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Rescheduling Production and Outbound Deliveries When Transportation Service Is Disrupted

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Abstract

Unexpected service disruptions in transportation systems caused by accidents, system breakdowns, poor weather conditions, etc., are quite common. When disruptions occur, rescheduling of vehicles is often needed in order to mitigate the damage caused by the disruptions. In integrated production and outbound distribution systems, a disruption in outbound distribution operation affects not only the delivery plan but also the production schedule. In this paper, we consider a simple integrated scheduling model of production and outbound deliveries with a minimum headway constraint between vehicle departures, and study the situation where an optimal solution of the integrated scheduling model has been obtained but the delivery service is suddenly unavailable for a certain time period due to some unexpected incidents. We would like to determine a new production and delivery schedule in which no delivery takes place during the unavailable period. The objective is to simultaneously maintain a low cost schedule and control the magnitude of changes in the delivery times of the finished goods. We consider three different ways to control the time disruption, and develop polynomial-time algorithms for the corresponding problems.

Keywords: Scheduling; rescheduling; outbound delivery; transportation disruption; dynamic programming

1 Introduction

Unexpected service disruptions in transportation systems caused by accidents, system breakdowns, poor weather conditions, etc., are quite common. When disruptions occur, rescheduling of vehicles is often needed in order to mitigate the damage caused by the disruptions. In some systems, production scheduling and outbound distribution planning are performed simultaneously. This kind of integrated planning is typically used in make-to-order environments in which there is a need to deliver finished goods to customers with a very short lead time. In these integrated production and outbound distribution systems, a disruption in outbound distribution operation affects not only the delivery plan but also the production schedule, since a revised job sequence that groups the jobs into a smaller number of shipments often enables the finished jobs to be delivered to different customers more effectively when the disruption period is over. This increases the complexity of the rescheduling decisions. A common phenomenon after a disruption period is that a large group of backlogged finished jobs needs to be delivered to customers. However, in many transportation systems, simultaneous departure of delivery vehicles from the production facility is impossible. This occurs, for example, when the loading dock of the outbound vehicles has limited capacity, or when there is a safety clearance requirement between vehicles, etc. Hence, a minimum headway constraint often exists between departure times of vehicles, and this further increases the difficulty of the rescheduling decisions.

Rescheduling of deliveries will affect the arrival times of the finished orders at the customers' locations. Both early and late arrivals are undesirable, as they will disrupt the customers' operations. Orders arriving earlier than planned will increase customers' inventories, while orders arriving later than planned will slow down customers' operations. A simple disruption control mechanism is to penalize the deviation of the finished order arrival time from the original arrival time at a constant rate per unit time. However, in some operations where the products are time-sensitive, an excessive amount of change in finished order arrival time occurring to some deliveries is highly undesirable. In such a case, controlling the maximum deviation of finished order arrival time from the original arrival time is more appropriate. In some applications, a promise of on-time delivery is given to customers. For those applications, the amount of deviation of finished order arrival time from the original arrival time is subject to a maximum tolerance.

In this paper, we consider a simple integrated scheduling model of production and outbound deliveries, and consider the situation where an optimal solution of the integrated scheduling model has been obtained but the delivery service is suddenly unavailable for a certain time period due to some unexpected incidents. We would like to determine a new production and delivery schedule in which no delivery takes place during the unavailable period and there is a minimum headway constraint between vehicle departures. The objective is to simultaneously maintain a low cost schedule and control the magnitude of changes in the delivery times of the finished goods. We consider three possible ways to control the magnitude of changes in the delivery times. The goal of this research is to develop methods for rescheduling the production and delivery efficiently.

Integrated scheduling of production and outbound distribution has been studied extensively. Different studies have considered models with different machine environments in the production facility, different constraints on the jobs, different inventory characteristics, different forms of delivery, different numbers of customers, and different performance measures. A number of surveys on these studies have appeared in the literature; see, for example, Chen (2004, 2010), Wang et al. (2015), and Moons et al. (2017). Various applications of integrated production and outbound distribution scheduling models have also been reported. Wang et al. (2005) studied a United States Postal Service mail processing and distribution center's operations and presented an optimization model to determine the processing sequence of the incoming mail that best matches a given outbound truck delivery schedule. Motivated by the operations of a make-to-order based PC assembly manufacturer, Li et al. (2004, 2005, 2006) analyzed several problems with the aim to synchronize the assembly schedule and the allocation of available flight capacity for delivery. Motivated by the operations of Dell, which adopts a "commit-to-delivery" business mode in a make-to-order environment, Stecke and Zhao (2007) analyzed production scheduling problems that minimize the total shipping cost of customer orders, where the delivery service is performed by a third-party logistics company. Geismar and Murthy (2015) considered the operations of a paper manufacturing plant, where delivery is made by railcars. They analyzed the cost reduction obtained through the coordination of production and distribution. Motivated by steel coil production, Li et al. (2017) and Tang et al. (2019) studied different integrated production, inventory, and delivery problems in which finished jobs need to be delivered to customer sites by transporters. Lmariouh et al. (2019) studied the production and distribution process of a bottled-water company, and they developed a mathematical programming model for integrated production, inventory, and transportation decisions.

Rescheduling of production operations has been studied for different causes of scheduling disruptions. These include studies of rescheduling a disrupted schedule caused by machine breakdown (see, e.g., Yin et al. 2016), arrivals of new orders (see, e.g., Hall and Potts 2004), job unavailability (see, e.g., Hall and Potts 2010), job rework (see, e.g., Liu and Zhou 2013), etc. Herrmann (2006) provided a review of basic concepts about rescheduling and discussed different rescheduling strategies, policies, and methods. Rescheduling of transportation services has also been studied extensively. Visentini et al. (2014) provided a recent survey on these works. Some studies specifically focus on particular types of transportation services. Clausen et al. (2010) conducted a review on the literature on disruption and recovery of aircraft schedules, while Cacchiani et al. (2014) conducted a review on the literature on railway rescheduling.

Despite the great deal of rescheduling research on operations scheduling models and transportation models, very few studies have considered rescheduling issues in integrated production and outbound distribution systems. Cai and Zhou (2014) considered a problem involving a firm which produces fresh products to supply to an export market as well as a local market. When the transportation service to the export market is disrupted with an uncertain time period of unavailability, the firm needs to decide whether it should let the finished products wait and increase the risk of decay, or put them for sale in the local market. For unfinished products, the firm needs to decide on the production start times and processing sequence. Unlike Cai and Zhou's work, our rescheduling model on integrated production and outbound distribution focuses on rescheduling both the production operation and the delivery plan when the outbound delivery service is disrupted for a known period of time. Other works that involve disruptions in integrated production and distribution systems include, for example, Hishamuddin et al. (2013) who considered lot-sizing decisions in a production and inventory system when the transportation service is disrupted, Sawik (2016) who studied the scheduling of multi-stage supply chains subject to disruption risks, and Giri and Sarkar (2017) who developed mechanisms for coordinating a supply chain with a third-party logistics service provider when there is a possibility of disruption in the production operation. Unlike these works, the production part of our integrated production and outbound distribution system is modeled as a single machine that processes jobs for different customers. For a recent survey of disruption recovery research in supply chains, see Ivanov et al. (2017).

The underlying integrated production and outbound delivery scheduling model that we consider in this paper can be described as follows. There is a production facility, which we refer to as a "machine," that processes customer orders, which we refer to as "jobs," one by one. The jobs belong to different customers, and finished jobs are delivered to the customers by vehicles. There are a small number of customers at different locations, and there are sufficient homogeneous vehicles available. Each vehicle can carry multiple finished jobs that belong to the same customer in each delivery, and each delivery incurs a constant cost. There is a minimum headway requirement between two consecutive vehicle departures. The objective of the underlying model is a weighted sum of the total arrival time of finished jobs at the customer locations and the total cost of delivery. In our rescheduling model, we assume that an optimal solution of the underlying integrated production and outbound delivery scheduling problem is known, but there is an unexpected disruption causing the delivery service to shut down completely for a certain time period. Our rescheduling model's decisions include resequencing the jobs in the production facility and re-determining the departure times of the finished jobs from the production facility. We use the same "time disruption" measurement as in Hall and Potts (2004, 2010) and Hall et al. (2007) to measure the damage caused by the disruption. As mentioned earlier, different applications require different ways to control the delivery time disruptions. Hence, we consider three different approaches: (i) to impose a penalty on the total delivery time disruption of the jobs; (ii) to impose a constraint on the maximum delivery time disruption among the jobs; and (iii) to impose a penalty on the maximum delivery time disruption among the jobs. These result in three variants of the rescheduling model.

The rest of the paper is organized as follows. In Section 2, we provide a mathematical description and some important properties of our model. In Section 3, we propose methods for determining the optimal solutions to the three variants of the rescheduling model. Section 4 discusses the special case with no minimum headway requirement. Section 5 concludes this study and suggests some future research directions. The proofs of all lemmas are presented in the Appendix (see Supplementary Materials).

2 Model Definitions and Properties

In this section, we provide a mathematical description and some important properties of our model. The underlying integrated production and outbound delivery problem is discussed in Section 2.1, while the corresponding rescheduling problems are discussed in Section 2.2.

