both energy generation and the regulation reserve. Even for the single-generator part, this paper covers more general constraints, such as ramping constraints considering start-up/shut-down ramping rates, that are not considered in Pan and Guan (2016b). As a result, the computational complexity studied in this paper is largely augmented because of reserve restrictions, the general physical constraints for each generator, and coupling constraints such as demand balance constraints that link different generators.

By investigating the physical characteristics of the multistage stochastic UC polytope (including the minimum-up/-down time, ramping rate, capacity upper/lower bound, regulation-up/-down reserves, and load balance requirements) and the scenario tree structure, we explore several families of strong valid inequalities to strengthen the original formulation. We first investigate the single-generator polytope and then the multi-generator polytope to derive strong valid inequalities. We summarize our main contributions as follows:

1. We propose a multistage stochastic programming model for system operators to co-optimize power generation and the regulation reserve under uncertainty. A scenario tree is utilized to represent uncertain parameters and capture uncertainty dynamics over time periods.
2. For the single-generator polytope of our proposed model, we first develop convex hull descriptions for certain special cases (e.g., the two-period case), where the number of inequalities for each convex hull representation is polynomial with respect to the number of scenarios. These derived inequalities for the convex hull descriptions can also be applied to strengthen the original formulation. Then, we derive strong valid inequalities covering multiple time periods and different scenarios and correspondingly develop efficient polynomial-time separation algorithms to speed up the branch-and-cut algorithm.
3. We extend the study for the multi-generator polytope and derive strong valid inequalities linking different generators and covering multiple time periods, with efficient polynomial-time separation algorithms also developed.
4. The final numerical experiments demonstrate that our proposed inequalities can speed up the branch-and-cut algorithm remarkably, which indicates that they can be used to solve large-scale problems in industry. Sensitivity analyses are conducted to show the performance of our proposed inequalities in terms of different levels of regulation reserve requirements.

The remainder of this paper is organized as follows. In Section 2, we describe the multistage SUC formulation that co-optimizes power generation and the regulation reserve. Convex hull descriptions for the two-period case under different parameter settings are provided in Section 3. In Section 4, we derive multi-period strong valid inequalities for the general scenario tree setting, and in Section 5, we derive strong valid inequalities for the multi-generator polytope. In Section 6, we perform computational experiments to verify the effectiveness of our proposed convex hulls and strong valid inequalities. We conclude this paper in Section 7.

2. Mathematical Formulation

To describe the mathematical formulation, we first introduce the following notation and then develop the multistage stochastic network-constrained UC model that co-optimizes power generation and the regulation reserve (denoted by MSUC-AS) with uncertain net load (i.e., actual electricity load minus renewable generation).

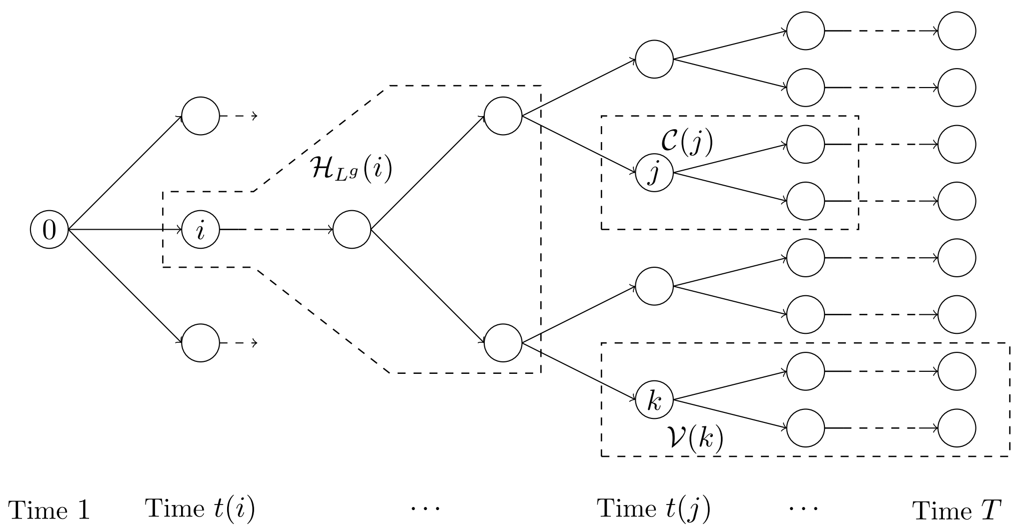

Figure 1 Multistage stochastic scenario tree (e.g., $L^g = t(j) - t(i)$**)**

We use $\mathcal{G}$, $\mathcal{B}$, and $\mathcal{E}$ to denote the sets of generators, buses, and transmission lines, respectively, with $|\mathcal{G}| = G$, $|\mathcal{B}| = B$, and $|\mathcal{E}| = E$. For each bus $b \in \mathcal{B}$, we let $\mathcal{G}_b \subseteq \mathcal{G}$ represent the set of generators located at bus b. For each transmission line $(j,h) \in \mathcal{E}$, we let C_{jh} denote the line capacity and K^b_{jh} denote the line flow distribution factor for the transmission line (j,h) due to the net injection at bus b, $\forall b \in \mathcal{B}$. For each generator $g \in \mathcal{G}$, we let $L^g(\ell^g)$ represent its minimum-up (-down) time limit, $\overline{C}^g(\underline{C}^g)$ its maximum (minimum) generation amount if it is online, V^g its ramping-up and ramping-down rate limit, $\overline{V}^g$ its start-up/shut-down ramping rate limit (which usually satisfies $\underline{C}^g < \overline{V}^g < \min\{\underline{C}^g + V^g, \overline{C}^g - V^g\}$), $\mathrm{U}^g(\mathrm{D}^g)$ its start-up (shut-down) cost, $\mathrm{RU}^g(\mathrm{RD}^g)$ its regulation-up (-down) reserve cost, and $f^g(\cdot)$, a nondecreasing convex function, the generation cost as a function of its electricity generation amount and online/offline status.

We consider net load uncertainty and adopt a scenario tree $\mathcal{T} = (\mathcal{V}, \mathcal{A})$ with T stages, as shown in Figure 1, to describe the underlying evolving process and possible realizations of the uncertain net load, with one stage representing one time period and the root node denoted by node 0. Each node $i \in \mathcal{V} \setminus \{0\}$ has a unique parent i^- in the previous period. We denote the period containing node i by $t(i)$ and use $\mathcal{P}(i)$ to represent the set of nodes along the unique path from the root node to node i. The set of immediate child nodes of node i is denoted by $\mathcal{C}(i)$, and the set of all descendants of node i, including itself, is denoted by $\mathcal{V}(i)$. Each node $i \in \mathcal{V}$ at time t represents a state of the system that can be distinguished by information available up to time t. We use p_i to denote the absolute probability associated with the state represented by node i. Node i at time T (i.e., the final period) corresponds to a realization of the uncertain data for the whole planning horizon, and thus the unique path from the root node to this node i is defined as a scenario. Finally, we use d^b_i to denote the net load of bus b at node i in the scenario tree and $W^+_i(W^-_i)$ to denote the minimum regulation-up (-down) reserve requirement at node i. Note that the decisions at time t are made after observing the data realization from the first period until t, and thus we associate the decisions with each node of the scenario tree. It follows that the decisions corresponding to each node i are nonanticipative with respect to future data realizations.

For the decision variables, corresponding to each node $i \in \mathcal{V}$ and each generator $g \in \mathcal{G}$, we let the binary variable y_i^g represent whether generator g is online (i.e., $y_i^g = 1$) or offline (i.e., $y_i^g = 0$) at node i, the binary variable u_i^g whether generator g starts up (i.e., $u_i^g = 1$) or not (i.e., $u_i^g = 0$), the continuous variable r_i^g the generation amount above the minimum generation amount, and the continuous variable $w_i^{+g}(w_i^{-g})$ the generation amount reserved from generator g for the regulation-up (-down) reserve requirement.

Based on the above description, the MSUC-AS formulation can be described as follows:

$$\min_{r,w^{\pm},y,u} \quad \sum_{i\in\mathcal{V}} p_i \left(\sum_{g=1}^{G} \left(\mathrm{U}^g u_i^g + \mathrm{D}^g \left(y_{i^-}^g - y_i^g + u_i^g \right) + \mathrm{RU}^g w_i^{+g} + \mathrm{RD}^g w_i^{-g} + f^g \left(r_i^g + \underline{C}^g y_i^g \right) \right) \right) \tag{1a}$$

$$\text{s.t.} \quad y_i^g - y_{i^-}^g \le y_k^g, \ \forall i \in \mathcal{V}\setminus\{0\}, \forall g \in \mathcal{G}, \forall k \in \mathcal{H}_{L^g}(i), \tag{1b}$$

$$y_{i^-}^g - y_i^g \le 1 - y_k^g, \ \forall i \in \mathcal{V}\setminus\{0\}, \forall g \in \mathcal{G}, \forall k \in \mathcal{H}_{\ell^g}(i), \tag{1c}$$

$$u_i^g \le \min\{y_i^g, 1 - y_{i^-}^g\}, \ \forall i \in \mathcal{V}\setminus\{0\}, \forall g \in \mathcal{G}, \tag{1d}$$

$$y_i^g - y_{i^-}^g \le u_i^g, \ \forall i \in \mathcal{V}\setminus\{0\}, \forall g \in \mathcal{G}, \tag{1e}$$

$$r_i^g - w_i^{-g} \ge 0, \ \forall i \in \mathcal{V}, \forall g \in \mathcal{G}, \tag{1f}$$

$$r_i^g + w_i^{+g} \le (\overline{C}^g - \underline{C}^g) y_i^g, \ \forall i \in \mathcal{V}, \forall g \in \mathcal{G}, \tag{1g}$$

$$r_i^g + w_i^{+g} - r_{i^-}^g \le \overline{V}^g + (\underline{C}^g + V^g - \overline{V}^g) y_{i^-}^g - \underline{C}^g y_i^g, \ \forall i \in \mathcal{V}\setminus\{0\}, \forall g \in \mathcal{G}, \tag{1h}$$

$$r_{i^-}^g - r_i^g + w_i^{-g} \le \overline{V}^g + (\underline{C}^g + V^g - \overline{V}^g) y_i^g - \underline{C}^g y_{i^-}^g, \ \forall i \in \mathcal{V}\setminus\{0\}, \forall g \in \mathcal{G}, \tag{1i}$$

$$\sum_{g=1}^{G} (r_i^g + \underline{C}^g y_i^g) = \sum_{b=1}^{B} d_i^b, \ \forall i \in \mathcal{V}, \tag{1j}$$

$$\sum_{g=1}^{G} w_i^{+g} \ge W_i^+, \ \forall i \in \mathcal{V}, \tag{1k}$$

$$\sum_{g=1}^{G} w_i^{-g} \ge W_i^-, \ \forall i \in \mathcal{V}, \tag{1l}$$

$$-C_{jh} \le \sum_{b=1}^{B} K_{jh}^b \left(\sum_{g=1}^{G_b} (r_i^g + \underline{C}^g y_i^g) - d_i^b \right) \le C_{jh}, \ \forall i \in \mathcal{V}, \forall (j,h) \in \mathcal{E}, \tag{1m}$$

$$r_i^g \ge 0, w_i^{+g} \ge 0, w_i^{-g} \ge 0, \ \forall i \in \mathcal{V}, \forall g \in \mathcal{G}, \tag{1n}$$

$$y_i^g \in \{0,1\}, \ \forall i \in \mathcal{V}, \ u_i^g \in \{0,1\}, \ \forall i \in \mathcal{V}\setminus\{0\}; \ \forall g \in \mathcal{G}, \tag{1o}$$

where the objective function (1a) is to minimize the total expected cost, including the start-up, shut-down, regulation reserve, and power generation costs. Constraints (1b) and (1c) represent the minimum-up and -down time restrictions, respectively, where $\mathcal{H}_r(i) = \{k \in \mathcal{V}(i) : 0 \le t(k) - t(i) \le r-1\}$ and it collects all of the scenario nodes of a scenario tree with node i as the root node and having r stages. Note that this scenario tree is a part of the whole scenario tree $\mathcal{T}$ in Figure 1. If generator g starts up at node i, then it has to stay online for at least L^g time periods, and thus it stays online at all of the nodes in $\mathcal{H}_{L^g}(i)$. Constraints (1d) and (1e) describe the relationship between online/offline status and start-up decisions. Constraints (1f) and (1g) describe the regulation-down and regulation-up reserve amount restrictions, respectively, together with the minimum and maximum generation amount requirements. In particular, at node i, an adequate amount of generation from generator g should be available when regulation-down reserve (i.e., w_i^{-g}) is required and generator g cannot produce too much generation when regulation-up reserve (i.e., w_i^{+g}) is required. Constraints (1h) and (1i) describe the generator ramping-up and ramping-down rate restrictions, i.e., the maximum generation increment and decrement, respectively. Note that the regulation reserve requirements are also considered so that the ramping rate limits can be satisfied even when providing regulation reserve (Carlson et al. 2012). Constraints (1j) guarantee the load balance at each node $i \in \mathcal{V}$. Constraints (1k) and (1l) describe the system-wide regulation-up and regulation-down reserve requirements at each node i, respectively. Finally, constraints (1m) describe the capacity limit of each transmission line (j,h).

Remark 1. The consideration of ancillary services together with energy power generation leads to significantly increased complexity in scheduling both power generation and regulation-up/-down reserves in terms of each generator's online/offline status, generation amount, and reserve amount. More importantly, the computational difficulty due to coupling constraints such as (1j) - (1m) is augmented by the introduction of ancillary services. Technically, the inclusion of additional continuous variables (e.g., w_i^{+g} and w_i^{-g}) challenges power system operators when solving the resulting mixed-integer programming model. In the following, we use a small example to further show that the inclusion of ancillary services also leads to the change in the optimal integer solution (corresponding to the binary decision variables y_i^g and u_i^g).

Example 1. We consider two generators (i.e., $G=2$) at the same bus without a transmission line (i.e., $B=1$) and a scenario tree with $T=3$, where each node in the first and second periods has two child nodes corresponding to the same conditional probability, leading to seven scenario nodes in total. It follows that $\mathcal{V} = \{0,1,\ldots,6\}$ with root node 0, $1^- = 2^- = 0$, $3^- = 4^- = 1$, and $5^- = 6^- = 2$. The generator data are listed in Table 1, where a^g, b^g, and c^g are the coefficients of the generation cost function $f^g(r_i^g + \underline{C}^g y_i^g) = a^g(r_i^g + \underline{C}^g y_i^g)^2 + b^g(r_i^g + \underline{C}^g y_i^g) + c^g y_i^g$, $\forall i \in \mathcal{V}$, $g = 1,2$. The loads at each node are $d_0 = 10.8$, $d_1 = 15.7$, $d_2 = 9.5$, $d_3 = 24.5$, $d_4 = 33.3$, $d_5 = 27.8$, $d_6 = 21.2$, and $W_i^+ = W_i^- = 0.1 d_i$, $\forall i \in \mathcal{V}$. By solving Problem (1) under two cases, i.e., without and with reserve

Table 1 Example 1 Generator Data

Generator	$\overline{C}^g$	$\underline{C}^g$	V^g	$\overline{V}^g$	L^g	ℓ^g	U^g	D^g	RU^g	RD^g	a^g	b^g	c^g
$g=1$	45	6	7.5	9.75	1	1	180	0	56.2	56.2	0.0697	26.2438	31.6700
$g=2$	60	9.3	10	14.3	1	1	350	0	46.61	46.61	0.0098	22.9422	58.8101

requirements, we obtain the corresponding optimal online/offline decisions (i.e., y_i^g) as shown in Figures 2 and 3, respectively. The two numbers in brackets are the values of $[y_i^1, y_i^2], \forall i \in \mathcal{V}$.

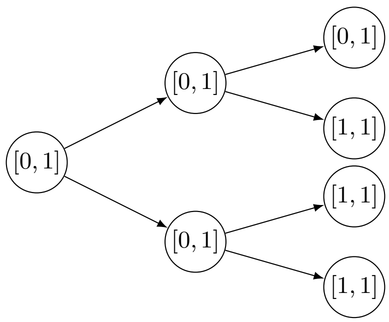

Figure 2 Without Reserve Requirements

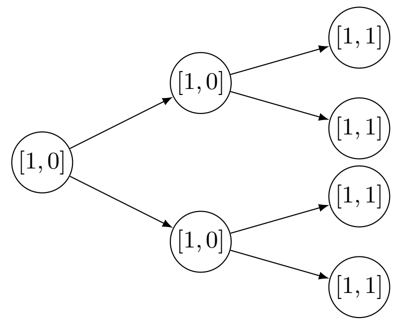

Figure 3 With Reserve Requirements

As we can observe from Figures 2 and 3, adding reserve requirements, together with regulation-up/-down reserve variables, results in a different generator online/offline schedule compared to the case without reserve requirements and thus without regulation-up/-down reserve variables. The reasons for this difference are multi-fold because of the complex physical and network characteristics. For example, these two generators have to coordinate with each other to satisfy the load and reserve requirements. Note that each generator has its own minimum generation output restriction and that if both generators are online, then the minimum generation output of the whole system will be equal to $\underline{C}^1 + \underline{C}^2 = 15.3$, which is larger than d_0 and d_2, i.e., the loads at nodes 0 and 2. It follows that at these two nodes, only one generator can be online because of load balance constraints (1j). Because of Generator 2's low generation cost, it is naturally chosen to be online in the case without reserve requirements. However, adding reserve requirements changes the choice of generators to be online. That is, if Generator 2 is online at node 2 again in the case with reserve requirements, then by the load balance equation, we have $r_2^2 = d_2 - \underline{C}^2 = 9.5 - 9.3 = 0.2$, which is less than the regulation-down requirement (i.e., $W_2^- = 0.1d_2 = 0.95$). Thus, Generator 2 cannot be online at node 2 because it fails to satisfy the restriction $r_2^2 \geq w_2^{-2}$. This is the reason why only Generator 1 is online at node 2 in the case with reserve requirements.

Therefore, in this paper, we study how we can improve the computational efficiency of solving Problem (1) by investigating the polyhedral structure induced by the constraints therein. Our derived valid inequalities are applied to strengthen the formulation and improve the computational performance through our derived branch-and-cut scheme. By approximating the generation cost function $f^g(\cdot)$ with a piecewise linear function (Carrión and Arroyo 2006), we reformulate MSUC-AS as a mixed-integer linear program and provide the corresponding strong formulations. More specifically, we perform the polyhedral study in two steps. First, we explore all of the physical constraints of a single generator in Sections 3 – 4. Second, we extend our study to consider the polytope that includes the physical constraints of all of the generators and the coupling constraints in Section 5.

Before closing this section, we describe the single-generator polytope (i.e., conv(P)) for the first step mentioned above, by omitting the superscript g for each decision variable and parameter and defining conv(P) to represent the convex hull of set P, where $\mathbb{B} := \{0,1\}$ and

$$
\begin{aligned}
P := \Big\{ \; & (r, w^+, w^-, y, u) \in \mathbb{R}_+^{|\mathcal{V}|} \times \mathbb{R}_+^{|\mathcal{V}|} \times \mathbb{R}_+^{|\mathcal{V}|} \times \mathbb{B}^{|\mathcal{V}|} \times \mathbb{B}^{|\mathcal{V}|-1} : \\
& y_i - y_{i^-} \leq y_k, \; \forall i \in \mathcal{V} \setminus \{0\}, \forall k \in \mathcal{H}_L(i), && \text{(2a)} \\
& y_{i^-} - y_i \leq 1 - y_k, \; \forall i \in \mathcal{V} \setminus \{0\}, \forall k \in \mathcal{H}_\ell(i), && \text{(2b)} \\
& u_i \leq \min\{y_i, 1 - y_{i^-}\}, \; \forall i \in \mathcal{V} \setminus \{0\}, && \text{(2c)} \\
& y_i - y_{i^-} \leq u_i, \; \forall i \in \mathcal{V} \setminus \{0\}, && \text{(2d)} \\
& r_i - w_i^- \geq 0, \; \forall i \in \mathcal{V}, && \text{(2e)} \\
& r_i + w_i^+ \leq (\overline{C} - \underline{C}) y_i, \; \forall i \in \mathcal{V}, && \text{(2f)} \\
& r_i + w_i^+ - r_{i^-} \leq \overline{V} + (\underline{C} + V - \overline{V}) y_{i^-} - \underline{C} y_i, \; \forall i \in \mathcal{V} \setminus \{0\}, && \text{(2g)} \\
& r_{i^-} - r_i + w_i^- \leq \overline{V} + (\underline{C} + V - \overline{V}) y_i - \underline{C} y_{i^-}, \; \forall i \in \mathcal{V} \setminus \{0\} \Big\}. && \text{(2h)}
\end{aligned}
$$

In Sections 3 and 4, strong valid inequalities to strengthen P will be derived and the corresponding properties will be described in detail.

Remark 2. Note that the strong valid inequalities derived for conv(P) can be used to tighten any other problems with P embedded. Thus, the improvement made for polytope conv(P) can also benefit other operations in the power system.

3. Convex Hulls for Special Cases

In this section, we strengthen the single-generator formulation and derive tighter constraints by investigating a special case of polytope conv(P), i.e., a polytope based on a two-period scenario tree with a root node in the first period followed by several scenario nodes in the second period. One significant advantage here is that the strong valid inequalities derived for this two-period case can be applied to any two consecutive time periods of polytope conv(P). Accordingly, they can help ideally formulate some original physical requirements, such as capacity and ramping constraints.

For the two-period case, without loss of generality, we assume the minimum-up/-down time limit to be 1. We collect all of the leaf nodes in the scenario tree in set $\mathcal{N}$ with $|\mathcal{N}| = n$, i.e., n scenario nodes in the second period. The corresponding original constraint set (denoted by P_2) can be described as follows:

$$P_2 := \Big\{ (r, w^+, w^-, y, u) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{B}^{n+1} \times \mathbb{B}^n :$$

$$u_i - y_i \leq 0, \ \forall i \in \mathcal{N}, \tag{3a}$$

$$u_i + y_{i^-} \leq 1, \ \forall i \in \mathcal{N}, \tag{3b}$$

$$y_i - u_i \leq y_{i^-}, \ \forall i \in \mathcal{N}, \tag{3c}$$

$$w_i^+ \geq 0, \ w_i^- \geq 0, \ \forall i \in \mathcal{N} \cup \{i^-\}, \tag{3d}$$

$$w_i^- \leq r_i, \ \forall i \in \mathcal{N} \cup \{i^-\}, \tag{3e}$$

$$r_{i^-} + w_{i^-}^+ \leq (\overline{C} - \underline{C}) y_{i^-}, \tag{3f}$$

$$r_i + w_i^+ \leq (\overline{C} - \underline{C}) y_i, \ \forall i \in \mathcal{N}, \tag{3g}$$

$$r_i + w_i^+ - r_{i^-} \leq \overline{V} + (\underline{C} + V - \overline{V}) y_{i^-} - \underline{C} y_i, \ \forall i \in \mathcal{N}, \tag{3h}$$

$$r_{i^-} - r_i + w_i^- \leq \overline{V} + (\underline{C} + V - \overline{V}) y_i - \underline{C} y_{i^-}, \ \forall i \in \mathcal{N} \Big\}. \tag{3i}$$

By considering the minimum-up/-down time, ramping constraints, and regulation-up/-down restrictions together, we can tighten the right-hand side (RHS) of each ramping constraint. That is, by considering when to start up/shut down the generator and the regulation reserve restrictions, we can further shrink the range of generation amounts, which are shown as tighter RHSs of the corresponding constraints. We first show that the following inequalities are valid and facet-defining for conv(P_2) in Propositions 1 and EC.2, respectively, and then provide the corresponding convex hull representation with a detailed proof in Theorem 1.

Proposition 1. *The inequalities*

$$r_{i^-} \leq (\overline{V} - \underline{C}) y_{i^-} + (\overline{C} - \overline{V})(y_i - u_i), \ \forall i \in \mathcal{N}, \tag{4a}$$

$$r_i + w_i^+ \leq (\overline{V} + V - \underline{C}) y_i - V u_i + (\overline{C} - \overline{V} - V)(y_j - u_j), \ \forall i, j \in \mathcal{N}, \tag{4b}$$

$$w_i^+ + w_i^- \leq (\overline{V} + V - \underline{C}) y_i - V u_i + (\underline{C} + V - \overline{V})(y_j - u_j), \ \forall i, j \in \mathcal{N}, \tag{4c}$$

$$r_i + w_i^+ - r_{i^-} \leq V y_i - (\underline{C} + V - \overline{V}) u_i, \ \forall i \in \mathcal{N}, \tag{4d}$$

$$r_{i^-} - r_i + w_i^- \leq (\overline{V} - \underline{C}) y_{i^-} + (\underline{C} + V - \overline{V})(y_j - u_j), \ \forall i, j \in \mathcal{N}, \tag{4e}$$

$$r_i + w_i^+ - r_j + w_j^- \leq (\overline{V} + V - \underline{C}) y_i - V u_i + (\underline{C} + V - \overline{V})(y_k - u_k), \ \forall i, j, k \in \mathcal{N}, i \neq j, \tag{4f}$$

are valid for conv(P_2) when $\overline{C} - \underline{C} - 2V \geq 0$ and $\overline{C} - \overline{V} - V \geq 0$.

Proof. See Online Supplement EC.1.1 for the detailed proof. □

We can observe that for each scenario node $i \in \mathcal{N}$, the inequalities in Proposition 1 provide explicit upper bounds for various combinations of continuous variables by utilizing the generator status in node i and other nodes (e.g., j) in $\mathcal{N}$. For instance, inequalities (4a) provide explicit upper bounds for individual power generation (i.e., r_{i^-}), inequalities (4b) provide explicit upper bounds for the aggregation of power generation and the regulation reserve (i.e., $r_i + w_i^+$), inequalities (4c) provide explicit upper bounds for the summation of two types of regulation reserves (i.e., $w_i^+ + w_i^-$), and inequalities (4d) and (4e) provide explicit upper bounds for the ramping-up/-down rates. Meanwhile, we develop a family of strong valid inequalities (4f) to link the power generation and regulation reserve at two different scenario nodes in the second period, whereas this type of cross-scenario relationship is not indicated in the original set P_2. Note that the insights from generating these inequalities will help develop strong valid inequalities for more complex cases, such as three-period and multi-period polytopes. In addition, by adding minimum-up/-down time constraints and other nonnegativity restrictions, we have a compact polytope Q_2.

$$Q_2 := \Big\{(r, w^+, w^-, y, u) \in \mathbb{R}^{5n+4} : (3a) - (3f), (4a) - (4f), \\ u_i \geq 0,\ \forall i \in \mathcal{N}\Big\}. \tag{5}$$

Because inequalities (4a) through (4f) are valid for $\text{conv}(P_2)$, we can conclude that $\text{conv}(P_2) \subseteq Q_2$. In the following, we show that Q_2 represents the convex hull of P_2 by proving that a) Q_2 is full-dimensional, as shown in Proposition EC.1 in Online Supplement EC.1.2, b) all of the inequalities in Q_2 are facet-defining for $\text{conv}(P_2)$, as shown in Proposition EC.2 in Online Supplement EC.1.3, c) all of the inequalities in P_2 are dominated by those in Q_2, as shown in Proposition EC.3 in Online Supplement EC.1.4, and d) all of the extreme points of Q_2 are integral in y and u, as shown in Proposition EC.4 in Online Supplement EC.1.6.

THEOREM 1. *When $\overline{C} - \underline{C} - 2V > 0$ and $\overline{C} - \overline{V} - V > 0$, we have $Q_2 = conv(P_2)$.*

Proof. As polytope Q_2 is compact and full-dimensional, as shown in Proposition EC.1, we conclude that $\text{conv}(P_2) \subseteq Q_2$ from Propositions 1 and EC.2. In addition, based on the dominance relationship described in Proposition EC.3 and the integral extreme points of Q_2 described in Proposition EC.4, we have $Q_2 = \text{conv}(P_2)$. □

REMARK 3. In Theorem 1, we present the convex hull representation of P_2 under the condition that $\overline{C} - \underline{C} - 2V > 0$ and $\overline{C} - \overline{V} - V > 0$. We can also derive convex hull representations of P_2 under other conditions, which are constructed by a subset of inequalities in Q_2. In particular, when $\overline{C} - \underline{C} - 2V \leq 0$ and $\overline{C} - \overline{V} - V > 0$, the corresponding convex hull representation is

$\text{conv}(P_2) = \{(r, w^+, w^-, y, u) \in \mathbb{R}^{5n+4} : (3a) - (3f), (4a), (4b), (4d), (4e), (5)\}$, and when $\overline{C} - \overline{V} - V \leq 0$, the corresponding convex hull representation is $\text{conv}(P_2) = \{(r, w^+, w^-, y, u) \in \mathbb{R}^{5n+4} : (3a) - (3f), (4a), (4b)$ with $j = i, (4d), (4e), (5)\}$.

Remark 4. In addition to the two-period case studied in this section, we also derive convex hull descriptions for three-period cases, i.e. $T = 3$ in P. See Online Supplement EC.2 for detailed results. Note that both two- and three-period cases consider the basic structures of the complete scenario tree, as shown in Figure 1, and thus the corresponding convex hulls and strong valid inequalities can be applied to strengthen the general multi-period formulation. Meanwhile, we can observe that the number of derived inequalities in each convex hull is a polynomial function of the input size of the scenario tree.

4. Multi-Period Strong Valid Inequalities

In this section, we derive several families of strong valid inequalities covering multi-period scenario nodes to further strengthen the formulation. We first derive strong valid inequalities by considering a special multi-period structure case. We then extend the study to consider the most general scenario tree setting. For notational brevity, we define $\sum_{h=m}^{n} r_{i_h^-} = \sum_{h=m}^{n} w^+_{i_h^-} = \sum_{h=m}^{n} w^-_{i_h^-} = \sum_{h=m}^{n} y_{i_h^-} = \sum_{h=m}^{n} u_{i_h^-} = 0$ if $n < m$, where i_h^- is the h-fold parent of node i, with $i_0^- = i$ and $i_1^- = i^-$.

4.1. Scenario Tree Substructure Case

In this subsection, we derive strong valid inequalities covering scenario nodes in a substructure of the complete scenario tree. In this substructure, the uncertain parameters are realized in the first through $(T-1)$th time periods (leading to $T-1$ scenario nodes), and multiple scenario nodes (collected in set $\mathcal{N}$) are explored in the Tth period. That is, all of the sample paths considered in this substructure differ only in the final stage. Without loss of generality, we label each node in $\mathcal{N}$ (i.e., all the leaf nodes of this substructure) from 1 to n, i.e., $\mathcal{N} = \{1, \ldots, n\}$, and thus we have n scenarios (or paths) and $n + T - 1$ scenario nodes in total in this substructure.

First, we derive inequalities by considering the power generation and regulation reserve amounts at node i^-, i.e., the shared parent node of each node i in $\mathcal{N}$. The power generation at node i^- is affected not only by the generator status at this node, but also by the status at other nodes along the unique path from i^- to the root node, as illustrated by inequality (6). Furthermore, the amount (e.g., $w^+_{i^-}$) committed to the regulation-up requirement depends on the available generation capacity when power generation r_{i^-} is committed. It follows that the summation of the power generation and the regulation-up reserve at node i^-, which will be the highest possible power output in the real-time operation, will also be affected by the generator status along the path from i^- to the root node, as illustrated by inequality (7).

Proposition 2. *For each $i \in \mathcal{N}$ when $\overline{C} - \overline{V} - (L-2)V > 0$, the inequality*

$$r_{i^-} \leq (\overline{V} - \underline{C})y_{i^-} + (\overline{C} - \overline{V})\left(y_i - \sum_{h=0}^{L-1} u_{i_h^-}\right) + \sum_{h=1}^{L-1}(h-1)Vu_{i_h^-} \tag{6}$$

is valid and facet-defining for conv(P).

Proof. See Online Supplement EC.3.1 for the detailed proof. □

Proposition 3. *For each $i \in \mathcal{N}$ when $L \geq 3$ and $\overline{C} - \overline{V} - LV > 0$, the inequality*

$$r_{i^-} + w_{i^-}^+ \leq (\overline{V} - \underline{C})y_{i^-} + 2V\left(y_{i^-} - \sum_{h=1}^{L-1} u_{i_h^-}\right) + (\overline{C} - \overline{V} - 2V)\left(y_i - \sum_{h=0}^{L-1} u_{i_h^-}\right) + \sum_{h=1}^{L-1}(h-1)Vu_{i_h^-} \tag{7}$$

is valid and facet-defining for conv(P).

Proof. The proof is similar to that of Proposition 2 and thus is omitted here. □

From inequalities (6) and (7), we can observe that the generation status at node i^- relates to all of its child nodes directly, and thus there are $|\mathcal{N}|$ inequalities in total for each.

Second, we derive inequalities to tighten the upper bounds of the power generation and regulation reserve amounts at each leaf node i in $\mathcal{N}$. Although the original constraints in P indicate that the power generation at each leaf node is only decided by the generator status at this node (through capacity upper bound constraints) and its parent node (through ramping constraints), the generator status at any two leaf nodes (e.g., $i \in \mathcal{N}$ and $j \in \mathcal{N}$) are correlated with each other through their shared parent (e.g., $i^- = j^-$). We further explore this finding to derive two families of strong valid inequalities to provide better upper bounds to limit the summation of power generation and regulation-up amounts and the summation of the two types of regulation reserves at each leaf node in inequalities (8) and (9), respectively.

Proposition 4. *For each pair $(i,j) \in \mathcal{N}$, when $k = \min\{\lfloor(\overline{C} - \overline{V})/V\rfloor, L-1\}$, the inequality*

$$r_i + w_i^+ \leq (\overline{V} + V - \underline{C})y_i - Vu_i + (\overline{C} - \overline{V} - V)\left(y_j - \sum_{h=0}^{k} u_{j_h^-}\right) + \sum_{h=1}^{k} hVu_{j_h^-} \tag{8}$$

is valid and facet-defining for conv(P).

Proof. See Online Supplement EC.3.2 for the detailed proof. □

Proposition 5. *For each pair $(i,j) \in \mathcal{N}$, when $k = \min\{\lfloor(\overline{C} - \overline{V})/V\rfloor, L-1\}$, the inequality*

$$w_i^+ + w_i^- \leq (\overline{V} + V - \underline{C})y_i - Vu_i + (\underline{C} + V - \overline{V})\left(y_j - \sum_{h=0}^{k} u_{j_h^-}\right) + \sum_{h=1}^{k} hVu_{j_h^-} \tag{9}$$

is valid and facet-defining for conv(P).

Proof. The proof is similar to that of Proposition 4 and thus is omitted here. □

In contrast to inequalities (6) and (7), inequalities (8) and (9) provide the upper bounds for the generation amount at any leaf node $i \in \mathcal{N}$ while considering its siblings, e.g., node $j \in \mathcal{N}$ ($j \neq i$). Therefore, the number of inequalities here is in the order of $|\mathcal{N}|^2$, i.e., $\mathcal{O}(|\mathcal{N}|^2)$.

Third, we derive inequalities to tighten the generation difference (i.e., the ramping-up bound) between two scenario nodes on the same path (i.e., the same scenario). The inequalities incorporate the generation status of the other scenario nodes on this path.

PROPOSITION 6. *For each $i \in \mathcal{N}$ and $k \in [2, T-1]_{\mathbb{Z}}$ such that $\overline{C} - \overline{V} - kV > 0$, the inequality*

$$r_i + w_i^+ - r_{i_k^-} \leq kVy_i - \sum_{h=0}^{\min\{k-1,L-1\}} \left(\underline{C} + (k-h)V - \overline{V}\right) u_{i_h^-} \tag{10}$$

is valid and facet-defining for conv(P).

Proof. See Online Supplement EC.3.3 for the detailed proof. □

Finally, we derive inequalities to tighten the ramping-up bounds by incorporating scenario nodes on different paths or scenarios, e.g., nodes i_k^- $(= j_k^-)$, $i \in \mathcal{N}$, and $j \in \mathcal{N}$.

PROPOSITION 7. *For each pair $(i,j) \in \mathcal{N}$ and each $k \in [2, T-1]_{\mathbb{Z}}$ such that $\overline{C} - \overline{V} - kV > 0$, the inequality*

$$\begin{aligned} r_i + w_i^+ - r_{i_k^-} + w_{i_k^-}^- \leq & (\overline{V} + V - \underline{C}) y_i - V u_i + (\underline{C} + (k+1)V - \overline{V}) \left(y_j - \sum_{h=1}^{\min\{k-1,L-1\}} u_{j_h^-} \right) \\ & + \sum_{h=1}^{\min\{k-1,L-1\}} (h-1) V u_{j_h^-} \end{aligned} \tag{11}$$

is valid and facet-defining for conv(P).

Proof. The proof is similar to that of Proposition 6 and thus is omitted here. □

The intuition for inequality (11) can be shown as follows. The generation status at node i is affected by the status at its siblings, e.g., j without loss of generality and $j \neq i$, because node j influences the status at node j^-, which is also the parent of node i. That is, the generation/regulation reserve status and amount at node i are affected by its siblings (e.g., node j) through their shared parent j^-. In particular, if the generator is offline at node j, then r_{j^-} will be at most $\overline{V} - \underline{C}$ because of the ramping-down constraint, and therefore r_i will be bounded from above by $\overline{V} + V - \underline{C}$ because of the ramping-up constraint.