2.1 The Underlying Integrated Production and Outbound Delivery Problem

The underlying integrated production and outbound delivery problem being considered can be described mathematically as follows. There are a given set of n jobs $\mathcal{J} = \{J_1, J_2, \ldots, J_n\}$ and a given set of k customers $\mathcal{K} = \{K_1, K_2, \ldots, K_k\}$, where k is fixed. The jobs need to go through a production operation, and the completed jobs need to be delivered to their customers located at different locations. In the production operation, jobs are processed by a single machine without preemption. All jobs are available for processing at time 0. For $i = 1, 2, \ldots, k$, let $\mathcal{J}_i \subseteq \mathcal{J}$ be the subset of jobs that need to be delivered to customer K_i, and let $\mathcal{J}_i = \{J_{i1}, J_{i2}, \ldots, J_{in_i}\}$, where $n_i = |\mathcal{J}_i|$. Then, $\mathcal{J}_1 \cup \mathcal{J}_2 \cup \cdots \cup \mathcal{J}_k = \mathcal{J}$ and $n_1 + n_2 + \cdots + n_k = n$. For $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, n_i$, let $p_{ij} > 0$ be the processing time of J_{ij} in the production operation. For notational convenience, we assume the jobs are indexed in such a way that $p_{i1} \leq p_{i2} \leq \cdots \leq p_{in_i}$ for all $i = 1, 2, \ldots, k$. Sufficient delivery vehicles are available, and each vehicle can carry up to c completed jobs per shipment to a customer location, where $c \leq n$. Each shipment going to customer K_i incurs a fixed cost $\phi_i \geq 0$, and the travel time between the production facility and customer K_i is $\tau_i \geq 0$, for $i = 1, 2, \ldots, k$. Jobs that belong to different customers cannot be delivered together in the same shipment. A feasible solution π of the problem comprises a production schedule and a delivery schedule of the jobs. Let $E_{ij}(\pi)$ denote the departure time of J_{ij} from the production facility after J_{ij} completes its production operation, $D_{ij}(\pi)$ denote the time when J_{ij} arrives at customer K_i, and $r_i(\pi)$ denote the number of shipments used for delivering the jobs to customer K_i. The departure times of any two consecutive shipments must be at least $\delta \geq 0$ time units apart. Thus, for any J_{ij} and J_{hl}, a feasible solution π requires $|E_{ij}(\pi) - E_{hl}(\pi)|$ to be either 0 or at least δ. We refer to δ as the "minimum headway" of departures. Note that $D_{ij}(\pi) = E_{ij}(\pi) + \tau_i$. The time-based performance, which is a measurement of customer service, is given by $\sum_{i=1}^{k} \sum_{j=1}^{n_i} D_{ij}(\pi)$, while the total cost of delivery is $\sum_{i=1}^{k} r_i(\pi)\phi_i$. The objective of this problem is to determine a solution π such that

$$\Gamma_0(\pi) = \alpha \sum_{i=1}^{k} \sum_{j=1}^{n_i} D_{ij}(\pi) + \beta \sum_{i=1}^{k} r_i(\pi)\phi_i$$

is minimized, where $\alpha, \beta \geq 0$ are input parameters that represent the importance of the two performance measures. Let

$$\tilde{\Gamma}_0(\pi) = \alpha \sum_{i=1}^{k} \sum_{j=1}^{n_i} E_{ij}(\pi) + \beta \sum_{i=1}^{k} r_i(\pi)\phi_i.$$

Since $\tilde{\Gamma}_0(\pi)$ differs from $\Gamma_0(\pi)$ by a constant $\alpha \sum_{i=1}^{k} n_i \tau_i$ for any solution π, minimizing $\Gamma_0(\pi)$ is equivalent to minimizing $\tilde{\Gamma}_0(\pi)$. We denote the problem as problem $\mathbf{P}_0$.

In many applications, the minimum headway δ is very small compared to job processing times. If δ is no greater than the processing time of any job in the production operation (i.e., $\delta \leq \min_{J_{ij} \in \mathcal{J}} \{p_{ij}\}$), then any feasible solution of problem $\mathbf{P}_0$ satisfies the minimum headway constraint, and problem $\mathbf{P}_0$ becomes the same as problem (P2) in Chen and Vairaktarakis (2005), except we assume k is fixed. Problem (P2) in Chen and Vairaktarakis (2005), if expressed in Chen's (2010) notation, is problem $1||V(\infty, c), \text{direct}|k| \sum D_j + TC$. Hall and Potts (2003, sec. 3.1) have presented a dynamic programming algorithm for the problem $1||V(\infty, \infty), \text{direct}|k| \sum D_j + TC$ with an $O(n^{k+1})$ running time, and this computational complexity remains valid for problem $1||V(\infty, c), \text{direct}|k| \sum D_j + TC$ (see Table 5 in Chen 2010). Hence, under the condition "$\delta \leq \min_{J_{ij} \in \mathcal{J}} \{p_{ij}\}$," problem $\mathbf{P}_0$ is solvable in $O(n^{k+1})$ time. Note that when k is not fixed, whether or not problem $1||V(\infty, \infty), \text{direct}|k| \sum D_j + TC$ can be solved in polynomial time remains an open question (see Table 5 in Chen 2010).

Hall and Potts (2003) have shown that problem $1||V(\infty,\infty),\text{direct}|k|\sum D_j + TC$ possesses the following optimality properties:

(C1) There is no idle time between jobs in the production schedule.

(C2) Jobs that belong to the same shipment are processed consecutively by the machine.

(C3) A job which is processed earlier is delivered no later than a job which is processed later.

(C4) Jobs that belong to the same customer are processed by the machine in SPT order (i.e., nondecreasing order of processing times).

We will show that properties (C1)–(C4), as well as the following optimality property, also holds for problem $\mathbf{P}_0$:

(C5) The departure time of each shipment is either equal to the completion time of production of the last job in that shipment, or equal to the departure time of the previous shipment plus δ (if the current shipment is not the first shipment).

Lemma 1 *There exists an optimal solution of problem $\mathbf{P}_0$ which satisfies properties (C1)–(C5).*

To enhance the applicability of our model, we do not restrict our analysis to the case where $\delta \leq \min_{J_{ij}\in\mathcal{J}}\{p_{ij}\}$. In the following, we present an algorithm that determines an optimal solution of problem $\mathbf{P}_0$ for any $\delta \geq 0$. Let

$$\mathcal{E}_0 = \left\{ \sum_{i=1}^{k}\sum_{j=1}^{q_i} p_{ij} + y\delta \;\middle|\; q_i = 0,1,\ldots,n_i \text{ for } i=1,\ldots,k;\; y=0,1,\ldots,n-1 \right\}.$$

The set $\mathcal{E}_0$ contains all possible shipment departure times in an optimal solution that satisfies properties (C1)–(C5). The following dynamic program determines an optimal solution of $\mathbf{P}_0$ that satisfies these properties.

Algorithm $\mathbf{A}_0$

Preprocessing:

For each $(q_1,\ldots,q_k)$, where $q_i = 0,1,\ldots,n_i$ for $i=1,2,\ldots,k$, determine $\sum_{i=1}^{k}\sum_{j=1}^{q_i} p_{ij}$. Determine the elements of $\mathcal{E}_0$ and arrange them in ascending order.

Value function:

$f(q_1, \ldots, q_k; t)$ = minimum total cost $\tilde{\Gamma}_0(\pi')$ of a partial schedule π' for processing and delivering jobs $J_{i1}, J_{i2}, \ldots, J_{iq_i}$, for $i = 1, 2, \ldots, k$, such that the departure time of the last shipment is equal to t.

Recursive relation: For $q_i = 0, 1, \ldots, n_i$ $(i = 1, 2, \ldots, k)$ such that $\sum_{i=1}^{k} q_i \geq 1$, and for $t \in \mathcal{E}_0$ such that $t \geq \sum_{i=1}^{k} \sum_{j=1}^{q_i} p_{ij}$:

$$f(q_1, \ldots, q_k; t) = \begin{cases} \min\big\{ f(q_1, \ldots, q_{h-1}, q'_h, q_{h+1}, \ldots, q_k; t - \delta) + \alpha(q_h - q'_h)t + \beta\phi_h \;\big|\; h = 1, \ldots, k; \\ \qquad 0 \leq q'_h < q_h \text{ such that } q_h - q'_h \leq c \big\}, & \text{if } t > \sum_{i=1}^{k} \sum_{j=1}^{q_i} p_{ij}; \\ \min\big\{ f(q_1, \ldots, q_{h-1}, q'_h, q_{h+1}, \ldots, q_k; t') + \alpha(q_h - q'_h)t + \beta\phi_h \;\big|\; h = 1, \ldots, k; \\ \qquad 0 \leq q'_h < q_h \text{ such that } q_h - q'_h \leq c;\; t' \in \mathcal{E}_0 \text{ such that} \\ \qquad \sum_{i=1}^{k} \sum_{j=1}^{q_i} p_{ij} - \sum_{j=q'_h+1}^{q_h} p_{hj} \leq t' \leq t - \delta \big\}, & \text{if } t = \sum_{i=1}^{k} \sum_{j=1}^{q_i} p_{ij}. \end{cases}$$

Boundary conditions:

$f(0, \ldots, 0; 0) = 0$;

$f(0, \ldots, 0; t) = +\infty$ if $t \in \mathcal{E}_0$;

$f(q_1, \ldots, q_k; t) = +\infty$ if $t \notin \mathcal{E}_0$ or ($t \in \mathcal{E}_0$ and $t < \sum_{i=1}^{k} \sum_{j=1}^{q_i} p_{ij}$).

Optimal solution value: $\min\big\{ f(n_1, \ldots, n_k; t) \;\big|\; t \in \mathcal{E}_0;\; t \geq \sum_{i=1}^{k} \sum_{j=1}^{n_i} p_{ij} \big\}$.

Theorem 1 *Algorithm* $\mathbf{A}_0$ *finds an optimal solution for problem* $\mathbf{P}_0$ *in* $O(cn^{2k+1})$ *time.*

Proof. Suppose that a partial production and delivery schedule comprising the first q_i jobs of customer K_i, for $i = 1, 2, \ldots, k$, has been formed. Let t be the departure time of the last shipment in this partial schedule, and suppose the last shipment is for customer K_h and contains $q_h - q'_h$ jobs. We consider two different scenarios. The first scenario is $t > \sum_{i=1}^{k} \sum_{j=1}^{q_i} p_{ij}$. Under this scenario, by property (C5), the departure time of the second last shipment is equal to $t - \delta$, and the recursive relation enumerates all possible values of h and q'_h of this partial schedule. The second scenario is $t = \sum_{i=1}^{k} \sum_{j=1}^{q_i} p_{ij}$. Under this scenario, the recursive relation not only enumerates all possible values of h and q'_h of the partial schedule, but it also enumerates all possible departure times of the second last shipment. The minimum headway constraint implies that the departure time t' of the second last shipment is no greater than $t - \delta$, and t' must be no less than the completion time of production of the second last shipment. Thus, $\sum_{i=1}^{k} \sum_{j=1}^{q_i} p_{ij} - \sum_{j=q'_h+1}^{q_h} p_{hj} \leq t' \leq t - \delta$. Hence, algorithm $\mathbf{A}_0$ compares all possible solutions that satisfy the properties in Lemma 1. Therefore, algorithm $\mathbf{A}_0$ finds an optimal solution of problem $\mathbf{P}_0$.

In the preprocessing step, the quantity "$\sum_{i=1}^{k}\sum_{j=1}^{q_i} p_{ij}$" can be computed recursively for all possible $q_1, q_2, \ldots, q_k$ values in $O(n^k)$ time. Because $\mathcal{E}_0$ contains $O(n^{k+1})$ elements and there are $O(n^k)$ combinations of $q_1, q_2, \ldots, q_k$, the recurrence relation of $\mathbf{A}_0$ is executed $O(n^{2k+1})$ times. Among these $O(n^{2k+1})$ executions of the recurrence relation, only $O(n^k)$ executions are for the second scenario. Each execution of the first scenario requires $O(c)$ time, which covers the enumeration of $O(1)$ possible customer K_h for the last shipment and the enumeration of $O(c)$ possible values of q'_h. Each execution of the second scenario requires $O(cn^{k+1})$ time, which covers the enumeration of $O(1)$ possible customers K_h for the last shipment, $O(c)$ possible values of q'_h, and $O(n^{k+1})$ possible values of t'. Hence, algorithm $\mathbf{A}_0$ requires $O(n^{2k+1} \cdot c + n^k \cdot cn^{k+1}) = O(cn^{2k+1})$ time. ▮

Algorithm $\mathbf{A}_0$ has a similar structure as some dynamic programs developed for other integrated production and distribution models, where the value functions make use of multiple parameters to keep track of the number of completed jobs for each customer (see, e.g., algorithm SF in Hall and Potts 2003, algorithm DP2 in Chen and Vairaktarakis 2005, and algorithm A1 in Li et al. 2017). The main difference between algorithm $\mathbf{A}_0$ and these dynamic programs is that algorithm $\mathbf{A}_0$ also enumerates the shipment departure times in the set $\mathcal{E}_0$.

2.2 The Rescheduling Problems

In the rescheduling model, we consider the situation where an optimal solution π^* of $\mathbf{P}_0$ that satisfies (C1)–(C5) has been obtained. However, due to some unexpected incidents, the delivery service is unavailable during the time period $[0, T)$. As a result, both production and delivery need to be rescheduled. As mentioned in Section 1, we consider three possible ways to limit the time disruption. Thus, we consider three variants of the rescheduling model.

The first variant is to determine a solution σ with $E_{ij}(\sigma) \geq T$ for $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, n_i$, such that

$$\Gamma_1(\sigma) = \alpha \sum_{i=1}^{k}\sum_{j=1}^{n_i} D_{ij}(\sigma) + \beta \sum_{i=1}^{k} r_i(\sigma)\phi_i + \gamma \sum_{i=1}^{k}\sum_{j=1}^{n_i} |D_{ij}(\sigma) - D_{ij}(\pi^*)|$$

is minimized, and that any two consecutive shipments are at least δ time units apart. In this cost function, $|D_{ij}(\sigma) - D_{ij}(\pi^*)|$ is the delivery time disruption of J_{ij}, and $\gamma \geq 0$ is an input parameter that represents the importance of the additional performance measure "$\sum_{i=1}^{k} \sum_{j=1}^{n_i} |D_{ij}(\sigma) - D_{ij}(\pi^*)|$." We denote this variant as problem $\mathbf{P}_1$, which penalizes the delivery time disruption at a constant rate per unit time for each job. Let

$$\tilde{\Gamma}_1(\sigma) = \alpha \sum_{i=1}^{k} \sum_{j=1}^{n_i} E_{ij}(\sigma) + \beta \sum_{i=1}^{k} r_i(\sigma)\phi_i + \gamma \sum_{i=1}^{k} \sum_{j=1}^{n_i} |E_{ij}(\sigma) - E_{ij}(\pi^*)|.$$

Note that $\Gamma_1(\sigma) = \tilde{\Gamma}_1(\sigma) + \alpha \sum_{i=1}^{k} n_i \tau_i$ for any solution σ. Hence, minimizing $\Gamma_1(\sigma)$ is equivalent to minimizing $\tilde{\Gamma}_1(\sigma)$.

The second variant is to determine a solution σ with $E_{ij}(\sigma) \geq T$ for $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, n_i$, such that

$$\max_{i=1,\ldots,k;\, j=1,\ldots,n_i} |D_{ij}(\sigma) - D_{ij}(\pi^*)| \leq \theta, \tag{1}$$

and that

$$\Gamma_2(\sigma) = \alpha \sum_{i=1}^{k} \sum_{j=1}^{n_i} D_{ij}(\sigma) + \beta \sum_{i=1}^{k} r_i(\sigma)\phi_i$$

is minimized, and that any two consecutive shipments are at least δ time units apart, where $\theta \geq 0$ is an input parameter. We denote this variant as problem $\mathbf{P}_2$. This variant is applicable to the situation where a promise of on-time delivery is given to customers, and thus a maximum tolerance θ is imposed on the delivery time disruption of every job (see Hall and Potts 2004 for a similar "maximum time disruption" constraint in one of their models). Let

$$\tilde{\Gamma}_2(\sigma) = \alpha \sum_{i=1}^{k} \sum_{j=1}^{n_i} E_{ij}(\sigma) + \beta \sum_{i=1}^{k} r_i(\sigma)\phi_i.$$

Note that minimizing $\Gamma_2(\sigma)$ is equivalent to minimizing $\tilde{\Gamma}_2(\sigma)$, and constraint (1) is equivalent to the constraint

$$\max_{i=1,\ldots,k;\, j=1,\ldots,n_i} |E_{ij}(\sigma) - E_{ij}(\pi^*)| \leq \theta. \tag{2}$$

The third variant is to determine a solution σ with $E_{ij}(\sigma) \geq T$ for $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, n_i$, such that

$$\Gamma_3(\sigma) = \alpha \sum_{i=1}^{k} \sum_{j=1}^{n_i} D_{ij}(\sigma) + \beta \sum_{i=1}^{k} r_i(\sigma)\phi_i + \gamma \max_{i=1,\ldots,k;\, j=1,\ldots,n_i} |D_{ij}(\sigma) - D_{ij}(\pi^*)|$$

is minimized, and that any two consecutive shipments are at least δ time units apart, where $\gamma \geq 0$ is an input parameter that represents the importance of the additional performance measure "$\max_{i=1,\ldots,k;\, j=1,\ldots,n_i} |D_{ij}(\sigma) - D_{ij}(\pi^*)|$." We denote this problem as problem $\mathbf{P}_3$. This variant is applicable to the situation where an excessive change in delivery time occurring to some jobs needs to be discouraged. Let

$$\tilde{\Gamma}_3(\sigma) = \alpha \sum_{i=1}^{k} \sum_{j=1}^{n_i} E_{ij}(\sigma) + \beta \sum_{i=1}^{k} r_i(\sigma)\phi_i + \gamma \max_{i=1,\ldots,k;\, j=1,\ldots,n_i} |E_{ij}(\sigma) - E_{ij}(\pi^*)|.$$

Note that minimizing $\Gamma_3(\sigma)$ is equivalent to minimizing $\tilde{\Gamma}_3(\sigma)$.

As will be discussed in Section 3.3, our proposed method for solving problem $\mathbf{P}_3$ requires the solving of problem $\mathbf{P}_2$ for different values of parameter θ. Thus, problem $\mathbf{P}_2$ can be viewed as a stepping stone for solving problem $\mathbf{P}_3$. Note that problems $\mathbf{P}_1$ and $\mathbf{P}_3$ are always feasible, while problem $\mathbf{P}_2$ may be infeasible.

Lemma 2 *For $m = 1, 2, 3$, if problem $\mathbf{P}_m$ is feasible, then there exists an optimal solution of $\mathbf{P}_m$ that satisfies properties (C1)–(C3) in which the jobs belonging to each customer are processed by the machine in the same SPT sequence as in π^*.*

Note that the optimal solution of $\mathbf{P}_m$ mentioned in Lemma 2 not only satisfies property (C4), but its SPT processing sequence is the same as that of π^*. However, this optimal solution may not satisfy property (C5); see the numerical example presented in Section 3.1, which shows that property (C5) does not necessarily hold for problem $\mathbf{P}_1$.

Remark 1 *Problems $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$ can be applied to the situation where some jobs have been completed before time zero but have not been delivered. Under this situation, we replace those completed jobs by dummy jobs with zero processing times. It is easy to check that Lemmas 1 and 2 remain valid after replacing the processing times of the completed jobs by zero.*

3 Solution Methods

We consider the situation where an optimal solution π^* of $\mathbf{P}_0$ that satisfies (C1)–(C5) has been obtained. In the following subsections, we present solution methods for the rescheduling problems $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$. Results of a computational study are also reported.

3.1 Solution Method for Problem $\mathbf{P}_1$

The following lemma provides some optimality properties of problem $\mathbf{P}_1$.

Lemma 3 *There exists an optimal solution of* $\mathbf{P}_1$ *which satisfies the properties in Lemma 2 such that the departure time of each shipment is equal to one of the following values: (i)* T*; (ii) one of the values in* $\mathcal{E}_0$ *minus* $y\delta$*, for some* $y = 0, 1, \ldots, n-1$*; or (iii) departure time of the previous shipment plus* δ *(if the current shipment is not the first shipment).*

Define

$$\mathcal{E}_1 = \left[\left\{ \sum_{i=1}^{k} \sum_{j=1}^{q_i} p_{ij} + y\delta \;\middle|\; q_i = 0, 1, \ldots, n_i \text{ for } i = 1, \ldots, k;\; y = -n+1, \ldots, -1, 0, 1, \ldots, 2n-2 \right\} \cup \left\{ T + y\delta \mid y = 0, 1, \ldots, n-1 \right\} \right] \cap [T, +\infty).$$

The set $\mathcal{E}_1$ contains all possible shipment departure times in an optimal solution that satisfies the properties in Lemmas 2 and 3. The following dynamic programming algorithm, which is an extension of algorithm $\mathbf{A}_0$, determines an optimal solution of $\mathbf{P}_1$ that satisfies these properties.

Algorithm $\mathbf{A}_1$

Preprocessing:

Re-index the jobs in such a way that $\{J_{ij} \mid j = 1, \ldots, n_i\}$ are processed in increasing order of j in solution π^* for all $i = 1, 2, \ldots, k$. For each $(q_1, \ldots, q_k)$, where $q_i = 0, 1, \ldots, n_i$ for $i = 1, 2, \ldots, k$, determine $\sum_{i=1}^{k} \sum_{j=1}^{q_i} p_{ij}$. Determine the elements of $\mathcal{E}_1$ and arrange them in ascending order.

Value function:

$f(q_1, \ldots, q_k; t)$ = minimum total cost $\tilde{\Gamma}_1(\sigma')$ of a partial schedule σ' for processing and delivering jobs $J_{i1}, J_{i2}, \ldots, J_{iq_i}$, for $i = 1, 2, \ldots, k$, such that the departure time of the last shipment is equal to t.