REMARK 5. Because the number of derived inequalities (6) – (11) corresponds to the number of different pairs $(i,j) \in \mathcal{N}$ and $k \leq \min\{T-2, \lfloor(\overline{C} - \overline{V})/V\rfloor\}$ is a bounded parameter, we conclude that the number of derived inequalities above is polynomial in the number of nodes in the scenario tree in the order of $|\mathcal{N}|^2$, i.e., $\mathcal{O}(|\mathcal{N}|^2)$.

4.2. General Scenario Tree Structure Case

In this subsection, we explore the polyhedral structure for the general scenario tree structure (i.e., $\mathcal{T}$) by deriving inequalities covering all possible scenario nodes in $\mathcal{V}$ in Figure 1. Thus, our derived inequalities are applicable to cases where $t(i) \neq t(j)$ for any two nodes i and j in $\mathcal{V}$. Here, we let $\mathcal{N}$ be the set of leaf nodes of the general scenario tree, i.e., the scenario nodes at time T.

We focus on deriving strong valid inequalities that incorporate the power generation and regulation reserve amounts at the scenario nodes on different paths (scenarios). Similar to the analyses in Section 4.1, we find that power generation amounts, regulation-up reserve, and regulation-down reserve at the scenario nodes on different paths (i.e., cross-scenario nodes) are actually affected by each other, although their cross-scenario relationships are not explicitly represented by the original constraints in P. To better illustrate our derived strong valid inequalities that incorporate these relationships, we define the following notation. For any two nodes, e.g., nodes i and j, in the scenario tree as shown in Figure 4, we use $\text{dist}(i,j)$ to denote the distance between them. We define the path between nodes i and p as $\mathcal{P}(i,p) = \mathcal{P}(i) \setminus \mathcal{P}(p)$ and $\text{dist}(i,j) = |\mathcal{P}(i,p)| + |\mathcal{P}(j,p)|$, where node p is the shared ancestor of nodes i and j at the largest time period, i.e., $p = \text{argmax}\{t(n) : n \in \mathcal{P}(i) \cap \mathcal{P}(j)\}$. Accordingly, we derive strong valid inequalities that incorporate the generation/regulation reserve differences between nodes i and j in the following propositions.

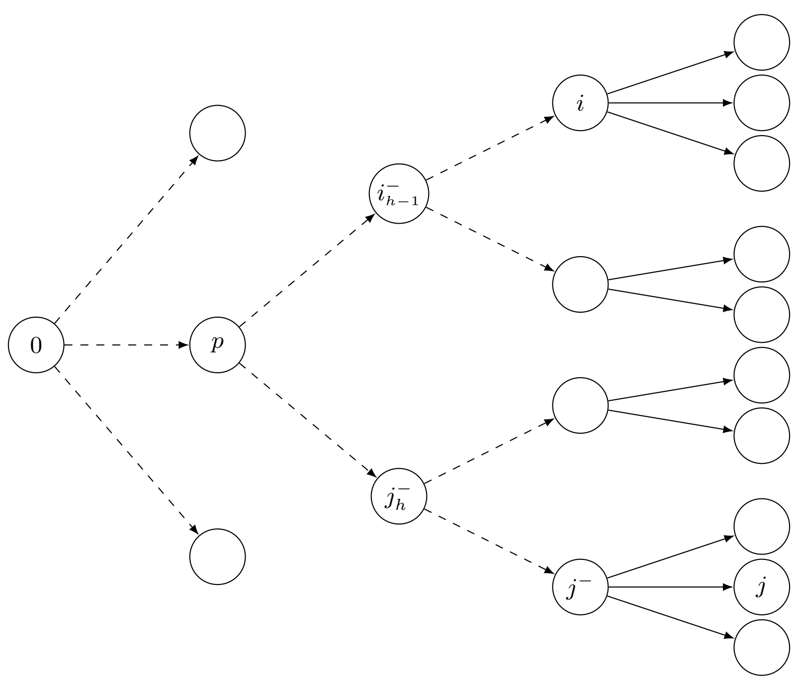

Figure 4 General Multi-Period Scenario Tree

PROPOSITION 8. *For each pair $(i,j)\in\mathcal{V}$ with i and j on different paths, the inequality*

$$r_i + w_i^+ - r_j + w_j^- \le kVy_i - \sum_{h=0}^{\min\{L-1,k-1\}} \left(\underline{C} + (k-h)V - \overline{V}\right) u_{i_h^-} \tag{12}$$

is valid and facet-defining for conv(P) when $\overline{C} - \overline{V} - kV > 0$ with $k = dist(i,j)$.

Proof. See Online Supplement EC.3.4 for the detailed proof. □

PROPOSITION 9. *For each pair $(i,j)\in\mathcal{N}$ with i^- and j on different paths, the inequality*

$$r_{i^-} - r_j + w_j^- \le (\overline{V} - \underline{C})y_{i^-} + (\underline{C} + kV - \overline{V}) \left(y_i - \sum_{h=0}^{\min\{L-1,k-1\}} u_{i_h^-}\right) + \sum_{h=1}^{\min\{L-1,k-1\}} (h-1)Vu_{i_h^-} \tag{13}$$

is valid and facet-defining for conv(P) when $\overline{C} - \overline{V} - kV > 0$ with $k = dist(i^-,j)$.

Proof. The proof is similar to that of Proposition 8 and thus is omitted here. □

PROPOSITION 10. *For each pair $(i,j)\in\mathcal{N}$ with i^- and j on different paths, the inequality*

$$\begin{aligned} r_{i^-} + w_{i^-}^+ - r_j + w_j^- \le &(\overline{V} - \underline{C})y_{i^-} + (k-1)V(y_i - u_i) - \sum_{h=1}^{\min\{L-1,k-1\}} (k-h)Vu_{i_h^-} \\ &+ (\underline{C} + V - \overline{V}) \left(y_j - \sum_{h=0}^{L-1} u_{j_h^-}\right) \end{aligned} \tag{14}$$

is valid and facet-defining for conv(P) when $\overline{C} - \overline{V} - kV > 0$ with $k = dist(i^-,j)$.

Proof. The proof is similar to that of Proposition 8 and thus is omitted here. □

Note that in inequalities (12) through (14), we generalize the basic idea of inequality (11) for the substructure case in Section 4.1 to that for the complete scenario tree structure here, which leads to more complicated forms. The condition $\overline{C} - \overline{V} - kV > 0$ is required to guarantee that the generator is able to ramp up (resp. ramp down) k_1 times along the path from nodes p to i (or i^-) as well as ramp down (resp. ramp up) k_2 times along the path from node p to node j with $k = k_1 + k_2$, where node p is the shared ancestor of nodes i (or i^-) and j at the largest time period.

PROPOSITION 11. *For each pair $(i,j)\in\mathcal{N}$ with i^- and j^- on different paths, when $\overline{C} - \overline{V} - kV > 0$ with $k = dist(i^-,j^-)$ and $L \le 3$, the inequalities*

$$\begin{aligned} r_{i^-} - r_{j^-} + r_j + w_j^+ \le &(\overline{V} - \underline{C})y_{i^-} + (\underline{C} + kV - \overline{V}) \left(y_i - \sum_{h=0}^{\min\{L-1,k-1\}} u_{i_h^-}\right) + (\overline{V} + V - \underline{C})y_j \\ &- Vu_j - (\overline{V} - \underline{C})u_{j^-} + (\overline{C} - \overline{V} - 2V) \left(y_n - \sum_{h=0}^{\min\{L-1,k-1\}} u_{n_h^-}\right), \end{aligned} \tag{15}$$

$$r_{i^-} - r_{j^-} + w_j^+ + w_j^- \le (\overline{V} - \underline{C})y_{i^-} + (\underline{C} + kV - \overline{V}) \left(y_i - \sum_{h=0}^{\min\{L-1,k-1\}} u_{i_h^-}\right)$$

$$+2Vy_j - (\underline{C} + 2V - \overline{V})u_j - Vu_{j^-} \tag{16}$$

are valid and facet-defining for conv(P), where node n has the same parent as node j.

Proof. The proof is similar to that of Proposition 8 and thus is omitted here. □

All of the proposed inequalities above in this section are in polynomial size in terms of the input size of the scenario tree and are at most in the order of $|\mathcal{V}|^2$, i.e., $\mathcal{O}(|\mathcal{V}|^2)$. In the following, we derive a family of more general inequalities in exponential size to bound from above the generation difference between two cross-scenario nodes in $\mathcal{V}$ plus the regulation-down reserve amount at one of them to further strengthen the original formulation.

PROPOSITION 12. *For each pair $(i,j) \in \mathcal{N}$ such that i_k^- and j are on different paths and $k = dist(i_k^-, j) \in \{[2, 2T-2]_{\mathbb{Z}} : \overline{C} - \underline{C} - kV > 0\}$, if $\min\{t(i_k^-), t(j)\} \geq 2$, then the inequality*

$$r_{i_k^-} - r_j + w_j^- \leq (\overline{V} - \underline{C})y_{i_k^-} + V \sum_{n \in S_0} \left(y_{i_{k-n}^-} - \sum_{m=0}^{\min\{L-1, n+\omega\}} u_{i_{k-n+m}^-} \right)$$
$$+ V \sum_{n \in S \cup \{\hat{n}\}} \Big(g_n - n \Big) \left(y_{i_{k-n}^-} - \sum_{m=0}^{L-1} u_{i_{k-n+m}^-} \right) + \psi(y,u) + \phi(u) \tag{17}$$

is valid and facet-defining for conv(P), where $S_0 = [1, \hat{n}-1]_{\mathbb{Z}}$ and $S \subseteq [\hat{n}+1, k-1]_{\mathbb{Z}}$ with $\hat{n} = \min\{t(i_k^-) - 2, L-2\}$ if $\min\{t(i_k^-) - 2, L-2\} \geq 2$ and $\hat{n} = \max\{1, L+1-t(i_k^-)\}$ otherwise, $g_n = \min\{a \in S \cup \{k\} : a > n\}$, ω is a nonnegative integer such that $t(i_{k+\omega}^-) = 2$, $\psi(y,u) = (\underline{C} + V - \overline{V})(y_i - \sum_{m=0}^{L-1} u_{i_m^-})$ or $(\underline{C} + V - \overline{V})(y_j - \sum_{m=0}^{L-1} u_{j_m^-})$, and $\phi(u) = V \sum_{m: m \geq 1, t(i_{k+m}^-) \geq T-L+1} m u_{i_{k+m}^-} + V \sum_{\{\forall m: 2 \leq t(i_{k+m}^-) \leq T-L,\, m \leq L-1\}} \min\{L-1-m, m\} u_{i_{k+m}^-}$.

Proof. See Online Supplement EC.3.5 for the detailed proof. □

(Separation) Because the number of inequalities (17) is exponential in terms of the input size of the scenario tree, we develop a separation scheme to find the most violated inequality in polynomial time. For any given point $(\hat{r}, \hat{w}^+, \hat{w}^-, \hat{y}, \hat{u}) \in \mathbb{R}_+^{5|\mathcal{V}|-1}$, to find the most violated inequality in (17) corresponding to each combination of (i,j,k), we propose a shortest path problem on a directed acyclic graph $\mathbb{G} = (\mathbb{V}, \mathbb{A})$, as shown in Figure 5, where the node and arc sets are defined as follows:

1. Node set $\mathbb{V} = \{s, t\} \cup \mathbb{V}'$, where s is the source node, t is the destination node, and $\mathbb{V}' = \{\hat{n}, \hat{n}+1, \ldots, k-1, k\}$ corresponding to nodes from $i_{k-\hat{n}}^-$ to i along the same path.
2. Arc set $\mathbb{A} = \{a_{s\hat{n}}, a_{kt}\} \cup \mathbb{A}'$, where $\mathbb{A}' = \bigcup_{\hat{n} \leq n_1 \leq n_2 \leq k} a_{n_1 n_2}$. Accordingly, we define the cost ω_{ij} of arc (i,j) for all possible (i,j)'s as follows:
   (a) $\omega_{s\hat{n}} = (\overline{V} - \underline{C})\hat{y}_{i_k^-} + V \sum_{n \in S_0} (\hat{y}_{i_{k-n}^-} - \sum_{m=0}^{\min\{L-1, n+\omega\}} \hat{u}_{i_{k-n+m}^-}) - \hat{r}_{i_k^-} + \hat{r}_j - \hat{w}_j^-$;
   (b) $\omega_{n_1 n_2} = V(n_2 - n_1)(\hat{y}_{i_{k-n_1}^-} - \sum_{j=0}^{L-1} \hat{u}_{k-n_1-j})$;
   (c) $\omega_{kt} = \psi(\hat{y}, \hat{u}) + \phi(\hat{u})$.

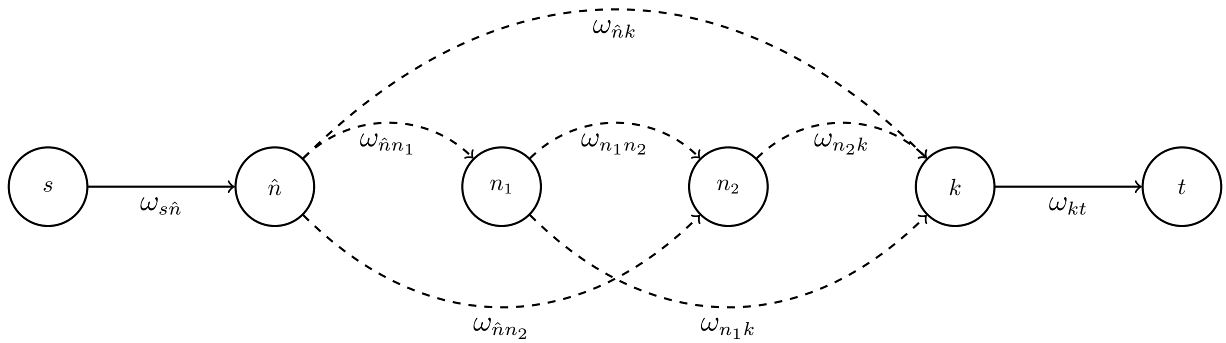

Figure 5 Acyclic Digraph of the Separation Scheme for Inequalities (17)

The shortest path from source s to destination t corresponds to one of the inequalities (17) with maximum violation if the objective value is negative, and the nodes on the shortest path determine the set S corresponding to this inequality. As the numbers of arcs and nodes for this shortest path problem are $\mathcal{O}(T^2)$ and $\mathcal{O}(T)$, respectively, we can use the topological sorting algorithm to solve the shortest path problem in $\mathcal{O}(T^2)$ time for each combination of (i,j,k). Note that $k = dist(i_k^-, j)$ is bounded by $\lfloor(\overline{C} - \overline{V})/V\rfloor$. Therefore, there is an $\mathcal{O}(|\mathcal{V}|^2T^2)$-time algorithm to solve the separation problem for all (i,j) with $dist(i_k^-, j) = k$.

PROPOSITION 13. *Given any point $(\hat{r}, \hat{w}^+, \hat{w}^-, \hat{y}, \hat{u}) \in \mathbb{R}_+^{5|\mathcal{V}|-1}$, there exists a polynomial-time separation algorithm running in $\mathcal{O}(|\mathcal{V}|^2T^2)$ time to find the most violated inequality (17), if any.*

5. Multi-Generator Strong Valid Inequalities

In this section, we extend our study to consider multiple generators together in an integrated polytope. Different from a power generation scheduling problem that considers only a single generator or one that is separable in terms of each individual generator, the co-optimization model MSUC-AS couples multiple generators through system-wide requirements. To further strengthen the MSUC-AS formulation, we derive strong valid inequalities linking different generators by exploring both physical characteristics and load balance requirements. That is, we study the multi-generator polytope conv(Ψ), where Ψ includes all of the physical constraints of each single generator and the load balance constraints linking all of them together, i.e.,

$$\Psi := \left\{(r, w^+, w^-, y, u) \in \mathbb{R}_+^{|\mathcal{G}||\mathcal{V}|} \times \mathbb{R}_+^{|\mathcal{G}||\mathcal{V}|} \times \mathbb{R}_+^{|\mathcal{G}||\mathcal{V}|} \times \mathbb{B}^{|\mathcal{G}||\mathcal{V}|} \times \mathbb{B}^{|\mathcal{G}|(|\mathcal{V}|-1)} : (1b) - (1j)\right\}.$$

Note that (1b) - (1i) enforce physical constraints on each individual generator in $\mathcal{G}$ and that the load balance constraints (1j) couple all of the generators in $\mathcal{G}$ together. For notational brevity, for each $i \in \mathcal{V}$, we denote $\sum_{b=1}^{B} d_i^b$ by $\overline{D}_i$. For simplicity, we assume that Ψ is not empty (i.e., Problem

(1) is feasible) and full-dimensional. We aim to derive several families of strong valid inequalities for conv(Ψ).

First, we derive strong valid inequalities by focusing on a subset of generators (e.g., $\mathcal{S}$) in $\mathcal{G}$. We mainly explore the load balance requirements at different scenario nodes, e.g., i^- and i, together with the start-up/shut-down actions, ramping rate limits, generation capacity limits, and regulation reserve requirements emphasized in Sections 3 and 4 to provide tighter upper bounds for the power generation amounts and/or regulation reserve amounts. For instance, we provide an explicit upper bound for the total summation over a subset of generators in terms of the power generation difference at two consecutive time periods and the regulation-up/-down reserve amounts, as illustrated by inequality (18).

Proposition 14. *For each $\mathcal{S} \subseteq \mathcal{G}$ and $i \in \mathcal{V} \setminus \{0\}$, the inequality*

$$\sum_{g\in\mathcal{S}}\left(r_i^g + w_i^{+g} - r_{i^-}^g\right) + \sum_{g\in\mathcal{G}} w_i^{-g} \le \overline{D}_i - \overline{D}_{i^-} + \sum_{g\in\mathcal{S}}\left(\underline{C}^g y_{i^-}^g + \left(2V^g - \underline{C}^g\right) y_i^g - \left(\underline{C}^g + 2V^g - \overline{V}^g\right) u_i^g\right)$$
$$+ \sum_{g\in\mathcal{G}\setminus\mathcal{S}}\left(\overline{V}^g y_{i^-}^g - \underline{C}^g y_i^g + \left(V^g + \underline{C}^g - \overline{V}^g\right)\left(y_i^g - u_i^g\right)\right) \tag{18}$$

is valid for conv(Ψ). Furthermore, it is facet-defining for the two-period case of conv(Ψ).

Proof. See Online Supplement EC.4.1 for the detailed proof. □

(Separation) Because the number of inequalities (18) is exponential in terms of the number of generators, we develop a separation scheme to find the most violated inequality in polynomial time. For any given point $(\hat{r}, \hat{w}^+, \hat{w}^-, \hat{y}, \hat{u}) \in \mathbb{R}_+^{|\mathcal{G}|(5|\mathcal{V}|-1)}$, to find the most violated inequalities (18) corresponding to each $i \in \mathcal{V} \setminus \{0\}$, we propose a shortest path problem on a directed acyclic graph $\mathbb{G} = (\mathbb{V}, \mathbb{A})$, as shown in Figure 6, where the node and arc sets are defined as follows:

1. Node set $\mathbb{V} = \{s,t\} \cup \mathbb{V}_1 \cup \mathbb{V}_2$, where s is the source node, t is the destination node, $\mathbb{V}_1 = \{1, 2, \cdots, |\mathcal{G}|\}$ with each number corresponding to each individual generator in $\mathcal{G}$, and $\mathbb{V}_2 = \{|\mathcal{G}|+1, |\mathcal{G}|+2, \cdots, 2|\mathcal{G}|\}$ with each individual number also corresponding to the same set of generators in $\mathcal{G}$. Thus, each generator $g \in \mathcal{G}$ is labeled by two numbers, and these two numbers' difference is $|\mathcal{G}|$.
2. Arc set $\mathbb{A} = \{a_{s1}, a_{s(|\mathcal{G}|+1)}, a_{|\mathcal{G}|t}, a_{(2|\mathcal{G}|)t}\} \cup \mathbb{A}_1 \cup \mathbb{A}_2$, where $\mathbb{A}_1 = \bigcup_{1\le n\le|\mathcal{G}|-1}\{a_{n(n+1)}, a_{n(n+|\mathcal{G}|+1)}\}$ and $\mathbb{A}_2 = \bigcup_{|\mathcal{G}|+1\le n\le 2|\mathcal{G}|-1}\{a_{n(n+1)}, a_{n(n-|\mathcal{G}|+1)}\}$. Any arc that goes to a node n with $1 \le n \le |\mathcal{G}|$ is illustrated as a dashed arc. Any arc that goes to a node n with $|\mathcal{G}|+1 \le n \le 2|\mathcal{G}|$ is illustrated as a solid arc. These two arcs that go to destination node t are illustrated as dotted arcs. Accordingly, we define the cost ω_{ij} of arc (i,j) for all possible (i,j)'s as follows:
   (a) If $1 \le j \le |\mathcal{G}|$, then $\omega_{ij} = \hat{r}_i^g + \hat{w}_i^{+g} - \hat{r}_{i^-}^g + \hat{w}_i^{-g} - \underline{C}^g \hat{y}_{i^-}^g - (2V^g - \underline{C}^g)\hat{y}_i^g + (\underline{C}^g + 2V^g - \overline{V}^g)\hat{u}_i^g$, where generator g is labeled by number j;

(b) If $|\mathcal{G}|+1 \leq j \leq 2|\mathcal{G}|$, then $\omega_{ij} = \hat{w}_i^{-g} - \overline{V}^g \hat{y}_{i^-}^g + \underline{C}^g \hat{y}_i^g - (V^g + \underline{C}^g - \overline{V}^g)(\hat{y}_i^g - \hat{u}_i^g)$, where generator g is labeled by number j;

(c) $\omega_{|\mathcal{G}|t} = \omega_{(2|\mathcal{G}|)t} = \overline{D}_{i^-} - \overline{D}_i$.

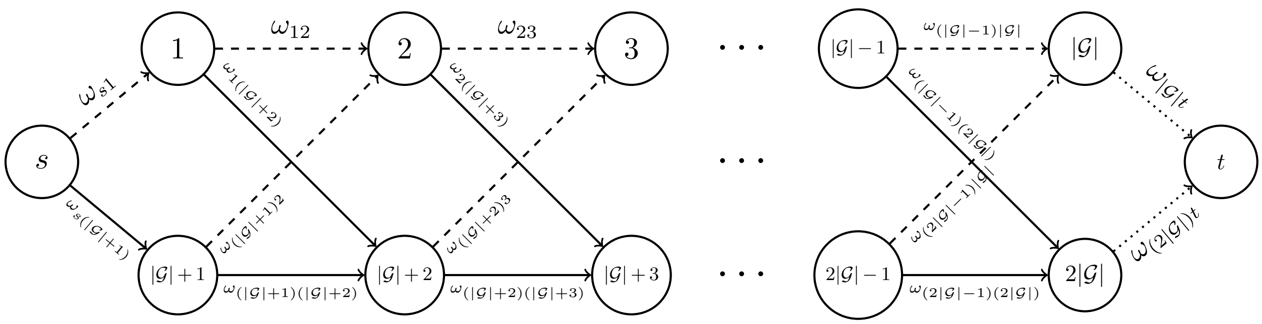

Figure 6 Acyclic Digraph of the Separation Scheme for Inequalities (18)

The shortest path from source s to destination t corresponds to one of the inequalities (18) with the maximum violation if the objective value is positive, and the nodes on the shortest path determine set S corresponding to this inequality. As the numbers of arcs and nodes for this shortest path problem are $4|\mathcal{G}|$ and $2|\mathcal{G}|+2$, respectively, we can use the topological sorting algorithm to solve the shortest path problem in $\mathcal{O}(|\mathcal{G}|)$ time for each $i \in \mathcal{V} \setminus \{0\}$. Therefore, there is an $\mathcal{O}(|\mathcal{V}||\mathcal{G}|)$-time algorithm to solve the separation problem for all $i \in \mathcal{V} \setminus \{0\}$.

Proposition 15. *Given any point $(\hat{r}, \hat{w}^+, \hat{w}^-, \hat{y}, \hat{u}) \in \mathbb{R}_+^{|\mathcal{G}|(5|\mathcal{V}|-1)}$, there exists a polynomial-time separation algorithm running in $\mathcal{O}(|\mathcal{V}||\mathcal{G}|)$ time to find the most violated inequality* (18)*, if any.*

Proposition 16. *For each $i \in \mathcal{V} \setminus \{0\}$, the inequality*

$$\sum_{g \in \mathcal{S}} r_i^g \geq \left(\overline{D}_{i^-} - \sum_{g \in \mathcal{S}} \left(\overline{V}^g + \underline{C}^g\right)\right)\left(1 - \sum_{g \in \mathcal{G} \setminus \mathcal{S}} \left(y_i^g - u_i^g\right)\right) \tag{19}$$

is valid for conv(Ψ), where $\mathcal{S} \subseteq \mathcal{G}$ such that $\overline{D}_{i^-} - \sum_{g \in \mathcal{S}} (V^g + \underline{C}^g) \geq 0$ and $\overline{D}_i - \overline{D}_{i^-} \geq \sum_{g \in \mathcal{G} \setminus \mathcal{S}} \overline{V}^g - \sum_{g \in \mathcal{S}} V^g$. Furthermore, it is facet-defining for the two-period case of conv(Ψ) if $\sum_{g \in \mathcal{S}} \underline{C}^g + \overline{C}^{\bar{g}} + \sum_{g \in \mathcal{G} \setminus \{\mathcal{S} \cup \{\bar{g}\}\}} \overline{V}^g \geq \overline{D}_i$ and $\sum_{g \in \mathcal{S}} (\underline{C}^g + V^g) + \overline{C}^{\bar{g}} \geq \overline{D}_{i^-}$, where generator $\bar{g}$ is the one with the highest capacity upper bound.

Proof. See Online Supplement EC.4.2 for the detailed proof. □

Similar to Proposition 15, we can also derive polynomial-time separation algorithms for inequality (19) and the following inequality (21), and thus we omit them.

Second, we focus on the online/offline status of each generator. Whether a generator is required to be online is determined by several factors, including its physical characteristics, the load requirement at the current time period, and whether this generator is required to be online or offline in the last and/or next time period. Based on this insight, we derive inequality (20) to enforce a minimum number of generators that have to be online at each node $i \in \mathcal{V}$.

PROPOSITION 17. *For each $i \in \mathcal{V}$, the inequality*

$$\sum_{g \in \mathcal{G}} y_i^g \geq q_i + 1 \tag{20}$$

is valid for conv(Ψ), where q_i is a nonnegative integer satisfying the condition that $\sum_{g=|\mathcal{G}|-q_i+2}^{|\mathcal{G}|} \overline{C}^{[g]} < d_i \leq \sum_{g=|\mathcal{G}|-q_i+1}^{|\mathcal{G}|} \overline{C}^{[g]}$, $\sum_{g=|\mathcal{G}|-q_i+1}^{|\mathcal{G}|} \overline{C}^{[g]} + \nu < d_j < \sum_{g=|\mathcal{G}|-q_i}^{|\mathcal{G}|} \overline{C}^{[g]}$ for $j = i^-$ or $j^- = i$, $\overline{C}^{[1]} \leq \overline{C}^{[2]} \leq \ldots \leq \overline{C}^{[|\mathcal{G}|]}$ is a sorted nondecreasing order of $\{\overline{C}^g : g \in \mathcal{G}\}$, and $\nu = \max\{\overline{V}^g : g \in \mathcal{G}\}$ with $\nu < \overline{C}^{[1]}$. Furthermore, it is facet-defining for the two-period case of conv(Ψ).

Proof. See Online Supplement EC.4.3 for the detailed proof. □

Finally, we conduct a further study to consider not only multiple generators but also multiple time periods, leading to inequality (21), which provides an upper bound for the summation over a subset of generators in terms of the power generation difference at nodes i_k^- and i.

PROPOSITION 18. *For each $\mathcal{S} \subseteq \mathcal{G}$, $k \in \{s \in [1, T-1]_{\mathbb{Z}} : \overline{C}^g - \underline{C}^g - sV^g > 0,\ \forall g \in \mathcal{G} \setminus \mathcal{S}\}$, and $i \in \mathcal{V}$ such that $t(i) \geq k+1$, the inequality*

$$\begin{aligned}
\sum_{g \in \mathcal{S}} \left(r_{i_k^-}^g - r_i^g \right) + \sum_{g \in \mathcal{G} \setminus \mathcal{S}} w_i^{+g} \leq \overline{D}_{i_k^-} - \overline{D}_i + \sum_{g \in \mathcal{G}} \left(\underline{C}^g y_i^g - \underline{C}^g y_{i_k^-}^g \right) \\
+ \sum_{g \in \mathcal{G} \setminus \mathcal{S}} \left(kV^g y_i^g - \sum_{m=0}^{\min\{k-1, L^g-1\}} \left(\underline{C}^g + (k-m)V^g - \overline{V}^g \right) u_{i_m^-}^g \right) \\
- \left\{ \Gamma - \left(\overline{D}_i - \sum_{g \in \mathcal{S}} \underline{C}^g \right) \right\} \left(1 - \sum_{g \in \mathcal{S}} \left(y_{i^-}^g - y_i^g + u_i^g \right) \right)
\end{aligned} \tag{21}$$

is valid for conv(Ψ), where $\overline{D}_i \geq \max\{\sum_{g \in \mathcal{G}} \underline{C}^g, \sum_{g \in \mathcal{S}} \overline{C}^g, \sum_{g \in \mathcal{G} \setminus \mathcal{S}} \overline{C}^g\}$, and $\Gamma = \min\{\sum_{g \in \mathcal{G} \setminus \mathcal{S}} \{(\underline{C}^g + kV^g) y_i^g - w_i^{+g} - \sum_{m=0}^{\min\{k-1, L^g-1\}} (\underline{C}^g + (k-m)V^g - \overline{V}^g) u_{i_m^-}^g\} : \sum_{g \in \mathcal{G} \setminus \mathcal{S}} x_i^g = \overline{D}_i - \sum_{g \in \mathcal{S}} \underline{C}^g,\ \underline{C}^g y_i^g \leq x_i^g \leq \overline{C}^g y_i^g, (2a) - (2d), \forall g \in \mathcal{G} \setminus \mathcal{S}\}$.

Proof. See Online Supplement EC.4.4 for the detailed proof. □

6. Computational Experiments

In this section, we test the performance of our proposed strong valid inequalities presented in Sections 3 – 5 by solving randomly generated instances. All of the numerical experiments were performed on a computer node with two AMD Opteron 2378 Quad Core Processors at 2.4GHz and 8GB memory. IBM ILOG CPLEX 12.3 with a single thread is utilized as the MIP solver. The running time limit is set as one hour and the optimality gap is set as 0.05%. All of the experiments were coded in C++.

6.1. Problem Setting

We perform computational experiments on a multistage stochastic network-constrained UC model for co-optimizing power generation and regulation reserve, with the instances modified based on the IEEE 118-bus system available at `motor.ece.iit.edu/data/SCUC_118`. The test system contains 118 buses, 186 transmission lines, 54 thermal generators, and 91 load buses. We assume that the system net load is stochastic and varies within the interval $[0, 2\bar{N}]$, where $\bar{N}$ is the nominal value of the system net load. The system-wide regulation requirements at each scenario node, i.e., (W_i^+, W_i^-) for $i \in \mathcal{V}$, are set as a proportion of the system-wide total load at this node. We define ρ to represent this proportion and let $W_i^+ = W_i^- = \rho \sum_{b=1}^{B} d_i^b$, $\forall i \in \mathcal{V}$. Here, we let $\rho = 10\%$ to show the performance of our proposed inequalities, and we discuss different levels of reserve requirements in Section 6.3. In the experiments, we first consider different numbers of generators in the system. We create six groups of instances by taking subsets of the 54 thermal generators so that we have 15, 20, 25, 30, 35, and 40 generators in the six groups, respectively. Next, we use K to denote the number of branches for each non-leaf node in the underlying scenario tree, and we consider three different types of scenario tree structures, i.e., $K = 2$, 3, and 4. We use T to denote the number of time periods and set $T \in \{8, 9\}$ when $K = 2$, $T \in \{5, 6\}$ when $K = 3$, and $T \in \{4, 5\}$ when $K = 4$. Based on these settings, the numbers of the scenario tree nodes of each combination (G, K, T) are comparable. For example, there are 121 nodes for combination $(G, K, T) = (15, 3, 5)$ and 85 nodes for combination $(G, K, T) = (15, 4, 4)$. Thus, we have created 54 combinations of G, K, and T with different numbers of generators, scenario tree structures, and time periods. For each combination, we test three randomly generated instances[1], and we provide the average result over these three instances in the following subsection.

6.2. Computational Results

Our proposed inequalities are added as cutting planes in the branch-and-cut algorithm, and we present the computational results from two perspectives: the strength of the problem formulation and the algorithm performance (see Tables 2 and 3). To highlight our focus on co-optimizing energy generation and the regulation reserve, here we only use the inequalities in Sections 3 – 5 that include regulation-up/-down reserve variables (i.e., w_i^+ and w_i^- for $i \in \mathcal{V}$). Most of the derived inequalities satisfy this requirement. For instance, in Section 4, only inequality (6) does not include regulation-up/-down reserve variables.

In addition, some families of strong valid inequalities are added in whole and some are not. First, for a family of inequalities whose size is a polynomial function of the input size of the scenario tree, specifically including inequalities (4a) - (4f), (6) - (16), and (20), they are added as a whole. For instance, inequalities (4a) in the two-period convex hull representation are applied to every scenario node in the scenario tree except root node 0. Second, for a family of inequalities whose size is an exponential function of the input size of the scenario tree, specifically including inequalities (17) - (19) and (21), we limit the number of inequalities added by using an offline selection process. It is well known that adding too many inequalities will potentially increase the computational time because the resulting model will become increasingly large. From each family of exponential-sized inequalities, we select a subset of them by heuristically restricting the validity condition to a small region where the validity condition is more restrictive (i.e., harder to satisfy) than the rest of the region. For instance, for inequalities (17), we select those satisfying $k \in [3, \lfloor(\overline{C} - \underline{C})/V\rfloor]_{\mathbb{Z}}$ by removing those satisfying $k = 2$. Note that inequalities (17) that satisfy $k = 2$ will be the same as inequalities (4e). For inequalities (18), we select those satisfying $|\mathcal{S}| \in [2,5]_{\mathbb{Z}}$ because of their good performance. For inequalities (19), we select all of them because the total number of inequalities in this family is not large due to the relatively strong condition. For inequalities (21), we select those satisfying $|\mathcal{S}| \in [2,5]_{\mathbb{Z}}$ and $k \in [3, \lfloor(\overline{C} - \underline{C})/V\rfloor]_{\mathbb{Z}}$ because of their good performance. Third, for those inequalities in the three-period convex hulls, e.g., (EC.5a) through (EC.5v), because they are polynomial-sized, we add all of them as user cuts by applying them to every scenario node in the scenario tree whenever appropriate.

Table 2 Root Node Results

		G=15			G=20		
K	T	LP Gap (%)	Cut Gap (%)	Percentage (%)	LP Gap (%)	Cut Gap (%)	Percentage (%)
2	8	2.049	0.106	94.825	0.780	0.063	91.940
	9	2.028	0.064	96.849	0.738	0.044	94.022
3	5	1.800	0.111	93.857	0.725	0.090	87.546
	6	2.213	0.343	84.508	0.765	0.071	90.672
4	4	0.518	0.050	90.369	0.417	0.141	66.240
	5	1.500	0.038	97.478	0.589	0.069	88.278
		G=25			G=30		
K	T	LP Gap (%)	Cut Gap (%)	Percentage (%)	LP Gap (%)	Cut Gap (%)	Percentage (%)
2	8	0.757	0.023	96.958	0.712	0.015	97.921
	9	0.723	0.027	96.206	0.720	0.020	97.288
3	5	0.621	0.043	93.156	0.652	0.041	93.778
	6	0.686	0.014	97.890	0.685	0.028	95.853
4	4	0.315	0.049	84.406	0.231	0.030	86.897
	5	0.526	0.040	92.339	0.538	0.025	95.371
		G=35			G=40		
K	T	LP Gap (%)	Cut Gap (%)	Percentage (%)	LP Gap (%)	Cut Gap (%)	Percentage (%)
2	8	0.695	0.018	97.367	0.686	0.017	97.488
	9	0.693	0.023	96.697	0.696	0.025	96.462
3	5	0.622	0.023	96.262	0.625	0.030	95.191
	6	0.655	0.023	96.527	0.746	0.033	95.515
4	4	0.193	0.023	88.302	0.192	0.017	91.261
	5	0.539	0.024	95.526	0.586	0.027	95.398

In Table 2, we show the effectiveness of our proposed strong valid inequalities in tightening the LP relaxation at the root node. The column labeled "LP Gap (%)" represents the relative LP relaxation gap of the original formulation with respect to the best integer solution that we can find from the default CPLEX and our branch-and-cut scheme. "LP Gap (%)" is defined as $(Z_{\text{MILP}} - Z_{\text{LP}})/Z_{\text{MILP}}$, where Z_{LP} is the objective value of the LP relaxation and Z_{MILP} represents the objective value of the MSUC-AS problem with the best integer solution. The column labeled "Cut Gap (%)" is the LP relaxation gap after adding our strong valid inequalities in Sections 3 – 5. These two columns indicate the significant decrease in LP relaxation gap as shown in Table 2. We use the column labeled "Percentage (%)" to show how much the LP relaxation gap is decreased based on "LP Gap (%)," i.e., Percentage (%) = (LP Gap (%) − Cut Gap (%))/LP Gap (%). As we can observe from Table 2, adding our proposed strong valid inequalities can help close most of the LP relaxation gap. In particular, in terms of instances with longer and denser (i.e., more stages and more branches) scenario trees and instances with more generators in the problem, the LP relaxation gap can be reduced by around 90%.