Recursive relation: For $q_i = 0, 1, \ldots, n_i$ $(i = 1, 2, \ldots, k)$ such that $\sum_{i=1}^{k} q_i \geq 1$, and for $t \in \mathcal{E}_1$ such that $t \geq \sum_{i=1}^{k} \sum_{j=1}^{q_i} p_{ij}$,

$$\begin{aligned} &f(q_1, \ldots, q_k; t) \\ &= \min \Bigg\{ f(q_1, \ldots, q_{h-1}, q'_h, q_{h+1}, \ldots, q_k; t') + \alpha(q_h - q'_h)t + \beta\phi_h + \gamma \sum_{j=q'_h+1}^{q_h} |t - E_{hj}(\pi^*)| \;\Bigg| \\ &\qquad\qquad h = 1, \ldots, k;\ 0 \leq q'_h < q_h \text{ such that } q_h - q'_h \leq c;\ t' \in \mathcal{E}_1 \cup \{0\} \text{ such that } t' \leq t - \delta \Bigg\}. \end{aligned}$$

Boundary conditions:

$f(0, \ldots, 0; 0) = 0$;

$f(0, \ldots, 0; t) = +\infty$ if $t \in \mathcal{E}_1$;

$f(q_1, \ldots, q_k; t) = +\infty$ if $t \in \mathcal{E}_1 \cup \{0\}$ and $t < \sum_{i=1}^{k} \sum_{j=1}^{q_i} p_{ij}$.

Optimal solution value: $\min \{ f(n_1, \ldots, n_k; t) \mid t \in \mathcal{E}_1;\ t \geq \sum_{i=1}^{k} \sum_{j=1}^{n_i} p_{ij} \}$.

Theorem 2 *Algorithm $\mathbf{A}_1$ finds an optimal solution for problem $\mathbf{P}_1$ in $O(cn^{3k+2})$ time.*

Proof. Suppose that a partial production and delivery schedule comprising the first q_i jobs of customer K_i, for $i = 1, 2, \ldots, k$, has been formed. Let t be the departure time of the last shipment in this partial schedule, and suppose the last shipment is for customer K_h and contains $q_h - q'_h$ jobs. Then, the departure time t' of the second last shipment must be equal to one of the values in $\mathcal{E}_1$ (if the last shipment is not the only shipment in the partial schedule). Property (C3) and the minimum headway constraint imply that $t' \leq t - \delta$. Thus, the recursive relation enumerates all possible departure times t' of the second last shipment of this partial schedule. In addition, the recursive relation enumerates all possible values of h and q'_h. Hence, algorithm $\mathbf{A}_1$ compares all possible solutions that satisfy the properties in Lemmas 2 and 3.

Because $\mathcal{E}_1$ contains $O(n^{k+1})$ elements and there are $O(n^k)$ combinations of $q_1, q_2, \ldots, q_k$, the recurrence relation of $\mathbf{A}_1$ is executed $O(n^{2k+1})$ times. Each execution of the recurrence relation enumerates $O(1)$ possible customers K_h for the last shipment, $O(c)$ possible values of q'_h, and $O(n^{k+1})$ possible values of t'. In each execution of the recurrence relation, the summation "$\sum_{j=q'_h+1}^{q_h} |t - E_{hj}(\pi^*)|$" can be determined recursively. Thus, each execution of the recurrence relation requires $O(cn^{k+1})$ time. Hence, algorithm $\mathbf{A}_1$ requires $O(cn^{3k+2})$ time. ∎

Note that the computational complexity of algorithm $\mathbf{A}_1$ is higher than that of algorithm $\mathbf{A}_0$. This is because problem $\mathbf{P}_0$ possesses more optimality properties than problem $\mathbf{P}_1$. Specifically, property (C5) does not apply to problem $\mathbf{P}_1$. Hence, algorithm $\mathbf{A}_1$ needs to enumerate more possible departure times for the shipments than algorithm $\mathbf{A}_0$.

To see why property (C5) does not apply to $\mathbf{P}_1$, consider the following example with a single customer and zero minimum headway: $\delta = 0$, $k = 1$, $n_1 = 56$, $p_{1,1} = p_{1,2} = \cdots = p_{1,56} = 1$, $c = 8$, $\alpha = 20$, $\beta = 10$, $\gamma = 41$, $\phi_1 = 55$, and $T = 48$. An optimal solution π^* of problem $\mathbf{P}_0$ obtained by algorithm $\mathbf{A}_0$ is shown in Figure 1(a), where the jobs $\{J_{1j} \mid j = 1, \ldots, 56\}$ are processed in increasing order of j. In this solution, $r_1(\pi^*) = 8$, $E_{1j}(\pi^*) = 7\lceil j/7 \rceil$ $(j = 1, 2, \ldots, 56)$, and $\tilde{\Gamma}_0(\pi^*) = \alpha \sum_{j=1}^{n_1} E_{1j}(\pi^*) + \beta r_1(\pi^*)\phi_1 = (20)(1764) + (10)(8)(55) = 39680$. When the delivery service is unavailable during the time period $[0, 48)$, a unique solution σ^* of problem $\mathbf{P}_1$ obtained

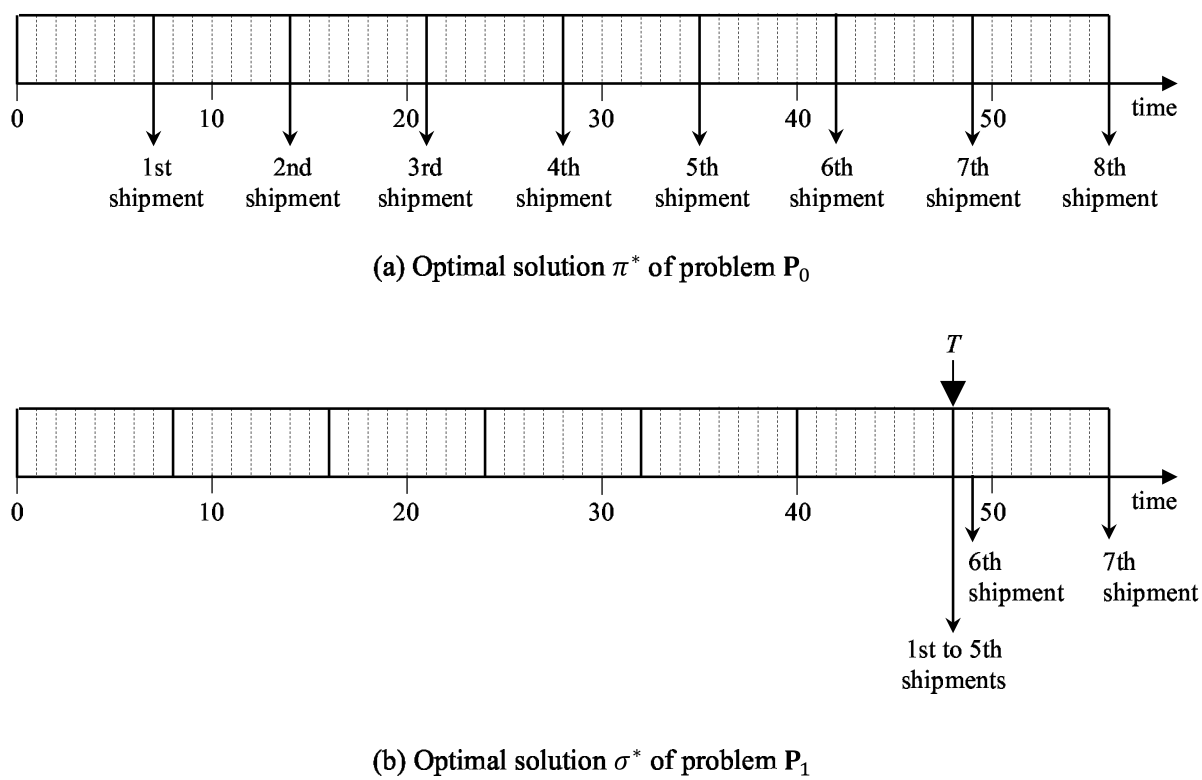

Figure 1: An example of problem $\mathbf{P}_1$.

by algorithm $\mathbf{A}_1$ is shown in Figure 1(b), where $r_1(\sigma^*) = 7$,

$$E_{1j}(\sigma^*) = \begin{cases} 48, & \text{for } j = 1, 2, \ldots, 40; \\ 49, & \text{for } j = 41, 42, \ldots, 48; \\ 56, & \text{for } j = 49, 50, \ldots, 56; \end{cases}$$

and $\tilde{\Gamma}_1(\sigma^*) = \alpha \sum_{j=1}^{n_1} E_{1j}(\sigma^*) + \beta r_1(\sigma^*)\phi_1 + \gamma \sum_{j=1}^{n_1} |E_{1j}(\sigma^*) - E_{1j}(\pi^*)| = (20)(2760) + (10)(7)(55) + (41)(996) = 99886$. In this optimal solution, the 6th delivery shipment departs from the production facility at time 49, even though this shipment is ready for delivery at time 48. This is because

$$E_{1j}(\pi^*) = \begin{cases} 42, & \text{for } j = 41, 42; \\ 49, & \text{for } j = 43, 44, \ldots, 48; \end{cases}$$

which implies that jobs $41, 42, \ldots, 48$ have a total delivery time disruption of $2 \cdot |48-42| + 6 \cdot |48-49| = 18$ if the 6th shipment departs at time 48, and have a smaller total delivery time disruption of $2 \cdot |49-42| + 6 \cdot |49-49| = 14$ if the 6th shipment departs at time 49. In this example, the optimal solution σ^* does not satisfy property (C5).

3.2 Solution Method for Problem $\mathbf{P}_2$

The following lemma provides some optimality properties of problem $\mathbf{P}_2$.

Lemma 4 *If the given instance of problem $\mathbf{P}_2$ is feasible, then there exists an optimal solution which satisfies the properties in Lemma 2 such that the departure time of each shipment is equal to one of the following values: (i) T; (ii) one of the values in $\mathcal{E}_0$; (iii) one of the values in $\mathcal{E}_0$ minus θ; or (iv) the departure time of the previous shipment plus δ (if the current shipment is not the first shipment).*

Define

$$\mathcal{E}_2 = \left[\left\{ \sum_{i=1}^{k} \sum_{j=1}^{q_i} p_{ij} - z\theta + y\delta \;\middle|\; q_i = 0, 1, \ldots, n_i \text{ for } i = 1, \ldots, k;\; z = 0, 1;\; y = 0, 1, \ldots, n-1 \right\} \right. \\ \left. \cup \left\{ T - z\theta + y\delta \;\middle|\; z = 0, 1;\; y = 0, 1, \ldots, n-1 \right\} \right] \cap [T, +\infty).$$

The set $\mathcal{E}_2$ contains all possible shipment departure times in an optimal solution that satisfies the properties in Lemma 4. The following dynamic programming algorithm, which is an extension of algorithm $\mathbf{A}_0$, solves problem $\mathbf{P}_2$ when the problem is feasible and returns an infinite optimal solution value when the problem is infeasible.