In Table 3, we present the average performance of our proposed strong valid inequalities in the branch-and-cut scheme by comparing the performance of our approach with that of the default CPLEX. The derived inequalities are added as user cuts in our branch-and-cut scheme following the selection process mentioned above. In the table, for each combination (G, K, T), the column labeled "Gap (%)" indicates the average terminating gaps over the instances solved to feasible solutions but not solved to default optimality when reaching the time limit. Each pair of numbers in the square bracket, e.g., $[a, b]$, represent the number of instances (e.g., a) (out of three) not solved to a feasible solution and the number of instances (e.g., b) solved to feasible solutions but not solved to our predefined optimality within the time limit, respectively. Thus, when this square bracket is indicated, there are $3 - a - b$ instances (out of three) solved to the optimality within the time limit. For each combination (G, K, T), the average running time over those instances leading to optimality is reported in the column labeled "CPU secs," where 3600 is given if all three instances cannot be solved to the default optimality or feasibility. The column labeled "# of Nodes" provides the number of branch-and-bound nodes that CPLEX explored and the column labeled "# of Cuts" gives the number of our derived inequalities used in the branch-and-cut scheme to solve the instances. As shown in Table 3, our approach performs better than the default CPLEX does for every case. Specifically, 1) some instances cannot be solved to feasibility by the default CPLEX but can be solved to feasibility by our approach, 2) some instances cannot be solved to optimality by the default CPLEX but can be solved to optimality by our approach, 3) our approach spends less time finding the optimal solution if both approaches can find optimal solutions within the time limit, and 4) our approach leads to a smaller optimality gap within the time limit if neither of the approaches can find an optimal solution within the time limit.

Table 3 Branch-and-Cut Scheme Results

			Default CPLEX			Branch-and-Cut			
G	K	T	Gap(%)	CPU secs	# of Nodes	Gap(%)	CPU secs	# of Nodes	# of Cuts
15	2	8	0.00	1656	5506	0.00	547	1543	259
		9	0.00	2397	2981	0.00	312	696	150
	3	5	0.00	175	2838	0.00	77	1610	109
		6	0.29[2, 1]	3600	6873	0.27[0, 3]	3600	10722	595
	4	4	0.00	7	53	0.00	4	68	98
		5	0.00	2033	9193	0.00	781	5324	380
20	2	8	0.00	886	2732	0.00	622	3944	446
		9	0.00	2524	3143	0.00	438	774	465
	3	5	0.00	86	971	0.00	40	571	145
		6	0.08[0, 2]	3495	7301	0.00	2789	10114	962
	4	4	0.00	74	1573	0.00	24	391	360
		5	0.00	2562	9229	0.00	1937	10113	704
25	2	8	0.00	396	1069	0.00	240	889	420
		9	0.00	805	924	0.00	298	496	583
	3	5	0.00	43	384	0.00	17	175	127
		6	0.00[1, 0]	3368	10044	0.00	2145	10157	1210
	4	4	0.00	32	607	0.00	27	291	428
		5	0.00	926	2265	0.00	441	1336	508
30	2	8	0.00	546	1511	0.00	229	564	423
		9	0.00	1212	1216	0.00	946	1086	554
	3	5	0.00	136	1516	0.00	17	104	151
		6	0.59[0, 1]	2863	5706	0.00	2433	6780	1050
	4	4	0.00	20	321	0.00	7	0	224
		5	0.06[0, 1]	2728	6022	0.00	1260	4300	671
35	2	8	0.00	639	1373	0.00	359	794	440
		9	0.00	461	408	0.00	139	302	441
	3	5	0.00	139	1042	0.00	61	426	93
		6	0.00	1931	4566	0.00	1328	3697	831
	4	4	0.00	7	0	0.00	4	0	76
		5	0.44[0, 1]	2094	4746	0.00	1138	1590	359
40	2	8	0.00	222	515	0.00	201	383	433
		9	0.00	746	602	0.00	454	500	486
	3	5	0.00	91	468	0.00	18	70	111
		6	0.08[0, 1]	2537	2805	0.00	1547	3623	1174
	4	4	0.00	4	0	0.00	3	0	74
		5	0.40[0, 1]	2964	5805	0.00	1394	3405	423

6.3. Sensitivity Analysis

In this subsection, we perform a sensitivity analysis to illustrate the effectiveness of our proposed inequalities corresponding to different reserve requirement levels. We test randomly generated instances by considering three different reserve requirement levels (i.e., $\rho = 3\%$, 5%, and 10%) that follow common practices in industry as indicated by ISO New England (2019) and Ma et al. (2013). We follow the same settings described in Sections 6.1 and 6.2 to randomly generate instances and to apply the branch-and-cut scheme, respectively. To show the main insights, we focus on two performance indicators: (1) how our proposed inequalities tighten the LP relaxation problem, which is defined as Percentage (%) = (LP Gap (%) − Cut Gap (%))/LP Gap (%) in Table 2 and is also labeled in Table 4, and (2) how many inequalities are used by CPLEX, which is labeled as "# of Cuts" in Table 4. We consider three different combinations of (G, K, T) in terms of the number of generators, scenario tree structures, and time periods, namely $(15, 2, 8), (15, 3, 5)$, and $(15, 4, 4)$. For each combination of (G, K, T), we randomly generate 10 test cases for a given reserve requirement level (which is labeled as "Reserve Rate(%)" in Table 4) and report the average results.

Table 4 **Sensitivity Analysis Results,** $G = 15$

K	T	Reserve Rate(%)	Percentage(%)	# of Cuts
2	8	3	80.82	285
		5	82.07	284
		10	98.25	205
3	5	3	59.25	126
		5	58.78	136
		10	93.29	100
4	4	3	62.54	140
		5	71.69	161
		10	91.98	127

From Table 4, we can observe that the root node gap improvement (i.e., Percentage (%)) becomes more significant when the reserve requirement level (i.e., ρ) increases. It indicates that when ρ increases, our proposed inequalities become more effective in tightening the original formulation. Meanwhile, the number of inequalities used by CPLEX is fairly stable when comparing the cases using different levels of reserve requirements.

7. Conclusions

The recent introduction of ancillary service market, which helps manage significant uncertainties brought by increasing renewable energy, has presented power system operators with a huge challenge in coordinating the traditional energy market and the new ancillary service market under uncertainty. To overcome this challenge, in this paper, we proposed a multistage stochastic optimization model for system operators to schedule power generation assets under uncertainty so that the power generation and regulation reserve service are co-optimized, which has received limited attention in the literature. Our approach utilized a multi-period scenario tree to describe the net load uncertainty and incorporated the physical constraints for each generator and the network constraints (e.g., demand and reserve requirements) that couple all of the generators, leading to a large-scale deterministic equivalent formulation. To efficiently solve the resulting large-scale mixed-integer program, we explored the polyhedral structure of the proposed model by taking advantage of the scenario tree structure, the physical characteristics of the individual generators, and the system-wide requirements that couple all of the generators to derive strong valid inequalities. First, we considered the single-generator polytope, for which we derived convex hull descriptions for certain special cases. For the general multi-period cases, we derived facet-defining inequalities linking different scenarios, with both power generation and regulation-up/-down reserves simultaneously considered in every inequality. Next, we considered the multi-generator polytope and derived strong valid inequalities linking different generators and covering multiple time periods. For the exponential number of inequalities in this paper, we proposed corresponding polynomial time separation algorithms. Finally, the computational experiments demonstrated the effectiveness of our proposed strong valid inequalities, which are embedded in the branch-and-cut framework as user cuts.

Acknowledgments

The authors thank the area editor, associate editor, and two anonymous referees for their helpful suggestions, which significantly improved the quality of this paper. The work of Kai Pan was supported in part by The Hong Kong Polytechnic University [Grant G-UAFD] and in part by The Research Grants Council of Hong Kong [Grant PolyU 155077/18B].

Endnotes

1. All the instance data are available for download at https://drive.google.com/file/d/1RotIkF_AbmdXdCbtPwWsYgOROqvQEkUf/view.

References

Ahmed, S., M. Tawarmalani, N. V. Sahinidis. 2004. A finite branch-and-bound algorithm for two-stage stochastic integer programs. *Mathematical Programming* **100**(2) 355–377.

Bendotti, P., P. Fouilhoux, C. Rottner. 2018. The min-up/min-down unit commitment polytope. *Journal of Combinatorial Optimization* **36**(3) 1024–1058.

Birge, J. R., F. Louveaux. 2011. *Introduction to Stochastic Programming*. Springer Science & Business Media.

Carlson, B., Y. Chen, M. Hong, R. Jones, K. Larson, X. Ma, P. Nieuwesteeg, H. Song, K. Sperry, M. Tackett, et al. 2012. MISO unlocks billions in savings through the application of operations research for energy and ancillary services markets. *Interfaces* **42**(1) 58–73.

Carøe, C. C., R. Schultz. 1998. *A two-stage stochastic program for unit commitment under uncertainty in a hydro-thermal power system*. Citeseer.

Carpentier, P., G. Gohen, J.-C. Culioli, A. Renaud. 1996. Stochastic optimization of unit commitment: a new decomposition framework. *IEEE Transactions on Power Systems* **11**(2) 1067–1073.

Carrión, M., J. M. Arroyo. 2006. A computationally efficient mixed-integer linear formulation for the thermal unit commitment problem. *IEEE Transactions on Power Systems* **21**(3) 1371–1378.

Cerisola, S., Á. Baíllo, J. M. Fernández-López, A. Ramos, R. Gollmer. 2009. Stochastic power generation unit commitment in electricity markets: A novel formulation and a comparison of solution methods. *Operations Research* **57**(1) 32–46.

Cheung, K., D. Gade, C. Silva-Monroy, S. M. Ryan, J.-P. Watson, R. J.-B. Wets, D. L. Woodruff. 2015. Toward scalable stochastic unit commitment. *Energy Systems* **6**(3) 417–438.

Cheung, K. W. 2008. Ancillary service market design and implementation in North America: from theory to practice. *Third International Conference on Electric Utility Deregulation and Restructuring and Power Technologies*. IEEE, 66–73.

Damcı-Kurt, P., S. Küçükyavuz, D. Rajan, A. Atamtürk. 2016. A polyhedral study of production ramping. *Mathematical Programming* **158**(1-2) 175–205.

Gade, D., G. Hackebeil, S. M. Ryan, J.-P. Watson, R. J.-B. Wets, D. L. Woodruff. 2016. Obtaining lower bounds from the progressive hedging algorithm for stochastic mixed-integer programs. *Mathematical Programming* **157**(1) 47–67.

Gentile, C., G. Morales-España, A. Ramos. 2017. A tight MIP formulation of the unit commitment problem with start-up and shut-down constraints. *EURO Journal on Computational Optimization* **5**(1-2) 177–201.

Gröwe, N., W. Römisch, R. Schultz. 1995. A simple recourse model for power dispatch under uncertain demand. *Annals of Operations Research* **59**(1) 135–164.

Guan, Y., S. Ahmed, G. L. Nemhauser. 2009. Cutting planes for multistage stochastic integer programs. *Operations Research* **57**(2) 287–298.

Guan, Y., S. Ahmed, G. L. Nemhauser, A. J. Miller. 2006. A branch-and-cut algorithm for the stochastic uncapacitated lot-sizing problem. *Mathematical Programming* **105**(1) 55–84.

Guan, Y., K. Pan, K. Zhou. 2018. Polynomial time algorithms and extended formulations for unit commitment problems. *IISE Transactions* **50**(8) 735–751.

Huang, K., S. Ahmed. 2009. The value of multistage stochastic programming in capacity planning under uncertainty. *Operations Research* **57**(4) 893–904.

ISO New England. 2019. Daily regulation requirement. https://www.iso-ne.com/isoexpress/web/reports/operations/-/tree/daily-regulation-requirement. Accessed: 2019-04-01.

Knueven, B., J. Ostrowski, J. Wang. 2018. The ramping polytope and cut generation for the unit commitment problem. *INFORMS Journal on Computing* **30**(4) 625–786.

Lee, J., J. Leung, F. Margot. 2004. Min-up/min-down polytopes. *Discrete Optimization* **1**(1) 77–85.

Li, Z., M. Shahidehpour. 2005. Security-constrained unit commitment for simultaneous clearing of energy and ancillary services markets. *IEEE Transactions on Power Systems* **20**(2) 1079–1088.

Liu, X., S. Küçükyavuz, J. Luedtke. 2016. Decomposition algorithms for two-stage chance-constrained programs. *Mathematical Programming* **157**(1) 219–243.

Løkketangen, A., D. L. Woodruff. 1996. Progressive hedging and tabu search applied to mixed integer (0, 1) multistage stochastic programming. *Journal of Heuristics* **2**(2) 111–128.

Luedtke, J. 2014. A branch-and-cut decomposition algorithm for solving chance-constrained mathematical programs with finite support. *Mathematical Programming* **146**(1-2) 219–244.

Luedtke, J., S. Ahmed, G. L. Nemhauser. 2010. An integer programming approach for linear programs with probabilistic constraints. *Mathematical Programming* **122**(2) 247–272.

Ma, O., N. Alkadi, P. Cappers, P. Denholm, J. Dudley, et al. 2013. Demand response for ancillary services. *IEEE Transactions on Smart Grid* **4**(4) 1988–1995.

Morales-España, G., J. M. Latorre, A. Ramos. 2013. Tight and compact MILP formulation for the thermal unit commitment problem. *IEEE Transactions on Power Systems* **28**(4) 4897–4908.

Nowak, M. P., W. Römisch. 2000. Stochastic Lagrangian relaxation applied to power scheduling in a hydro-thermal system under uncertainty. *Annals of Operations Research* **100**(1-4) 251–272.

Ostrowski, J., M. F. Anjos, A. Vannelli. 2012. Tight mixed integer linear programming formulations for the unit commitment problem. *IEEE Transactions on Power Systems* **1**(27) 39–46.

Ozturk, U. A., M. Mazumdar, B. A. Norman. 2004. A solution to the stochastic unit commitment problem using chance constrained programming. *IEEE Transactions on Power Systems* **19**(3) 1589–1598.

Pan, K., Y. Guan. 2016a. A polyhedral study of the integrated minimum-up/-down time and ramping polytope. Available online http://arxiv.org/pdf/1604.02184v1.pdf.

Pan, K., Y. Guan. 2016b. Strong formulations for multistage stochastic self-scheduling unit commitment. *Operations Research* **64**(6) 1482–1498.

Papavasiliou, A., S. S. Oren. 2013. Multiarea stochastic unit commitment for high wind penetration in a transmission constrained network. *Operations Research* **61**(3) 578–592.

Powell, W. B. 2007. *Approximate Dynamic Programming: Solving the Curses of Dimensionality*. John Wiley & Sons.

Rajan, D., S. Takriti. 2005. Minimum up/down polytopes of the unit commitment problem with start-up costs. Research Report RC23628, IBM.

Rockafellar, R. T., R. J.-B. Wets. 1991. Scenarios and policy aggregation in optimization under uncertainty. *Mathematics of Operations Research* **16**(1) 119–147.

Schumacher, K. M., A. E. M. Cohn, R. L.-Y. Chen. 2017. Algorithm for the N-2 security-constrained unit commitment problem with transmission switching. *INFORMS Journal on Computing* **29**(4) 645–659.

Sen, S., J. L. Higle. 2005. The C^3 theorem and a D^2 algorithm for large scale stochastic mixed-integer programming: set convexification. *Mathematical Programming* **104**(1) 1–20.

Sen, S., L. Yu, T. Genc. 2006. A stochastic programming approach to power portfolio optimization. *Operations Research* **54**(1) 55–72.

Takriti, S., J. Birge, E. Long. 1996. A stochastic model for the unit commitment problem. *IEEE Transactions on Power Systems* **11**(3) 1497–1508.

Takriti, S., B. Krasenbrink, L. S.-Y. Wu. 2000. Incorporating fuel constraints and electricity spot prices into the stochastic unit commitment problem. *Operations Research* **48**(2) 268–280.

Wang, J., M. Shahidehpour, Z. Li. 2008. Security-constrained unit commitment with volatile wind power generation. *IEEE Transactions on Power Systems* **23**(3) 1319–1327.

Wu, L., M. Shahidehpour, T. Li. 2007. Stochastic security-constrained unit commitment. *IEEE Transactions on Power Systems* **22**(2) 800–811.

Zhang, M., S. Küçükyavuz, S. Goel. 2014. A branch-and-cut method for dynamic decision making under joint chance constraints. *Management Science* **60**(5) 1317–1333.

Zhang, W., D. Nikovski. 2011. State-space approximate dynamic programming for stochastic unit commitment. *North American Power Symposium (NAPS), 2011*. IEEE, 1–7.

Zheng, Q., J. Wang, P. M. Pardalos, Y. Guan. 2013. A decomposition approach to the two-stage stochastic unit commitment problem. *Annals of Operations Research* **210**(1) 387–410.

Zhou, Z., T. Levin, G. Conzelmann. 2016. Survey of US ancillary services markets-revised. Tech. rep., Argonne National Laboratory (ANL).

Zou, J., S. Ahmed, X. A. Sun. 2018. Multistage stochastic unit commitment using stochastic dual dynamic integer programming. *IEEE Transactions on Power Systems* **34**(3) 1814–1823.

Zou, J., S. Ahmed, X. A. Sun. 2019. Stochastic dual dynamic integer programming. *Mathematical Programming* **175**(1-2) 461–502.

Online Supplement for "Multistage Stochastic Power Generation Scheduling Co-Optimizing Energy and Ancillary Services"

The detailed proofs for the theoretical results in Sections 3 – 5 are provided in this online supplement as follows.

EC.1. Proofs for Two-Period Formulations

EC.1.1. Proof for Proposition 1

Proof. **Inequality** (4a). Inequality (4a) for each i is clearly valid when $y_{i^-} = 0$ since $r_{i^-} = 0$ due to constraints (3g) and $y_i - u_i \geq 0$ due to constraints (3a). Thus, we only need to consider the case in which $y_{i^-} = 1$, leading to $u_i = 0$, $\forall i \in \mathcal{N}$, due to minimum-down time constraints (3b). It follows that inequality (4a) converts to $r_{i^-} \leq \overline{V} - \underline{C} + (\overline{C} - \overline{V})y_i, \forall i \in \mathcal{N}$. If there exists some $i \in \mathcal{N}$ such that $y_i = 0$, then inequality (4a) becomes $r_{i^-} \leq \overline{V} - \underline{C}$, which is valid due to constraints (3i). Otherwise, $y_i = 1$, $\forall i \in \mathcal{N}$, then inequality (4a) becomes $r_{i^-} \leq \overline{C} - \underline{C}$, which is valid due to constraints (3g).

Inequality (4b). We show the validity of inequality (4b) by discussing the following three possible cases in terms of the values of y_i and u_i.

1) If $y_i = 0$, then inequality (4b) for each pair (i, j) is clearly valid due to minimum-up time constraints (3a).
2) If $y_i = 1$ and $u_i = 0$, then $y_{i^-} = 1$ and $u_j = 0$, $\forall j \in \mathcal{N}$, due to constraints (3c). Inequality (4b) becomes $r_i + w_i^+ \leq (\overline{V} + V - \underline{C}) + (\overline{C} - \overline{V} - V)y_j$, $\forall i, j \in \mathcal{N}$. If there exists some $j \in \mathcal{N}$ such that $y_j = 0$, then inequality (4b) becomes $r_i + w_i^+ \leq \overline{V} - \underline{C} + V$, which is valid because $r_i + w_i^+ \leq r_{i^-} + V$ due to ramping-up constraints (3h) and $r_{i^-} \leq \overline{V} - \underline{C}$ due to ramping-down constraints (3i). Otherwise, $y_j = 1$, $\forall j \in \mathcal{N}$, then inequality (4b) converts to $r_i + w_i^+ \leq \overline{C} - \underline{C}$, which is valid due to constraints (3g).
3) If $y_i = u_i = 1$, then $y_{i^-} = 1$ due to constraints (3c) and $y_j = u_j$, $\forall j \in \mathcal{N}$, due to constraints (3a) and (3c). Inequality (4b) becomes $r_i + w_i^+ \leq \overline{V} - \underline{C}$, which is valid due to ramping-up constraints (3h).

Inequality (4c). This inequality illustrates an explicit relationship between the regulation-up and -down reserves by providing a tight upper bound for the summation of these two types of reserves at each scenario node. We show its validity by following the similar proof for inequality (4b) as follows.

1) If $y_i = 0$, then inequality (4c) for each pair (i, j) is clearly valid due to minimum-up time constraints (3a).

2) If $y_i = 1$ and $u_i = 0$, then $y_{i^-} = 1$ and $u_j = 0$, $\forall j \in \mathcal{N}$, due to constraints (3c). Inequality (4c) becomes $w_i^+ + w_i^- \leq (\overline{V} + V - \underline{C}) + (\underline{C} + V - \overline{V})y_j$, $\forall i, j \in \mathcal{N}$. If there exists some $j \in \mathcal{N}$ such that $y_j = 0$, then inequality (4c) converts to $w_i^+ + w_i^- \leq \overline{V} + V - \underline{C}$, which is valid because $w_i^+ + w_i^- \leq r_i + w_i^+$ due to constraints (3e) and $r_i + w_i^+ \leq \overline{V} + V - \underline{C}$ due to constraints (3h) and (3i). Otherwise, $y_j = 1$, $\forall j \in \mathcal{N}$, then inequality (4c) becomes $w_i^+ + w_i^- \leq 2V$, which is valid due to constraints (3h) and (3i).

3) If $y_i = u_i = 1$, then $y_{i^-} = 1$ due to constraints (3c), and $y_j = u_j$, $\forall j \in \mathcal{N}$, due to constraints (3a) and (3c). Inequality (4c) becomes $w_i^+ + w_i^- \leq \overline{V} - \underline{C}$, which is valid because $w_i^+ + w_i^- \leq r_i + w_i^+ \leq \overline{V} - \underline{C}$ due to constraints (3e) and (3h).

For **inequality** (4d), we show how we obtain inequality (4d) for each given $i \in \mathcal{N}$, which evidently shows that inequality (4d) is valid. From constraints (3h), we have $r_i + w_i^+ - r_{i^-} \leq \overline{V}(1 - y_{i^-}) + (\underline{C} + V)y_{i^-} - \underline{C}y_i$. Since $u_i \leq 1 - y_{i^-}$ due to constraints (3b), we tighten the RHS of constraints (3h) by replacing $1 - y_{i^-}$ with u_i. It follows that

$$r_i + w_i^+ - r_{i^-} \leq \overline{V}u_i + (\underline{C} + V)y_{i^-} - \underline{C}y_i, \tag{EC.1}$$

which is clearly valid if $u_i = 1 - y_{i^-}$. Otherwise we have $u_i = 0$ and $y_{i^-} = 0$, indicating that (EC.1) is also valid. Next, we continue to tighten inequality (EC.1) by replacing y_{i^-} with $y_i - u_i$, as $y_{i^-} \geq y_i - u_i$ due to constraints (3c). It follows that we obtain inequality (4d), which is clearly valid if $y_{i^-} = y_i - u_i$. Otherwise we have $y_{i^-} = 1$ and $y_i - u_i = 0$, indicating $u_i = 0$, $\forall i \in \mathcal{N}$, due to constraints (3b). Thus inequality (4d) becomes $r_{i^-} \geq 0$, which is clearly valid.

Inequality (4e). We show the validity of inequality (4e) for each pair (i, j) by discussing the following two possible cases.

1) If $y_{i^-} = 0$, then $r_{i^-} = 0$ and $y_i = u_i$, $\forall i \in \mathcal{N}$. As a result, inequality (4e) becomes $-r_i + w_i^- \leq 0$, which is constraint (3e) and thus valid.

2) If $y_{i^-} = 1$, then $u_i = 0$, $\forall i \in \mathcal{N}$, due to constraints (3b). Inequality (4e) converts to $r_{i^-} - r_i + w_i^- \leq (\overline{V} - \underline{C}) + (\underline{C} + V - \overline{V})y_j$, $\forall j \in \mathcal{N}$. If there exists some $j \in \mathcal{N}$ such that $y_j = 0$, then inequality (4e) becomes $r_{i^-} - r_i + w_i^- \leq \overline{V} - \underline{C}$, which is valid due to (3e) and ramp-down constraints (3i). Otherwise, $y_j = 1$, $\forall j \in \mathcal{N}$, then inequality (4e) becomes $r_{i^-} - r_i + w_i^- \leq V$, which is valid due to ramping-down constraints (3i).

For **inequality** (4f), we show how we derive inequality (4f) from valid inequalities (4d) and (4e). First, we derive a valid inequality (i.e., (EC.2) in the following) for conv(P_2) from inequalities (4d) and (4e). In fact, inequality (4e) can be written as $r_{j^-} - r_j + w_j^- \leq (\overline{V} - \underline{C})y_{i^-} + (\underline{C} + V - \overline{V})(y_k - u_k)$, $\forall j, k \in \mathcal{N}$, and by summing up (4d) and (4e), we obtain inequality (EC.2) as follows:

$$r_i + w_i - r_j + w_j^- \leq (\overline{V} - \underline{C})y_{i^-} + Vy_i - (\underline{C} + V - \overline{V})u_i + (\underline{C} + V - \overline{V})(y_k - u_k), \ \forall i, j, k \in \mathcal{N}, i \neq j. \tag{EC.2}$$

Next, we continue to tighten inequality (EC.2) by replacing y_{i^-} with $y_i - u_i$ and accordingly we obtain inequality (4f), which is clearly valid if $y_{i^-} = y_i - u_i$. Otherwise we have $y_{i^-} = 1$ and $y_i - u_i = 0$, indicating $u_k = 0, \ \forall k \in \mathcal{N}$, due to constraints (3b). Thus inequality (4f) converts to $-r_j + w_j^- \leq (\underline{C} + V - \overline{V})y_k, \ \forall j, k \in \mathcal{N}$, which is valid because $r_j \geq w_j^-$ due to constraints (3e) and $y_k \geq 0$. □

EC.1.2. Proof for Proposition EC.1

Proposition EC.1. *The polytope Q_2 is full-dimensional.*

Proof. We show that Q_2 is a full-dimensional polytope by providing $5n+4$ linearly independent points in Q_2 here.

(1) For each $i \in \mathcal{N}$ (totally n points), $y_i = u_i = 1$ and $y_j = u_j = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$. $r_i = w_i^+ = w_i^- = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(2) For each $i \in \mathcal{N}$ (totally n points), $y_i = u_i = 1$ and $y_j = u_j = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$. $r_i = \overline{V} - \underline{C}, r_j = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$ and $w_i^+ = w_i^- = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(3) For each $i \in \mathcal{N}$ (totally n points), $y_i = u_i = 1$ and $y_j = u_j = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$. $w_i^+ = \overline{V} - \underline{C}, w_j^+ = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$ and $r_i = w_i^- = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(4) For each $i \in \mathcal{N}$ (totally n points), $y_i = u_i = 1$ and $y_j = u_j = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$. $r_i = w_i^- = \overline{V} - \underline{C}, r_j = w_j^- = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$ and $w_i^+ = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(5) For each $i \in \mathcal{N}$ (totally n points). $y_{i^-} = y_i = 1$ and $y_j = 0, \forall j \in \mathcal{N}, j \neq i$. $u_i = 0, \forall i \in \mathcal{N} \cup \{i^-\}$, and $r_i = w_i^+ = w_i^- = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(6) (Totally one point). $y_{i^-} = 1$ and $y_i = u_i = 0, \forall i \in \mathcal{N}$. $r_i = w_i^+ = w_i^- = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(7) (Totally one point). $y_{i^-} = 1$ and $y_i = u_i = 0, \forall i \in \mathcal{N}$. $r_{i^-} = \overline{V} - \underline{C}, w_{i^-}^+ = w_{i^-}^- = 0$, and $r_i = w_i^+ = w_i^- = 0, \forall i \in \mathcal{N}$.

(8) (Totally one point). $y_{i^-} = 1$ and $y_i = u_i = 0, \forall i \in \mathcal{N}$. $r_{i^-} = 0, w_{i^-}^+ = \overline{V} - \underline{C}, w_{i^-}^- = 0$, and $r_i = w_i^+ = w_i^- = 0, \forall i \in \mathcal{N}$.

(9) (Totally one point). $y_{i^-} = 1$ and $y_i = u_i = 0, \forall i \in \mathcal{N}$. $r_{i^-} = \overline{V} - \underline{C}, w_{i^-}^+ = 0, w_{i^-}^- = \overline{V} - \underline{C}$, and $r_i = w_i^+ = w_i^- = 0, \forall i \in \mathcal{N}$.

We can easily observe that with Gaussian elimination, the points in Groups (1) - (9) can construct a lower-triangular matrix and thus they are linearly independent. It follows that the statement is proved. □

EC.1.3. Proof for Proposition EC.2

Proposition EC.2. *All of the inequalities in Q_2 are facet-defining for conv(P_2).*

Proof. We show each inequality in Q_2 is facet-defining for $\text{conv}(P_2)$ by generating $5n+4$ affinely independent points in $\text{conv}(P_2)$. Since $\vec{0} \in \text{conv}(P_2)$, we only need to construct $5n+3$ linearly independent points in $\text{conv}(P_2)$ with each satisfying one inequality of Q_2 at equation.

For inequality (3a)**:** $y_j \geq u_j, \forall j \in \mathcal{N}$**.**

For any specific $j \in \mathcal{N}$, we generate same groups of points as in Online Supplement EC.1.2 excluding the point with $y_j = 1$ from Group (5).

For inequality (3b)**:** $u_j + y_{i^-} \leq 1, \forall j \in \mathcal{N}$**.**

For any specific $j \in \mathcal{N}$, we can pick $5n+3$ points from the points generated in Online Supplement EC.1.2, as all of them satisfy inequality (3b) at equality.

For inequality (3c)**:** $y_j - u_j \leq y_{j^-}, \forall j \in \mathcal{N}$**.**

For any specific $j \in \mathcal{N}$, we can pick $5n+3$ points from the points generated in Online Supplement EC.1.2, as all of them satisfy inequality (3c) at equality.

For inequality (3e)**:** $r_i \geq w_i^-, \forall i \in \mathcal{N}$**.**

For any specific $i \in \mathcal{N}$, without loss of generality, we assume that node i is the first node in $\mathcal{N}$ due to symmetry, i.e., $i = 1$, then we can generate the following groups of points.

(**1**) For each $i \in \mathcal{N}$ (totally n points), $y_i = u_i = 1$ and $y_j = u_j = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$. $r_i = w_i^+ = w_i^- = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(**2**) For each $i \in \mathcal{N} \setminus \{1\}$ (totally $n-1$ points), $y_i = u_i = 1$ and $y_j = u_j = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$. $r_i = \overline{V} - \underline{C}, r_j = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$ and $w_i^+ = w_i^- = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(**3**) For each $i \in \mathcal{N}$ (totally n points), $y_i = u_i = 1$ and $y_j = u_j = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$. $w_i^+ = \overline{V} - \underline{C}, w_j^+ = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$ and $r_i = w_i^- = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(**4**) For each $i \in \mathcal{N}$ (totally n points), $y_i = u_i = 1$ and $y_j = u_j = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$. $r_i = w_i^- = \overline{V} - \underline{C}, r_j = w_j^- = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$ and $w_i^+ = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(**5**) For each $i \in \mathcal{N}$ (totally n points). $y_{i^-} = y_i = 1$ and $y_j = 0, \forall j \in \mathcal{N}, j \neq i$. $u_i = 0, \forall i \in \mathcal{N} \cup \{i^-\}$, and $r_i = w_i^+ = w_i^- = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(**6**) (Totally one point). $y_{i^-} = 1$ and $y_i = u_i = 0, \forall i \in \mathcal{N}$. $r_i = w_i^+ = w_i^- = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(**7**) (Totally one point). $y_{i^-} = 1$ and $y_i = u_i = 0, \forall i \in \mathcal{N}$. $r_{i^-} = \overline{V} - \underline{C}, w_{i^-}^+ = w_{i^-}^- = 0$, and $r_i = w_i^+ = w_i^- = 0, \forall i \in \mathcal{N}$.

(**8**) (Totally one point). $y_{i^-} = 1$ and $y_i = u_i = 0, \forall i \in \mathcal{N}$. $r_{i^-} = 0, w_{i^-}^+ = \overline{V} - \underline{C}, w_{i^-}^- = 0$, and $r_i = w_i^+ = w_i^- = 0, \forall i \in \mathcal{N}$.

(**9**) (Totally one point). $y_{i^-} = 1$ and $y_i = u_i = 0, \forall i \in \mathcal{N}$. $r_{i^-} = \overline{V} - \underline{C}, w_{i^-}^+ = 0, w_{i^-}^- = \overline{V} - \underline{C}$, and $r_i = w_i^+ = w_i^- = 0, \forall i \in \mathcal{N}$.

As the $5n+3$ generated points above are picked from the ones in Online Supplement EC.1.2, they are clearly linearly independent.

For inequality (4a)**.** For any specific $i \in \mathcal{N}$, without loss of generality, e.g., $i = 1$, we can generate the following groups of points.

(**1**) For each $i \in \mathcal{N}$ (totally n points), $y_i = u_i = 1$ and $y_j = u_j = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$. $r_i = w_i^+ = w_i^- = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(**2**) For each $i \in \mathcal{N}$ (totally n points), $y_i = u_i = 1$ and $y_j = u_j = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$. $r_i = \overline{V} - \underline{C}, r_j = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$ and $w_i^+ = w_i^- = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(**3**) For each $i \in \mathcal{N}$ (totally n points), $y_i = u_i = 1$ and $y_j = u_j = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$. $w_i^+ = \overline{V} - \underline{C}, w_j^+ = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$ and $r_i = w_i^- = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(**4**) For each $i \in \mathcal{N}$ (totally n points), $y_i = u_i = 1$ and $y_j = u_j = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$. $r_i = w_i^- = \overline{V} - \underline{C}, r_j = w_j^- = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$ and $w_i^+ = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(**5**) For each $i \in [2, n]_{\mathbb{Z}}$ (totally $n-1$ points). $y_{i^-} = y_i = 1$ and $y_j = 0, \forall j \in \mathcal{N}, j \neq i$. $u_i = 0, \forall i \in \mathcal{N} \cup \{i^-\}$, $r_{i^-} = \overline{V} - \underline{C}, r_i = 0, \forall i \in \mathcal{N}$ and $w_i^+ = w_i^- = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(**6**) (Totally one point). $y_{i^-} = 1$ and $y_i = u_i = 0, \forall i \in \mathcal{N}$. $r_i = w_i^+ = w_i^- = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(**7**) (Totally one point). $y_{i^-} = 1$ and $y_i = u_i = 0, \forall i \in \mathcal{N}$. $r_{i^-} = \overline{V} - \underline{C}, w_{i^-}^+ = w_{i^-}^- = 0$, and $r_i = w_i^+ = w_i^- = 0, \forall i \in \mathcal{N}$.

(**8**) (Totally one point). $y_{i^-} = 1$ and $y_i = u_i = 0, \forall i \in \mathcal{N}$. $r_{i^-} = \overline{V} - \underline{C}, w_{i^-}^+ = \overline{C} - \overline{V}, w_{i^-}^- = 0$, and $r_i = w_i^+ = w_i^- = 0, \forall i \in \mathcal{N}$.

(**9**) (Totally one point). $y_{i^-} = 1$ and $y_i = u_i = 0, \forall i \in \mathcal{N}$. $r_{i^-} = \overline{V} - \underline{C}, w_{i^-}^+ = 0, w_{i^-}^- = \overline{V} - \underline{C}$, and $r_i = w_i^+ = w_i^- = 0, \forall i \in \mathcal{N}$.

For inequality (4b)**.** We prove that inequality (4b) is facet-defining for conv(P_2). Without loss of generality, we assume $i = 1$ and $j = n$.