Algorithm $\mathbf{A}_2$

Preprocessing:

Re-index the jobs in such a way that $\{J_{ij} \mid j = 1, \ldots, n_i\}$ are processed in increasing order of j in solution π^* for all $i = 1, 2, \ldots, k$. For each $(q_1, \ldots, q_k)$, where $q_i = 0, 1, \ldots, n_i$ for $i = 1, 2, \ldots, k$, determine $\sum_{i=1}^{k} \sum_{j=1}^{q_i} p_{ij}$. Determine the elements of $\mathcal{E}_2$ and arrange them in ascending order.

Value function:

$f(q_1, \ldots, q_k; t) =$ minimum total cost $\tilde{\Gamma}_2(\sigma')$ of a partial schedule σ' for processing and delivering jobs $J_{i1}, J_{i2}, \ldots, J_{iq_i}$, for $i = 1, 2, \ldots, k$, such that the departure time of the last shipment is equal to t.

Recursive relation: For $q_i = 0, 1, \ldots, n_i$ $(i = 1, 2, \ldots, k)$ such that $\sum_{i=1}^{k} q_i \geq 1$, and for $t \in \mathcal{E}_2$ such that $t \geq \sum_{i=1}^{k} \sum_{j=1}^{q_i} p_{ij}$ and $\min_{h=1,\ldots,k \text{ s.t. } q_h > 0} |t - E_{hq_h}(\pi^*)| \leq \theta$,

$$\begin{aligned} f(q_1, \ldots, q_k; t) = \min \Big\{ & f(q_1, \ldots, q_{h-1}, q_h', q_{h+1}, \ldots, q_k; t') + \alpha(q_h - q_h')t + \beta\phi_h \ \Big| \\ & h = 1, \ldots, k \text{ and } 0 \leq q_h' < q_h \text{ such that } q_h - q_h' \leq c,\ |t - E_{hq_h}(\pi^*)| \leq \theta, \\ & \text{and } |t - E_{h,q_h'+1}(\pi^*)| \leq \theta;\ t' \in \mathcal{E}_2 \cup \{0\} \text{ such that } t' \leq t - \delta \Big\}. \end{aligned}$$

Boundary conditions:

$f(0, \ldots, 0; 0) = 0$;

$f(0, \ldots, 0; t) = +\infty$ if $t \in \mathcal{E}_2$;

$f(q_1, \ldots, q_k; t) = +\infty$ if $t \in \mathcal{E}_2 \cup \{0\}$ and $t < \sum_{i=1}^{k} \sum_{j=1}^{q_i} p_{ij}$;

$f(q_1, \ldots, q_k; t) = +\infty$ if $\min_{h=1,\ldots,k \text{ s.t. } q_h > 0} |t - E_{hq_h}(\pi^*)| > \theta$.

Optimal solution value: $\min \big\{ f(n_1, \ldots, n_k; t) \ \big|\ t \in \mathcal{E}_2;\ t \geq \sum_{i=1}^{k} \sum_{j=1}^{n_i} p_{ij};\ \min_{h=1,\ldots,k} |t - E_{hn_h}(\pi^*)| \leq \theta \big\}$.

Theorem 3 *Algorithm $\mathbf{A}_2$ either detects infeasibility of or finds an optimal solution for problem $\mathbf{P}_2$ in $O(cn^{3k+2})$ time.*

Proof. Suppose that a partial production and delivery schedule comprising the first q_i jobs of customer K_i, for $i = 1, 2, \ldots, k$, has been formed. If the last shipment is for customer K_h and contains $q_h - q'_h$ jobs, then by Lemma 4, the departure time t' of the second last shipment must be equal to one of the values in $\mathcal{E}_2$ (if the last shipment is not the only shipment in the partial schedule). Property (C3) and the minimum headway constraint imply that $t' \leq t - \delta$. Thus, the recursive relation enumerates all possible values of t'. Since $E_{h,q'_h+1}(\pi^*) \leq E_{h,q'_h+2}(\pi^*) \leq \cdots \leq E_{hq_h}(\pi^*)$, the conditions "$|t - E_{hq_h}(\pi^*)| \leq \theta$" and "$|t - E_{h,q'_h+1}(\pi^*)| \leq \theta$" in the recurrence relation are satisfied if and only if $|t - E_{hj}(\pi^*)| \leq \theta$ for $j = q'_h + 1, q'_h + 2, \ldots, q_h$, or equivalently, $\max_{j=q'_h+1,\ldots,q_h} |t - E_{hj}(\pi^*)| \leq \theta$. Hence, the recursive relation enumerates all possible values of h and q'_h that satisfy constraint (2). The inequality "$\min_{h=1,\ldots,k \text{ s.t. } q_h>0} |t - E_{hq_h}(\pi^*)| \leq \theta$" ensures that there exist some h and q'_h values which satisfy constraint (2). If this inequality is violated, then the boundary condition sets $f(q_1, \ldots, q_k; t)$ to $+\infty$. Therefore, algorithm $\mathbf{A}_2$ compares all possible feasible solutions that satisfy the properties in Lemma 4, and it finds an optimal solution of problem $\mathbf{P}_2$ if the optimal solution value is finite.

Because $\mathcal{E}_2$ contains $O(n^{k+1})$ elements and there are $O(n^k)$ combinations of $q_1, q_2, \ldots, q_k$, the recurrence relation of $\mathbf{A}_2$ is executed $O(n^{2k+1})$ times. Each execution of the recurrence relation enumerates $O(1)$ possible customers K_h for the last shipment, $O(c)$ possible values of q'_h, and $O(n^{k+1})$ possible values of t'. Thus, each execution of the recurrence relation requires $O(cn^{k+1})$ time. Hence, algorithm $\mathbf{A}_2$ requires $O(cn^{3k+2})$ time. ∎

3.3 Solution Method for Problem $\mathbf{P}_3$

The main idea of our solution method for problem $\mathbf{P}_3$ is to develop a set $\mathcal{S}$ that contains all possible maximum delivery time disruptions of an optimal solution. Then, we can restrict our search for an optimal maximum delivery time disruption to the elements of $\mathcal{S}$. Note that in any solution of problem $\mathbf{P}_3$, there is a "bottleneck job" whose delivery time disruption is the largest among all jobs. To develop the set $\mathcal{S}$, we first construct a set $\mathcal{E}_3$ that contains all possible shipment departure times of a bottleneck job in an optimal solution of problem $\mathbf{P}_3$.

For notational convenience, we assume, without loss of generality, that the jobs $\{J_{ij} \mid j =$

$1,\ldots,n_i\}$ are processed in increasing order of j in solution π^* for all $i=1,2,\ldots,k$. Under this assumption, $E_{i1}(\pi^*)\le E_{i2}(\pi^*)\le\cdots\le E_{in_i}(\pi^*)$. Define

$$\begin{aligned}\mathcal{E}_3=\Bigg[&\left\{\sum_{i=1}^{k}\sum_{j=1}^{q_i}p_{ij}\pm y\delta\;\middle|\;q_i=0,1,\ldots,n_i\text{ for }i=1,\ldots,k;\ y=0,1,\ldots,n-1\right\}\\&\cup\left\{\frac{E_{ij}(\pi^*)+E_{hl}(\pi^*)-y\delta}{2}\;\middle|\;i,h=1,\ldots,k;\,j=1,\ldots,n_i;\,l=1,\ldots,n_h;\,y=0,1,\ldots,n-1\right\}\\&\cup\big\{T+y\delta\;\big|\;y=0,1,\ldots,n-1\big\}\Bigg]\cap[T,+\infty)\end{aligned}$$

The rationale behind the definition of $\mathcal{E}_3$ is as follows. Any solution of problem $\mathbf{P}_3$ contains different groups of consecutive shipments, where the departure times of two consecutive shipments within each group is δ time units apart. In the definition of $\mathcal{E}_3$, the term "$\sum_{i=1}^{k}\sum_{j=1}^{q_i}p_{ij}\pm y\delta$" represents the departure time of the bottleneck job when this bottleneck job belongs to a group of consecutive shipments in which one of the shipments' departure time is $\sum_{i=1}^{k}\sum_{j=1}^{q_i}p_{ij}$. Let θ denote the maximum delivery time disruption of the solution. The term "$\frac{E_{ij}(\pi^*)+E_{hl}(\pi^*)-y\delta}{2}$" represents the departure time of the bottleneck job when this bottleneck job belongs to a shipment with departure time $E_{ij}(\pi^*)-\theta$, while another bottleneck job with a later departure time $E_{hl}(\pi^*)+\theta$ exists in the same group of consecutive shipments, and the departure time of these two shipments are $y\delta$ time units apart (thus, $[E_{hl}(\pi^*)+\theta]-[E_{ij}(\pi^*)-\theta]=y\delta$, or equivalently, $E_{ij}(\pi^*)-\theta=\frac{E_{ij}(\pi^*)+E_{hl}(\pi^*)-y\delta}{2}$). The term "$T+y\delta$" represents the departure time of the bottleneck job when this bottleneck job belongs to a group of consecutive shipments in which the first shipment's departure time is T.

Define

$$\mathcal{S}=\big\{|t-E_{ij}(\pi^*)|\;\big|\;t\in\mathcal{E}_3;\ i=1,\ldots,k;\ j=1,\ldots,n_i\big\}.$$

Because $\mathcal{E}_3$ contains all possible shipment departure times of a bottleneck job in an optimal solution of $\mathbf{P}_3$, set $\mathcal{S}$ contains all possible maximum delivery time disruptions of this optimal solution. Lemma 5 below and its proof provide a formal argument of this idea.

For any solution σ of problem $\mathbf{P}_3$, let

$$\Delta_{ij}(\sigma)=|E_{ij}(\sigma)-E_{ij}(\pi^*)|$$

denote the delivery time disruption of job J_{ij} in this solution, and let

$$\Delta_{\max}(\sigma) = \max_{i=1,\ldots,k;\, j=1,\ldots,n_i} \{\Delta_{ij}(\sigma)\}$$

denote the maximum delivery time disruption of this solution.

Lemma 5 *There exists an optimal solution σ^* of problem $\mathbf{P}_3$ such that $\Delta_{\max}(\sigma^*) \in \mathcal{S}$.*

We now present an algorithm which determines an optimal solution of $\mathbf{P}_3$.

Algorithm $\mathbf{A}_3$

Step 1: For each $s \in \mathcal{S}$,

(a) solve the problem using algorithm $\mathbf{A}_2$ with θ set equal to s (and ignore the cost term $\gamma \max_{i=1,\ldots,k;\, j=1,\ldots,n_i} |D_{ij}(\sigma) - D_{ij}(\pi^*)|$ in the objective function $\Gamma_3(\gamma)$), let $\sigma(s)$ be the solution obtained, and $Z(s)$ be the objective value of this solution;

(b) let $Z'(s) = Z(s) + \gamma s$.

Step 2: Among $\{\sigma(s) \mid s \in \mathcal{S}\}$, select the solution with the smallest $Z'(s)$ value.

Theorem 4 *Algorithm $\mathbf{A}_3$ finds an optimal solution of problem $\mathbf{P}_3$ in $O(cn^{4k+4})$ time.*

Proof. By Lemma 5, it suffices to consider candidate solutions with a maximum delivery time disruption in set $\mathcal{S}$. Suppose there exists such an optimal solution with a maximum delivery time disruption $s \in \mathcal{S}$. Then, an optimal solution of problem $\mathbf{P}_3$ can be obtained by applying algorithm $\mathbf{A}_2$ with θ set equal to s, and the optimal objective value of $\mathbf{P}_3$ is equal to the objective value of the solution generated by algorithm $\mathbf{A}_2$ plus γs. Hence, algorithm $\mathbf{A}_3$, which enumerates all possible s values, finds an optimal solution of $\mathbf{P}_3$.

Set $\mathcal{E}_3$ contains $O(n^{k+1})$ elements. Thus, $\mathcal{S}$ contains $O(n^{k+2})$ elements. Hence, Step 1 of algorithm $\mathbf{A}_3$ is executed $O(n^{k+2})$ times. By Theorem 3, each execution of Step 1 requires $O(cn^{3k+2})$ time. Therefore, the running time of algorithm $\mathbf{A}_3$ is $O(cn^{4k+4})$. ∎

Remark 2 *Algorithm $\mathbf{A}_3$ can be implemented more efficiently as follows. First, we precompute $Z(\infty)$; that is, the optimal value of $\tilde{\Gamma}_2(\sigma)$ when constraint* (2) *is relaxed. Then, we execute Step 1* *of* $\mathbf{A}_3$ *in increasing order of* s. *In each iteration, we keep track of the best* s *value obtained so far. Let* s_0 *denote the* s *value being considered. Then,* $s^* = \arg\min\{Z'(s) \mid s \leq s_0;\ s \in \mathcal{S}\}$ *is the best* s *value obtained so far. Note that* $Z'(s) \geq Z(\infty) + \gamma s$ *and that* $Z(\infty) + \gamma s$ *is nondecreasing in* s. *Hence, Step 1 can be terminated immediately once* $Z(\infty) + \gamma s \geq Z(s^*)$, *or equivalently* $s \geq [Z(s^*) - Z(\infty)]/\gamma$. *Another way to improve the efficiency of algorithm* $\mathbf{A}_3$ *is to consider only those* s *values that satisfy the condition "*$s \geq T - \min_{i=1,\ldots,k}\{E_{i1}(\pi^*)\}$.*" If this condition is violated, then in Step 1(a), algorithm* $\mathbf{A}_2$ *will return an infinite solution value when* θ *is set equal to* s. *Thus, it suffices to consider those* s *values that satisfy this condition.*

3.4 Computational Study

When the delivery service is shut down unexpectedly for the time period $[0, T)$, our rescheduling model aims to determine a revised production and delivery plan, with a mechanism to avoid serious disruption in the arrival time of the finished jobs at the customers' locations. On the other hand, managers who seek simplicity often make use of solution approaches that are easy to implement. Hence, we conduct computational experiments to investigate by how much the solutions generated by our rescheduling methods improve over the solution obtained by a simple solution approach which only adjusts the departure times of the delivery shipments without revising the production schedule. Specifically, we compare the optimal solutions of our models with a solution $\bar{\sigma}$ obtained as follows:

(i) Solution $\bar{\sigma}$ has the same processing sequence as the original schedule π^*.

(ii) Solution $\bar{\sigma}$ has the same delivery shipments as the original schedule π^*.

(iii) The departure time of the ℓth shipment in solution $\bar{\sigma}$ is $\min\{T + (\ell - 1)\delta, E^*_\ell\}$, where E^*_ℓ is the departure time of the ℓth shipment in the original schedule π^*.

In other words, solution $\bar{\sigma}$ uses the same processing sequence and delivery shipments as schedule π^*, and it attempts to deliver all delayed shipments as early as possible after the disruption period.

In this computational study, we analyze the improvement of our solution methods over the simple solution approach when solving the rescheduling problems $\mathbf{P}_1$ and $\mathbf{P}_3$. These two rescheduling problems impose penalties on the total time disruption of jobs and maximum time disruption of jobs, respectively, and are always feasible. For $h = 1, 3$, we let σ_h^* denote the optimal solution of problem $\mathbf{P}_h$, and let

$$I_h = \frac{\Gamma_h(\bar{\sigma}) - \Gamma_h(\sigma_h^*)}{\Gamma_h(\bar{\sigma})} \times 100\%,$$

which is the percentage improvement of solution σ_h^* over solution $\bar{\sigma}$.

The test data are selected as follows. We set $\alpha = \beta = 1$, $\delta = 10$, $c = 3$, $\phi_1 = \phi_2 = \cdots = \phi_k = 10$, and $T = 30$. Job processing times are randomly generated, with each p_{ij} being uniformly distributed in $\{1, 2, \ldots, 10\}$. For each job $J_j \in \mathcal{J}$, we randomly assign a customer from $\{K_1, K_2, \ldots, K_k\}$ to J_j with equal probability. Since $\alpha \sum_{i=1}^{k} n_i \tau_i$ is a constant, for simplicity, we set $\tau_1 = \tau_2 = \cdots = \tau_k = 0$. For problem $\mathbf{P}_1$, we consider $n \in \{10, 20, 40, 80\}$, $k \in \{1, 2, 3\}$, and $\gamma \in \{\frac{1}{4}, 1, 4\}$. Thus, there are 36 combinations of n, k, and γ values for $\mathbf{P}_1$. For problem $\mathbf{P}_3$, because the penalty γ is applied to the maximum of n jobs instead of the sum of n jobs, we consider $n \in \{10, 20, 40, 80\}$, $k \in \{1, 2, 3\}$, and $\gamma \in \{\frac{n}{16}, \frac{n}{4}, n\}$, so that a heavier unit delivery time disruption penalty is imposed on larger instances. Thus, there are also 36 combinations of n, k, and γ values for $\mathbf{P}_3$. Different n and k values correspond to different problem sizes, while different γ values correspond to different levels of control of delivery time disruption. For each combination of n and k, we generate 10 random test instances. For each test instance, we determine σ_1^* using algorithm $\mathbf{A}_1$, determine σ_3^* using algorithm $\mathbf{A}_3$, determine $\bar{\sigma}$, and then compute I_1 and I_3. Algorithms $\mathbf{A}_1$ and $\mathbf{A}_3$ are coded in C++, and the experiments are run on a computer with an Intel Core i7-7700HQ 2.80-GHz CPU and 32 GB of RAM. The efficiency improvement methods described in Remark 2 are used when $\mathbf{A}_3$ is implemented.

Table 1 summarizes the results of the computational study, where each row reports the mean result of the 10 random instances. The running times of algorithms $\mathbf{A}_1$ and $\mathbf{A}_3$ are reported in the "Time" columns. From the computational results, we observe that I_1 and I_3 decrease as n increases. In other words, the effectiveness of solution $\bar{\sigma}$ increases as the number of jobs increases. Thus, compared to the simple solution approach, algorithms $\mathbf{A}_1$ and $\mathbf{A}_3$ can obtain significant cost saving when the number of jobs is small. When the number of jobs is large, solution $\bar{\sigma}$ is effective. Hence, for large size problems, the simple solution approach is an alternative method for obtaining

Table 1: Computational results.

n	k	Problem $\mathbf{P}_1$						Problem $\mathbf{P}_3$					
		$\gamma = \frac{1}{4}$		$\gamma = 1$		$\gamma = 4$		$\gamma = \frac{n}{16}$		$\gamma = \frac{n}{4}$		$\gamma = n$	
		I_1 (%)	Time (sec.)	I_1 (%)	Time (sec.)	I_1 (%)	Time (sec.)	I_3 (%)	Time (sec.)	I_3 (%)	Time (sec.)	I_3 (%)	Time (sec.)
10	1	9.70	0.0	11.51	0.0	15.01	0.0	8.62	0.0	7.92	0.0	5.96	0.0
	2	11.39	0.0	12.56	0.0	15.00	0.0	9.88	0.0	8.21	0.0	6.27	0.0
	3	10.83	0.1	12.02	0.1	14.80	0.1	9.64	0.1	8.41	0.0	6.43	0.0
20	1	5.70	0.1	7.37	0.1	11.04	0.1	5.00	0.0	4.69	0.0	3.75	0.0
	2	5.20	0.7	6.04	0.7	8.39	0.7	4.40	0.4	4.07	0.2	3.23	0.2
	3	7.30	2.4	8.19	2.4	10.99	2.4	6.12	1.5	4.60	1.1	3.67	0.7
40	1	2.22	0.7	2.99	0.7	5.05	0.7	1.94	0.2	1.86	0.2	1.58	0.2
	2	2.89	10.8	3.83	10.8	6.37	10.9	2.40	4.0	2.28	3.1	1.95	2.9
	3	2.34	75.5	2.88	74.8	4.69	75.9	1.72	48.0	1.57	26.3	1.31	21.5
80	1	0.86	6.5	1.24	6.5	2.38	6.5	0.73	1.6	0.71	1.6	0.65	1.6
	2	1.18	172.2	1.67	173.0	3.21	173.0	0.90	55.3	0.86	46.9	0.78	44.7
	3	1.21	1072.6	1.68	1068.5	3.18	1072.6	0.88	365.0	0.86	301.2	0.78	285.4

approximation solutions efficiently.

We also observe that I_1 increases as γ increases, while I_3 decreases as γ increases. This is because in problem $\mathbf{P}_1$ the quantity $\sum_{i=1}^{k}\sum_{j=1}^{n_i}|E_{ij}(\sigma^*)-E_{ij}(\pi^*)|$ obtained by algorithm $\mathbf{A}_1$ is significantly smaller than the quantity $\sum_{i=1}^{k}\sum_{j=1}^{n_i}|E_{ij}(\bar{\sigma})-E_{ij}(\pi^*)|$ obtained by the simple solution approach, while in problem $\mathbf{P}_3$ the quantity $\max_{i=1,\ldots,k;\, j=1,\ldots,n_i}|E_{ij}(\sigma^*)-E_{ij}(\pi^*)|$ obtained by algorithm $\mathbf{A}_3$ is quite close to the quantity $\max_{i=1,\ldots,k;\, j=1,\ldots,n_i}|E_{ij}(\bar{\sigma})-E_{ij}(\pi^*)|$. Thus, the numerator of I_1 increases significantly as γ increases, while the numerator of I_3 is relatively insensitive to an increase in γ.