(**1**) For each $i \in [2, n]_{\mathbb{Z}}$ (totally $n-1$ points), $y_i = u_i = 1$ and $y_j = u_j = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$. $r_i = w_i^+ = w_i^- = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(**2**) For each $i \in \mathcal{N}$ (totally n points), $y_i = u_i = 1$ and $y_j = u_j = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$. $r_i = \overline{V} - \underline{C}, r_j = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$ and $w_i^+ = w_i^- = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(**3**) For each $i \in \mathcal{N}$ (totally n points), $y_i = u_i = 1$ and $y_j = u_j = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$. $w_i^+ = \overline{V} - \underline{C}, w_j^+ = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$ and $r_i = w_i^- = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(**4**) For each $i \in \mathcal{N}$ (totally n points), $y_i = u_i = 1$ and $y_j = u_j = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$. $r_i = w_i^- = \overline{V} - \underline{C}, r_j = w_j^- = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$ and $w_i^+ = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(**5**) For each $i \in [2, n-1]_{\mathbb{Z}}$ (totally $n-2$ points). $y_{i^-} = y_i = 1$ and $y_j = 0, \forall j \in \mathcal{N}, j \neq i$. $u_i = 0, \forall i \in \mathcal{N} \cup \{i^-\}$, and $r_i = w_i^+ = w_i^- = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(**6**) (Totally one point). $y_{i^-} = y_1 = 1$ and $y_i = u_i = 0, \forall i \in [2, n]_{\mathbb{Z}}$. $r_{i^-} = \overline{V} - \underline{C}, w_{i^-}^+ = w_{i^-}^- = 0$, $w_1^+ = \overline{V} + V - \underline{C}, w_i^+ = 0, \forall i \in [2, n]_{\mathbb{Z}}$ and $r_i = w_i^- = 0, \forall i \in \mathcal{N}$.

(**7**) (Totally one point). $y_i = 1, \forall i \in \mathcal{N} \cup \{i^-\}$ and $u_i = 0, \forall i \in \mathcal{N} \cup \{i^-\}$. $r_{i^-} = \overline{C} - \underline{C} - V, w_{i^-}^+ = w_{i^-}^- = 0$, and $r_i = \overline{C} - \underline{C} - 2V, \forall i \in \mathcal{N}$, $w_1^+ = 2V, w_i^+ = 0, \forall i \in [2, n]_{\mathbb{Z}}, w_i^- = 0, \forall i \in \mathcal{N}$.

(**8**) (Totally one point). $y_{i^-} = 1$ and $y_i = u_i = 0, \forall i \in \mathcal{N}$. $r_i = w_i^+ = w_i^- = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(**9**) (Totally one point). $y_{i^-} = 1$ and $y_i = u_i = 0, \forall i \in \mathcal{N}$. $r_{i^-} = \overline{V} - \underline{C}, w_{i^-}^+ = w_{i^-}^- = 0$, and $r_i = w_i^+ = w_i^- = 0, \forall i \in \mathcal{N}$.

(**10**) (Totally one point). $y_{i^-} = 1$ and $y_i = u_i = 0, \forall i \in \mathcal{N}$. $r_{i^-} = 0, w^+_{i^-} = \overline{C} - \underline{C}, w^-_{i^-} = 0$, and $r_i = w^+_i = w^-_i = 0, \forall i \in \mathcal{N}$.

(**11**) (Totally one point). $y_{i^-} = 1$ and $y_i = u_i = 0, \forall i \in \mathcal{N}$. $r_{i^-} = \overline{V} - \underline{C}, w^+_{i^-} = 0, w^-_{i^-} = \overline{V} - \underline{C}$, and $r_i = w^+_i = w^-_i = 0, \forall i \in \mathcal{N}$.

For inequality (4c)**.** We prove that inequality (4c) is facet-defining for conv(P_2). Without loss of generality, we assume $i = 1$ and $j = n$.

(**1**) For each $i \in [2, n]_{\mathbb{Z}}$ (totally $n-1$ points), $y_i = u_i = 1$ and $y_j = u_j = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$. $r_i = w^+_i = w^-_i = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(**2**) For each $i \in [2, n]_{\mathbb{Z}}$ (totally $n-1$ points), $y_i = u_i = 1$ and $y_j = u_j = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$. $r_i = \overline{V} - \underline{C}, r_j = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$ and $w^+_i = w^-_i = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(**3**) For each $i \in \mathcal{N}$ (totally n points), $y_i = u_i = 1$ and $y_j = u_j = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$. $w^+_i = \overline{V} - \underline{C}, w^+_j = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$ and $r_i = w^-_i = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(**4**) For each $i \in \mathcal{N}$ (totally n points), $y_i = u_i = 1$ and $y_j = u_j = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$. $r_i = w^-_i = \overline{V} - \underline{C}, r_j = w^-_j = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$ and $w^+_i = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(**5**) For each $i \in [2, n-1]_{\mathbb{Z}}$ (totally $n-2$ points). $y_{i^-} = y_i = 1$ and $y_j = 0, \forall j \in \mathcal{N}, j \neq i$. $u_i = 0, \forall i \in \mathcal{N} \cup \{i^-\}$, and $r_i = w^+_i = w^-_i = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(**6**) (Totally one point). $y_{i^-} = y_1 = 1$ and $y_i = u_i = 0, \forall i \in [2, n]_{\mathbb{Z}}$. $r_{i^-} = \overline{V} - \underline{C}, w^+_{i^-} = w^-_{i^-} = 0$, $w^+_1 = \overline{V} + V - \underline{C}, w^+_i = 0, \forall i \in [2, n]_{\mathbb{Z}}$ and $r_i = w^-_i = 0, \forall i \in \mathcal{N}$.

(**7**) (Totally one point). $y_i = 1, \forall i \in \mathcal{N} \cup \{i^-\}$ and $u_i = 0, \forall i \in \mathcal{N} \cup \{i^-\}$. $r_{i^-} = V, w^+_{i^-} = w^-_{i^-} = 0$, and $w^+_1 = 2V, w^+_i = 0, \forall i \in [2, n]_{\mathbb{Z}}, r_i = w^-_i = 0, \forall i \in \mathcal{N}$.

(**8**) (Totally one point). $y_i = 1, \forall i \in \mathcal{N} \cup \{i^-\}$ and $u_i = 0, \forall i \in \mathcal{N} \cup \{i^-\}$. $r_{i^-} = \overline{C} - \underline{C} - V, w^+_{i^-} = w^-_{i^-} = 0$, and $r_i = \overline{C} - \underline{C} - 2V, \forall i \in \mathcal{N}$, $w^+_1 = 2V, w^+_i = 0, \forall i \in [2, n]_{\mathbb{Z}}, w^-_i = 0, \forall i \in \mathcal{N}$.

(**9**) (Totally one point). $y_{i^-} = 1$ and $y_i = u_i = 0, \forall i \in \mathcal{N}$. $r_i = w^+_i = w^-_i = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(**10**) (Totally one point). $y_{i^-} = 1$ and $y_i = u_i = 0, \forall i \in \mathcal{N}$. $r_{i^-} = \overline{V} - \underline{C}, w^+_{i^-} = w^-_{i^-} = 0$, and $r_i = w^+_i = w^-_i = 0, \forall i \in \mathcal{N}$.

(**11**) (Totally one point). $y_{i^-} = 1$ and $y_i = u_i = 0, \forall i \in \mathcal{N}$. $r_{i^-} = 0, w^+_{i^-} = \overline{C} - \underline{C}, w^-_{i^-} = 0$, and $r_i = w^+_i = w^-_i = 0, \forall i \in \mathcal{N}$.

(**12**) (Totally one point). $y_{i^-} = 1$ and $y_i = u_i = 0, \forall i \in \mathcal{N}$. $r_{i^-} = \overline{V} - \underline{C}, w^+_{i^-} = 0, w^-_{i^-} = \overline{V} - \underline{C}$, and $r_i = w^+_i = w^-_i = 0, \forall i \in \mathcal{N}$.

For inequality (4d)**.** Without loss of generality, we assume $i = 1$.

(**1**) For each $i \in [2, n]_{\mathbb{Z}}$ (totally $n-1$ points), $y_i = u_i = 1$ and $y_j = u_j = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$. $r_i = w^+_i = w^-_i = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(**2**) For each $i \in \mathcal{N}$ (totally n points), $y_i = u_i = 1$ and $y_j = u_j = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$. $r_i = \overline{V} - \underline{C}, r_j = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$ and $w^+_i = w^-_i = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(**3**) For each $i \in \mathcal{N}$ (totally n points), $y_i = u_i = 1$ and $y_j = u_j = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$. $w_i^+ = \overline{V} - \underline{C}, w_j^+ = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$ and $r_i = w_i^- = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(**4**) For each $i \in \mathcal{N}$ (totally n points), $y_i = u_i = 1$ and $y_j = u_j = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$. $r_i = w_i^- = \overline{V} - \underline{C}, r_j = w_j^- = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$ and $w_i^+ = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(**5**) For each $i \in [2, n]_{\mathbb{Z}}$ (totally $n-1$ points). $y_{i^-} = y_i = 1$ and $y_j = 0, \forall j \in \mathcal{N}, j \neq i$. $u_i = 0, \forall i \in \mathcal{N} \cup \{i^-\}$, and $r_i = w_i^+ = w_i^- = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(**6**) (Totally one point). $y_{i^-} = y_1 = 1$ and $y_i = u_i = 0, \forall i \in [2, n]_{\mathbb{Z}}$. $r_{i^-} = w_{i^-}^+ = w_{i^-}^- = 0$, $w_1^+ = V, w_i^+ = 0, \forall i \in [2, n]_{\mathbb{Z}}$ and $r_i = w_i^- = 0, \forall i \in \mathcal{N}$.

(**7**) (Totally one point). $y_{i^-} = y_1 = 1$ and $y_i = u_i = 0, \forall i \in [2, n]_{\mathbb{Z}}$. $r_{i^-} = \overline{V} - \underline{C}, w_{i^-}^+ = w_{i^-}^- = 0$, $w_1^+ = \overline{V} + V - \underline{C}, w_i^+ = 0, \forall i \in [2, n]_{\mathbb{Z}}$ and $r_i = w_i^- = 0, \forall i \in \mathcal{N}$.

(**8**) (Totally one point). $y_{i^-} = y_1 = 1$ and $y_i = u_i = 0, \forall i \in [2, n]_{\mathbb{Z}}$. $r_{i^-} = w_{i^-}^- = \overline{V} - \underline{C}, w_{i^-}^+ = 0$, $w_1^+ = \overline{V} + V - \underline{C}, w_i^+ = 0, \forall i \in [2, n]_{\mathbb{Z}}$ and $r_i = w_i^- = 0, \forall i \in \mathcal{N}$.

(**9**) (Totally one point). $y_{i^-} = 1$ and $y_i = u_i = 0, \forall i \in \mathcal{N}$. $r_i = w_i^+ = w_i^- = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(**10**) (Totally one point). $y_{i^-} = 1$ and $y_i = u_i = 0, \forall i \in \mathcal{N}$. $r_{i^-} = 0, w_{i^-}^+ = \overline{C} - \underline{C}, w_{i^-}^- = 0$, and $r_i = w_i^+ = w_i^- = 0, \forall i \in \mathcal{N}$.

For inequality (4e)**.** Without loss of generality, we assume $i = 1$ and $j = n$.

(**1**) For each $i \in \mathcal{N}$ (totally n points), $y_i = u_i = 1$ and $y_j = u_j = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$. $r_i = w_i^+ = w_i^- = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(**2**) For each $i \in [2, n]_{\mathbb{Z}}$ (totally $n-1$ points), $y_i = u_i = 1$ and $y_j = u_j = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$. $r_i = \overline{V} - \underline{C}, r_j = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$ and $w_i^+ = w_i^- = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(**3**) For each $i \in \mathcal{N}$ (totally n points), $y_i = u_i = 1$ and $y_j = u_j = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$. $w_i^+ = \overline{V} - \underline{C}, w_j^+ = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$ and $r_i = w_i^- = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(**4**) For each $i \in \mathcal{N}$ (totally n points), $y_i = u_i = 1$ and $y_j = u_j = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$. $r_i = w_i^- = \overline{V} - \underline{C}, r_j = w_j^- = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$ and $w_i^+ = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(**5**) For each $i \in [1, n-1]_{\mathbb{Z}}$ (totally $n-1$ points). $y_{i^-} = y_i = 1$ and $y_j = 0, \forall j \in \mathcal{N}, j \neq i$. $u_i = 0, \forall i \in \mathcal{N} \cup \{i^-\}$, $r_{i^-} = \overline{V} - \underline{C}, r_i = 0, \forall i \in \mathcal{N}$, and $w_i^+ = w_i^- = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(**6**) (Totally one point). $y_i = 1, \forall i \in \mathcal{N} \cup \{i^-\}$ and $u_i = 0, \forall i \in \mathcal{N} \cup \{i^-\}$. $r_{i^-} = \overline{C} - \underline{C} - V, w_{i^-}^+ = w_{i^-}^- = 0$, and $r_i = \overline{C} - \underline{C} - 2V, \forall i \in \mathcal{N}$, $w_1^+ = 2V, w_i^+ = 0, \forall i \in [2, n]_{\mathbb{Z}}, w_i^- = 0, \forall i \in \mathcal{N}$.

(**7**) (Totally one point). $y_{i^-} = 1$ and $y_i = u_i = 0, \forall i \in \mathcal{N}$. $r_i = w_i^+ = w_i^- = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(**8**) (Totally one point). $y_{i^-} = 1$ and $y_i = u_i = 0, \forall i \in \mathcal{N}$. $r_{i^-} = \overline{V} - \underline{C}, w_{i^-}^+ = w_{i^-}^- = 0$, and $r_i = w_i^+ = w_i^- = 0, \forall i \in \mathcal{N}$.

(**9**) (Totally one point). $y_{i^-} = 1$ and $y_i = u_i = 0, \forall i \in \mathcal{N}$. $r_{i^-} = \overline{V} - \underline{C}, w_{i^-}^+ = \overline{C} - \overline{V}, w_{i^-}^- = 0$, and $r_i = w_i^+ = w_i^- = 0, \forall i \in \mathcal{N}$.

(**10**) (Totally one point). $y_{i^-} = 1$ and $y_i = u_i = 0, \forall i \in \mathcal{N}$. $r_{i^-} = \overline{V} - \underline{C}, w_{i^-}^+ = 0, w_{i^-}^- = \overline{V} - \underline{C}$, and $r_i = w_i^+ = w_i^- = 0, \forall i \in \mathcal{N}$.

For inequality (4f)**.** Without loss of generality, we assume $i=1$, $j=n$, and $k=2$.

(**1**) For each $i \in [2,n]_{\mathbb{Z}}$ (totally $n-1$ points), $y_i = u_i = 1$ and $y_j = u_j = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$. $r_i = w_i^+ = w_i^- = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(**2**) For each $i \in [1,n-1]_{\mathbb{Z}}$ (totally $n-1$ points), $y_i = u_i = 1$ and $y_j = u_j = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$. $r_i = \overline{V} - \underline{C}, r_j = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$ and $w_i^+ = w_i^- = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(**3**) For each $i \in \mathcal{N}$ (totally n points), $y_i = u_i = 1$ and $y_j = u_j = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$. $w_i^+ = \overline{V} - \underline{C}, w_j^+ = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$ and $r_i = w_i^- = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(**4**) For each $i \in \mathcal{N}$ (totally n points), $y_i = u_i = 1$ and $y_j = u_j = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$. $r_i = w_i^- = \overline{V} - \underline{C}, r_j = w_j^- = 0, \forall j \in \mathcal{N} \cup \{i^-\}, j \neq i$ and $w_i^+ = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(**5**) For each $i \in [2,n-1]_{\mathbb{Z}}$ (totally $n-2$ points). $y_{i^-} = y_i = 1$ and $y_j = 0, \forall j \in \mathcal{N}, j \neq i$. $u_i = 0, \forall i \in \mathcal{N} \cup \{i^-\}$, and $r_i = w_i^+ = w_i^- = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(**6**) (Totally one point). $y_{i^-} = y_1 = 1$ and $y_j = 0, \forall j \in [2,n]_{\mathbb{Z}}$. $u_i = 0, \forall i \in \mathcal{N} \cup \{i^-\}$, $r_{i^-} = \overline{V} - \underline{C}, r_1 = \overline{V} + V - \underline{C}, r_i = 0, \forall i \in [2,n]_{\mathbb{Z}}$ and $w_i^+ = w_i^- = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(**7**) (Totally one point). $y_i = 1, \forall i \in \mathcal{N} \cup \{i^-\}$ and $u_i = 0, \forall i \in \mathcal{N} \cup \{i^-\}$. $r_{i^-} = V, w_{i^-}^+ = w_{i^-}^- = 0$, and $r_i = 0, \forall i \in \mathcal{N}$, $w_1^+ = 2V, w_i^+ = 0, \forall i \in [2,n]_{\mathbb{Z}}, w_i^- = 0, \forall i \in \mathcal{N}$.

(**8**) (Totally one point). $y_i = 1, \forall i \in \mathcal{N} \cup \{i^-\}$ and $u_i = 0, \forall i \in \mathcal{N} \cup \{i^-\}$. $r_{i^-} = \overline{C} - \underline{C} - V, w_{i^-}^+ = w_{i^-}^- = 0$, and $r_i = \overline{C} - \underline{C} - 2V, \forall i \in \mathcal{N}$, $w_1^+ = 2V, w_i^+ = 0, \forall i \in [2,n]_{\mathbb{Z}}, w_i^- = 0, \forall i \in \mathcal{N}$.

(**9**) (Totally one point). $y_{i^-} = 1$ and $y_i = u_i = 0, \forall i \in \mathcal{N}$. $r_i = w_i^+ = w_i^- = 0, \forall i \in \mathcal{N} \cup \{i^-\}$.

(**10**) (Totally one point). $y_{i^-} = 1$ and $y_i = u_i = 0, \forall i \in \mathcal{N}$. $r_{i^-} = \overline{V} - \underline{C}, w_{i^-}^+ = w_{i^-}^- = 0$, and $r_i = w_i^+ = w_i^- = 0, \forall i \in \mathcal{N}$.

(**11**) (Totally one point). $y_{i^-} = 1$ and $y_i = u_i = 0, \forall i \in \mathcal{N}$. $r_{i^-} = \overline{V} - \underline{C}, w_{i^-}^+ = \overline{C} - \overline{V}, w_{i^-}^- = 0$, and $r_i = w_i^+ = w_i^- = 0, \forall i \in \mathcal{N}$.

(**12**) (Totally one point). $y_{i^-} = 1$ and $y_i = u_i = 0, \forall i \in \mathcal{N}$. $r_{i^-} = \overline{V} - \underline{C}, w_{i^-}^+ = 0, w_{i^-}^- = \overline{V} - \underline{C}$, and $r_i = w_i^+ = w_i^- = 0, \forall i \in \mathcal{N}$.

Finally, we can follow the similar way to easily show that trivial inequalities (3d) and (5) are also facet-defining and thus we omit the corresponding proof here. □

EC.1.4. Proof for Proposition EC.3

Proposition EC.3. *All of the inequalities in P_2 are dominated by those in Q_2.*

Proof. We prove the proposition by showing that all the inequalities in P_2 can be represented as an affine combination of inequalities in Q_2. Since inequalities (3a) - (3f) belong to Q_2, we only need to consider inequalities (3g) - (3i).

Inequality (3g). By picking $i = j$ for inequality (4b) in Q_2, we can obtain $r_i + w_i^+ \leq (\overline{C} - \underline{C})y_i - (\overline{C} - \overline{V})u_i$ which implies inequality (3g) in P_2.

Inequality (3h). We can derive inequality (3h) by rewriting inequality (4d) as $r_i + w_i^+ - r_{i^-} \le Vy_i - (\underline{C} + V - \overline{V})u_i = (\underline{C} + V - \overline{V})(y_i - u_i) + \overline{V}y_i - \underline{C}u_i \le (\underline{C} + V - \overline{V})y_{i^-} + \overline{V} - \underline{C}u_i$, where the last inequality holds because of $y_i \le 1$ and constraints (3c).

Inequality (3i). Inequality (3i) is dominated, because we can pick $i = j \in \mathcal{N}$ in inequality (4e) such that $r_{i^-} - r_i + w_i^- \le (\overline{V} - \underline{C})y_{i^-} + (\underline{C} + V - \overline{V})(y_i - u_i) = \overline{V}y_{i^-} + (\underline{C} + V - \overline{V})y_i - \underline{C}y_{i^-} - (\underline{C} + V - \overline{V})u_i \le \overline{V} + (\underline{C} + V - \overline{V})y_i - \underline{C}y_{i^-}$, where the last inequality holds because of $y_{i^-} \le 1$ and $u_i \ge 0$. □

EC.1.5. Proof for Lemma EC.1

Lemma EC.1. *For the following two-period multistage stochastic self-scheduling problem co-optimizing power generation and regulation reserve service*

$$z^* = \max\left\{\sum_{i=0}^{n} a_i r_i + \sum_{i=0}^{n} b_i w_i^+ + \sum_{i=0}^{n} c_i w_i^- + \sum_{i=0}^{n} d_i y_i + \sum_{i=1}^{n} e_i u_i : (r, w^+, w^-, y, u) \in P_2\right\}, \quad \text{(EC.3)}$$

where $(a, b, c, d, e) \in \mathbb{R}^{5n+4}$, there exists at least one optimal solution satisfying one of the following six conditions:

1) *$r_0 = w_0^+ = w_0^- = y_0 = 0$, $(r_i,\ w_i^+,\ w_i^-,\ y_i) \in \{(0,\ 0,\ 0,\ 0),\ (0,\ 0,\ 0,\ 1),\ (\overline{V} - \underline{C},\ 0,\ 0,\ 1),\ (0,\ \overline{V} - \underline{C},\ 0,\ 1),\ (\overline{V} - \underline{C},\ 0,\ \overline{V} - \underline{C},\ 1)\}$, $\forall i = 1, \cdots, n$, and binary variables u are uniquely decided following the constraints in P_2;*
2) *$(r_0,\ w_0^+,\ w_0^-,\ y_0) \in \{(0,\ 0,\ 0,\ 1),\ (0,\ \overline{C} - \underline{C}, 0, 1)\}$, $(r_i,\ w_i^+,\ w_i^-,\ y_i) \in \{(0,\ 0,\ 0,\ 0),\ (0,\ 0,\ 0,\ 1),\ (V,\ 0,\ 0,\ 1),\ (0,\ V,\ 0,\ 1),\ (V,\ 0,\ V,\ 1)\}$, $\forall i = 1, \cdots, n$, and binary variables y and u are uniquely decided following the constraints in P_2;*
3) *$(r_0,\ w_0^+,\ w_0^-) \in \{(\overline{V} - \underline{C},\ 0,\ 0),\ (\overline{V} - \underline{C},\ 0,\ \overline{V} - \underline{C}),\ (\overline{V} - \underline{C},\ \overline{C} - \overline{V},\ 0),\ (\overline{V} - \underline{C},\ \overline{C} - \overline{V},\ \overline{V} - \underline{C})\}$, $(r_i,\ w_i^+,\ w_i^-,\ y_i) \in \{(0,\ 0,\ 0,\ 0),\ (0,\ 0,\ 0,\ 1),\ (\overline{V} + V - \underline{C},\ 0,\ 0,\ 1),\ (0,\ \overline{V} + V - \underline{C},\ 0,\ 1),\ (\overline{V} + V - \underline{C},\ 0,\ \overline{V} + V - \underline{C},\ 1)\}$, $\forall i = 1, \cdots, n$, and binary variables y and u are uniquely decided following the constraints in P_2;*
4) *$(r_0,\ w_0^+,\ w_0^-) \in \{(V,\ 0,\ 0),\ (V,\ 0,\ V),\ (V,\ \overline{C} - \underline{C} - V,\ 0),\ (V,\ \overline{C} - \underline{C} - V,\ V)\}$, $(r_i,\ w_i^+,\ w_i^-) \in \{(0,\ 0,\ 0),\ (2V,\ 0,\ 0),\ (0,\ 2V,\ 0),\ (2V,\ 0,\ 2V)\}$, $\forall i = 1, \cdots, n$, and binary variables y and u are uniquely decided following the constraints in P_2;*
5) *$(r_0,\ w_0^+,\ w_0^-) \in \{(\overline{C} - \underline{C} - V,\ 0,\ 0),\ (\overline{C} - \underline{C} - V,\ 0,\ \overline{C} - \underline{C} - V),\ (\overline{C} - \underline{C} - V,\ V,\ 0),\ (\overline{C} - \underline{C} - V,\ V,\ \overline{C} - \underline{C} - V)\}$, $(r_i,\ w_i^+,\ w_i^-) \in \{(\overline{C} - \underline{C} - 2V,\ 0,\ 0),\ (\overline{C} - \underline{C} - 2V,\ 2V,\ 0),\ (\overline{C} - \underline{C},\ 0,\ 0),\ (\overline{C} - \underline{C},\ 0,\ 2V)\}$, $\forall i = 1, \cdots, n$, and binary variables y and u are uniquely decided following the constraints in P_2;*
6) *$(r_0,\ w_0^+,\ w_0^-) \in \{(\overline{C} - \underline{C},\ 0,\ 0),\ (\overline{C} - \underline{C},\ 0,\ \overline{C} - \underline{C})\}$, $(r_i,\ w_i^+,\ w_i^-) \in \{(\overline{C} - \underline{C},\ 0,\ 0),\ (\overline{C} - \underline{C},\ 0,\ V),\ (\overline{C} - \underline{C} - V,\ 0,\ 0),\ (\overline{C} - \underline{C} - V,\ V,\ 0)\}$, $\forall i = 1, \cdots, n$, and binary variables y and u are uniquely decided following the constraints in P_2.*

Proof. Let $A^+=\{i\in[1,n]_{\mathbb{Z}}:a_i\geq 0\}$, $A^-=\{i\in[1,n]_{\mathbb{Z}}:a_i<0\}$, $B^+=\{i\in[1,n]_{\mathbb{Z}}:b_i\geq 0\}$, $B^-=\{i\in[1,n]_{\mathbb{Z}}:b_i<0\}$, $C^+=\{i\in[1,n]_{\mathbb{Z}}:c_i\geq 0\}$, $C^-=\{i\in[1,n]_{\mathbb{Z}}:c_i<0\}$. We discuss two different cases based on the unit commitment status ("online" or "offline") at the root node.

1) The generator is offline at the root node, i.e., $r_0=w_0^+=w_0^-=y_0=0$. For this case, problem (EC.3) becomes separable in terms of each scenario node $i\in\mathcal{N}$. Thus, we further discuss the following eight situations based on if $i\in A^+$ or $i\in A^-$; if $i\in B^+$ or $i\in B^-$; if $i\in C^+$ or $i\in C^-$, for each $i\in[1,n]_{\mathbb{Z}}$:

   1.1) If $i\in A^+\cap B^+\cap C^+$, then to maximize objective function (EC.3), the generator should be scheduled online at node i with (i) $r_i=w_i^-=\overline{V}-\underline{C}$, i.e., $(r_i,w_i^+,w_i^-,y_i,u_i)=(\overline{V}-\underline{C},0,\overline{V}-\underline{C},1,1)$, following constraints (3h) and (3e) if $a_i(\overline{V}-\underline{C})+c_i(\overline{V}-\underline{C})+d_i+e_i\geq 0$ and $a_i+c_i\geq b_i$, or (ii) $w_i^+=\overline{V}-\underline{C}$, i.e., $(r_i,w_i^+,w_i^-,y_i,u_i)=(0,\overline{V}-\underline{C},0,1,1)$, if $b_i(\overline{V}-\underline{C})+d_i+e_i\geq 0$ and $a_i+c_i<b_i$, or (iii) offline otherwise, i.e., $(r_i,w_i^+,w_i^-,y_i,u_i)=(0,0,0,0,0)$.

   1.2) If $i\in A^+\cap B^-\cap C^+$, then to maximize objective function (EC.3), the generator should be scheduled online at node i with (i) $r_i=w_i^-=\overline{V}-\underline{C}$, i.e., $(r_i,w_i^+,w_i^-,y_i,u_i)=(\overline{V}-\underline{C},0,\overline{V}-\underline{C},1,1)$, following constraints (3h) and (3e) if $a_i(\overline{V}-\underline{C})+c_i(\overline{V}-\underline{C})+d_i+e_i\geq 0$, or (ii) offline otherwise, i.e., $(r_i,w_i^+,w_i^-,y_i,u_i)=(0,0,0,0,0)$.

   1.3) If $i\in A^+\cap B^+\cap C^-$, then to maximize objective function (EC.3), the generator should be scheduled online at node i with (i) $r_i=\overline{V}-\underline{C}$, i.e., $(r_i,w_i^+,w_i^-,y_i,u_i)=(\overline{V}-\underline{C},0,0,1,1)$, following constraints (3h) and (3e) if $a_i(\overline{V}-\underline{C})+d_i+e_i\geq 0$ and $a_i\geq b_i$, or (ii) the generator should be scheduled online at node i with $w_i^+=\overline{V}-\underline{C}$, i.e., $(r_i,w_i^+,w_i^-,y_i,u_i)=(0,\overline{V}-\underline{C},0,1,1)$, following constraints (3h) and (3e) if $b_i(\overline{V}-\underline{C})+d_i+e_i\geq 0$ and $a_i<b_i$, or (iii) the generator at node i should be offline otherwise, i.e., $(r_i,w_i^+,w_i^-,y_i,u_i)=(0,0,0,0,0)$.

   1.4) If $i\in A^+\cap B^-\cap C^-$, then to maximize objective function (EC.3), the generator should be scheduled online at node i with (i) $r_i=\overline{V}-\underline{C}$, i.e., $(r_i,w_i^+,w_i^-,y_i,u_i)=(\overline{V}-\underline{C},0,0,1,1)$, following constraints (3h) if $a_i(\overline{V}-\underline{C})+d_i+e_i\geq 0$, or (ii) offline otherwise, i.e., $(r_i,w_i^+,w_i^-,y_i,u_i)=(0,0,0,0,0)$.

   1.5) If $i\in A^-\cap B^+\cap C^+$, then to maximize objective function (EC.3), the generator should be scheduled online at node i with (i) $w_i^+=\overline{V}-\underline{C}$, i.e., $(r_i,w_i^+,w_i^-,y_i,u_i)=(0,\overline{V}-\underline{C},0,1,1)$, following constraints (3h) and (3e) if $b_i(\overline{V}-\underline{C})+d_i+e_i\geq 0$ and $a_i+c_i<b_i$, or (ii) $r_i=w_i^-=\overline{V}-\underline{C}$, i.e., $(r_i,w_i^+,w_i^-,y_i,u_i)=(\overline{V}-\underline{C},0,\overline{V}-\underline{C},1,1)$, following constraints (3h) and (3e) if $a_i(\overline{V}-\underline{C})+c_i(\overline{V}-\underline{C})+d_i+e_i\geq 0$ and $a_i+c_i\geq b_i$, or (iii) offline otherwise, i.e., $(r_i,w_i^+,w_i^-,y_i,u_i)=(0,0,0,0,0)$.

   1.6) If $i\in A^-\cap B^+\cap C^-$, then to maximize objective function (EC.3), the generator should be scheduled online at node i with (i) $w_i^+=\overline{V}-\underline{C}$, i.e., $(r_i,w_i^+,w_i^-,y_i,u_i)=(0,\overline{V}-\underline{C},0,1,1)$, following constraints (3h) and (3e) if $b_i(\overline{V}-\underline{C})+d_i+e_i\geq 0$, or (ii) offline otherwise, i.e., $(r_i,w_i^+,w_i^-,y_i,u_i)=(0,0,0,0,0)$.

1.7) If $i \in A^- \cap B^- \cap C^+$, then to maximize objective function (EC.3), the generator should be scheduled online at node i with (i) $r_i = w_i^- = \overline{V} - \underline{C}$, i.e., $(r_i, w_i^+, w_i^-, y_i, u_i) = (\overline{V} - \underline{C}, 0, \overline{V} - \underline{C}, 1, 1)$, following constraints (3h) and (3e) if $a_i(\overline{V} - \underline{C}) + c_i(\overline{V} - \underline{C}) + d_i + e_i \geq 0$, or (ii) $r_i = w_i^+ = w_i^- = 0$, i.e., $(r_i, w_i^+, w_i^-, y_i, u_i) = (0, 0, 0, 1, 1)$, following constraints (3h) and (3e) if $a_i(\overline{V} - \underline{C}) + c_i(\overline{V} - \underline{C}) + d_i + e_i < 0$ and $d_i + e_i \geq 0$, or (iii) offline otherwise, i.e., $(r_i, w_i^+, w_i^-, y_i, u_i) = (0, 0, 0, 0, 0)$.

1.8) If $i \in A^- \cap B^- \cap C^-$, then to maximize objective function (EC.3), the generator should be scheduled online at node i with (i) $r_i = w_i^+ = w_i^- = 0$, i.e., $(r_i, w_i^+, w_i^-, y_i, u_i) = (0, 0, 0, 1, 1)$, following constraints (3h) and (3e) if $d_i + e_i \geq 0$, or (ii) offline otherwise, i.e., $(r_i, w_i^+, w_i^-, y_i, u_i) = (0, 0, 0, 0, 0)$.

From the above 1.1) to 1.8), we verified Claim (1).

2) The generator is scheduled online at the root node, i.e., $y_0 = 1$. It follows that $u_i = 0$ for all $i = 1, \cdots, n$. First, we determine the values of w_0^+ and w_0^- based on coefficients b_0 and c_0 respectively. Note that $0 \leq w_0^+ \leq \overline{C} - \underline{C} - r_0$ due to constraints (3f) and $0 \leq w_0^- \leq r_0$ due to constraints (3e). It follows that, to maximize objective function (EC.3), $w_0^+ = \overline{C} - \underline{C} - r_0$ if $b_0 \geq 0$, and $w_0^+ = 0$ otherwise. Meanwhile, $w_0^- = r_0$ if $c_0 \geq 0$, and $w_0^- = 0$ otherwise. Next, we further discuss the following two cases in terms of the value of r_0:

2.1) If $0 \leq r_0 \leq \overline{V} - \underline{C}$, then similar to (1) above, we discuss the following eight cases in terms of each $i \in \mathcal{N}$. It follows that $r_i + w_i^+ \leq r_0 + V$, $r_i - w_i^- \geq 0, \forall i \in \mathcal{N}$, with $y_i = 1$ due to constraints (3e), (3h), and (3i), and $r_i = w_i^+ = w_i^- = 0$ with $y_i = 0$.

2.1.1) If $i \in A^+ \cap B^+ \cap C^+$, then to maximize objective function (EC.3), the generator should be scheduled online at node i with (i) $r_i = w_i^- = r_0 + V$, i.e., $(r_i, w_i^+, w_i^-, y_i, u_i) = (r_0 + V, 0, r_0 + V, 1, 0)$, following constraints (3h) and (3e) if $a_i(r_0 + V) + c_i(r_0 + V) + d_i \geq 0$ and $a_i + c_i \geq b_i$, or (ii) $w_i^+ = r_0 + V$, i.e., $(r_i, w_i^+, w_i^-, y_i, u_i) = (0, r_0 + V, 0, 1, 0)$, following constraints (3h) and (3e) if $b_i(r_0 + V) + d_i \geq 0$ and $a_i + c_i < b_i$, or (iii) offline otherwise, i.e., $(r_i, w_i^+, w_i^-, y_i, u_i) = (0, 0, 0, 0, 0)$.

2.1.2) If $i \in A^+ \cap B^- \cap C^+$, then to maximize objective function (EC.3), the generator should be scheduled online at node i with (i) $r_i = w_i^- = r_0 + V$, i.e., $(r_i, w_i^+, w_i^-, y_i, u_i) = (r_0 + V, 0, r_0 + V, 1, 0)$, following constraints (3h) and (3e) if $a_i(r_0 + V) + c_i(r_0 + V) + d_i \geq 0$, or (ii) offline otherwise, i.e., $(r_i, w_i^+, w_i^-, y_i, u_i) = (0, 0, 0, 0, 0)$.

2.1.3) If $i \in A^+ \cap B^+ \cap C^-$, then to maximize objective function (EC.3), the generator should be scheduled online at node i with (i) $r_i = r_0 + V$, i.e., $(r_i, w_i^+, w_i^-, y_i, u_i) = (r_0 + V, 0, 0, 1, 0)$, following constraints (3h) and (3e) if $a_i(r_0 + V) + d_i \geq 0$ and $a_i \geq b_i$, or (ii) the generator should be scheduled online at node i with $w_i^+ = r_0 + V$, i.e., $(r_i, w_i^+, w_i^-, y_i, u_i) = (0, r_0 + V, 0, 1, 0)$, following constraints (3h) and (3e) if $b_i(r_0 +$

$V) + d_i \geq 0$ and $a_i < b_i$, or (iii) the generator at node i should be offline otherwise, i.e., $(r_i, w_i^+, w_i^-, y_i, u_i) = (0,0,0,0,0)$.

2.1.4) If $i \in A^+ \cap B^- \cap C^-$, then to maximize objective function (EC.3), the generator should be scheduled online at node i with (i) $r_i = r_0 + V$, i.e., $(r_i, w_i^+, w_i^-, y_i, u_i) = (r_0 + V, 0, 0, 1, 0)$, following constraints (3h) if $a_i(r_0 + V) + d_i \geq 0$, or (ii) offline otherwise, i.e., $(r_i, w_i^+, w_i^-, y_i, u_i) = (0,0,0,0,0)$.

2.1.5) If $i \in A^- \cap B^+ \cap C^+$, then to maximize objective function (EC.3), the generator should be scheduled online at node i with (i) $w_i^+ = r_0 + V$, i.e., $(r_i, w_i^+, w_i^-, y_i, u_i) = (0, r_0 + V, 0, 1, 0)$, following constraints (3h) and (3e) if $b_i(r_0 + V) + d_i \geq 0$ and $a_i + c_i < b_i$, or (ii) $r_i = w_i^- = r_0 + V$, i.e., $(r_i, w_i^+, w_i^-, y_i, u_i) = (r_0 + V, 0, r_0 + V, 1, 0)$, following constraints (3h) and (3e) if $a_i(r_0 + V) + c_i(r_0 + V) + d_i \geq 0$ and $a_i + c_i \geq b_i$, or (iii) offline otherwise, i.e., $(r_i, w_i^+, w_i^-, y_i, u_i) = (0,0,0,0,0)$.

2.1.6) If $i \in A^- \cap B^+ \cap C^-$, then to maximize objective function (EC.3), the generator should be scheduled online at node i with (i) $w_i^+ = r_0 + V$, i.e., $(r_i, w_i^+, w_i^-, y_i, u_i) = (0, r_0 + V, 0, 1, 0)$, following constraints (3h) and (3e) if $b_i(r_0 + V) + d_i \geq 0$, or (ii) offline otherwise, i.e., $(r_i, w_i^+, w_i^-, y_i, u_i) = (0,0,0,0,0)$.

2.1.7) If $i \in A^- \cap B^- \cap C^+$, then to maximize objective function (EC.3), the generator should be scheduled online at node i with (i) $r_i = w_i^- = r_0 + V$, i.e., $(r_i, w_i^+, w_i^-, y_i, u_i) = (r_0 + V, 0, r_0 + V, 1, 0)$, following constraints (3h) and (3e) if $a_i(r_0 + V) + c_i(r_0 + V) + d_i \geq 0$ and $a_i + c_i \geq 0$, or (ii) $r_i = w_i^+ = w_i^- = 0$, i.e., $(r_i, w_i^+, w_i^-, y_i, u_i) = (0,0,0,1,0)$, following constraints (3h) and (3e) if $a_i(r_0 + V) + c_i(r_0 + V) + d_i < 0$ and $d_i \geq 0$, or (iii) offline otherwise, i.e., $(r_i, w_i^+, w_i^-, y_i, u_i) = (0,0,0,0,0)$.

2.1.8) If $i \in A^- \cap B^- \cap C^-$, then to maximize objective function (EC.3), the generator should be scheduled online at node i with (i) $r_i = w_i^+ = w_i^- = 0$, i.e., $(r_i, w_i^+, w_i^-, y_i, u_i) = (0,0,0,1,0)$, following constraints (3h) and (3e) if $d_i \geq 0$, or (ii) offline otherwise, i.e., $(r_i, w_i^+, w_i^-, y_i, u_i) = (0,0,0,0,0)$.

From 2.1.1) to 2.1.8), we can represent the optimal objective value of (EC.3) with a given set of $(a_0, b_0, c_0, a_i, b_i, c_i, d_i), \forall i \in \mathcal{N}$, as a function of r_0, i.e., $g(r_0) = (a_0 r_0 + \max\{b_0, 0\}(\overline{C} - \underline{C} - r_0) + \max\{c_0, 0\} r_0 + d_0) + \sum_{i \in \mathcal{N}_1}(a_i(r_0 + V) + c_i(r_0 + V) + d_i) + \sum_{i \in \mathcal{N}_2}(a_i(r_0 + V) + d_i) + \sum_{i \in \mathcal{N}_3}(b_i(r_0 + V) + d_i) + \sum_{i \in \mathcal{N}_4} d_i$, where $\mathcal{N}_i, i = 1,2,3,4$, based on the arguments derived above from 2.1.1) to 2.1.8), are defined as follows: $\mathcal{N}_1 = \{i \in \mathcal{N} : (a_i + c_i)(r_0 + V) + d_i \geq 0, a_i + c_i \geq b_i, a_i + c_i \geq 0, c_i \geq 0\}$, $\mathcal{N}_2 = \{i \in \mathcal{N} : a_i(r_0 + V) + d_i \geq 0, a_i \geq b_i, a_i \geq 0, c_i < 0\}$, $\mathcal{N}_3 = \{i \in \mathcal{N} : b_i(r_0 + V) + d_i \geq 0, b_i \geq 0, b_i \geq a_i, a_i + c_i < b_i\}$, $\mathcal{N}_4 = \{i \in \mathcal{N} : (a_i + c_i)(r_0 + V) + d_i < 0, d_i \geq 0, a_i < 0, a_i + c_i < 0, b_i < 0\}$. Furthermore, we can simplify the representation of $g(r_0)$ as $g(r_0) = (a_0 r_0 + \max\{b_0, 0\}(\overline{C} - \underline{C} - r_0) + \max\{c_0, 0\} r_0 + d_0) +$

$\sum_{i\in\mathcal{N}}\left[[\max\{a_i+c_i,a_i,b_i\}]^+(r_0+V)+d_i\right]^+$, where $[x]^+$ represents the positive part of any real number x, i.e., $[x]^+=\max\{x,0\}$. Thus, $g(r_0)$ is a convex function with respect to $r_0\in[0,\overline{V}-\underline{C}]$. It follows that the optimal solutions are achieved when $r_0=0$ or $r_0=\overline{V}-\underline{C}$.

i. If $r_0=0$, then $w_0^+,w_0^-,r_i,w_i^+,w_i^-,y_i,u_i,\forall i\in\mathcal{N}$, can be obtained based on 2.1.1) to 2.1.8). Thus, Claim (2) is proved.

ii. If $r_0=\overline{V}-\underline{C}$, then there exists at least one node i for some $i\in\mathcal{N}$ such that the generator is offline, i.e., $y_i=0$ for some $i\in\mathcal{N}$. If such node i does not exist, then generators are online at each node $i,\forall i\in\mathcal{N}$, i.e., $y_i=1,u_i=0,\forall i\in\mathcal{N}$, and r_i,w_i^+,w_i^- are determined following 2.1.1) to 2.1.8) above. w_0^+,w_0^- can be set as follows:

$$w_0^+=\begin{cases}\overline{C}-\overline{V}, & b_0\geq 0\\ 0, & b_0<0\end{cases},\quad w_0^-=\begin{cases}\overline{V}-\underline{C}, & c_0\geq 0\\ 0, & c_0<0\end{cases}.$$

Without loss of generality, we let $(r_i,w_i^+,w_i^-)=(\overline{V}+V-\underline{C},0,\overline{V}+V-\underline{C}),\forall i\in\mathcal{N}_1$, $(r_i,w_i^+,w_i^-)=(\overline{V}+V-\underline{C},0,0),\forall i\in\mathcal{N}_2$, $(r_i,w_i^+,w_i^-)=(0,\overline{V}+V-\underline{C},0),\forall i\in\mathcal{N}_3$ and $(r_i,w_i^+,w_i^-)=(0,0,0),\forall i\in\mathcal{N}_4$. However, it is easy to observe that such (r,w^+,w^-,y,u) can be represented as a linear combination of the following two points belonging to P_2:

$$(r,w^+,w^-,y,u)=\frac{1}{2}(\hat{r},\hat{w}^+,\hat{w}^-,\hat{y},\hat{u})+\frac{1}{2}(\tilde{r},\tilde{w}^+,\tilde{w}^-,\tilde{y},\tilde{u}),$$

where $\hat{y}=\tilde{y}=y,\hat{u}=\tilde{u}=u,\hat{r}_0=r_0+\varepsilon,\tilde{r}_0=r_0-\varepsilon$ and

$$\hat{w}_0^+=\begin{cases}w_0^+-\varepsilon, & b_0\geq 0\\ w_0^+, & b_0<0\end{cases},\qquad \tilde{w}_0^+=\begin{cases}w_0^++\varepsilon, & b_0\geq 0\\ w_0^+, & b_0<0\end{cases},$$

$$\hat{w}_0^-=\begin{cases}w_0^-+\varepsilon, & c_0\geq 0\\ w_0^-, & c_0<0\end{cases},\qquad \tilde{w}_0^-=\begin{cases}w_0^--\varepsilon, & c_0\geq 0\\ w_0^-, & c_0<0\end{cases},$$

$$\hat{r}_i=\begin{cases}r_i+\varepsilon, & i\in\mathcal{N}_1\cup\mathcal{N}_2\\ r_i, & o.w.\end{cases},\qquad \hat{r}_i=\begin{cases}r_i-\varepsilon, & i\in\mathcal{N}_1\cup\mathcal{N}_2\\ r_i, & o.w.\end{cases},$$

$$\hat{w}_i^+=\begin{cases}w_i^++\varepsilon, & i\in\mathcal{N}_1\cup\mathcal{N}_3\\ w_i^+, & o.w.\end{cases},\qquad \tilde{w}_i^+=\begin{cases}w_i^+-\varepsilon, & i\in\mathcal{N}_1\cup\mathcal{N}_3\\ w_i^+, & o.w.\end{cases},$$

$$\hat{w}_i^-=\begin{cases}w_i^-+\varepsilon, & i\in\mathcal{N}_1\\ w_i^-, & o.w.\end{cases},\qquad \tilde{w}_i^-=\begin{cases}w_i^--\varepsilon, & i\in\mathcal{N}_1\\ w_i^-, & o.w.\end{cases},$$

with $\varepsilon\in(0,\min\{\overline{V}-\underline{C},\overline{C}-\overline{V}-V\})$. This is a contradiction since (r,w^+,w^-,y,u) should be an extreme point of $\text{conv}(P_2)$ if there is only one optimal solution for (EC.3). Thus, Claim (3) is verified.

2.2) If $\overline{V} - \underline{C} < r_0 \le \overline{C} - \underline{C}$, then the generator should be scheduled online at node $i, \forall i \in \mathcal{N}$, due to ramping-down constraints (3i), i.e., $y_i = 1, u_i = 0, \forall i \in \mathcal{N}$. It follows that $r_i + w_i^+ \le \min\{\overline{C} - \underline{C}, r_0 + V\}$, $r_i - w_i^- \ge \max\{0, r_0 - V\}, \forall i \in \mathcal{N}$, due to constraints (3e), (3h), and (3i).

2.2.1) If $a_i + c_i \ge 0, a_i + c_i \ge b_i$, and $c_i \ge 0$, then to maximize objective function (EC.3), the generator should be scheduled online at node i with $r_i = \min\{\overline{C} - \underline{C}, r_0 + V\}, w_i^+ = 0, w_i^- = \min\{\overline{C} - \underline{C}, 2V, r_0 + V, \overline{C} - \underline{C} + V - r_0\}$.

2.2.2) If $c_i < 0, a_i \ge 0$, and $a_i \ge b_i$, then to maximize objective function (EC.3), the generator should be scheduled online at node i with $r_i = \min\{\overline{C} - \underline{C}, r_0 + V\}, w_i^+ = w_i^- = 0$.

2.2.3) If $b_i > a_i + c_i, b_i > a_i$, and $b_i \ge 0$, then to maximize objective function (EC.3), the generator should be scheduled online at node i with $r_i = \max\{0, r_0 - V\}, w_i^+ = \min\{\overline{C} - \underline{C}, 2V, r_0 + V, \overline{C} - \underline{C} + V - r_0\}, w_i^- = 0$.

2.2.4) If $a_i < 0, a_i + c_i < 0$, and $b_i < 0$, then to maximize objective function (EC.3), the generator should be scheduled online at node i with $r_i = \max\{0, r_0 - V\}, w_i^+ = w_i^- = 0$.

From the above 2.2.1) to 2.2.4), we can represent the optimal objective value of (EC.3) with a given set of $(a_0, b_0, c_0, a_i, b_i, c_i, d_i), \forall i \in \mathcal{N}$, as a continuous function of r_0, i.e., $g(r_0) = (a_0 r_0 + \max\{b_0, 0\}(\overline{C} - \underline{C} - r_0) + \max\{c_0, 0\} r_0 + d_0) + \sum_{i \in \mathcal{N}_1'}(a_i \min\{\overline{C} - \underline{C}, r_0 + V\} + c_i \min\{\overline{C} - \underline{C}, 2V, r_0 + V, \overline{C} - \underline{C} + V - r_0\} + d_i) + \sum_{i \in \mathcal{N}_2'}(a_i \min\{\overline{C} - \underline{C}, r_0 + V\} + d_i) + \sum_{i \in \mathcal{N}_3'}(a_i \max\{0, r_0 - V\} + b_i \min\{\overline{C} - \underline{C}, 2V, r_0 + V, \overline{C} - \underline{C} + V - r_0\} + d_i) + \sum_{i \in \mathcal{N}_4'}(a_i \max\{0, r_0 - V\} + d_i)$, where $\mathcal{N}_i', i = 1,2,3,4$, are defined as follows: $\mathcal{N}_1' = \{i \in \mathcal{N}: a_i + c_i \ge 0, a_i + c_i \ge b_i, c_i \ge 0\}$, $\mathcal{N}_2' = \{i \in \mathcal{N}: a_i \ge b_i, a_i \ge 0, c_i < 0\}$, $\mathcal{N}_3' = \{i \in \mathcal{N}: b_i \ge 0, b_i \ge a_i, a_i + c_i < b_i\}$, and $\mathcal{N}_4' = \{i \in \mathcal{N}: a_i < 0, a_i + c_i < 0, b_i < 0\}$. We further discuss the following three cases in terms of r_0.

i. If $\overline{V} - \underline{C} < r_0 \le V$, then $\min\{\overline{C} - \underline{C}, r_0 + V\} = r_0 + V, \min\{\overline{C} - \underline{C}, 2V, r_0 + V, \overline{C} - \underline{C} + V - r_0\} = r_0 + V, \max\{0, r_0 - V\} = 0$. Thus, $g(r_0) = (a_0 r_0 + \max\{b_0, 0\}(\overline{C} - \underline{C} - r_0) + \max\{c_0, 0\} r_0 + d_0) + \sum_{i \in \mathcal{N}_1'}(a_i(r_0 + V) + c_i(r_0 + V) + d_i) + \sum_{i \in \mathcal{N}_2'}(a_i(r_0 + V) + d_i) + \sum_{i \in \mathcal{N}_3'}(b_i(r_0 + V) + d_i) + \sum_{i \in \mathcal{N}_4'} d_i$, which is a convex function with respect to $r_0 \in (\overline{V} - \underline{C}, V]$. It follows that the optimal solutions are achieved when $r_0 = V$ or $r_0 = \overline{V} - \underline{C} + \epsilon$, where ϵ is an arbitrarily small positive real number. When $r_0 = \overline{V} - \underline{C} + \epsilon$, a similar contradiction argument as that in 2.1) can be applied to show such point is not an optimal solution. When $r_0 = V$, (r_i, w_i^+, w_i^-) are determined based on the above 2.2.1) to 2.2.4) and there cannot exist the case where $r_i = 2V, w_i^+ = w_i^- = 0, \forall i \in \mathcal{N}$. Otherwise, such (r, w^+, w^-, y, u) can be represented as a linear combination of the following two points belonging to P_2:

$$(r, w^+, w^-, y, u) = \frac{1}{2}(\hat{r}, \hat{w}^+, \hat{w}^-, \hat{y}, \hat{u}) + \frac{1}{2}(\tilde{r}, \tilde{w}^+, \tilde{w}^-, \tilde{y}, \tilde{u}),$$

where $\hat{y} = \tilde{y} = y, \hat{u} = \tilde{u} = u, \hat{w}^+ = \tilde{w}^+ = w^+, \hat{w}^- = \tilde{w}^- = w^-, \hat{r} = r + \varepsilon, \tilde{r} = r - \varepsilon$, with $\varepsilon \in (0, V)$. Thus, we have verified Claim (4).

ii. If $V \le r_0 \le \overline{C} - \underline{C} - V$, then $\min\{\overline{C} - \underline{C}, r_0 + V\} = r_0 + V, \min\{\overline{C} - \underline{C}, 2V, r_0 + V, \overline{C} - \underline{C} + V - r_0\} = 2V, \max\{0, r_0 - V\} = r_0 - V$. Thus, $g(r_0) = (a_0 r_0 + \max\{b_0, 0\}(\overline{C} - \underline{C} - r_0) + \max\{c_0, 0\} r_0 + d_0) + \sum_{i \in \mathcal{N}_1'} (a_i(r_0 + V) + c_i(2V) + d_i) + \sum_{i \in \mathcal{N}_2'} (a_i(r_0 + V) + d_i) + \sum_{i \in \mathcal{N}_3'} (a_i(r_0 - V) + b_i(2V) + d_i) + \sum_{i \in \mathcal{N}_4'} (a_i(r_0 - V) + d_i)$, which is a convex function with respect to $r_0 \in [V, \overline{C} - \underline{C} - V]$. It follows that the optimal solutions are achieved when $r_0 = V$ or $r_0 = \overline{C} - \underline{C} - V$. Here we only need to discuss the case $r_0 = \overline{C} - \underline{C} - V$, where (r_i, w_i^+, w_i^-) are determined based on the above 2.2.1) to 2.2.4) and there cannot exist the case where $r_i = \overline{C} - \underline{C} - 2V, w_i^+ = w_i^- = 0, \forall i \in \mathcal{N}$. Otherwise, such (r, w^+, w^-, y, u) can be represented as a linear combination of the following two points belonging to P_2:

$$(r, w^+, w^-, y, u) = \frac{1}{2}(\hat{r}, \hat{w}^+, \hat{w}^-, \hat{y}, \hat{u}) + \frac{1}{2}(\tilde{r}, \tilde{w}^+, \tilde{w}^-, \tilde{y}, \tilde{u}),$$

where $\hat{y} = \tilde{y} = y, \hat{u} = \tilde{u} = u, \hat{w}^+ = \tilde{w}^+ = w^+, \hat{w}^- = \tilde{w}^- = w^-, \hat{r} = r + \varepsilon, \tilde{r} = r - \varepsilon$, with $\varepsilon \in (0, \overline{C} - \underline{C} - 2V)$. Thus, we have verified Claim (5).

iii. If $\overline{C} - \underline{C} - V \le r_0 \le \overline{C} - \underline{C}$, then $\min\{\overline{C} - \underline{C}, r_0 + V\} = \overline{C} - \underline{C}, \min\{\overline{C} - \underline{C}, 2V, r_0 + V, \overline{C} - \underline{C} + V - r_0\} = 2V, \max\{0, r_0 - V\} = r_0 - V$. Thus, $g(r_0) = (a_0 r_0 + \max\{b_0, 0\}(\overline{C} - \underline{C} - r_0) + \max\{c_0, 0\} r_0 + d_0) + \sum_{i \in \mathcal{N}_1'} (a_i(\overline{C} - \underline{C}) + c_i(2V) + d_i) + \sum_{i \in \mathcal{N}_2'} (a_i(\overline{C} - \underline{C}) + d_i) + \sum_{i \in \mathcal{N}_3'} (a_i(r_0 - V) + b_i(2V) + d_i) + \sum_{i \in \mathcal{N}_4'} (a_i(r_0 - V) + d_i)$, which is a convex function with respect to $r_0 \in [V, \overline{C} - \underline{C} - V]$. It follows that the optimal solutions are achieved when $r_0 = \overline{C} - \underline{C} - V$ or $r_0 = \overline{C} - \underline{C}$. Here we only need to discuss the case $r_0 = \overline{C} - \underline{C}$, and (r_i, w_i^+, w_i^-) are determined based on the above 2.2.1) to 2.2.4). Thus, we have verified Claim (6).

This completes the proof. $\square$

EC.1.6. Proof for Proposition EC.4

PROPOSITION EC.4. *All of the extreme points of Q_2 are integral in y and u.*

Proof. We prove the claim by showing that every point in the six groups of extreme points described in Lemma EC.1 satisfies $5n + 4$ linearly independent inequalities at equation, which indicates that they are also extreme points of Q_2. Thus, our claim is verified under the facet-defining conditions in Proposition EC.2 and dominance conditions in Proposition EC.3. In the following, we prove the claim in the following six possible cases:

1) For Group (1) points, we have $r_0 = w_0^+ = w_0^- = y_0 = 0$ and let

$$(r_i, w_i^+, w_i^-, y_i, u_i) = \begin{cases} (\overline{V} - \underline{C}, 0, \overline{V} - \underline{C}, 1, 1), & \forall i \in \mathcal{N}_1, \\ (\overline{V} - \underline{C}, 0, 0, 1, 1), & \forall i \in \mathcal{N}_2, \\ (0, \overline{V} - \underline{C}, 0, 1, 1), & \forall i \in \mathcal{N}_3, \\ (0, 0, 0, 1, 1), & \forall i \in \mathcal{N}_4, \\ (0, 0, 0, 0, 0), & \forall i \in \mathcal{N} \setminus \bigcup_{k=1}^4 \mathcal{N}_k, \end{cases}$$

where $\mathcal{N}_k, k = 1,2,3,4$ are defined as $\mathcal{N}_1 = \{i \in \mathcal{N}: (a_i + c_i)(\overline{V} - \underline{C}) + d_i + e_i \geq 0, a_i + c_i \geq b_i, a_i + c_i \geq 0, c_i \geq 0\}$, $\mathcal{N}_2 = \{i \in \mathcal{N}: a_i(\overline{V} - \underline{C}) + d_i + e_i \geq 0, a_i \geq b_i, c_i < 0\}$, $\mathcal{N}_3 = \{i \in \mathcal{N}: b_i(\overline{V} - \underline{C}) + d_i + e_i \geq 0, b_i \geq 0, b_i \geq a_i, a_i + c_i < b_i\}$, $\mathcal{N}_4 = \{i \in \mathcal{N}: (a_i + c_i)(\overline{V} - \underline{C}) + d_i + e_i < 0, d_i + e_i \geq 0, a_i < 0, a_i + c_i < 0, b_i < 0\}$. Without loss of generality, we assume $\mathcal{N}_i \neq \emptyset, \forall i = 1,2,3,4$ and $(\mathcal{N} \setminus \bigcup_{k=1}^4 \mathcal{N}_k) \neq \emptyset$. The following $5n + 4$ linearly independent inequalities are satisfied at equality: (3a) (for each $i \in \mathcal{N}$), (3b) (for each $i \in \bigcup_{k=1}^4 \mathcal{N}_k$), (3c) (for each $i \in \mathcal{N}$), (4a) (for each $i \in \mathcal{N}$), (4d) (for each $i \in \mathcal{N} \setminus \bigcup_{k=1}^4 \mathcal{N}_k$), (4e) (for each $i \in \mathcal{N}$ and some $j \in \mathcal{N}$), $w_0^+ = 0$, $w_0^- = 0$, (3e) (for $i = 0$), (3f).

2) For Group (2) points, we have $y_0 = 1$, $r_0 = w_0^- = 0$ and $w_0^+ = 0$ or $\overline{C} - \underline{C}$. We let

$$(r_i, w_i^+, w_i^-, y_i, u_i) = \begin{cases} (V, 0, V, 1, 0), & \forall i \in \mathcal{N}_1, \\ (V, 0, 0, 1, 0), & \forall i \in \mathcal{N}_2, \\ (0, V, 0, 1, 0), & \forall i \in \mathcal{N}_3, \\ (0, 0, 0, 1, 0), & \forall i \in \mathcal{N}_4, \\ (0, 0, 0, 0, 0), & \forall i \in \mathcal{N} \setminus \bigcup_{k=1}^4 \mathcal{N}_k, \end{cases}$$

where $\mathcal{N}_k, k = 1,2,3,4$ are defined as $\mathcal{N}_1 = \{i \in \mathcal{N}: (a_i + c_i)V + d_i + e_i \geq 0, a_i + c_i \geq b_i, a_i + c_i \geq 0, c_i \geq 0\}$, $\mathcal{N}_2 = \{i \in \mathcal{N}: a_i V + d_i + e_i \geq 0, a_i \geq b_i, c_i < 0\}$, $\mathcal{N}_3 = \{i \in \mathcal{N}: b_i V + d_i + e_i \geq 0, b_i \geq 0, b_i \geq a_i, a_i + c_i < b_i\}$, $\mathcal{N}_4 = \{i \in \mathcal{N}: (a_i + c_i)V + d_i + e_i < 0, d_i + e_i \geq 0, a_i < 0, a_i + c_i < 0, b_i < 0\}$. Without loss of generality, we assume $\mathcal{N}_i \neq \emptyset, \forall i = 1,2,3,4$ and $(\mathcal{N} \setminus \bigcup_{k=1}^4 \mathcal{N}_k) \neq \emptyset$. The following $5n + 3$ linearly independent inequalities are satisfied at equality: $u_i = 0$ (for each $i \in \mathcal{N}$), $w_i^+ = 0$ (for each $i \in \mathcal{N} \setminus \mathcal{N}_3$), $w_i^- = 0$ (for each $i \in \{0\} \cup (\mathcal{N} \setminus \mathcal{N}_1)$), (3b) (for some $i \in \mathcal{N}$), (3c) (for each $i \in \bigcup_{k=1}^4 \mathcal{N}_k$), (3e) (for $i \in \{0\} \cup (\mathcal{N} \setminus \mathcal{N}_2)$), (4d) (for each $i \in \mathcal{N} \setminus \mathcal{N}_4$). The last linearly independent inequality satisfied at equality is (3f) when $w_0^+ = \overline{C} - \underline{C}$, or $w_0^+ = 0$ otherwise. Thus, we have found $5n + 4$ linearly independent inequalities satisfied at equality.

3) For Group (3) points, we have $y_0 = 1$ and $r_0 = \overline{V} - \underline{C}$. We let

$$(r_i, w_i^+, w_i^-, y_i, u_i) = \begin{cases} (\overline{V} + V - \underline{C}, 0, \overline{V} + V - \underline{C}, 1, 0), & \forall i \in \mathcal{N}_1, \\ (\overline{V} + V - \underline{C}, 0, 0, 1, 0), & \forall i \in \mathcal{N}_2, \\ (0, \overline{V} + V - \underline{C}, 0, 1, 0), & \forall i \in \mathcal{N}_3, \\ (0, 0, 0, 1, 0), & \forall i \in \mathcal{N}_4, \\ (0, 0, 0, 0, 0), & \forall i \in \mathcal{N} \setminus \bigcup_{k=1}^4 \mathcal{N}_k, \end{cases}$$

where $\mathcal{N}_k, k=1,2,3,4$ are defined as $\mathcal{N}_1 = \{i \in \mathcal{N}:\ (a_i+c_i)(\overline{V}+V-\underline{C})+d_i+e_i \geq 0, a_i+c_i \geq b_i, a_i+c_i \geq 0, c_i \geq 0\}$, $\mathcal{N}_2 = \{i \in \mathcal{N}:\ a_i(\overline{V}+V-\underline{C})+d_i+e_i \geq 0, a_i \geq b_i, c_i < 0\}$, $\mathcal{N}_3 = \{i \in \mathcal{N}:\ b_i(\overline{V}+V-\underline{C})+d_i+e_i \geq 0, b_i \geq 0, b_i \geq a_i, a_i+c_i < b_i\}$, $\mathcal{N}_4 = \{i \in \mathcal{N}:\ (a_i+c_i)(\overline{V}+V-\underline{C})+d_i+e_i < 0, d_i+e_i \geq 0, a_i < 0, a_i+c_i < 0, b_i < 0\}$. Based on the proof in Online Supplement EC.1.5, we have $(\mathcal{N} \setminus \bigcup_{k=1}^4 \mathcal{N}_k) \neq \emptyset$. Without loss of generality, we prove under the conditions $\mathcal{N}_i \neq \emptyset, \forall i = 1,2,3,4$ and $(\mathcal{N} \setminus \bigcup_{k=1}^4 \mathcal{N}_k) \neq \emptyset$. The following $5n+2$ linearly independent inequalities are satisfied at equality: $u_i = 0$ (for each $i \in \mathcal{N}$), $w_i^+ = 0$ (for each $i \in \mathcal{N} \setminus \mathcal{N}_3$), $w_i^- = 0$ (for each $i \in \mathcal{N} \setminus \mathcal{N}_1$), (3a) (for each $i \in \bigcup_{k=1}^4 \mathcal{N}_k$), (3b) (for each $i \in \mathcal{N}$), (3c) (for each $i \in \mathcal{N} \setminus \bigcup_{k=1}^4 \mathcal{N}_k$), (4a) (for some $i \in \mathcal{N} \setminus \bigcup_{k=1}^4 \mathcal{N}_k$), (4d) (for each $i \in \mathcal{N}_1 \cup \mathcal{N}_3$), (4e) (for some $i \in \mathcal{N} \setminus \mathcal{N}_2$ and some $j \in \mathcal{N}$). The last two linearly independent inequalities satisfied at equality are (i) (3e) (for $i=0$) and (3f) when $w_0^+ = \overline{C} - \overline{V}, w_0^- = \overline{V} - \underline{C}$, or (ii) $w_0^- = 0$ and (3f) when $w_0^+ = \overline{C} - \overline{V}, w_0^- = 0$, or (iii) $w_0^+ = 0$ and (3e) (for $i=0$) when $w_0^+ = 0, w_0^- = \overline{V} - \underline{C}$, or (iv) $w_0^+ = 0$ and $w_0^- = 0$ otherwise. Thus, we have found $5n+4$ linearly independent inequalities satisfied at equality.

4) For Group (4) points, we have $y_0 = 1$ and $r_0 = V$. It follows that $y_i = 1$ and $u_i = 0$. We let

$$(r_i, w_i^+, w_i^-, y_i, u_i) = \begin{cases} (2V, 0, 2V, 1, 0), & \forall i \in \mathcal{N}_1', \\ (2V, 0, 0, 1, 0), & \forall i \in \mathcal{N}_2', \\ (0, 2V, 0, 1, 0), & \forall i \in \mathcal{N}_3', \\ (0, 0, 0, 1, 0), & \forall i \in \mathcal{N}_4', \end{cases}$$

where $\mathcal{N}_k', k=1,2,3,4$, are defined as follows: $\mathcal{N}_1' = \{i \in \mathcal{N}:\ a_i+c_i \geq 0, a_i+c_i \geq b_i, c_i \geq 0\}$, $\mathcal{N}_2' = \{i \in \mathcal{N}:\ a_i \geq b_i, a_i \geq 0, c_i < 0\}$, $\mathcal{N}_3' = \{i \in \mathcal{N}:\ b_i \geq 0, b_i \geq a_i, a_i+c_i < b_i\}$, and $\mathcal{N}_4' = \{i \in \mathcal{N}:\ a_i < 0, a_i+c_i < 0, b_i < 0\}$ with $\bigcup_{i=1}^4 \mathcal{N}_i' = \mathcal{N}$. Based on the proof in Online Supplement EC.1.5, we have $\mathcal{N}_2' \subset \mathcal{N}$. Without loss of generality, we prove under the conditions $\mathcal{N}_i' \neq \emptyset, \forall i = 1,2,3,4$. The following $5n+2$ linearly independent inequalities are satisfied at equality: $u_i = 0$ (for each $i \in \mathcal{N}$), $w_i^+ = 0$ (for each $i \in \mathcal{N} \setminus \mathcal{N}_3'$), $w_i^- = 0$ (for each $i \in \mathcal{N} \setminus \mathcal{N}_1'$), (3b) (for each $i \in \mathcal{N}$), (3c) (for each $i \in \mathcal{N}$), (4c) (for each $i \in \mathcal{N}_1' \cup \mathcal{N}_3'$ and some $j \in \mathcal{N}'$), (4d) (for some $i \in \mathcal{N} \setminus \mathcal{N}_4'$), (4e) (for some $i \in \mathcal{N} \setminus \mathcal{N}_2'$ and some $j \in \mathcal{N}$). The last two linearly independent inequalities satisfied at equality are (i) (3e) (for $i=0$) and (3f) when $w_0^+ = \overline{C} - \underline{C} - V, w_0^- = V$, or (ii) $w_0^- = 0$ and (3f) when $w_0^+ = \overline{C} - \underline{C} - V, w_0^- = 0$, or (iii) $w_0^+ = 0$ and (3e) (for $i=0$) when $w_0^+ = 0, w_0^- = V$, or (iv) $w_0^+ = 0$ and $w_0^- = 0$ otherwise. Thus, we have found $5n+4$ linearly independent inequalities satisfied at equality.

5) For Group (5) points, we have $y_0 = 1$ and $r_0 = \overline{C} - \underline{C} - V$. It follows that $y_i = 1$ and $u_i = 0$. We let

$$(r_i, w_i^+, w_i^-, y_i, u_i) = \begin{cases} (\overline{C} - \underline{C}, 0, 2V, 1, 0), & \forall i \in \mathcal{N}_1', \\ (\overline{C} - \underline{C}, 0, 0, 1, 0), & \forall i \in \mathcal{N}_2', \\ (\overline{C} - \underline{C} - 2V, 2V, 0, 1, 0), & \forall i \in \mathcal{N}_3', \\ (\overline{C} - \underline{C} - 2V, 0, 0, 1, 0), & \forall i \in \mathcal{N}_4', \end{cases}$$

where $\mathcal{N}'_k, k=1,2,3,4$, are defined as follows: $\mathcal{N}'_1=\{i\in\mathcal{N}:\ a_i+c_i\geq 0, a_i+c_i\geq b_i, c_i\geq 0\}$, $\mathcal{N}'_2=\{i\in\mathcal{N}:\ a_i\geq b_i, a_i\geq 0, c_i<0\}$, $\mathcal{N}'_3=\{i\in\mathcal{N}:\ b_i\geq 0, b_i\geq a_i, a_i+c_i<b_i\}$, and $\mathcal{N}'_4=\{i\in\mathcal{N}:\ a_i<0, a_i+c_i<0, b_i<0\}$ with $\bigcup_{i=1}^4\mathcal{N}'_i=\mathcal{N}$. Based on the proof in Online Supplement EC.1.5, we have $\mathcal{N}'_4\subset\mathcal{N}$. Without loss of generality, we prove under the conditions $\mathcal{N}'_i\neq\emptyset, \forall i=1,2,3,4$. The following $5n+2$ linearly independent inequalities are satisfied at equality: $u_i=0$ (for each $i\in\mathcal{N}$), $w_i^+=0$ (for each $i\in\mathcal{N}\setminus\mathcal{N}'_3$), $w_i^-=0$ (for each $i\in\mathcal{N}\setminus\mathcal{N}'_1$), (3b) (for each $i\in\mathcal{N}$), (3c) (for each $i\in\mathcal{N}$), (4c) (for each $i\in\mathcal{N}'_1\cup\mathcal{N}'_3$ and some $j\in\mathcal{N}$), (4d) (for some $i\in\mathcal{N}\setminus\mathcal{N}'_4$), (4e) (for some $i\in\mathcal{N}\setminus\mathcal{N}'_2$ and some $j\in\mathcal{N}$). The last two linearly independent inequalities satisfied at equality are (i) (3e) (for $i=0$) and (3f) when $w_0^+=V, w_0^-=\overline{C}-\underline{C}-V$, or (ii) $w_0^-=0$ and (3f) when $w_0^+=V, w_0^-=0$, or (iii) $w_0^+=0$ and (3e) (for $i=0$) when $w_0^+=0, w_0^-=\overline{C}-\underline{C}-V$, or (iv) $w_0^+=0$ and $w_0^-=0$ otherwise. Thus, we have found $5n+4$ linearly independent inequalities satisfied at equality.

6) For Group (6) points, we have $y_0=1$ and $r_0=\overline{C}-\underline{C}$. It follows that $y_i=1$ and $u_i=0$. We let

$$(r_i,w_i^+,w_i^-,y_i,u_i)=\begin{cases}(\overline{C}-\underline{C},0,V,1,0), & \forall i\in\mathcal{N}'_1,\\(\overline{C}-\underline{C},0,0,1,0), & \forall i\in\mathcal{N}'_2,\\(\overline{C}-\underline{C}-V,V,0,1,0), & \forall i\in\mathcal{N}'_3,\\(\overline{C}-\underline{C}-V,0,0,1,0), & \forall i\in\mathcal{N}'_4,\end{cases}$$

where $\mathcal{N}'_k, k=1,2,3,4$, are defined as follows: $\mathcal{N}'_1=\{i\in\mathcal{N}:\ a_i+c_i\geq 0, a_i+c_i\geq b_i, c_i\geq 0\}$, $\mathcal{N}'_2=\{i\in\mathcal{N}:\ a_i\geq b_i, a_i\geq 0, c_i<0\}$, $\mathcal{N}'_3=\{i\in\mathcal{N}:\ b_i\geq 0, b_i\geq a_i, a_i+c_i<b_i\}$, and $\mathcal{N}'_4=\{i\in\mathcal{N}:\ a_i<0, a_i+c_i<0, b_i<0\}$. Without loss of generality, we assume $\mathcal{N}'_i\neq\emptyset, \forall i=1,2,3,4$. The following $5n+2$ linearly independent inequalities are satisfied at equality: $u_i=0$ (for each $i\in\mathcal{N}$), (3b) (for each $i\in\mathcal{N}$), (3c) (for each $i\in\mathcal{N}$), (3f), (4a) (for each $i\in\mathcal{N}$), (4b) (for each $i\in\mathcal{N}$ and some $j\in\mathcal{N}$), (4e) (for some $i\in\mathcal{N}\setminus\mathcal{N}'_2$ and some $j\in\mathcal{N}$). The last two linearly independent inequalities satisfied at equality are (i) $w_0^+=0$ and (3e) (for $i=0$) when $w_0^+=0, w_0^-=\overline{C}-\underline{C}$, or (ii) $w_0^+=0$ and $w_0^-=0$ otherwise. Thus, we have found $5n+4$ linearly independent inequalities satisfied at equality.