The running times of both algorithms $\mathbf{A}_1$ and $\mathbf{A}_3$ increase as n and k increase. According to Theorems 2 and 4, algorithm $\mathbf{A}_3$ has significantly higher computational complexity than algorithm $\mathbf{A}_1$. However, as shown in Table 1, algorithm $\mathbf{A}_3$ can be executed more efficiently than algorithm $\mathbf{A}_1$ by using the implementation techniques presented in Remark 2. Note that the running time of algorithm $\mathbf{A}_3$ decreases as γ increases. This is because when γ gets larger, the first implementation method presented in Remark 2 becomes more effective in reducing the number of s values that need to be enumerated.

4 Special Case with No Minimum Headway Constraint

If there is no minimum headway constraint (i.e., $\delta = 0$), the rescheduling problems can be solved more efficiently. In this section, we discuss the computational complexities of problems $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$ when $\delta = 0$.

When $\delta = 0$, it is easy to show that there exists an optimal solution of $\mathbf{P}_1$ which satisfies the properties in Lemma 2 such that the departure time of each shipment is equal to one of the following values: (i) T; (ii) completion time of production of the shipment's last job; or (iii) departure time of one of the shipment's jobs in solution π^*. The following dynamic program determines such an optimal solution.

Algorithm $\mathbf{A}_1'$

Preprocessing:

Re-index the jobs in such a way that $\{J_{ij} \mid j = 1, \ldots, n_i\}$ are processed in increasing order of j in solution π^* for all $i = 1, 2, \ldots, k$. For each $(q_1, \ldots, q_k)$, where $q_i = 0, 1, \ldots, n_i$ for $i = 1, 2, \ldots, k$, determine $\sum_{i=1}^{k} \sum_{j=1}^{q_i} p_{ij}$.

Value function:

$f(q_1, \ldots, q_k)$ = minimum total cost $\tilde{\Gamma}_1(\sigma')$ of a partial schedule σ' for processing and delivering jobs $J_{i1}, J_{i2}, \ldots, J_{iq_i}$, for $i = 1, 2, \ldots, k$.

Recursive relation: For $q_i = 0, 1, \ldots, n_i$ $(i = 1, 2, \ldots, k)$ such that $\sum_{i=1}^{k} q_i \geq 1$,

$$f(q_1, \ldots, q_k) = \min \Bigg\{ f(q_1, \ldots, q_{h-1}, q_h', q_{h+1}, \ldots, q_k) + \alpha(q_h - q_h')s + \beta\phi_h + \gamma \sum_{j=q_h'+1}^{q_h} |s - E_{hj}(\pi^*)| \ \Bigg| \ h = 1, \ldots, k;\ 0 \leq q_h' < q_h \text{ such that } q_h - q_h' \leq c; \ s \in S_h(q_1, \ldots, q_k; q_h') \text{ such that } s \geq \tilde{s}(q_1, \ldots, q_k) \Bigg\},$$

where $\tilde{s}(q_1, \ldots, q_k) = \max\{T, \sum_{i=1}^{k} \sum_{j=1}^{q_i} p_{ij}\}$ and $S_h(q_1, \ldots, q_k; q_h') = \big\{\tilde{s}(q_1, \ldots, q_k), E_{h,q_h'+1}(\pi^*), E_{h,q_h'+2}(\pi^*), \ldots, E_{hq_h}(\pi^*)\big\}$.

Boundary condition: $f(0, \ldots, 0) = 0$.

Optimal solution value: $f(n_1, \ldots, n_k)$.

Suppose that a partial production and delivery schedule comprising the first q_i jobs of customer K_i, for $i = 1, 2, \ldots, k$, has been formed. If the last shipment is for customer K_h and contains $q_h - q'_h$ jobs, then the departure time of the last shipment must be equal to one of the values in $S_h(q_1, \ldots, q_k; q'_h)$. The departure time of the last shipment must also be at least $\tilde{s}(q_1, \ldots, q_k)$. Thus, the above recursive relation enumerates all possible departure times s of the last shipment of this partial schedule. In addition, the recursive relation enumerates all possible values of h and q'_h. The recurrence relation is executed $O(n^k)$ times, and each execution of the recurrence relation enumerates $O(1)$ possible customers K_h for the last shipment, $O(c)$ possible values of q'_h, and $O(c)$ elements of $S_h(q_1, \ldots, q_k; q'_h)$. Therefore, we have the following result.

Theorem 5 *When $\delta = 0$, problem $\mathbf{P}_1$ can be solved in $O(c^2 n^k)$ time.*

When $\delta = 0$, it is easy to determine if a given instance of problem $\mathbf{P}_2$ is feasible by checking whether or not $\theta \geq T - \min_{i=1,\ldots,k;\, j=1,\ldots,n_i}\{E_{ij}(\pi^*)\}$. It is easy to show that if $\delta = 0$ and the given instance of $\mathbf{P}_2$ is feasible, then there exists an optimal solution of $\mathbf{P}_2$ which satisfies the properties in Lemma 2 such that the departure time of each shipment is equal to the largest of the following values: (i) T; (ii) completion time of production of the shipment's last job; or (iii) departure time of the shipment's last job in solution π^* minus θ. The following dynamic program either determines such an optimal solution or returns an infinite optimal solution value.

Algorithm $\mathbf{A}'_2$

Preprocessing:

Re-index the jobs in such a way that $\{J_{ij} \mid j = 1, \ldots, n_i\}$ are processed in increasing order of j in solution π^* for all $i = 1, 2, \ldots, k$. For each $(q_1, \ldots, q_k)$, where $q_i = 0, 1, \ldots, n_i$ for $i = 1, 2, \ldots, k$, determine $\sum_{i=1}^{k} \sum_{j=1}^{q_i} p_{ij}$.

Value function:

$f(q_1, \ldots, q_k) =$ minimum total cost $\tilde{\Gamma}_2(\sigma')$ of a partial schedule σ' for processing and delivering jobs $J_{i1}, J_{i2}, \ldots, J_{iq_i}$, for $i = 1, 2, \ldots, k$.

Recursive relation: For $q_i = 0, 1, \ldots, n_i$ $(i = 1, 2, \ldots, k)$ such that $\sum_{i=1}^{k} q_i \geq 1$ and that

$\max_{h=1,\ldots,k \text{ s.t. } q_h>0}\{E_{hq_h}(\pi^*)+\theta\} \geq \max\{T, \sum_{i=1}^{k}\sum_{j=1}^{q_i} p_{ij}\}$,

$$f(q_1,\ldots,q_k) = \min\Big\{f(q_1,\ldots,q_{h-1},q'_h,q_{h+1},\ldots,q_k)+\alpha(q_h-q'_h)s_h(q_1,\ldots,q_k)+\beta\phi_h \;\Big|\; h=1,\ldots,k; \\ 0 \leq q'_h < q_h \text{ such that } q_h-q'_h \leq c \text{ and } E_{h,q'_h+1}(\pi^*)+\theta \geq s_h(q_1,\ldots,q_k)\Big\},$$

where $s_h(q_1,\ldots,q_k) = \max\{T, \sum_{i=1}^{k}\sum_{j=1}^{q_i} p_{ij}, E_{hq_h}(\pi^*)-\theta\}$.

Boundary conditions:

$f(0,\ldots,0)=0$;

$f(q_1,\ldots,q_k)=+\infty$ if $\max_{h=1,\ldots,k \text{ s.t. } q_h>0}\{E_{hq_h}(\pi^*)+\theta\} < \max\{T, \sum_{i=1}^{k}\sum_{j=1}^{q_i} p_{ij}\}$.

Optimal solution value: $f(n_1,\ldots,n_k)$.

Suppose that a partial production and delivery schedule comprising the first q_i jobs of customer K_i, for $i=1,2,\ldots,k$, has been formed. If the last shipment is for customer K_h and contains $q_h-q'_h$ jobs, then the departure time of the last shipment must be equal to $s_h(q_1,\ldots,q_k)$. Note that $s_h(q_1,\ldots,q_k) \geq E_{hq_h}(\pi^*)-\theta \geq E_{hj}(\pi^*)-\theta$ for $j=q'_h+1,q'_h+2,\ldots,q_h$. Thus, the condition "$E_{h,q'_h+1}(\pi^*)+\theta \geq s_h(q_1,\ldots,q_k)$" in the recurrence relation is satisfied if and only if $E_{hj}(\pi^*)-\theta \leq s_h(q_1,\ldots,q_k) \leq E_{hj}(\pi^*)+\theta$ for $j=q'_h+1,\ldots,q_h$, or equivalently, $\max_{j=q'_h+1,\ldots,q_h}|s_h(q_1,\ldots,q_k)-E_{hj}(\pi^*)| \leq \theta$. Hence, the recursive relation enumerates all possible values of h and q'_h such that the partial schedule satisfies constraint (2). The inequality "$\max_{h=1,\ldots,k \text{ s.t. } q_h>0}\{E_{hq_h}(\pi^*)+\theta\} \geq \max\{T, \sum_{i=1}^{k}\sum_{j=1}^{q_i} p_{ij}\}$" ensures that there exist some h and q'_h values such that constraint (2) is satisfied. If this inequality is violated, then the boundary condition sets $f(q_1,\ldots,q_k)$ to $+\infty$. The recurrence relation is executed $O(n^k)$ times and each execution of the recurrence relation requires $O(c)$ time. Therefore, we have the following result.

Theorem 6 *When $\delta=0$, problem $\mathbf{P}_2$ can be solved in $O(cn^k)$ time.*

Algorithms $\mathbf{A}'_1$ and $\mathbf{A}'_2$ have a similar structure as some dynamic programs developed for other integrated production and distribution models (see, e.g., Hall and Potts 2003, Chen and Vairaktarakis 2005, and Li et al. 2017). However, in the recursive relation of $\mathbf{A}'_1$ there is a need to search the values of s in the set $S_h(q_1,\ldots,q_k;q'_h)$, and in the recursive relation of $\mathbf{A}'_2$ the search of q'_h is restricted by the constraint "$E_{h,q'_h+1}(\pi^*)+\theta \geq s_h(q_1,\ldots,q_k)$."