Thus, the proof is complete. □

EC.2. Three-Period Convex Hulls

In this section, we perform the polyhedral study for the three-period formulation, i.e., $T=3$ in P, and derive convex hull descriptions for the cases with different minimum-up/-down time limits for a special scenario tree setting, as shown in Figure EC.1. In this case, the uncertain parameters are realized in the first and second time periods and multiple scenario nodes are explored in the third period. Note that this scenario tree is a basic structure appearing in the complete scenario tree as shown in Figure 1.

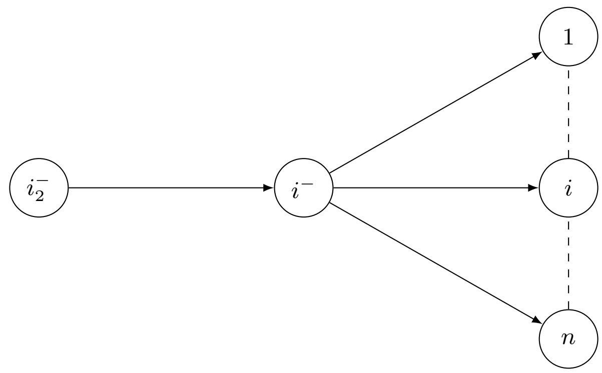

Figure EC.1 Three-Period Scenario Tree

First, we study the case in which $L=\ell=2$ in the original constraint set, which is the most representative one that includes all of the capacity, ramping rate, and minimum-up/-down time requirements among all possible cases: 1) $L=\ell=1$, 2) $L=1$ and $\ell=2$, 3) $L=2$ and $\ell=1$, and 4) $L=\ell=2$. Under this setting, the original constraint set can be described as follows:

$$P_3^2 := \Big\{ (r,w^+,w^-,y,u)\in\mathbb{R}^{n+2}\times\mathbb{R}^{n+2}\times\mathbb{R}^{n+2}\times\mathbb{B}^{n+2}\times\mathbb{B}^{n+1}:$$

$$u_{i^-}+u_i\le y_i,\ \forall i\in\mathcal{N}, \tag{EC.4a}$$

$$y_{i_2^-}+u_{i^-}+u_i\le 1,\ \forall i\in\mathcal{N}, \tag{EC.4b}$$

$$u_{i^-}\ge y_{i^-}-y_{i_2^-},\ u_i\ge y_i-y_{i^-},\ \forall i\in\mathcal{N}, \tag{EC.4c}$$

$$w_i^+\ge 0,\ w_i^-\ge 0,\ \forall i\in\mathcal{N}\cup\{i^-,i_2^-\}, \tag{EC.4d}$$

$$r_i\ge w_i^-,\ \forall i\in\mathcal{N}\cup\{i^-,i_2^-\}, \tag{EC.4e}$$

$$r_i+w_i^+\le(\overline{C}-\underline{C})y_i,\ \forall i\in\mathcal{N}\cup\{i^-,i_2^-\}, \tag{EC.4f}$$

$$r_i+w_i^+-r_{i^-}\le\overline{V}+(\underline{C}+V-\overline{V})y_{i^-}-\underline{C}y_i,\ \forall i\in\mathcal{N}\cup\{i^-\}, \tag{EC.4g}$$

$$r_{i^-}-r_i+w_i^-\le\overline{V}+(\underline{C}+V-\overline{V})y_i-\underline{C}y_{i^-},\ \forall i\in\mathcal{N}\cup\{i^-\}\Big\}, \tag{EC.4h}$$

where $\mathcal{N}$ is the set of all leaf nodes in the third time period as shown in Figure EC.1 and $|\mathcal{N}|=n$.

Following the similar techniques illustrated in Proposition 1, we develop several families of strong valid inequalities to strengthen P_3^2 as follows.

Proposition EC.5. *The inequalities*

$$r_{i_2^-}\le(\overline{V}-\underline{C})y_{i_2^-}+V(y_{i^-}-u_{i^-})+(\overline{C}-\overline{V}-V)(y_i-u_i-u_{i^-}),\forall i\in\mathcal{N}, \tag{EC.5a}$$

$$r_{i^-}\le(\overline{V}-\underline{C})y_{i^-}+(\overline{C}-\overline{V})(y_i-u_i-u_{i^-}),\ \forall i\in\mathcal{N}, \tag{EC.5b}$$

$$w_{i^-}^-\le(\overline{V}-\underline{C})y_{i^-}+(\underline{C}+2V-\overline{V})(y_i-u_i-u_{i^-}),\ \forall i\in\mathcal{N}, \tag{EC.5c}$$

$$w_{i^-}^++w_{i^-}^-\le 2Vy_{i^-}-(\underline{C}+2V-\overline{V})u_{i^-}, \tag{EC.5d}$$

$$w_i^++w_i^-\le(\overline{V}+V-\underline{C})y_i-Vu_i+(\underline{C}+V-\overline{V})(y_j-u_j-u_{i^-}),\ \forall i,j\in\mathcal{N}, \tag{EC.5e}$$

$$
\begin{aligned}
r_{i^-} + w_{i^-}^+ &\le (\overline{V} + 2V - \underline{C}) y_{i^-} - 2V u_{i^-} + (\overline{C} - \overline{V} - 2V)(y_i - u_i - u_{i^-}), \forall i \in \mathcal{N}, && \text{(EC.5f)}\\
r_i + w_i^+ &\le (\overline{V} + V - \underline{C}) y_i - V u_i + (\overline{C} - \overline{V} - V)(y_j - u_j - u_{i^-}),\ \forall i, j \in \mathcal{N}, && \text{(EC.5g)}\\
r_{i^-} - r_{i_2^-} &\le (\overline{V} - \underline{C}) y_{i^-} + (\underline{C} + V - \overline{V})(y_i - u_i - u_{i^-}),\ \forall i \in \mathcal{N}, && \text{(EC.5h)}\\
r_{i_2^-} - r_{i^-} + w_{i^-}^- &\le (\overline{V} - \underline{C}) y_{i_2^-} + (\underline{C} + V - \overline{V})(y_{i^-} - u_{i^-}), && \text{(EC.5i)}\\
r_{i^-} - r_i + w_i^- &\le (\overline{V} - \underline{C}) y_{i^-} + (\underline{C} + V - \overline{V})(y_j - u_j - u_{i^-}),\ \forall i, j \in \mathcal{N}, && \text{(EC.5j)}\\
r_{i_2^-} - r_i + w_i^- &\le (\overline{V} - \underline{C}) y_{i_2^-} + V(y_{i^-} - u_{i^-}) + (\underline{C} + V - \overline{V})(y_j - u_j - u_{i^-}),\\
&\qquad \forall i, j \in \mathcal{N}, && \text{(EC.5k)}\\
r_i + w_i^+ - r_{i_2^-} &\le (\overline{V} + V - \underline{C}) y_i - V u_i + (\underline{C} + V - \overline{V})(y_j - u_j - u_{i^-}),\ \forall i, j \in \mathcal{N}, && \text{(EC.5l)}\\
r_i + w_i^+ - r_{i^-} &\le V y_i - (\underline{C} + V - \overline{V}) u_i,\ \forall i \in \mathcal{N} \cup \{i^-\}, && \text{(EC.5m)}\\
w_{i^-}^+ + r_i + w_i^+ &\le 2V(y_{i^-} - u_{i^-}) + (\overline{V} + V - \underline{C}) y_i - V u_i\\
&\quad + (\overline{C} - \overline{V} - 2V)(y_j - u_j - u_{i^-}),\ \forall i, j \in \mathcal{N}, && \text{(EC.5n)}\\
w_{i^-}^+ + w_i^+ + w_i^- &\le 2V(y_{i^-} - u_{i^-}) + (\overline{V} + V - \underline{C}) y_i - V u_i\\
&\quad + (\underline{C} + V - \overline{V})(y_j - u_j - u_{i^-}),\ \forall i, j \in \mathcal{N}, && \text{(EC.5o)}\\
r_{i^-} + w_{i^-}^+ - r_i + w_i^- &\le (\overline{V} - \underline{C}) y_{i^-} + 2V(y_{i^-} - u_{i^-}) + (\underline{C} + V - \overline{V})(y_j - u_j - u_{i^-}),\\
&\qquad \forall i, j \in \mathcal{N}, && \text{(EC.5p)}\\
r_i + w_i^+ - r_{i^-} + w_{i^-}^- &\le (\overline{V} + V - \underline{C}) y_i - V u_i + (\underline{C} + 2V - \overline{V})(y_j - u_j - u_{i^-}),\ \forall i, j \in \mathcal{N}, && \text{(EC.5q)}\\
r_i + w_i^+ - r_j + w_j^- &\le (\overline{V} + V - \underline{C}) y_i - V u_i + (\underline{C} + V - \overline{V})(y_k - u_k - u_{i^-}),\\
&\qquad \forall i, j, k \in \mathcal{N}, i \neq j, && \text{(EC.5r)}\\
r_{i_2^-} - r_{i^-} + r_i + w_i^+ &\le (\overline{V} - \underline{C}) y_{i_2^-} + (\underline{C} + V - \overline{V}) y_{i^-} - V u_{i^-} + (\overline{V} + V - \underline{C}) y_i\\
&\quad - V u_i + (\overline{C} - \overline{V} - V)(y_j - u_j - u_{i^-}),\ \forall i, j \in \mathcal{N}, && \text{(EC.5s)}\\
r_{i_2^-} - r_{i^-} + w_i^+ + w_i^- &\le (\overline{V} - \underline{C}) y_{i_2^-} + (\underline{C} + V - \overline{V}) y_{i^-} - V u_{i^-} + (\overline{V} + V - \underline{C}) y_i\\
&\quad - V u_i + (\underline{C} + V - \overline{V})(y_j - u_j - u_{i^-}),\ \forall i, j \in \mathcal{N}, && \text{(EC.5t)}\\
w_{i^-}^+ + r_i + w_i^+ - r_j + w_j^- &\le 2V(y_{i^-} - u_{i^-}) + (\overline{V} + V - \underline{C}) y_i - V u_i\\
&\quad + (\underline{C} + V - \overline{V})(y_k - u_k - u_{i^-}),\ \forall i, j, k \in \mathcal{N}, i \neq j, && \text{(EC.5u)}\\
r_{i_2^-} - r_{i^-} + r_i + w_i^+ - r_j + w_j^- &\le (\overline{V} - \underline{C}) y_{i_2^-} + (\underline{C} + V - \overline{V}) y_{i^-} - V u_{i^-} + (\overline{V} + V - \underline{C}) y_i\\
&\quad - V u_i + (\underline{C} + V - \overline{V})(y_k - u_k - u_{i^-}),\ \forall i, j, k \in \mathcal{N}, i \neq j, && \text{(EC.5v)}
\end{aligned}
$$

are valid for conv(P_3^2). Furthermore, they are facet-defining for conv(P_3^2) when $\overline{C} - \overline{V} \ge 2V$.

Proof. The proof is similar to that of Proposition EC.2 and thus is omitted here. □

Note that with one more time period and correspondingly one more scenario node (i.e., i_2^-) than the two-period case in Section 3, several families of inequalities with more complex structures (e.g., (EC.5s) - (EC.5v)) than those in Q_2 are developed to relate both power generation and regulation reserve at nodes i_2^-, i^-, and i, and possibly node i's sibling node $j \in \mathcal{N}$.

Remark EC.1. We can similarly derive strong valid inequalities for the case when $\overline{C} - \overline{V} \geq 2V$ in Proposition EC.5 does not hold. We adopt the condition $\overline{C} - \overline{V} \geq 2V$ here because it represents the most common generator characteristics and under this condition, the corresponding convex hull representation has relatively the largest number of inequalities.

Now, through utilizing inequalities (EC.5a) - (EC.5v), we introduce the linear programming description of conv(P_3^2) by adding several original constraints from P_3^2 and trivial inequalities as follows:

$$Q_3^2 := \Big\{(r, w^+, w^-, y, u) \in \mathbb{R}^{5n+9} : \text{(EC.4a)} - \text{(EC.4e)}, \text{(EC.5a)} - \text{(EC.5v)}, \\ r_{i_2^-} + w^+_{i_2^-} \leq (\overline{C} - \underline{C}) y_{i_2^-}, \ u_i \geq 0, \ \forall i \in \mathcal{N}\Big\}. \tag{EC.6}$$

Theorem EC.1. *$Q_3^2 = conv(P_3^2)$.*

Proof. The proof is similar to that of Theorem 1 and thus is omitted here. □

Next, we consider the case in which $L = \ell = 1$ and meanwhile we also assume $\overline{C} - \overline{V} - 2V \geq 0$. For this case, the original constraint set (denoted by P_3^1) can be described as follows:

$$P_3^1 := \Big\{ (r, w^+, w^-, y, u) \in \mathbb{R}^{n+2} \times \mathbb{R}^{n+2} \times \mathbb{R}^{n+2} \times \mathbb{B}^{n+2} \times \mathbb{B}^{n+1} : \text{(EC.4c)} - \text{(EC.4g)},$$

$$u_i \leq y_i, \ \forall i \in \mathcal{N} \cup \{i^-\}, \tag{EC.7a}$$

$$y_{i^-} + u_i \leq 1, \ \forall i \in \mathcal{N} \cup \{i^-\}\Big\}. \tag{EC.7b}$$

Accordingly, following the similar methods described above, we can derive the convex hull description as follows:

Theorem EC.2. *For the case when $L = \ell = 1$ and $\overline{C} - \overline{V} - 2V \geq 0$, the convex hull representation for the three-period problem is $Q_3^1 = conv(P_3^1) =$*

$$\Big\{(r, w^+, w^-, y, u) \in \mathbb{R}^{5n+9} : \text{(EC.4c)} - \text{(EC.4e)}, \text{(EC.5d)}, \text{(EC.5i)}, \text{(EC.5m)}, \text{(EC.6)}, \text{(EC.7a)} - \text{(EC.7b)},$$

$$r_{i_2^-} \leq (\overline{V} - \underline{C}) y_{i_2^-} + V(y_{i^-} - u_{i^-}) + (\overline{C} - \overline{V} - V)(y_i - u_i), \ \forall i \in \mathcal{N}, \tag{EC.8a}$$

$$r_{i_2^-} \leq (\overline{V} - \underline{C}) y_{i_2^-} + (\overline{C} - \overline{V})(y_{i^-} - u_{i^-}), \tag{EC.8b}$$

$$r_{i^-} \leq (\overline{V} - \underline{C}) y_{i^-} + (\overline{C} - \overline{V})(y_i - u_i), \ \forall i \in \mathcal{N}, \tag{EC.8c}$$

$$w^-_{i^-} \leq (\overline{V} - \underline{C}) y_{i^-} + (\underline{C} + 2V - \overline{V})(y_i - u_i), \ \forall i \in \mathcal{N}, \tag{EC.8d}$$

$$w_i^+ + w_i^- \leq (\overline{V} + V - \underline{C})y_i - Vu_i + (\underline{C} + V - \overline{V})(y_j - u_j), \ \forall i \in \mathcal{N}, \forall j \in \mathcal{N} \cup \{i^-\}, \tag{EC.8e}$$

$$r_{i^-} + w_{i^-}^+ \leq (\overline{V} - \underline{C})y_{i^-} + 2V(y_{i^-} - u_{i^-}) + (\overline{C} - \overline{V} - 2V)(y_j - u_j), \ \forall j \in \mathcal{N} \cup \{i^-\}, \tag{EC.8f}$$

$$r_i + w_i^+ \leq (\overline{V} + V - \underline{C})y_i - Vu_i + (\overline{C} - \overline{V} - V)(y_j - u_j), \ \forall i \in \mathcal{N}, \forall j \in \mathcal{N} \cup \{i^-\}, \tag{EC.8g}$$

$$r_{i^-} - r_{i_2^-} \leq (\overline{V} - \underline{C})y_{i^-} + (\underline{C} + V - \overline{V})(y_i - u_i), \ \forall i \in \mathcal{N}, \tag{EC.8h}$$

$$r_{i^-} - r_i + w_i^- \leq (\overline{V} - \underline{C})y_{i^-} + (\underline{C} + V - \overline{V})(y_j - u_j), \ \forall i \in \mathcal{N}, \forall j \in \mathcal{N} \cup \{i^-\}, \tag{EC.8i}$$

$$r_{i_2^-} - r_i + w_i^- \leq (\overline{V} - \underline{C})y_{i_2^-} + V(y_{i^-} - u_{i^-}) + (\underline{C} + V - \overline{V})(y_j - u_j), \ \forall i \in \mathcal{N}, \forall j \in \mathcal{N} \cup \{i^-\}, \tag{EC.8j}$$

$$r_i + w_i^+ - r_{i_2^-} \leq (\overline{V} + V - \underline{C})y_i - Vu_i + (\underline{C} + V - \overline{V})(y_j - u_j), \ \forall i \in \mathcal{N}, \forall j \in \mathcal{N} \cup \{i^-\}, \tag{EC.8k}$$

$$\begin{aligned} w_{i^-}^+ + r_i + w_i^+ \leq\ & 2Vy_{i^-} - (\underline{C} + 2V - \overline{V})u_{i^-} + (\overline{V} + V - \underline{C})y_i - Vu_i \\ & + (\overline{C} - \overline{V} - 2V)(y_j - u_j), \ \forall i, j \in \mathcal{N}, \end{aligned} \tag{EC.8l}$$

$$r_{i^-} + w_{i^-}^+ - r_i + w_i^- \leq 3Vy_{i^-} - 2Vu_{i^-} + (\underline{C} + V - \overline{V})(y_j - u_j - y_{i^-}), \ \forall i \in \mathcal{N}, \forall j \in \mathcal{N} \cup \{i^-\} \tag{EC.8m}$$

$$r_i + w_i^+ - r_{i^-} + w_{i^-}^- \leq 3Vy_i - Vu_i + (\underline{C} + 2V - \overline{V})(y_j - u_j - y_i), \ \forall i \in \mathcal{N}, \forall j \in \mathcal{N} \cup \{i^-\}, \tag{EC.8n}$$

$$\left. r_i + w_i^+ - r_j + w_j^- \leq 2Vy_i - Vu_i + (\underline{C} + V - \overline{V})(y_k - u_k - y_i), \forall i, j \in \mathcal{N}, i \neq j, \forall k \in \mathcal{N} \cup \{i^-\} \right\}. \tag{EC.8o}$$

Theorem EC.3. *For the case when $L = 1$ and $\ell = 2$ and $\overline{C} - \overline{V} - 2V \geq 0$, the convex hull representation of the original constraint set is the same as Q_3^1 except that (EC.7b) is replaced by (EC.4b). For the case when $L = 2$ and $\ell = 1$ and $\overline{C} - \overline{V} - 2V \geq 0$, the convex hull representation of the original constraint set is the same as Q_3^2 except that (EC.4b) is replaced by (EC.7b). In fact, when two original constraint sets have the same value of L, the corresponding convex hull descriptions are the same except that the minimum-down constraints are different.*

Proof. The proof is similar to that of Theorem 1 and thus is omitted here. □

EC.3. Proofs for Multi-Period Formulations

To show an inequality is facet-defining for conv(P), we create $5|\mathcal{V}| - 1$ affinely independent points in conv(P) that satisfy this inequality at equation. Since $\vec{0} \in \text{conv}(P)$, it is sufficient to create the remaining $5|\mathcal{V}| - 2$ linearly independent points. In this section, we label the nodes in the tree as follows for the convenience of generating points. Due to the symmetry of the scenario tree, we label the nodes in $\mathcal{V}$ following the breadth-first search rule, i.e., root node 0 is labeled as 0, the first node at the second stage, i.e., $t(1)$, is labeled as 1, the second node at $t(1)$ is labeled as 2, the first node at the third stage, i.e., $t(2)$, is labeled as $n+1$, the first node at $t(k)$ is labeled as $1 + n + \cdots + n^{k-1} = (n^k - 1)/(n-1)$, and the last node at stage T is $|\mathcal{V}| - 1$. Without loss of generality, we assume that i is the first node of all leaf nodes in the scenario tree and its ancestor node i^- is also the first node of all nodes at $t(i^-)$ as for each node considered in one inequality in the following proofs, we can always rearrange the scenario tree to achieve this. We use $\Gamma(t(i))$ to denote the set of all of the nodes in the same period with node i and we use $\Gamma(i^-)$ to collect all of the nodes with labeling less than that of node i^-.

EC.3.1. Proof for Proposition 2

Proof. **(Validity)** We discuss the following two possible cases in terms of the value of y_{i^-}.

1. If $y_{i^-}=0$, then $u_s=0, \forall s\in[i^-_L, i^-]$, due to minimum-up time constraints (2a). Thus, $r_{i^-}=0$ and $\sum_{h=1}^{L-1}(h-1)Vu_{i^-_h}=0$. Inequality (6) converts to $0\le(\overline{C}-\overline{V})(y_i-\sum_{h=0}^{L-1}u_{i^-_h})$, which is valid due to (2a).
2. If $y_{i^-}=1$, then we consider the following two possible cases in terms of when the generator starts up.
   (a) If $u_s=1$ for some $s\in[i^-_{L-1}, i^-]_{\mathbb{Z}}$, then inequality (6) converts to $r_{i^-}\le\overline{V}-\underline{C}+(t(i^-)-t(s))Vu_{i^-_h}$ due to minimum-up time constraints (2a). It follows that inequality (6) is valid since $y_i-\sum_{h=0}^{L-1}u_{i^-_h}\ge 0$ due to ramping-up constraints (2g).
   (b) If $u_s=1$ for some $s\in[1, i^-_L]_{\mathbb{Z}}$, then inequality (6) converts to $r_{i^-}\le(\overline{V}-\underline{C})+(\overline{C}-\overline{V})y_i$, which is valid due to capacity constraints (2f) and the fact that y_i is binary.

(Facet-defining) Here we prove the case where $L\ge 3$ because it is easier to prove the case in which $L\le 2$, where the last term on the RHS of inequality (6) is 0. We create $5|\mathcal{V}|-2$ points as follows.

1. For each $\alpha\in[0, |\Gamma(i^-)|-1]_{\mathbb{Z}}$, we create four points with $y_s=1, \forall s\in[0,\alpha]_{\mathbb{Z}}$ and $y_s=0$ otherwise, and $u_s=0, \forall s\in[0,|\mathcal{V}|-1]_{\mathbb{Z}}$, leading to $4|\Gamma(i^-)|$ points in total. In addition, for each $\alpha\in[0,|\Gamma(i^-)|-1]_{\mathbb{Z}}$, we assign different values of r_s, w_s^+, and w_s^- to those four points as follows.
   (a) $r_s=\overline{V}-\underline{C}, \forall s\in[0,\alpha]_{\mathbb{Z}}$ and $r_s=0, \forall s\in[\alpha+1,|\mathcal{V}|-1]_{\mathbb{Z}}$. $w_s^+=w_s^-=0, \forall s\in[0,|\mathcal{V}|-1]_{\mathbb{Z}}$.
   (b) $r_s=w_s^-=\overline{V}-\underline{C}, \forall s\in[0,\alpha]_{\mathbb{Z}}$ and $r_s=w_s^-=0, \forall s\in[\alpha+1,|\mathcal{V}|-1]_{\mathbb{Z}}$. $w_s^+=0, \forall s\in[0,|\mathcal{V}|-1]_{\mathbb{Z}}$.
   (c) $w_s^+=\overline{V}-\underline{C}, \forall s\in[0,\alpha]_{\mathbb{Z}}$ and $w_s^+=0, \forall s\in[\alpha+1,|\mathcal{V}|-1]_{\mathbb{Z}}$. $r_s=w_s^-=0, \forall s\in[0,|\mathcal{V}|-1]_{\mathbb{Z}}$.
   (d) $r_s=w_s^+=w_s^-=0, \forall s\in[0,|\mathcal{V}|-1]_{\mathbb{Z}}$.

   Note that these four points are obviously linearly independent, and for each α, since we have $y_s=1, \forall s\in[0,\alpha]_{\mathbb{Z}}$, we can conclude that these $4|\Gamma(i^-)|$ points are linearly independent.
2. For each $\alpha\in[|\Gamma(i^-)|, |\Gamma(i^-)|+|\Gamma(t(i^-))|-1]_{\mathbb{Z}}$, we create three points with $y_s=1, \forall s\in[0,\alpha]_{\mathbb{Z}}$, $y_s=0$ otherwise, and $u_s=0, \forall s\in[0,|\mathcal{V}|-1]_{\mathbb{Z}}$, leading to $3|\Gamma(t(i^-))|$ points in total. In addition, for each α, we assign different values of r_s, w_s^+, and w_s^- to those three points as follows.
   (a) $r_s=\overline{V}-\underline{C}, \forall s\in[0,\alpha]_{\mathbb{Z}}$ and $r_s=0, \forall s\in[\alpha+1,|\mathcal{V}|-1]_{\mathbb{Z}}$. $w_s^+=w_s^-=0, \forall s\in[0,|\mathcal{V}|-1]_{\mathbb{Z}}$.
   (b) $r_s=w_s^-=\overline{V}-\underline{C}, \forall s\in[0,\alpha]_{\mathbb{Z}}$ and $r_s=w_s^-=0, \forall s\in[\alpha+1,|\mathcal{V}|-1]_{\mathbb{Z}}$. $w_s^+=0, \forall s\in[0,|\mathcal{V}|-1]_{\mathbb{Z}}$.
   (c) $w_s^+=\overline{V}-\underline{C}, \forall s\in[0,\alpha]_{\mathbb{Z}}$ and $w_s^+=0, \forall s\in[\alpha+1,|\mathcal{V}|-1]_{\mathbb{Z}}$. $r_s=w_s^-=0, \forall s\in[0,|\mathcal{V}|-1]_{\mathbb{Z}}$.
3. For $\alpha=|\Gamma(i^-)|+|\Gamma(t(i^-))|$, we create one point with $y_s=1, u_s=0, \forall s\in[0,|\mathcal{V}|-1]_{\mathbb{Z}}$, $r_\alpha=\overline{C}-\underline{C}$ and $r_s=\overline{C}-\underline{C}-V, \forall s\in[0,|\mathcal{V}|-1]_{\mathbb{Z}}\setminus\{\alpha\}$, and $w_s^+=w_s^-=0, \forall s\in[0,|\mathcal{V}|-1]_{\mathbb{Z}}$.

4. For each $\alpha \in [|\Gamma(i^-)|+|\Gamma(t(i^-))|+1, |\mathcal{V}|-1]_{\mathbb{Z}}$, we create one point with $y_s = 1, \forall s \in [0, |\Gamma(i^-)| + |\Gamma(t(i^-))|-1]_{\mathbb{Z}} \cup \{\alpha\}$, $y_s = 0$ otherwise, and $u_s = 0, \forall s \in [0, |\mathcal{V}|-1]_{\mathbb{Z}}$, leading to $|\mathcal{V}| - |\Gamma(i^-)| - |\Gamma(t(i^-))| - 1$ points in total. In addition, for each $\alpha \in [|\Gamma(i^-)| + |\Gamma(t(i^-))| + 1, |\mathcal{V}| - 1]_{\mathbb{Z}}$, we assign $r_{i^-} = \overline{V} - \underline{C}$ and $r_s = 0$ otherwise. $w_s^+ = w_s^- = 0, \forall s \in [0, |\mathcal{V}|-1]_{\mathbb{Z}}$.

Thus, we have created $|\mathcal{V}| + 3|\Gamma(i^-)| + 2|\Gamma(t(i^-))|$ linearly independent points above. Next, we create another $4|\mathcal{V}| - 2 - 3|\Gamma(i^-)| - 2|\Gamma(t(i^-))|$ points in the following and sort them according to the values of u_s.

5. For each $\alpha \in [1, |\Gamma(i^-_{L-1})| - 1]_{\mathbb{Z}}$, we create one point with $u_\alpha = 1$ and $y_s = 1, r_s = \overline{V} - \underline{C}, \forall s \in \mathcal{H}_L(\alpha)$, and $w_s^+ = w_s^- = 0, \forall s \in [0, |\mathcal{V}|-1]_{\mathbb{Z}}$, leading to $|\Gamma(i^-_{L-1})| - 1$ points in total.

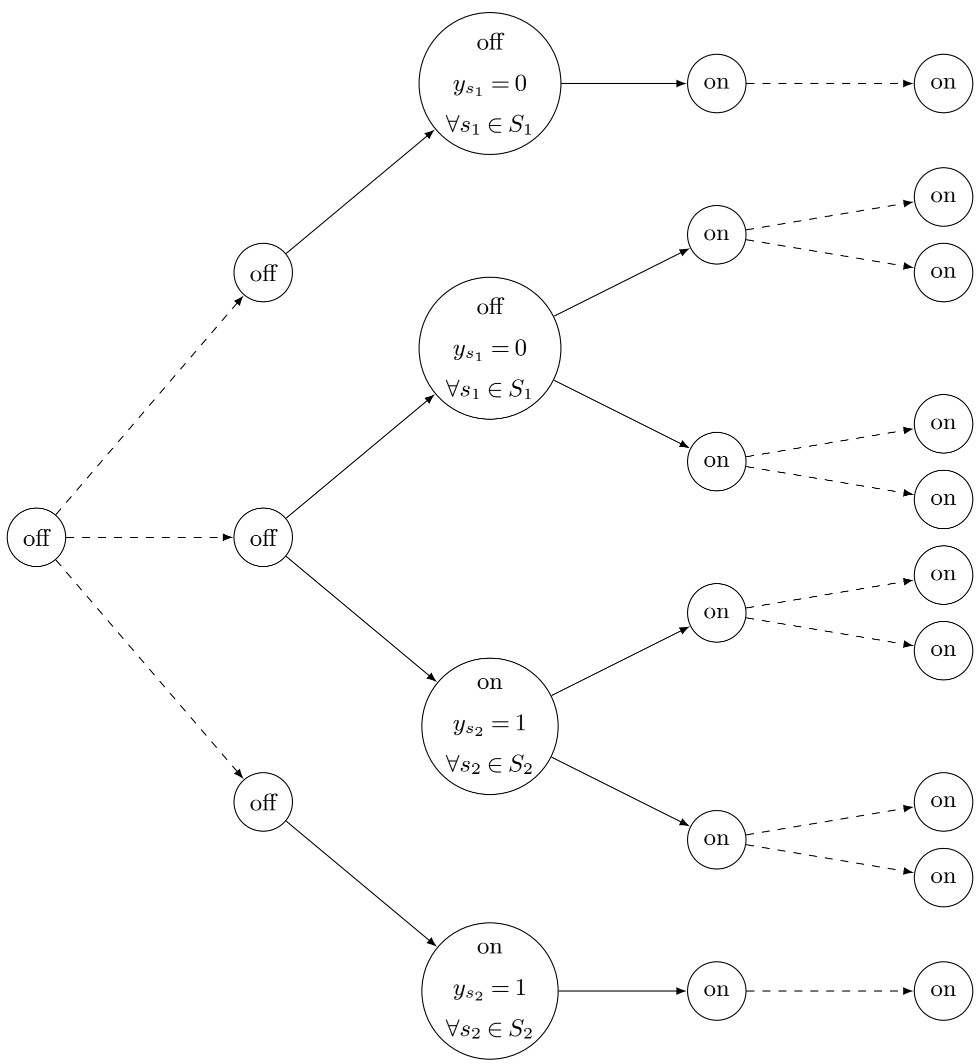

Figure EC.2 **Online/offline Status of the Scenario Tree for Case** 6 **of Online Supplement EC.3.1, where** $S_1 = [|\Gamma(i^-_{L-1})| + \sum_{h=1}^{k} |\Gamma(t(i^-_{L-h}))|, \alpha - 1]_{\mathbb{Z}}$ **and** $S_2 = [\alpha, |\Gamma(i^-_{L-1})| + \sum_{h=1}^{k+1} |\Gamma(t(i^-_{L-h}))| - 1]_{\mathbb{Z}}$

6. For each $k \in [0, L-3]_{\mathbb{Z}}$ and each $\alpha \in [|\Gamma(i^-_{L-1})| + \sum_{h=1}^{k} |\Gamma(t(i^-_{L-h}))|, |\Gamma(i^-_{L-1})| + \sum_{h=1}^{k+1} |\Gamma(t(i^-_{L-h}))| - 1]_{\mathbb{Z}}$, we create one point with $u_s = 1, \forall s \in [\alpha, |\Gamma(i^-_{L-1})| + \sum_{h=1}^{k+1} |\Gamma(t(i^-_{L-h}))| - 1]_{\mathbb{Z}}$ and $u_s = 0$ otherwise, leading to $|\Gamma(i^-)| - |\Gamma(i^-_{L-1})|$ points in total. Furthermore, as shown in Figure EC.2, we start up the generator at the child nodes of each node $s \in [|\Gamma(i^-_{L-1})| + \sum_{h=1}^{k} |\Gamma(t(i^-_{L-h}))|, \alpha - 1]_{\mathbb{Z}}$, i.e., $u_s = 1, \forall s \in [|\Gamma(i^-_{L-1})| + \sum_{h=1}^{k+1} |\Gamma(t(i^-_{L-h}))|, |\Gamma(i^-_{L-1})| + \sum_{h=1}^{k+2} |\Gamma(t(i^-_{L-h}))| - 1]_{\mathbb{Z}} \setminus \bigcup_{k=\alpha}^{|\Gamma(i^-_{L-1})| + \sum_{h=1}^{k+1} |\Gamma(t(i^-_{L-h}))| - 1} \mathcal{H}_L(k)$, $y_s = 1, \forall s \in [\alpha, |\mathcal{V}| - 1]_{\mathbb{Z}}$, and $y_s = 0$ otherwise. $r_s = \overline{V} - \underline{C} + (t(s) - t(\alpha))V$, if $s \in \mathcal{C}(\hat{s}), \forall \hat{s} \in [\alpha, |\Gamma(i^-_{L-1})| + \sum_{h=1}^{k+1} |\Gamma(t(i^-_{L-h}))| - 1]_{\mathbb{Z}}$ and $r_s = \overline{V} - \underline{C} + (t(s) - t(\alpha) - 1)V$, if $s \in \mathcal{C}(\hat{s}), \forall \hat{s} \in [|\Gamma(i^-_{L-1})| + \sum_{h=1}^{k} |\Gamma(t(i^-_{L-h}))|, \alpha - 1]_{\mathbb{Z}}$. $w^+_s = w^-_s = 0, \forall s \in [0, |\mathcal{V}| - 1]_{\mathbb{Z}}$.
7. For each $\alpha \in [|\Gamma(i^-)|, |\Gamma(i^-)| + |\Gamma(t(i^-))| - 1]_{\mathbb{Z}}$, we create two points with $u_s = 1, \forall s \in [\alpha, |\Gamma(i^-)| + |\Gamma(t(i^-))| - 1]_{\mathbb{Z}}$, $u_s = 0$ otherwise, and $y_{\hat{s}} = 1, \forall \hat{s} \in \mathcal{H}_L(s), \forall s \in [\alpha, |\Gamma(i^-)| + |\Gamma(t(i^-))| - 1]_{\mathbb{Z}}$, leading to $2|\Gamma(t(i^-))| - 2$ points in total. In addition, for each α, we assign different values of r_s, w^+_s, and w^-_s to those two points as follows.
   (a) $r_{\hat{s}} = \overline{V} - \underline{C}, \forall \hat{s} \in \mathcal{H}_L(s), \forall s \in [\alpha, |\Gamma(i^-)| + |\Gamma(t(i^-))| - 1]_{\mathbb{Z}}$ and $r_{\hat{s}} = 0$ otherwise. $w^+_s = w^-_s = 0, \forall s \in [0, |\mathcal{V}| - 1]_{\mathbb{Z}}$.
   (b) $r_{\hat{s}} = w^-_{\hat{s}} = \overline{V} - \underline{C}, \forall \hat{s} \in \mathcal{H}_L(s), \forall s \in [\alpha, |\Gamma(i^-)| + |\Gamma(t(i^-))| - 1]_{\mathbb{Z}}$ and $r_{\hat{s}} = w^-_{\hat{s}} = 0$ otherwise. $w^+_s = 0, \forall s \in [0, |\mathcal{V}| - 1]_{\mathbb{Z}}$.
8. For $\alpha = |\Gamma(i^-)|$, we create one point with $u_s = 1, \forall s \in [\alpha, |\Gamma(i^-)| + |\Gamma(t(i^-))| - 1]_{\mathbb{Z}}$, $u_s = 0$ otherwise, $y_s = 1, \forall s \in [\alpha, |\mathcal{V}| - 1]_{\mathbb{Z}}$, and $y_s = 0$ otherwise. $r_s = \overline{V} - \underline{C}, \forall s \in [\alpha, |\Gamma(i^-)| + |\Gamma(t(i^-))| - 1]_{\mathbb{Z}}$ and $r_s = \overline{V} + V - \underline{C}, \forall s \in [|\Gamma(i^-)| + |\Gamma(t(i^-))|, |\mathcal{V}| - 1]_{\mathbb{Z}}$. $w^+_s = w^-_s = 0, \forall s \in [0, |\mathcal{V}| - 1]_{\mathbb{Z}}$.
9. For each $\alpha \in [|\Gamma(i^-)| + |\Gamma(t(i^-))|, |\mathcal{V}| - 1]_{\mathbb{Z}}$, we create four points with $u_s = y_s = 1, \forall s \in [\alpha, |\mathcal{V}| - 1]_{\mathbb{Z}}$, leading to $4|\mathcal{V}| - 4|\Gamma(t(i^-))| - 4|\Gamma(i^-)|$ points in total. In addition, for each α, we assign different values of r_s, w^+_s, and w^-_s to those four points as follows.
   (a) $r_s = \overline{V} - \underline{C}, \forall s \in [\alpha, |\mathcal{V}| - 1]_{\mathbb{Z}}$ and $r_s = 0, \forall s \in [0, \alpha - 1]_{\mathbb{Z}}$. $w^+_s = w^-_s = 0, \forall s \in [0, |\mathcal{V}| - 1]_{\mathbb{Z}}$.
   (b) $r_s = w^-_s = \overline{V} - \underline{C}, \forall s \in [\alpha, |\mathcal{V}| - 1]_{\mathbb{Z}}$ and $r_s = w^-_s = 0, \forall s \in [0, \alpha - 1]_{\mathbb{Z}}$. $w^+_s = 0, \forall s \in [0, |\mathcal{V}| - 1]_{\mathbb{Z}}$.
   (c) $w^+_s = \overline{V} - \underline{C}, \forall s \in [\alpha, |\mathcal{V}| - 1]_{\mathbb{Z}}$ and $w^+_s = 0, \forall s \in [0, \alpha - 1]_{\mathbb{Z}}$. $r_s = w^-_s = 0, \forall s \in [0, |\mathcal{V}| - 1]_{\mathbb{Z}}$.
   (d) $r_s = w^+_s = w^-_s = 0, \forall s \in [0, |\mathcal{V}| - 1]_{\mathbb{Z}}$.

In summary, we have created $5|\mathcal{V}| - 2$ points and they are linearly independent since they can be easily transformed to a lower-triangular matrix by sorting them according to the values of α. □

EC.3.2. Proof for Proposition 4

Proof. The validity proof is trivially similar to that of Online Supplement EC.3.1, so we only present the proof for facet-defining in the following proof for brevity. Similar to Online Supplement

EC.3.1, we create $5|\mathcal{V}|-2$ linearly independent points satisfying inequality (8) at equality and sort these points according to the values of α, which forms a lower-triangular matrix.

Without loss of generality, we assume that node i is the first direct child node of node i^- and j is the second direct child node of node i^-. We create $5|\mathcal{V}|-2$ points as follows.

1. For each $\alpha \in [0, |\Gamma(i)|-1]_{\mathbb{Z}}$, we create four points with $y_s = 1, \forall s \in [0,\alpha]_{\mathbb{Z}}$, and $u_s = 0, \forall s \in [0,|\mathcal{V}|-1]_{\mathbb{Z}}$, leading to $4|\Gamma(i)|$ points in total. In addition, for each $\alpha \in [0, |\Gamma(i^-)|-1]_{\mathbb{Z}}$, we assign different values of r_s, w_s^+, and w_s^- to those four points as follows.
   (a) $r_s = \overline{V} - \underline{C}, \forall s \in [0,\alpha]_{\mathbb{Z}}$ and $r_s = 0, \forall s \in [\alpha+1, |\mathcal{V}|-1]_{\mathbb{Z}}$. $w_s^+ = w_s^- = 0, \forall s \in [0,|\mathcal{V}|-1]_{\mathbb{Z}}$.
   (b) $r_s = w_s^- = \overline{V} - \underline{C}, \forall s \in [0,\alpha]_{\mathbb{Z}}$ and $r_s = w_s^- = 0, \forall s \in [\alpha+1, |\mathcal{V}|-1]_{\mathbb{Z}}$. $w_s^+ = 0, \forall s \in [0,|\mathcal{V}|-1]_{\mathbb{Z}}$.
   (c) $w_s^+ = \overline{V} - \underline{C}, \forall s \in [0,\alpha]_{\mathbb{Z}}$ and $w_s^+ = 0, \forall s \in [\alpha+1, |\mathcal{V}|-1]_{\mathbb{Z}}$. $r_s = w_s^- = 0, \forall s \in [0,|\mathcal{V}|-1]_{\mathbb{Z}}$.
   (d) $r_s = w_s^+ = w_s^- = 0, \forall s \in [0,|\mathcal{V}|-1]_{\mathbb{Z}}$.
2. For $\alpha = |\Gamma(i)|$, we create one point with $y_s = 1, \forall s \in [0,\alpha]_{\mathbb{Z}}$, and $u_s = 0, \forall s \in [0,|\mathcal{V}|-1]_{\mathbb{Z}}$. $r_s = \overline{V} - \underline{C}, \forall s \in [0,\alpha]_{\mathbb{Z}}$ and $r_s = 0, \forall s \in [\alpha+1, |\mathcal{V}|-1]_{\mathbb{Z}}$. $w_s^+ = V$, if $s = \alpha$ and $w_s^+ = 0$ otherwise. $w_s^- = 0, \forall s \in [0,|\mathcal{V}|-1]_{\mathbb{Z}}$.
3. For $\alpha = |\Gamma(i)|+1$, we create one point with $y_s = 1, \forall s \in [0,|\mathcal{V}|-1]_{\mathbb{Z}}$, and $u_s = 0, \forall s \in [0,|\mathcal{V}|-1]_{\mathbb{Z}}$. $r_s = \overline{C} - \underline{C} - V$, if $t(s) = t(i^-)$ and $r_s = \overline{C} - \underline{C} - 2V$ otherwise. $w_s^+ = 2V$, if $s = \alpha$ and $w_s^+ = 0$ otherwise. $w_s^- = 0, \forall s \in [0,|\mathcal{V}|-1]_{\mathbb{Z}}$.
4. For each $\alpha \in [|\Gamma(i)|+2, |\mathcal{V}|-1]_{\mathbb{Z}}$, we create one point with $y_s = 1, \forall s \in [0,|\Gamma(i)|-1]_{\mathbb{Z}} \cup \{\alpha\}$, and $u_s = 0, \forall s \in [0,|\mathcal{V}|-1]_{\mathbb{Z}}$, leading to $|\mathcal{V}| - |\Gamma(i)| - 2$ points in total. In addition, we let $r_s = \overline{V} - \underline{C}, \forall s \in [0,|\Gamma(i)|-1]_{\mathbb{Z}} \cup \{\alpha\}$ and $r_s = 0$ otherwise. $w_s^+ = V$, if $s = \alpha$ and $w_s^+ = 0$ otherwise. $w_s^- = 0, \forall s \in [0,|\mathcal{V}|-1]_{\mathbb{Z}}$.

Now we have constructed $|\mathcal{V}| + 3|\Gamma(i)|$ linearly independent points. Thus, we only need to construct another $4|\mathcal{V}| - 3|\Gamma(i)| - 2$ linearly independent points as follows.

5. For each $\alpha \in [1, |\Gamma(j_k^-)|-1]_{\mathbb{Z}}$, we create one point with $u_\alpha = 1$ and $y_s = 1, r_s = \overline{V} - \underline{C}, \forall s \in \mathcal{H}_L(\alpha)$, and $w_s^+ = w_s^- = 0, \forall s \in [0,|\mathcal{V}|-1]_{\mathbb{Z}}$, leading to $|\Gamma(j_k^-)|-1$ points in total.
6. For each $n \in [0, k-1]_{\mathbb{Z}}$ and each $\alpha \in [|\Gamma(j_k^-)| + \sum_{h=1}^{n} |\Gamma(t(j_{k+1-h}^-))|, |\Gamma(j_k^-)| + \sum_{h=1}^{n+1} |\Gamma(t(j_{k+1-h}^-))| - 1]_{\mathbb{Z}}$, we create one point with $u_s = 1, \forall s \in [\alpha, |\Gamma(j_k^-)| + \sum_{h=1}^{n+1} |\Gamma(t(j_{k+1-h}^-))| - 1]_{\mathbb{Z}}$, leading to $|\Gamma(i^-)| - |\Gamma(j_k^-)|$ points in total. Furthermore, we start up the generator at the child nodes of each node $s \in [|\Gamma(j_k^-)| + \sum_{h=1}^{n} |\Gamma(t(j_{k+1-h}^-))|, \alpha - 1]$, i.e., $u_s = 1, \forall s \in [|\Gamma(j_k^-)| + \sum_{h=1}^{n+1} |\Gamma(t(j_{k+1-h}^-))|, |\Gamma(j_k^-)| + \sum_{h=1}^{n+2} |\Gamma(t(j_{k+1-h}^-))| - 1]_{\mathbb{Z}} \setminus \cup_{m=\alpha}^{|\Gamma(j_k^-)| + \sum_{h=1}^{n+1} |\Gamma(t(j_{k+1-h}^-))| - 1} \mathcal{H}_L(m)$. And $y_s = 1, \forall s \in [\alpha, |\mathcal{V}|-1]_{\mathbb{Z}}$. $r_s = \overline{V} - \underline{C} + (t(s) - t(\alpha))V$, if $s \in \mathcal{C}(\hat{s}), \forall \hat{s} \in [\alpha, |\Gamma(i_{L-1}^-)| + \sum_{h=1}^{k+1} |\Gamma(t(i_{L-h}^-))| - 1]_{\mathbb{Z}}$ and $r_s = \overline{V} - \underline{C} + (t(s) - t(\alpha) - 1)V$, if $s \in \mathcal{C}(\hat{s}), \forall \hat{s} \in [|\Gamma(i_{L-1}^-)| + \sum_{h=1}^{k} |\Gamma(t(i_{L-h}^-))|, \alpha - 1]_{\mathbb{Z}}$. $w_s^+ = w_s^- = 0, \forall s \in [0,|\mathcal{V}|-1]_{\mathbb{Z}}$.

7. For $\alpha = |\Gamma(i)|$, we create three points with $u_\alpha = 1$ and $y_\alpha = 1$, leading to three points in total. We assign different values of r_s, w_s^+, and w_s^- to those three points as follows.
   (a) $r_\alpha = \overline{V} - \underline{C}$ and $w_s^+ = w_s^- = 0, \forall s \in [0, |\mathcal{V}|-1]_{\mathbb{Z}}$.
   (b) $r_\alpha = w_\alpha^- = \overline{V} - \underline{C}$ and $w_s^+ = 0, \forall s \in [0, |\mathcal{V}|-1]_{\mathbb{Z}}$.
   (c) $w_\alpha^+ = \overline{V} - \underline{C}$ and $r_s = w_s^- = 0, \forall s \in [0, |\mathcal{V}|-1]_{\mathbb{Z}}$.
8. For each $\alpha \in [|\Gamma(i)|+1, |\mathcal{V}|-1]_{\mathbb{Z}}$, we create four points with $u_\alpha = 1$ and $y_\alpha = 1$, leading to $4|\mathcal{V}| - 4|\Gamma(i)| - 4$ points in total. In addition, for each α, we assign different values of r_s, w_s^+, and w_s^- to those four points as follows.
   (a) $r_\alpha = \overline{V} - \underline{C}$ and $w_s^+ = w_s^- = 0, \forall s \in [0, |\mathcal{V}|-1]_{\mathbb{Z}}$.
   (b) $r_\alpha = w_\alpha^- = \overline{V} - \underline{C}$ and $w_s^+ = 0, \forall s \in [0, |\mathcal{V}|-1]_{\mathbb{Z}}$.
   (c) $w_\alpha^+ = \overline{V} - \underline{C}$ and $r_s = w_s^- = 0, \forall s \in [0, |\mathcal{V}|-1]_{\mathbb{Z}}$.
   (d) $r_s = w_s^+ = w_s^- = 0, \forall s \in [0, |\mathcal{V}|-1]_{\mathbb{Z}}$.

In summary, we have created $5|\mathcal{V}| - 2$ points and they are linearly independent since they can be easily transformed to a lower-triangular matrix by sorting them according to the values of α. □

EC.3.3. Proof for Proposition 6

Proof. The validity proof is trivially similar to that of Online Supplement EC.3.1, so we only present the facet-defining proof as follows for brevity. Similar to Online Supplement EC.3.1, we create $5|\mathcal{V}| - 2$ linearly independent points satisfying inequality (10) at equality and sort these points according to the values of α, which forms a lower-triangular matrix. For notational brevity, we define $\beta = \min\{k-1, L-1\}$. Without loss of generality, we assume that node i is the first node of set $\mathcal{N}$.

First, we present $|\mathcal{V}| + 3|\Gamma(i)|$ points as follows, where each α represents the time period when the generator shuts down.

1. For $\alpha \in [0, |\Gamma(i)|-1]_{\mathbb{Z}}$, we create four points with $y_s = 1, \forall s \in [0, \alpha]_{\mathbb{Z}}$, $y_s = 0$ otherwise, and $u_s = 0, \forall s \in [0, |\mathcal{V}|-1]_{\mathbb{Z}}$, leading to $4|\Gamma(i)|$ points in total. In addition, for each $\alpha \in [0, |\Gamma(i)|-1]_{\mathbb{Z}}$, we assign different values of r_s, w_s^+, and w_s^- to those four points as follows.
   (a) $r_s = \overline{V} - \underline{C}, \forall s \in [0, \alpha]_{\mathbb{Z}}$ and $r_s = 0, \forall s \in [\alpha+1, |\mathcal{V}|-1]_{\mathbb{Z}}$. $w_s^+ = w_s^- = 0, \forall s \in [0, |\mathcal{V}|-1]_{\mathbb{Z}}$.
   (b) $r_s = w_s^- = \overline{V} - \underline{C}, \forall s \in [0, \alpha]_{\mathbb{Z}}$ and $r_s = w_s^- = 0, \forall s \in [\alpha+1, |\mathcal{V}|-1]_{\mathbb{Z}}$. $w_s^+ = 0, \forall s \in [0, |\mathcal{V}|-1]_{\mathbb{Z}}$.
   (c) $w_s^+ = \overline{V} - \underline{C}, \forall s \in [0, \alpha]_{\mathbb{Z}}$ and $w_s^+ = 0, \forall s \in [\alpha+1, |\mathcal{V}|-1]_{\mathbb{Z}}$. $r_s = w_s^- = 0, \forall s \in [0, |\mathcal{V}|-1]_{\mathbb{Z}}$.
   (d) $r_s = w_s^+ = w_s^- = 0, \forall s \in [0, |\mathcal{V}|-1]_{\mathbb{Z}}$.

   Note here that these four points are obviously linearly independent, and for each α, since we have $y_s = 1, \forall s \in [0, \alpha]_{\mathbb{Z}}$, we can conclude that these $4|\Gamma(i)|$ points are linearly independent.

2. For $\alpha = |\Gamma(i)|$, we create one point with $y_s = 1, \forall s \in [0, |\mathcal{V}|-1]_{\mathbb{Z}}$, and $u_s = 0, \forall s \in [0, |\mathcal{V}|-1]_{\mathbb{Z}}$. We assign values of r_s, w_s^+, and w_s^- as follows.

$$r_s = \begin{cases} \overline{V} - \underline{C}, & s \in [0, |\Gamma(i_k^-)|]_{\mathbb{Z}} \\ \overline{V} - \underline{C} + (t(s) - t(i_k^-))V, & s \in [|\Gamma(i_{k-1}^-)|, |\Gamma(i^-)|]_{\mathbb{Z}} \\ \overline{V} - \underline{C} + kV, & s \in [|\Gamma(i)|, |\mathcal{V}|-1]_{\mathbb{Z}} \end{cases}$$

and $w_s^+ = w_s^- = 0, \forall s \in [0, |\mathcal{V}|-1]_{\mathbb{Z}}$.

3. For $\alpha \in [|\Gamma(i)|+1, |\mathcal{V}|-1]_{\mathbb{Z}}$, we create one point with $y_s = 1, \forall s \in [0, |\Gamma(i)|-1]_{\mathbb{Z}} \cup [|\Gamma(i)|+1, \alpha]_{\mathbb{Z}}$, $y_s = 0$ otherwise, and $u_s = 0, \forall s \in [0, |\mathcal{V}|-1]_{\mathbb{Z}}$, leading to $|\mathcal{V}| - |\Gamma(i)| - 1$ points in total. In addition, for each α, we let $r_s = 0, \forall s \in [0, \alpha-1]_{\mathbb{Z}}$, $r_s = \overline{V} - \underline{C}, \forall s \in [\alpha, |\mathcal{V}|-1]_{\mathbb{Z}}$ and $w_s^+ = w_s^- = 0, \forall s \in [0, |\mathcal{V}|-1]_{\mathbb{Z}}$.

Now we have constructed $|\mathcal{V}| + 3|\Gamma(i)|$ linearly independent points. Thus, we only need to construct another $4|\mathcal{V}| - 3|\Gamma(i)| - 2$ linearly independent points as follows, where each α represents the time period when the generator starts up.

4. For each $\alpha \in [1, |\Gamma(i_\beta^-)|-1]_{\mathbb{Z}}$, we consider two possible cases in terms of β.
   (a) If $\beta = \min\{k-1, L-1\} = k-1$, then we create one point with $y_s = 1, \forall s \in [\alpha, |\mathcal{V}|-1]_{\mathbb{Z}}$, $y_s = 0, \forall s \in [0, \alpha-1]_{\mathbb{Z}}$, and $u_\alpha = 1, u_s = 0, \forall s \neq \alpha$, leading to $|\Gamma(i_\beta^-)| - 1$ points in total. In addition, for each α, we assign values of r_s, w_s^+, and w_s^- as follows.

$$r_s = \begin{cases} 0, & s \in [0, \alpha-1]_{\mathbb{Z}} \\ \overline{V} - \underline{C}, & s \in [\alpha, |\Gamma(i_k^-)|]_{\mathbb{Z}} \\ \overline{V} - \underline{C} + (t(s) - t(i_k^-))V, & s \in [|\Gamma(i_k^-)|+1, |\Gamma(i^-)|]_{\mathbb{Z}} \\ \overline{V} - \underline{C} + kV, & s \in [|\Gamma(i)|, |\mathcal{V}|-1]_{\mathbb{Z}} \end{cases}$$

   and $w_s^+ = w_s^- = 0, \forall s \in [0, |\mathcal{V}|-1]_{\mathbb{Z}}$.
   (b) If $\beta = \min\{k-1, L-1\} = L-1$, then we create one point with $y_s = 1, \forall s \in [\alpha, \alpha+L-1]_{\mathbb{Z}}$, $y_s = 0, \forall s \in [0, \alpha-1]_{\mathbb{Z}} \cup [\alpha+L, |\mathcal{V}|-1]_{\mathbb{Z}}$, and $u_\alpha = 1, u_s = 0, \forall s \neq \alpha$, leading to $|\Gamma(i_\beta^-)| - 1$ points in total. In addition, for each α, we let $r_s = \overline{V} - \underline{C}, \forall s \in [\alpha, \alpha+L-1]_{\mathbb{Z}} \setminus \{i_k^-\}$ and $r_s = 0$ otherwise. $w_s^+ = \overline{V} - \underline{C}$, if $\alpha = i_k^-$, and $w_s^+ = 0$ otherwise. $w_s^- = 0, \forall s \in [0, |\mathcal{V}|-1]_{\mathbb{Z}}$.
5. For each $\alpha \in [|\Gamma(i_\beta^-)|, |\Gamma(i)|-1]_{\mathbb{Z}}$, we create one point with $y_s = 1, \forall s \in [\alpha, |\mathcal{V}|-1]_{\mathbb{Z}}$, $y_s = 0, \forall s \in [0, \alpha-1]_{\mathbb{Z}}$, and $u_\alpha = 1, u_s = 0, \forall s \neq \alpha$, leading to $|\Gamma(i)| - |\Gamma(i_\beta^-)|$ points in total. In addition, for each α, we let $r_s = 0, \forall s \in [0, \alpha-1]_{\mathbb{Z}}$, $r_s = \overline{V} - \underline{C} + (t(s) - t(\alpha))V, \forall s \in [\alpha, |\mathcal{V}|-1]_{\mathbb{Z}}$, and $w_s^+ = w_s^- = 0, \forall s \in [0, |\mathcal{V}|-1]_{\mathbb{Z}}$.
6. For $\alpha = |\Gamma(i)|$, we create three points with $u_\alpha = y_\alpha = 1$, and $y_s = u_s = 0, \forall s \neq \alpha$, leading to three points in total. We assign different values of r_s, w_s^+, and w_s^- to those three points as follows.
   (a) $r_\alpha = \overline{V} - \underline{C}$ and $w_s^+ = w_s^- = 0, \forall s \in [0, |\mathcal{V}|-1]_{\mathbb{Z}}$.
   (b) $r_\alpha = w_\alpha^- = \overline{V} - \underline{C}$ and $w_s^+ = 0, \forall s \in [0, |\mathcal{V}|-1]_{\mathbb{Z}}$.

(c) $w^+_\alpha = \overline{V} - \underline{C}$ and $r_s = w^-_s = 0, \forall s \in [0, |\mathcal{V}|-1]_{\mathbb{Z}}$.

7. For each $\alpha \in [|\Gamma(i)|+1, |\mathcal{V}|-1]_{\mathbb{Z}}$, we create four points with $u_\alpha = y_\alpha = 1$, and $y_s = u_s = 0, \forall s \neq \alpha$, leading to $4|\mathcal{V}| - 4|\Gamma(i)| - 4$ points in total. In addition, for each α, we assign different values of r_s, w^+_s, and w^-_s to those four points as follows.
   (a) $r_\alpha = \overline{V} - \underline{C}$ and $w^+_s = w^-_s = 0, \forall s \in [0, |\mathcal{V}|-1]_{\mathbb{Z}}$.
   (b) $r_\alpha = w^-_\alpha = \overline{V} - \underline{C}$ and $w^+_s = 0, \forall s \in [0, |\mathcal{V}|-1]_{\mathbb{Z}}$.
   (c) $w^+_\alpha = \overline{V} - \underline{C}$ and $r_s = w^-_s = 0, \forall s \in [0, |\mathcal{V}|-1]_{\mathbb{Z}}$.
   (d) $r_s = w^+_s = w^-_s = 0, \forall s \in [0, |\mathcal{V}|-1]_{\mathbb{Z}}$.

In summary, we have created $5|\mathcal{V}|-2$ points and they are linearly independent since they can be easily transformed to a lower-triangular matrix by sorting them according to the values of α. □

EC.3.4. Proof for Proposition 8

Proof. Without loss of generality, we assume that node p is the shared ancestor node of nodes i and j at the largest time period, i.e., $p = \text{argmax}\{t(n) : n \in \mathcal{P}(i) \cap \mathcal{P}(j)\}$, and we denote the distances by $dist(i,p) \equiv k_1$ and $dist(j,p) \equiv k_2$, where $k = k_1 + k_2$. We let d be a leaf node on the path such that $j \in \mathcal{P}(p,d)$. Here we assume that $k_1 \geq k_2$, since the other case where $k_1 < k_2$ can be proved similarly and thus is omitted.

Note that due to the symmetry of the scenario tree, we only need to prove inequality (12) is facet-defining for conv($\bar{P}$), where $\bar{P}$ is constructed with the same constraints in P that are applied to the scenario structure $\bar{\mathcal{V}}$ in Figure EC.3. To simplify the process of creating linearly independent points, we re-index the nodes in $\bar{\mathcal{V}}$ as follows:

(1) For nodes $0, \ldots, i^-_{k+1}, i^-_k, \ldots, i^-_{k_1+1}, p$, we re-label them as $0, \ldots, n-k_2-1, n-k_2, \ldots, n-1, n$, where $n = t(p) - 1$.
(2) For nodes $j^-_{k_2-1}, \ldots, j, \ldots, d$, we re-label them as $n+1, \ldots, n+k_2, \ldots, n+k_1$.
(3) For nodes $i^-_{k_1-1}, \ldots, i$, we re-label them as $n+k_1+1, \ldots, n+2k_1$.

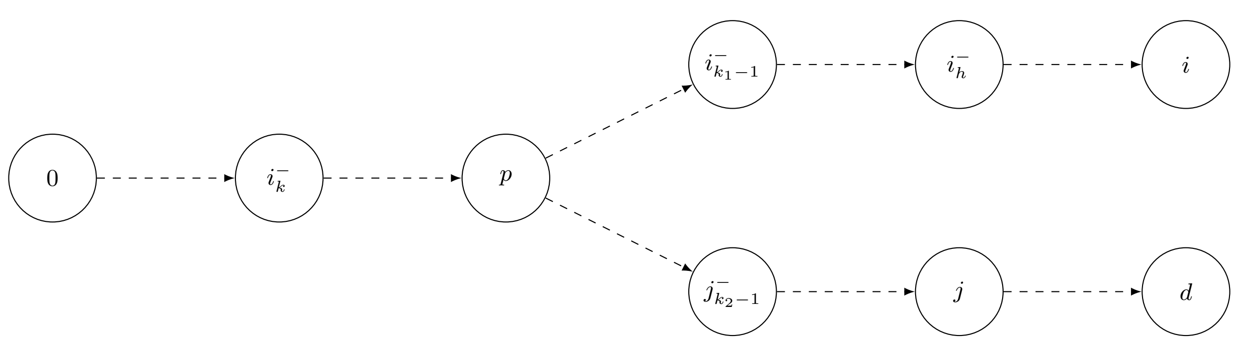

Figure EC.3 Re-Indexed Scenario Tree

To show that inequality (12) is facet-defining for $\mathrm{conv}(\bar{P})$, we will create $5(n+2k_1+1)-1$ linearly independent points while satisfying inequality (12) at equality. Since $\vec{0} \in \mathrm{conv}(\bar{P})$, we only need to create $5n+10k_1+3$ points as follows.

1. For each $\alpha \in [0, n+k_2-1]_{\mathbb{Z}}$, we create four points with $y_s = 1, \forall s \in [0,\alpha]_{\mathbb{Z}}$, and $u_s = 0, \forall s \in [0, 2T-n]_{\mathbb{Z}}$, leading to $4n+4k_2$ points in total. In addition, for each $\alpha \in [0, n+k_2-1]_{\mathbb{Z}}$, we assign different values of r_s, w_s^+, and w_s^- to those four points as follows.
   (a) $r_s = \overline{V} - \underline{C}, \forall s \in [0,\alpha]_{\mathbb{Z}}$ and $r_s = 0, \forall s \in [\alpha+1, n+2k_1]_{\mathbb{Z}}$. $w_s^+ = w_s^- = 0, \forall s \in [0, n+2k_1]_{\mathbb{Z}}$.
   (b) $r_s = w_s^- = \overline{V} - \underline{C}, \forall s \in [0,\alpha]_{\mathbb{Z}}$ and $r_s = w_s^- = 0, \forall s \in [\alpha+1, n+2k_1]_{\mathbb{Z}}$. $w_s^+ = 0, \forall s \in [0, n+2k_1]_{\mathbb{Z}}$.
   (c) $w_s^+ = \overline{V} - \underline{C}, \forall s \in [0,\alpha]_{\mathbb{Z}}$ and $w_s^+ = 0, \forall s \in [\alpha+1, n+2k_1]_{\mathbb{Z}}$. $r_s = w_s^- = 0, \forall s \in [0, n+2k_1]_{\mathbb{Z}}$.
   (d) $r_s = w_s^+ = w_s^- = 0, \forall s \in [0, n+2k_1]_{\mathbb{Z}}$.
2. For $\alpha = n+k_2$, we create three points with $y_s = 1, \forall s \in [0,\alpha]_{\mathbb{Z}}$ and $u_s = 0, \forall s \in [0, n+2k_1]_{\mathbb{Z}}$, leading to three points in total. In addition, we assign different values of r_s, w_s^+, and w_s^- to those three points as follows.
   (a) $r_s = w_s^- = \overline{V} - \underline{C}, \forall s \in [0,\alpha]_{\mathbb{Z}}$ and $r_s = w_s^- = 0, \forall s \in [\alpha+1, n+2k_1]_{\mathbb{Z}}$. $w_s^+ = 0, \forall s \in [0, n+2k_1]_{\mathbb{Z}}$.
   (b) $w_s^+ = \overline{V} - \underline{C}, \forall s \in [0,\alpha]_{\mathbb{Z}}$ and $w_s^+ = 0, \forall s \in [\alpha+1, n+2k_1]_{\mathbb{Z}}$. $r_s = w_s^- = 0, \forall s \in [0, n+2k_1]_{\mathbb{Z}}$.
   (c) $r_s = w_s^+ = w_s^- = 0, \forall s \in [0, n+2k_1]_{\mathbb{Z}}$.
3. For each $\alpha \in [n+k_2+1, n+2k_1-1]_{\mathbb{Z}}$, we create four points with $y_s = 1, \forall s \in [0,\alpha]_{\mathbb{Z}}$ and $u_s = 0, \forall s \in [0, n+2k_1]_{\mathbb{Z}}$, leading to $8k_1 - 4k_2 - 4$ points in total. In addition, for each α, we assign different values of r_s, w_s^+, and w_s^- to those four points as follows.
   (a) $r_s = \overline{V} - \underline{C}, \forall s \in [0, n+k_2-1]_{\mathbb{Z}} \cup [n+k_2+1, \alpha]_{\mathbb{Z}}$, and $r_s = 0$ otherwise. $w_s^+ = w_s^- = 0, \forall s \in [0, n+2k_1]_{\mathbb{Z}}$.
   (b) $r_s = w_s^- = \overline{V} - \underline{C}, \forall s \in [0,\alpha]_{\mathbb{Z}}$ and $r_s = w_s^- = 0, \forall s \in [\alpha+1, n+2k_1]_{\mathbb{Z}}$. $w_s^+ = 0, \forall s \in [0, n+2k_1]_{\mathbb{Z}}$.
   (c) $w_s^+ = \overline{V} - \underline{C}, \forall s \in [0,\alpha]_{\mathbb{Z}}$ and $w_s^+ = 0, \forall s \in [\alpha+1, n+2k_1]_{\mathbb{Z}}$. $r_s = w_s^- = 0, \forall s \in [0, n+2k_1]_{\mathbb{Z}}$.
   (d) $r_s = w_s^+ = w_s^- = 0, \forall s \in [0, n+2k_1]_{\mathbb{Z}}$.
4. For $\alpha = n+2k_1$, we create three points with $y_s = 1, u_s = 0, \forall s \in [0, n+2k_1]_{\mathbb{Z}}$. In addition, we assign different values of r_s, w_s^+, and w_s^- to those three points as follows.
   (a) $r_s = \overline{V} + k_2 V - \underline{C}, \forall s \in [0,n]_{\mathbb{Z}}$, $r_s = \overline{V} + (k_2 - s + n)V - \underline{C}, \forall s \in [n+1, n+k_2]_{\mathbb{Z}}$, $r_s = \overline{V} + (s - n - k_1 + k_2)V - \underline{C}, \forall s \in [n+k_1+1, n+2k_1]_{\mathbb{Z}}$ and $r_s = 0$ otherwise. $w_s^+ = w_s^- = 0, \forall s \in [0, n+2k_1]_{\mathbb{Z}}$.
   (b) $r_s = \overline{V} + k_2 V - \underline{C}, \forall s \in [0,n]_{\mathbb{Z}}$, $r_s = \overline{V} + (k_2 - s + n)V - \underline{C}, \forall s \in [n+1, n+k_2]_{\mathbb{Z}}$, $r_s = \overline{V} + (s - n - k_1 + k_2)V - \underline{C}, \forall s \in [n+k_1+1, n+2k_1-1]_{\mathbb{Z}}$, $r_{n+2k_1} = \overline{V} + (2k_1 - 1 + k_2)V - \underline{C}$ and $r_s = 0$ otherwise. $w_{n+2k_1}^+ = V$ and $w_s^+ = 0$ otherwise. $w_s^- = 0, \forall s \in [0, n+2k_1]_{\mathbb{Z}}$.

(c) $r_s = k_2V, \forall s \in [0,n]_{\mathbb{Z}}$, $r_s = (k_2 - s + n)V, \forall s \in [n+1, n+k_2]_{\mathbb{Z}}$, $r_s = (s - n - k_1 + k_2)V, \forall s \in [n+k_1+1, n+2k_1]_{\mathbb{Z}}$ and $r_s = 0$ otherwise. $w_s^+ = w_s^- = 0, \forall s \in [0, n+2k_1]_{\mathbb{Z}}$.

Now we have constructed $4n + 8k_1 + 2$ linearly independent points. Thus, we only need to create another $n + 2k_1 + 1$ linearly independent points and sort them according to the values of u_s. Here we assume $\min\{L-1, k-1\} \geq k_1$, otherwise the proof can be completed easily. Furthermore, we assume $k - 1 \leq L - 1$, i.e., $k \leq L$, since the case in which $L \leq k$ can be proved similarly.

5. For each $\alpha \in [1, n+k_1-L]_{\mathbb{Z}}$, we create one point with $u_\alpha = 1$ and $y_s = 1, r_s = w_s^- = \overline{V} - \underline{C}, \forall s \in [\alpha, \alpha+L-1]_{\mathbb{Z}} \cup [n+k_1+1, \alpha+L-1+k_1]_{\mathbb{Z}}$. $w_s^+ = 0, \forall s \in [0, n+2k_1]_{\mathbb{Z}}$, leading to $n + k_1 - L$ points in total.
6. For each $\alpha \in [n+k_1-L+1, n-k_2]_{\mathbb{Z}}$, we create one point with $u_\alpha = 1$ and $y_s = 1, \forall s \in [\alpha, n+2k_1]_{\mathbb{Z}}$. $r_s = \overline{V} + (s-\alpha)V - \underline{C}, \forall s \in [\alpha, \alpha+k_2]_{\mathbb{Z}}$, $r_s = \overline{V} + k_2V - \underline{C}, \forall s \in [\alpha+k_2+1, n]_{\mathbb{Z}}$, $r_s = \overline{V} + (k_2 - s + n)V - \underline{C}, \forall s \in [n+1, n+k_2]_{\mathbb{Z}}$, $r_s = \overline{V} + (k_2 + s - n - k_1)V - \underline{C}, \forall s \in [n+k_1+1, n+2k_1]_{\mathbb{Z}}$ and $r_s = 0$ otherwise. $w_s^+ = w_s^- = 0, \forall s \in [0, n+2k_1]_{\mathbb{Z}}$, leading to $L - k_1 - k_2$ points in total.
7. For $\alpha = n - k_2$, we create one point with $u_\alpha = 1$ and $y_s = 1, \forall s \in [\alpha, n+2k_1]_{\mathbb{Z}}$. $r_s = \overline{V} + (s-\alpha)V - \underline{C}, \forall s \in [\alpha, n]_{\mathbb{Z}}$, $r_s = \overline{V} + (k_2 - s + n)V - \underline{C}, \forall s \in [n+1, 2n-\alpha]_{\mathbb{Z}}$, $r_s = \overline{V} + (s - k_1 - \alpha)V - \underline{C}, \forall s \in [n+k_1+1, n+2k_1-1]_{\mathbb{Z}}$, $r_{n+2k_1} = \overline{V} + (k-2)V - \underline{C}$ and $r_s = 0$ otherwise. $w_{n+2k_1}^+ = 2V$ and $w_s^+ = 0$ otherwise. $w_s^- = 0, \forall s \in [0, n+2k_1]_{\mathbb{Z}}$.
8. For each $\alpha \in [n-k_2+1, n]_{\mathbb{Z}}$, we create one point with $u_\alpha = 1$ and $y_s = 1, \forall s \in [\alpha, n+2k_1]_{\mathbb{Z}}$. $r_s = \overline{V} + (s-\alpha)V - \underline{C}, \forall s \in [\alpha, n]_{\mathbb{Z}}$, $r_s = \overline{V} + (k_2 - s + n)V - \underline{C}, \forall s \in [n+1, 2n-\alpha]_{\mathbb{Z}}$, $r_s = \overline{V} + (s - k_1 - \alpha)V - \underline{C}, \forall s \in [n+k_1+1, n+2k_1]_{\mathbb{Z}}$ and $r_s = 0$ otherwise. $w_s^+ = w_s^- = 0, \forall s \in [0, n+2k_1]_{\mathbb{Z}}$, leading to k_2 points in total.
9. For each $\alpha \in [n+1, n+k_1]_{\mathbb{Z}}$, we create one point with $u_\alpha = 1$ and $y_s = 1, \forall s \in [\alpha, n+2k_1]_{\mathbb{Z}}$. $r_s = w_s^- = \overline{V} - \underline{C}, \forall s \in [\alpha, n+k_1]_{\mathbb{Z}}$ and $r_s = w_s^- = 0$ otherwise. $w_s^+ = 0, \forall s \in [0, n+2k_1]_{\mathbb{Z}}$, leading to k_1 points in total.
10. For each $\alpha \in [n+k_1+1, n+2k_1]_{\mathbb{Z}}$, we create one point with $u_\alpha = 1$ and $y_s = 1, \forall s \in [\alpha, n+2k_1]_{\mathbb{Z}}$. $r_s = \overline{V} + (s-\alpha)V - \underline{C}, \forall s \in [\alpha, n+2k_1]_{\mathbb{Z}}$ and $r_s = 0$ otherwise. $w_s^+ = w_s^- = 0, \forall s \in [0, n+2k_1]_{\mathbb{Z}}$, leading to k_1 points in total.

In summary, we have created $5n + 10k_1 + 3$ points and they are linearly independent since they can be easily transformed to a lower-triangular matrix by sorting them according to the values of α. □

EC.3.5. Proof for Proposition 12

Proof. **(Validity)** We show that inequality (17) is valid for conv(P) with $\psi(y,u) = (\underline{C} + V - \overline{V})(y_j - \sum_{m=0}^{L-1} u_{j_m^-})$ and the case $\psi(y,u) = (\underline{C} + V - \overline{V})(y_i - \sum_{m=0}^{L-1} u_{i_m^-})$ is similar and thus is omitted here. We can observe that inequality (17) is obviously valid when $y_{i_k^-} = 0$, so we focus on $y_{i_k^-} = 1$ in the following discussion.

First, we consider the case where $y_j = 0$, i.e., $r_j = w_j^- = 0$. We denote the latest start-up node by i_{k+s}^- where $s \geq 0$ and then we discuss the following two possible cases in terms of the value of s.

1) If $s \geq L-1$, then $\phi = 0$ and $i_{k+s-L+1}^- \in \mathcal{P}(i_k^-)$. Now we need to consider when the generator will shut down. We denote the shut-down node by h and thus we have $h \notin \mathcal{P}(i_k^-)$. We further discuss three possible cases of h as follows.
   1.1) If $h \in \mathcal{P}(i_{k-\hat{n}}^-)$, then we can let $h = i_{k-\alpha}^-$ for some $\alpha \in [1, \hat{n}]_{\mathbb{Z}}$ and $r_{i_k^-} \leq \overline{V} - \underline{C} + \min\{k-1, s, \alpha-1\}V \leq \overline{V} - \underline{C} + (\alpha-1)V \leq \overline{V} - \underline{C} + V\sum_{n=0}^{\alpha-1}(y_{i_{k-n}^-} - \sum_{m=0}^{\min\{L-1,n+\omega\}} u_{i_{k-n+m}^-})$, which is less than the RHS of inequality (17).
   1.2) If $h \in \mathcal{P}(i) \setminus \mathcal{P}(i_{k-\hat{n}}^-)$, then we can have $h = i_{k-\alpha}^-$ for some $\alpha \in [\hat{n}+1, k]_{\mathbb{Z}}$ and $r_{i_k^-} \leq \overline{V} - \underline{C} + \min\{k-1, s, \alpha-1\}V \leq \overline{V} - \underline{C} + (\alpha-1)V \leq \overline{V} - \underline{C} + V\sum_{n\in S_0}(y_{i_{k-n}^-} - \sum_{m=0}^{\min\{L-1,n+\omega\}} u_{i_{k-n+m}^-}) + V\sum_{n\in S\cup\{\hat{n}\}}(g_n - n)(y_{i_{k-n}^-} - \sum_{m=0}^{L-1} u_{i_{k-n+m}^-})$ indicating that inequality (17) is valid.
   1.3) If $h \in \mathcal{V}(i) \setminus \mathcal{P}(i)$, then inequality (17) becomes $r_{i_k^-} \leq kV$, which is also valid.
2) If $s \in [0, L-2]_{\mathbb{Z}}$, then $i_{k+s-L+1}^- \in \mathcal{V}(i_k^-) \setminus \mathcal{P}(i_k^-)$ and $y_{i_{k-n}^-} - \sum_{m=0}^{\min\{L-1,n+\omega\}} u_{i_{k-n+m}^-} = 0$ for each $n \in [1, L-1-s]_{\mathbb{Z}}$. Similarly, we let the shut-down node be h and discuss the following two possible cases in terms of the value of s.
   2.1) If $L-1-s \in [1, \hat{n}-1]_{\mathbb{Z}}$, then we discuss three possible cases in terms of the value of h as follows.
      2.1.1) If $h \in \mathcal{P}(i_{k-\hat{n}^-})$, then we let $h = i_{k-\alpha}^-$ for some $\alpha \in [L-1-s, \hat{n}]_{\mathbb{Z}}$ and similar to previous discussion in Subcase 1.1), we have $r_{i_k^-} \leq \overline{V} - \underline{C} + \min\{k-1, s, \alpha-1\}V \leq \overline{V} - \underline{C} + \min\{s, \alpha-1\}V \leq \overline{V} - \underline{C} + (\alpha-1-(L-1-s))V + \phi$ where $\phi = sV$ or $(L-1-s)V$ indicating that inequality (17) is valid.
      2.1.2) If $h \in \mathcal{P}(i) \setminus \mathcal{P}(i_{k-\hat{n}}^-)$, then we can have $h = i_{k-\alpha}^-$ for some $\alpha \in [\hat{n}+1, k]_{\mathbb{Z}}$ and $r_{i_k^-} \leq \overline{V} - \underline{C} + \min\{k-1, s, \alpha-1\}V$. Inequality (17) now becomes $r_{i_k^-} \leq \overline{V} - \underline{C} + (\hat{n}-1-(L-1-s))V + V\sum_{n\in(S\cup\{\hat{n}\})\cap[\hat{n},\alpha-1]_{\mathbb{Z}}}(g_n - n) + \phi$. Next, if $\phi = sV$, then clearly $r_{i_k^-} \leq \overline{V} - \underline{C} + \min\{k-1, s, \alpha-1\}V \leq \overline{V} - \underline{C} + (\hat{n}-1-(L-1-s))V + V\sum_{n\in(S\cup\{\hat{n}\})\cap[\hat{n},\alpha-1]_{\mathbb{Z}}}(g_n - n) + \phi$. If $\phi = (L-1-s)V$, then inequality (17) converts to $r_{i_k^-} \leq \overline{V} - \underline{C} + (\hat{n}-1)V + V\sum_{n\in(S\cup\{\hat{n}\})\cap[\hat{n},\alpha-1]_{\mathbb{Z}}}(g_n - n)$, which is also greater than $\overline{V} - \underline{C} + \min\{k-1, s, \alpha-1\}V$.
      2.1.3) If $h \in \mathcal{V}(i) \setminus \mathcal{P}(i)$, then we have $r_{i_k^-} \leq \overline{V} - \underline{C} + \min\{k-1, s, \alpha-1\}V$. Inequality (17) converts to $r_{i_k^-} \leq \overline{V} - \underline{C} + (\hat{n}-1-(L-1-s))V + V\sum_{n\in(S\cup\{\hat{n}\})}(g_n - n) + \phi = \overline{V} - \underline{C} + (\hat{n}-1-(L-1-s))V + (k-\hat{n})V + \phi$. So we only need to show that

$$r_{i_k^-} \leq \overline{V} - \underline{C} + \min\{k-1, s, \alpha-1\}V \leq \overline{V} - \underline{C} + (\hat{n}-1-(L-1-s))V + (k-\hat{n})V + \phi \tag{EC.9}$$

holds. Next, if $\phi = sV$, then clearly $r_{i_k^-} \leq \overline{V} - \underline{C} + \min\{k-1, s, \alpha-1\}V \leq \overline{V} - \underline{C} + (\hat{n} - 1 - (L-1-s))V + (k-\hat{n})V + \phi$, indicating (EC.9) holds. If $\phi = (L-1-s)V$, then inequality (17) converts to $r_{i_k^-} \leq \overline{V} - \underline{C} + (k-1)V$ which is clearly greater than $\overline{V} - \underline{C} + \min\{k-1, s, \alpha-1\}V$.

2.2) If $L-1-s \geq \hat{n}$, then it follows that $s \leq L-1-\hat{n} \leq L-2$. Here we assume $L-1-s < k$, as the other case in which $L-1-s \geq k$, indicating $i_{k+s-L+1}^- \in \mathcal{V}(i) \setminus \mathcal{P}(i)$ and thus $y_{i_{k-n}^-} - \sum_{m=0}^{\min\{L-1,n+\omega\}} u_{i_{k-n+m}^-} = 0$ for each $n \in [1,k]_{\mathbb{Z}}$, is easier to prove and thus is omitted here. Similar to the proof in 2.1), we also have $r_{i_k^-} \leq \overline{V} - \underline{C} + \min\{k-1, s, \alpha-1\}V$ and we only need to show that $r_{i_k^-} \leq \overline{V} - \underline{C} + \min\{k-1, s, \alpha-1\}V \leq \overline{V} - \underline{C} + V\sum_{n\in(S\cup\{\hat{n}\})\cap[0,L-s]_{\mathbb{Z}}}(g_n - n) + \phi$ holds. Next, we show that $\phi = sV$, where inequality (17) is clearly valid. Otherwise we have $\phi = (L-1-s)V$, then it follows that $L-1-s \leq s-1$, i.e., $2s \geq L$, which will not hold due to the following discussions.

i. If $t(i_k^-) \geq L$, then $\min\{t(i_k^-) - 2, L-2\} = L-2 \geq L/2$, where the last inequality holds because of $L/2 \leq s \leq L-2$, and it follows by the definition of $\hat{n}$ that $\hat{n} = L-2$. However, note that $L-1-s \leq L-1-L/2 = L/2-1 \leq L-2-1 = \hat{n}-1$ since $s \geq L/2$, which contradicts to the condition $L-1-s \geq \hat{n}$.

ii. If $t(i_k^-) \leq L-1$, then $t(i_{k+s-L+1}^-) = t(i_k^-) + L-1-s \leq 2(L-1) - s \leq -2$ since $s \geq L/2$, which contradicts to $L-1-s \geq \hat{n}$.

Therefore, $\phi = sV$ and $r_{i_k^-} \leq \overline{V} - \underline{C} + \min\{k-1, s, \alpha-1\}V \leq \overline{V} - \underline{C} + V\sum_{n\in(S\cup\{\hat{n}\})\cap[0,L-s]_{\mathbb{Z}}}(g_n - n) + \phi$ holds obviously.

Next, we consider the case in which $y_j = 1$. Let p be the shared ancestor node of i_k^- and j at the largest time period, i.e., $p = \operatorname{argmax}\{t(s) : s \in \mathcal{P}(i_k^-) \cap \mathcal{P}(j)\} = i_{k+k_1}^- = j_{k_2}^-$. Here we only discuss the case where there is neither a start-up between $i_{k+k_1}^-$ and i_k^- nor a start-up between $j_{k_2}^-$ and j, otherwise the proof would be similar to the discussion before. Furthermore, we discuss the case $k_1 \leq L-2$ since when $k_1 \geq L-1$, $\phi = 0$ is a simpler case. We denote the last start-up node by i_{k+s}^- where $s \geq 0$.

3) If $s \geq L-1$, then $\phi = 0$ and $i_{k+s-L+1}^- \in \mathcal{P}(i_k^-)$. Now we need to consider when the generator will shut-down. We denote it by h and $h \notin \mathcal{P}(i_k^-)$. We further discuss three possible cases of h.

3.1) If $h \in \mathcal{P}(i_{k-\hat{n}}^-)$, then we can let $h = i_{k-\alpha}^-$ for some $\alpha \in [1,\hat{n}]_{\mathbb{Z}}$ and $r_{i_k^-} - r_j + w_j^- \leq \min\{kV, \overline{V} - \underline{C} + \min\{k-1, s, \alpha-1\}V + (\underline{C} + V - \overline{V})\} \leq \overline{V} - \underline{C} + (\alpha-1)V + (\underline{C} + V - \overline{V}) \leq \overline{V} - \underline{C} + V\sum_{n=0}^{\alpha-1}(y_{i_{k-n}^-} - \sum_{m=0}^{\min\{L-1,n+\omega\}} u_{i_{k-n+m}^-}) + (\underline{C} + V - \overline{V})(y_j - \sum_{m=0}^{L-1} u_{j_m^-})$, which is less than the RHS of inequality (17).

3.2) If $h \in \mathcal{P}(i) \setminus \mathcal{P}(i_{k-\hat{n}}^-)$, then we can have $h = i_{k-\alpha}^-$ for some $\alpha \in [\hat{n}+1, k]_{\mathbb{Z}}$ and $r_{i_k^-} - r_j + w_j^- \leq \min\{kV, \overline{V} - \underline{C} + \min\{k-1, s, \alpha-1\}V + (\underline{C} + V - \overline{V})\} \leq \overline{V} - \underline{C} + (\alpha-1)V + (\underline{C} + V - \overline{V}) \leq \overline{V} - \underline{C} + V\sum_{n\in S_0}(y_{i_{k-n}^-} - \sum_{m=0}^{\min\{L-1,n+\omega\}} u_{i_{k-n+m}^-}) + V\sum_{n\in(S\cap[\hat{n}+1,k-\alpha+1]-\mathbb{Z})\cup\{\hat{n}\}}(g_n -$

$n)(y_{i^-_{k-n}} - \sum_{m=0}^{L-1} u_{i^-_{k-n+m}}) + (\underline{C} + V - \overline{V})(y_j - \sum_{m=0}^{L-1} u_{j^-_m})$, which is less than the RHS of inequality (17).

3.3) If $h \in \mathcal{V}(i) \setminus \mathcal{P}(i)$, then inequality (17) becomes $r_{i^-_k} \leq kV$, which is also valid.

4) If $s \in [k_1, L-2]_{\mathbb{Z}}$, the proof is similar to subcase 2) and thus omitted here.

(Facet-defining) The facet-defining proof is similar to that for Proposition 2 in Online Supplement EC.3.1 and thus is omitted here. □

EC.4. Proofs for Multi-Generator Formulations

EC.4.1. Proof for Proposition 14

Proof. (**Validity**) We have

$$\begin{aligned}
\sum_{g\in\mathcal{S}}\left(r_i^g + w_i^{+g} - r_{i^-}^g + w_i^{-g}\right) &= \overline{D}_i - \overline{D}_{i^-} - \sum_{g\in\mathcal{S}}\left(\underline{C}^g y_i^g - \underline{C}^g y_{i^-}^g\right) + \sum_{g\in\mathcal{S}}\left(w_i^{+g} + w_i^{-g}\right) \\
&\quad + \sum_{g\in\mathcal{G}\setminus\mathcal{S}}\left(r_{i^-}^g + \underline{C}^g y_{i^-}^g - r_i^g - \underline{C}^g y_i^g\right) \\
\leq \overline{D}_i - \overline{D}_{i^-} &- \sum_{g\in\mathcal{S}}\left(\underline{C}^g y_i^g - \underline{C}^g y_{i^-}^g\right) + \sum_{g\in\mathcal{S}}\left(w_i^{+g} + w_i^{-g}\right) \\
&+ \sum_{g\in\mathcal{G}\setminus\mathcal{S}}\left(\left(\overline{V}^g - \underline{C}^g\right)y_{i^-}^g + \left(\underline{C}^g + V^g - \overline{V}^g\right)\left(y_i^g - u_i^g\right) - w_i^{-g} + \underline{C}^g y_{i^-}^g - \underline{C}^g y_i^g\right) \\
\leq \overline{D}_i - \overline{D}_{i^-} &- \sum_{g\in\mathcal{S}}\left(\underline{C}^g y_i^g - \underline{C}^g y_{i^-}^g\right) + \sum_{g\in\mathcal{S}}\left(2V^g y_i^g - \left(\underline{C}^g + 2V^g - \overline{V}^g\right)u_i^g\right) \\
&+ \sum_{g\in\mathcal{G}\setminus\mathcal{S}}\left(\left(\overline{V}^g - \underline{C}^g\right)y_{i^-}^g + \left(\underline{C}^g + V^g - \overline{V}^g\right)\left(y_i^g - u_i^g\right) - w_i^{-g} + \underline{C}^g y_{i^-}^g - \underline{C}^g y_i^g\right) \\
= \overline{D}_i - \overline{D}_{i^-} &+ \sum_{g\in\mathcal{S}}\left(\underline{C}^g y_{i^-}^g + \left(2V^g - \underline{C}^g\right)y_i^g - \left(\underline{C}^g + 2V^g - \overline{V}^g\right)u_i^g\right) \\
&+ \sum_{g\in\mathcal{G}\setminus\mathcal{S}}\left(\overline{V}^g y_{i^-}^g - \underline{C}^g y_i^g + \left(\underline{C}^g + V^g - \overline{V}^g\right)\left(y_i^g - u_i^g\right) - w_i^{-g}\right),
\end{aligned}$$

which readily indicates that (18) is valid. In particular, in the above induction, the first equality is due to load balance (1j) at nodes i and i^-. The first inequality holds because $r_{i^-}^g - r_i^g + w_i^{-g} \leq (\overline{V}^g - \underline{C}^g)y_{i^-}^g + (\underline{C}^g + V^g - \overline{V}^g)(y_i^g - u_i^g)$, which can be proved easily by following the similar proof in Proposition 1. The second inequality holds because $w_i^{+g} + w_i^{-g} \leq 2V^g y_i^g - (\underline{C}^g + 2V^g - \overline{V}^g)u_i^g$.

(**Facet-defining**) The two-period case of conv(Ψ) is defined as the convex hull of $\Psi_2 = \{(r, w^+, w^-, y, u) \in \mathbb{R}_+^{2|\mathcal{G}|} \times \mathbb{R}_+^{2|\mathcal{G}|} \times \mathbb{R}_+^{2|\mathcal{G}|} \times \mathbb{B}^{2|\mathcal{G}|} \times \mathbb{B}^{|\mathcal{G}|} : (1b) - (1j)\}$. In the following, we create $9|\mathcal{G}|$ affinely independent points that satisfy inequality (18) at equation, where each point has the following components $(r_{i^-}^g, r_i^g, w_{i^-}^{+g}, w_i^{+g}, w_{i^-}^{-g}, w_i^{-g}, y_{i^-}^g, y_i^g, u_i^g), \forall g \in \mathcal{G}$.

First, we discuss the possible values that we can choose for $(y_{i-}^g, y_i^g, u_i^g), \forall g \in \mathcal{G}$. Note that for each given $g \in \mathcal{G}$, to create the affinely independence among the dimensions spanned by (y_{i-}^g, y_i^g, u_i^g), only three possible sets of values for (y_{i-}^g, y_i^g, u_i^g) can be used, i.e.,

$$(y_{i-}^g, y_i^g, u_i^g) = (1,0,0), \tag{EC.10}$$

$$(y_{i-}^g, y_i^g, u_i^g) = (1,1,0), \text{ and} \tag{EC.11}$$

$$(y_{i-}^g, y_i^g, u_i^g) = (0,1,1), \tag{EC.12}$$

which are affinely independent by themselves.

Next, we discuss the possible values that we can choose for $(r_{i-}^g, r_i^g, w_{i-}^{+g}, w_i^{+g}, w_{i-}^{-g}, w_i^{-g}), \forall g \in \mathcal{G}$. For each given g, to create the affinely independence among the dimensions spanned by $(r_{i-}^g, r_i^g, w_{i-}^{+g}, w_i^{+g}, w_{i-}^{-g}, w_i^{-g})$ and to satisfy (18) at equation, we can choose the following possible values for $(r_{i-}^g, r_i^g, w_{i-}^{+g}, w_i^{+g}, w_{i-}^{-g}, w_i^{-g})$ depending on the values of (y_{i-}^g, y_i^g, u_i^g):

- If $(y_{i-}^g, y_i^g, u_i^g) = (1,0,0)$, then $(r_{i-}^g, r_i^g, w_{i-}^{+g}, w_i^{+g}, w_{i-}^{-g}, w_i^{-g})$ can take $(0,0,0,0,0,0)$, $(\overline{V}^g - \underline{C}^g - s, 0,0,0,0,0)$, $(\overline{V}^g - \underline{C}^g - s, 0, \overline{V}^g - \underline{C}^g - s, 0,0,0)$, and $(0, \overline{V}^g - \underline{C}^g - s, 0,0,0,0)$ for some $s \in [0, \overline{V}^g - \underline{C}^g)$, which are affinely independent by themselves.
- If $(y_{i-}^g, y_i^g, u_i^g) = (1,1,0)$, then $(r_{i-}^g, r_i^g, w_{i-}^{+g}, w_i^{+g}, w_{i-}^{-g}, w_i^{-g})$ can take $(r_{i-}^{g*} + s, 0, 0, r_i^{g*} - s, V^g - s, V^g - s)$, $(r_{i-}^{g*}, s, 0, r_i^{g*} - s, V^g - s, V^g - s)$, and $(r_{i-}^{g*} + s, 0, s, r_i^{g*} - s, V^g - s, V^g - s)$ for some $s \in [0, V^g]$, which are affinely independent by themselves. The values of r_{i-}^{g*} and r_i^{g*} can be decided by considering the load balance at nodes i^- and i.
- If $(y_{i-}^g, y_i^g, u_i^g) = (0,1,1)$, then $(r_{i-}^g, r_i^g, w_{i-}^{+g}, w_i^{+g}, w_{i-}^{-g}, w_i^{-g})$ can take $(0,0,0,0,0,0)$, $(0,0,0,\overline{V}^g - \underline{C}^g - s, 0,0)$, $(0,0,0,\overline{V}^g - \underline{C}^g - s, 0, \overline{V}^g - \underline{C}^g - s)$, and $(0,0,0,0,\overline{V}^g - \underline{C}^g - s, 0)$ for some $s \in [0, \overline{V}^g - \underline{C}^g)$, which are affinely independent by themselves.

Meanwhile, by integrating the possible values of $(r_{i-}^g, r_i^g, w_{i-}^{+g}, w_i^{+g}, w_{i-}^{-g}, w_i^{-g})$ and (y_{i-}^g, y_i^g, u_i^g), it is clear that they are affinely independence by themselves with g given. Therefore, by appropriately choosing these values through adjusting the value of s as mentioned above and for different $g \in \mathcal{G}$, we can easily generate $9|\mathcal{G}|$ feasible points in $\text{conv}(\Psi_2)$ that satisfy (18) at equation. $\square$

EC.4.2. Proof for Proposition 16

Proof. (**Validity**) If $\sum_{g \in \mathcal{G} \setminus \mathcal{S}} (y_i^g - u_i^g) \geq 1$, then it is clear that (19) holds. If $\sum_{g \in \mathcal{G} \setminus \mathcal{S}} (y_i^g - u_i^g) = 0$, i.e., $y_i^g = u_i^g$ for all $g \in \mathcal{G} \setminus \mathcal{S}$, then we need to show that

$$\sum_{g \in \mathcal{S}} r_i^g \geq \overline{D}_{i^-} - \sum_{g \in \mathcal{S}} \left(\overline{V}^g + \underline{C}^g \right). \tag{EC.13}$$

Assume $S_0 \subseteq \mathcal{G} \setminus \mathcal{S}$ such that $y_i^g = u_i^g = 1$ for all $g \in S_0$ and $y_i^g = u_i^g = 0$ for all $g \in \mathcal{G} \setminus \{\mathcal{S} \cup S_0\}$. It follows that

$$\sum_{g \in \mathcal{S}} r_i^g = \overline{D}_i - \sum_{g \in S_0} (r_i^g + \underline{C}^g y_i^g) - \sum_{g \in \mathcal{S}} \underline{C}^g y_i^g$$

$$\geq \overline{D}_{i^-} + \sum_{g\in\mathcal{G}\setminus\mathcal{S}} \overline{V}^g - \sum_{g\in\mathcal{S}} V^g - \sum_{g\in S_0} \overline{V}^g - \sum_{g\in\mathcal{S}} \underline{C}^g$$
$$\geq \overline{D}_{i^-} - \sum_{g\in\mathcal{S}} \left(\overline{V}^g + \underline{C}^g\right),$$

where the first equality is due to load balance (1j) at node i, the first inequality holds because $\overline{D}_i - \overline{D}_{i^-} \geq \sum_{g\in\mathcal{G}\setminus\mathcal{S}} \overline{V}^g - \sum_{g\in\mathcal{S}} V^g$, $r_i^g + \underline{C}^g y_i^g \leq \overline{V}^g$ for $g\in S_0$, and $y_i^g \leq 1$ for $g\in\mathcal{S}$, and the second inequality is due to $S_0 \subseteq \mathcal{G}\setminus\mathcal{S}$. Thus, (EC.13) holds.

(**Facet-defining**) The two-period case of conv(Ψ) is defined as the convex hull of $\Psi_2 = \{(r, w^+, w^-, y, u) \in \mathbb{R}_+^{2|\mathcal{G}|} \times \mathbb{R}_+^{2|\mathcal{G}|} \times \mathbb{R}_+^{2|\mathcal{G}|} \times \mathbb{B}^{2|\mathcal{G}|} \times \mathbb{B}^{|\mathcal{G}|} : (1b) - (1j)\}$. In the following, we create $9|\mathcal{G}|$ affinely independent points that satisfy inequality (19) at equation, where each point has the following components $(r_{i^-}^g, r_i^g, w_{i^-}^{+g}, w_i^{+g}, w_{i^-}^{-g}, w_i^{-g}, y_{i^-}^g, y_i^g, u_i^g), \forall g\in\mathcal{G}$.

First, we discuss the possible values that we can choose for $(y_{i^-}^g, y_i^g, u_i^g), \forall g\in\mathcal{G}$. Note that for each given $g\in\mathcal{G}$, to create the affinely independence among the dimensions spanned by $(y_{i^-}^g, y_i^g, u_i^g)$, only three possible sets of values for $(y_{i^-}^g, y_i^g, u_i^g)$ can be used, i.e., (EC.10) - (EC.12), which are affinely independent by themselves. Note that we need to appropriately choose the value of $(y_{i^-}^g, y_i^g, u_i^g)$ to ensure that $\sum_{g\in\mathcal{G}\setminus\mathcal{S}} (y_i^g - u_i^g) \leq 1$.

Next, we discuss the possible values that we can choose for $(r_{i^-}^g, r_i^g, w_{i^-}^{+g}, w_i^{+g}, w_{i^-}^{-g}, w_i^{-g}), \forall g\in\mathcal{G}$. For each given g, to create the affinely independence among the dimensions spanned by $(r_{i^-}^g, r_i^g, w_{i^-}^{+g}, w_i^{+g}, w_{i^-}^{-g}, w_i^{-g})$ and to satisfy (19) at equation, we can choose the following possible values for $(r_{i^-}^g, r_i^g, w_{i^-}^{+g}, w_i^{+g}, w_{i^-}^{-g}, w_i^{-g})$ depending on the values of $(y_{i^-}^g, y_i^g, u_i^g)$:

- If $(y_{i^-}^g, y_i^g, u_i^g) = (1,0,0)$, then $(r_{i^-}^g, r_i^g, w_{i^-}^{+g}, w_i^{+g}, w_{i^-}^{-g}, w_i^{-g})$ can take $(0,0,0,0,0,0)$, $(\overline{V}^g - \underline{C}^g - s, 0,0,0,0,0)$, $(\overline{V}^g - \underline{C}^g - s, 0, \overline{V}^g - \underline{C}^g - s, 0,0,0)$, and $(0, \overline{V}^g - \underline{C}^g - s, 0,0,0,0)$ for some $s\in [0, \overline{V}^g - \underline{C}^g)$, which are affinely independent by themselves.
- If $(y_{i^-}^g, y_i^g, u_i^g) = (1,1,0)$, then we discuss the following possible cases:
  - —If $g\in\mathcal{S}$ and $\sum_{g\in\mathcal{G}\setminus\mathcal{S}} (y_i^g - u_i^g) = 1$ (i.e., there only exists $g\in\mathcal{G}\setminus\mathcal{S}$ such that $y_i^g = 1$ and $u_i^g = 0$), then $(r_{i^-}^g, r_i^g, w_{i^-}^{+g}, w_i^{+g}, w_{i^-}^{-g}, w_i^{-g})$ can take $(0,0,0,0,V^g - s, V^g - s)$, $(r_{i^-}^{g*} + s, 0,0,0, V^g - s, V^g - s)$, $(r_{i^-}^{g*}, s, 0,0, V^g - s, V^g - s)$, and $(r_{i^-}^{g*} + s, 0, s, 0, V^g - s, V^g - s)$ for some $s\in [0, V^g]$, which are affinely independent by themselves. The value of $r_{i^-}^{g*}$ can be decided by considering the load balance at node i^-.
  - —Otherwise, then $(r_{i^-}^g, r_i^g, w_{i^-}^{+g}, w_i^{+g}, w_{i^-}^{-g}, w_i^{-g})$ can take $(0,0,0,0,V^g - s, V^g - s)$, $(r_{i^-}^{g*}, s, 0, r_i^{g*} - s, V^g - s, V^g - s)$, and $(r_{i^-}^{g*} + s, 0, s, r_i^{g*} - s, V^g - s, V^g - s)$ for some $s\in [0, V^g]$, which are affinely independent by themselves. The value of $r_{i^-}^{g*}$ can be decided by considering the load balance at node i^-.
- If $(y_{i^-}^g, y_i^g, u_i^g) = (0,1,1)$, then $(r_{i^-}^g, r_i^g, w_{i^-}^{+g}, w_i^{+g}, w_{i^-}^{-g}, w_i^{-g})$ can take $(0,0,0,0,0,0)$, $(0,0,0,\overline{V}^g - \underline{C}^g - s, 0,0)$, $(0,0,0,\overline{V}^g - \underline{C}^g - s, 0, \overline{V}^g - \underline{C}^g - s)$, and $(0,0,0,0,\overline{V}^g - \underline{C}^g - s, 0)$ for some $s\in [0, \overline{V}^g - \underline{C}^g)$, which are affinely independent by themselves.

Meanwhile, by integrating the possible values of $(r^g_{i^-}, r^g_i, w^{+g}_{i^-}, w^{+g}_i, w^{-g}_{i^-}, w^{-g}_i)$ and $(y^g_{i^-}, y^g_i, u^g_i)$, it is clear that they are affinely independence by themselves with g given. Therefore, by appropriately choosing these values through adjusting the value of s as mentioned above and for different $g \in \mathcal{G}$, we can easily generate $9|\mathcal{G}|$ feasible points in $\mathrm{conv}(\Psi_2)$ that satisfy (19) at equation. □

EC.4.3. Proof for Proposition 17

Proof. (**Validity**) We only prove the case in which $j = i^-$ since the case in which $j^- = i$ follows a similar proof. We prove it by contradiction. Assume $\sum_{g\in\mathcal{G}} y^g_i \le q_i$. Then we have $\sum_{g\in\mathcal{G}} y^g_i = q_i$ due to the condition $\sum_{g=|\mathcal{G}|-q_i+2}^{|\mathcal{G}|} \overline{C}^{[g]} < d_i \le \sum_{g=|\mathcal{G}|-q_i+1}^{|\mathcal{G}|} \overline{C}^{[g]}$. Due to the condition $\sum_{g=|\mathcal{G}|-q_i+1}^{|\mathcal{G}|} \overline{C}^{[g]} + \nu < d_j < \sum_{g=|\mathcal{G}|-q_i}^{|\mathcal{G}|} \overline{C}^{[g]}$, we have $\sum_{g\in\mathcal{G}} y^g_{i^-} \ge q_i + 1$. Thus, at least one generator should shut down at node i, meaning $x^g_{i^-} \le \overline{V}^g$ for some g with $y^g_{i^-} = 1$ due to ramping-down rate restriction (1i). Since $d_{i^-} - d_i > (\sum_{g=|\mathcal{G}|-q_i+1}^{|\mathcal{G}|} \overline{C}^{[g]} + \nu) - \sum_{g=|\mathcal{G}|-q_i+1}^{|\mathcal{G}|} \overline{C}^{[g]} = \nu \ge \overline{V}^g$ for all $g \in \mathcal{G}$, which prevents any generator from shutting down at node i, a contradiction.

(Facet-defining) The facet-defining proof is similar to that for Proposition 14 in Online Supplement EC.4.1 and thus is omitted here. □

EC.4.4. Proof for Proposition 18

Proof. We first consider the case in which $\sum_{g\in\mathcal{S}} \left(y^g_{i^-} - y^g_i + u^g_i\right) \ge 1$. We have

$$
\begin{aligned}
\sum_{g\in\mathcal{S}} \left(r^g_{i^-_k} - r^g_i\right) + \sum_{g\in\mathcal{G}\setminus\mathcal{S}} w^{+g}_i &= \overline{D}_{i^-_k} - \overline{D}_i + \sum_{g\in\mathcal{G}} \left(\underline{C}^g y^g_i - \underline{C}^g y^g_{i^-_k}\right) + \sum_{g\in\mathcal{G}\setminus\mathcal{S}} \left(r^g_i + w^{+g}_i - r^g_{i^-_k}\right) \\
&\le \overline{D}_{i^-_k} - \overline{D}_i + \sum_{g\in\mathcal{G}} \left(\underline{C}^g y^g_i - \underline{C}^g y^g_{i^-_k}\right) \\
&\quad + \sum_{g\in\mathcal{G}\setminus\mathcal{S}} \left(kV^g y^g_i - \sum_{m=0}^{\min\{k-1, L^g-1\}} \left(\underline{C}^g + (k-m)V^g - \overline{V}^g\right) u^g_{i^-_m}\right) \\
&\le \overline{D}_{i^-_k} - \overline{D}_i + \sum_{g\in\mathcal{G}} \left(\underline{C}^g y^g_i - \underline{C}^g y^g_{i^-_k}\right) \\
&\quad + \sum_{g\in\mathcal{G}\setminus\mathcal{S}} \left(kV^g y^g_i - \sum_{m=0}^{\min\{k-1, L^g-1\}} \left(\underline{C}^g + (k-m)V^g - \overline{V}^g\right) u^g_{i^-_m}\right) \\
&\quad - \left\{\Gamma - \left(\overline{D}_i - \sum_{g\in\mathcal{S}} \underline{C}^g\right)\right\} \left(1 - \sum_{g\in\mathcal{S}} \left(y^g_{i^-} - y^g_i + u^g_i\right)\right),
\end{aligned}
$$

where the first equality is due to load balance (1j) at nodes i and i^-, the first inequality is due to (10), and the second inequality is due to the condition $\sum_{g\in\mathcal{S}} \left(y^g_{i^-} - y^g_i + u^g_i\right) \ge 1$.

Next, if $\sum_{g\in\mathcal{S}} \left(y^g_{i^-} - y^g_i + u^g_i\right) = 0$, then we have $y^g_{i^-} - y^g_i + u^g_i = 0$ for all $g \in \mathcal{S}$ due to (1e). It follows that for each $g \in \mathcal{S}$, there exists three possible cases: 1) $y^g_{i^-} = 1$, $y^g_i = 1$, and $u^g_i = 0$; 2)

$y^g_{i^-}=0$, $y^g_i=u^g_i=1$; and 3) $y^g_{i^-}=y^g_i=u^g_i=0$. Since $|\mathcal{G}\setminus\mathcal{S}|\le q_i-1$, the third case can be ruled out and we focus on the first two cases where $y^g_i=1$ for all $g\in\mathcal{S}$. It follows that

$$\begin{aligned}
\sum_{g\in\mathcal{S}}\left(r^g_{i_k^-}-r^g_i\right)+\sum_{g\in\mathcal{G}\setminus\mathcal{S}}w_i^{+g} &\le \sum_{g\in\mathcal{S}}\left(r^g_{i_k^-}-0\right)+\sum_{g\in\mathcal{G}\setminus\mathcal{S}}w_i^{+g}\\
&\le \sum_{g\in\mathcal{S}}r^g_{i_k^-}+\left(\overline{D}_{i_k^-}-\sum_{g\in\mathcal{S}}r^g_{i_k^-}-\sum_{g\in\mathcal{G}}\underline{C}^g y^g_{i_k^-}\right)+\sum_{g\in\mathcal{G}\setminus\mathcal{S}}w_i^{+g}\\
&\le \sum_{g\in\mathcal{S}}r^g_{i_k^-}+\left(\overline{D}_{i_k^-}-\sum_{g\in\mathcal{S}}r^g_{i_k^-}-\sum_{g\in\mathcal{G}}\underline{C}^g y^g_{i_k^-}\right)-\Gamma+\sum_{g\in\mathcal{G}\setminus\mathcal{S}}\underline{C}^g y^g_i\\
&\quad+\sum_{g\in\mathcal{G}\setminus\mathcal{S}}\left(kV^g y^g_i-\sum_{m=0}^{\min\{k-1,L^g-1\}}\left(\underline{C}^g+(k-m)V^g-\overline{V}^g\right)u^g_{i_m^-}\right)\\
&=\overline{D}_{i_k^-}-\overline{D}_i+\sum_{g\in\mathcal{G}}\left(\underline{C}^g y^g_i-\underline{C}^g y^g_{i_k^-}\right)\\
&\quad+\sum_{g\in\mathcal{G}\setminus\mathcal{S}}\left(kV^g y^g_i-\sum_{m=0}^{\min\{k-1,L^g-1\}}\left(\underline{C}^g+(k-m)V^g-\overline{V}^g\right)u^g_{i_m^-}\right)\\
&\quad-\left\{\Gamma-\left(\overline{D}_i-\sum_{g\in\mathcal{S}}\underline{C}^g\right)\right\}\left(1-\sum_{g\in\mathcal{S}}\left(y^g_{i^-}-y^g_i+u^g_i\right)\right),
\end{aligned}$$

where the first inequality is due to the fact that $y^g_i=1$ and $r^g_i\ge 0$ for all $g\in\mathcal{S}$, the second inequality is due to the fact that $\overline{D}_{i_k^-}-\sum_{g\in\mathcal{S}}r^g_{i_k^-}-\sum_{g\in\mathcal{G}}\underline{C}^g y^g_{i_k^-}\ge 0$, and the third inequality is due to the definition of Γ. □