If $\delta = 0$, then when applying algorithm $\mathbf{A}_3$ to solve problem $\mathbf{P}_3$, we may replace algorithm $\mathbf{A}_2$ by algorithm $\mathbf{A}'_2$ in Step 1(a) of algorithm $\mathbf{A}_3$. This can reduce the running time of each execution of Step 1 to $O(cn^k)$. Note that when $\delta = 0$, we have $|\mathcal{E}_3| = O(n^k)$ and $|\mathcal{S}| = O(n^{k+1})$, and thus Step 1 of algorithm $\mathbf{A}_3$ is executed $O(n^{k+1})$ times. Therefore, we have the following result.

Theorem 7 *When $\delta = 0$, problem $\mathbf{P}_3$ can be solved in $O(cn^{2k+1})$ time.*

Table 2 summarizes the complexity results of the case with $\delta > 0$ and the case with $\delta = 0$. Note that the computational complexities of the solution methods for problems $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$ are significantly higher when $\delta > 0$. This is because when δ is positive, the number of possible shipment departure times in the optimal solutions is significantly larger.

Table 2: Computational complexities of solution methods.

	Problem $\mathbf{P}_1$	Problem $\mathbf{P}_2$	Problem $\mathbf{P}_3$
$\delta > 0$	$O(cn^{3k+2})$	$O(cn^{3k+2})$	$O(cn^{4k+4})$
$\delta = 0$	$O(c^2n^k)$	$O(cn^k)$	$O(cn^{2k+1})$

5 Conclusions

This paper considers an integrated production and outbound distribution model and analyzes the job rescheduling decision in response to a disruption in the delivery service. We develop polynomial-time dynamic programming algorithms for three variants of this rescheduling problem. The main idea in the development of these algorithms is the construction of the sets $\mathcal{E}_1$, $\mathcal{E}_2$, and $\mathcal{E}_3$, which contain all the possible shipment departure times in the optimal solutions of the three problem variants, and the sizes of $\mathcal{E}_1$, $\mathcal{E}_2$, and $\mathcal{E}_3$ are polynomial in the input size of the problem. These sets restrict the search space of the optimal solutions, and thus enable us to develop polynomial-time algorithms. We also analyze the computational complexity of the special case with no minimum headway constraint.

Some future directions on this research are of interest. First, the polynomial-time solvability of our algorithms is based on the assumption that the number of customers, k, is fixed. Models with this assumption are applicable to situations where the finished jobs are delivered to customers located at a small number of fixed locations, or situations where the travel time between the production facility and a customer location has a fixed number of possibilities. In practice, it is common for different jobs to belong to different customers, with each customer having his/her own location. Thus, it would be worthwhile to develop efficient solution methods for the case where k is arbitrary. Second, in our model we made an assumption that sufficient delivery vehicles are available, and that jobs belonging to different customers cannot be delivered together. However, in practice, there could be situations where the availability of delivery vehicles is the bottleneck of the operation, while allowing more flexibility in the routing of the vehicles is important. Hence, it would be useful to extend our analysis to the case where the number of vehicles is limited and vehicles can make deliveries to multiple customers in each trip. Third, many other integrated production and outbound distribution models in the literature incorporate features such as vehicle routing, delivery due dates, multiple-machine production, etc. (see, e.g., Chen 2010). An extensive study on rescheduling issues occurred in other integrated production and outbound distribution models would be another interesting future research direction. Finally, note that in our model we have considered the situation where the entire delivery service is shut down for a certain time period. For systems with limited number of vehicles, there are situations where only a subset of delivery vehicles is unavailable due to incidents such as vehicle breakdown and driver unavailability. Analyzing how to reschedule production and delivery for these situations is also an important research direction.

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Supplementary Materials

Appendix

Proof of Lemma 1: Given any optimal solution of $\mathbf{P}_0$ where idle time exists between some consecutive jobs in the production schedule, we can eliminate all such idle time by moving the jobs earlier in the production schedule. This move is made without changing the delivery schedule, and the objective value is unaffected. The resulting solution is an alternative optimal solution that satisfies property (C1).

Suppose that this alternative optimal solution does not satisfy property (C2). Then, there exist two jobs J_{ij} and J_{il} in the same shipment such that J_{ij} is processed before J_{il}, and that there exist some job(s) belonging to a different shipment processed between J_{ij} and J_{il}. We can construct an alternative solution by moving job J_{ij} to the position immediately before job J_{il} in the production schedule (without inserting idle time between jobs). This move is made without changing the delivery schedule, and the objective value is unaffected. By repeatedly making this change in the production schedule, we obtain an alternative optimal solution that satisfies properties (C1) and (C2).

Suppose that this alternative optimal solution does not satisfy property (C3). Then, there exist two shipments with corresponding job subsets B_h and B_i, such that the jobs in B_h are processed immediately before the jobs in B_i, and that the departure time of B_h is later than that of B_i. We can construct an alternative solution by interchanging the processing of B_h and B_i in the production schedule. This move is made without changing the delivery schedule, and the objective value is unaffected. By repeatedly making this change in the production schedule, we obtain an alternative optimal solution that satisfies properties (C1)–(C3).

Suppose that this alternative optimal solution does not satisfy property (C4). Then, there exist two jobs J_{ij} and J_{ik} which belong to the same customer K_i, such that $p_{ij} < p_{ik}$, and that J_{ik} is processed before J_{ij} in the production schedule. We can construct an alternative solution by interchanging the processing of J_{ij} and J_{ik} in the production schedule (without inserting idle time between jobs). This move is made without changing the delivery schedule, and the objective value is unaffected. By repeatedly making this change in the production schedule, we obtain an alternative optimal solution that satisfies properties (C1)–(C4).

Suppose that this alternative optimal solution does not satisfy property (C5). Then, this solution contains a shipment, say the sth shipment, with a departure time which is neither equal to the completion time of production of the last job in the sth shipment, nor equal to the departure time of the $(s-1)$st shipment plus δ (if $s \neq 1$). Let C_s and E_s denote the production completion time and departure time, respectively, of the sth shipment. Let E_{s-1} denote the departure time of the $(s-1)$st shipment if $s \neq 1$, and let $E_0 = -\infty$. We can construct an alternative optimal solution by reducing E_s to $\max\{E_{s-1}+\delta, C_s\}$. By repeatedly applying this argument, we obtain an optimal solution of problem $\mathbf{P}_0$ that satisfies properties (C1)–(C5). ∎

Proof of Lemma 2: Consider any $m = 1, 2, 3$, and suppose $\mathbf{P}_m$ is feasible. Using the same argument as in the proof of Lemma 1, it is easy to check that there exists an optimal solution σ^* of problem $\mathbf{P}_m$ which satisfies properties (C1)–(C3). Suppose that in problem $\mathbf{P}_m$ there exists a customer K_i such that in solution σ^* the jobs belonging to K_i are not processed by the machine in the same sequence as in solution π^*. Then, there exist jobs J_{ij} and J_{il} such that J_{ij} is processed after J_{il} in the production schedule of σ^*, but J_{ij} is processed before J_{il} in the production schedule of π^*. Property (C3) of solution π^* implies that

$$E_{ij}(\pi^*) \leq E_{il}(\pi^*). \tag{A1}$$

In solution σ^*, let B_i denote the job subset associated with the delivery shipment which includes J_{il}, and B_i' denote the job subset associated with the delivery shipment which includes J_{ij}. Consider a new solution σ^{**} of $\mathbf{P}_m$ obtained by the following steps: (i) interchange the positions of J_{ij} and J_{il} in the production schedule (without inserting idle time between jobs); (ii) if $B_i = B_i'$, then do not change B_i and B_i'; and (iii) if $B_i \neq B_i'$, then let $B_i \leftarrow (B_i \backslash \{J_{il}\}) \cup \{J_{ij}\}$ and $B_i' \leftarrow (B_i' \backslash \{J_{ij}\}) \cup \{J_{il}\}$. Property (C4) of solution π^* implies that $p_{ij} \leq p_{il}$. Thus, this new solution can be constructed without changing the delivery schedule of the shipments. It is easy to see that solution σ^{**} satisfies properties (C1)–(C3). Note that $E_{iu}(\sigma^{**}) = E_{iu}(\sigma^*)$ for all $u \in \{1, 2, \ldots, n_i\} \setminus \{j, l\}$. In addition, we have the following relationships between σ^* and σ^{**}:

$$E_{ij}(\sigma^*) = E_{il}(\sigma^{**}); \tag{A2}$$

$$E_{il}(\sigma^*) = E_{ij}(\sigma^{**}); \tag{A3}$$

$$E_{ij}(\sigma^{**}) \leq E_{ij}(\sigma^*); \tag{A4}$$

$$E_{il}(\sigma^*) \leq E_{il}(\sigma^{**}). \tag{A5}$$

Next, we focus on problem $\mathbf{P}_1$. We let $G(\sigma^*)$ denote the total contribution of jobs J_{ij} and J_{il} to the objective value $\tilde{\Gamma}_1(\sigma^*)$ of solution σ^*, and let $G(\sigma^{**})$ denote the total contribution of jobs J_{ij} and J_{il} to the objective value $\tilde{\Gamma}_1(\sigma^{**})$ of solution σ^{**}, excluding their contributions to the total delivery cost $\beta \sum_{i=1}^{k} r_i(\sigma)\phi_i$. That is,

$$G(\sigma^*) = \alpha E_{ij}(\sigma^*) + \alpha E_{il}(\sigma^*) + \gamma|E_{ij}(\sigma^*) - E_{ij}(\pi^*)| + \gamma|E_{il}(\sigma^*) - E_{il}(\pi^*)| \tag{A6}$$

and

$$G(\sigma^{**}) = \alpha E_{ij}(\sigma^{**}) + \alpha E_{il}(\sigma^{**}) + \gamma|E_{ij}(\sigma^{**}) - E_{ij}(\pi^*)| + \gamma|E_{il}(\sigma^{**}) - E_{il}(\pi^*)|. \tag{A7}$$

We will show that $G(\sigma^{**}) \leq G(\sigma^*)$.

From (A2), (A3), (A6), and (A7), we have

$$\begin{aligned} G(\sigma^*) - G(\sigma^{**}) = \gamma\big[&|E_{ij}(\sigma^*) - E_{ij}(\pi^*)| - |E_{ij}(\sigma^{**}) - E_{ij}(\pi^*)| \\ &+ |E_{il}(\sigma^*) - E_{il}(\pi^*)| - |E_{il}(\sigma^{**}) - E_{il}(\pi^*)|\big]. \end{aligned} \tag{A8}$$

We divide the analysis into three cases.

Case 1: $E_{ij}(\sigma^{**}) \geq E_{ij}(\pi^*)$. In this case, by (A2), (A3), and (A5),

$$E_{ij}(\sigma^*) - E_{ij}(\pi^*) = E_{il}(\sigma^{**}) - E_{ij}(\pi^*) \geq E_{il}(\sigma^*) - E_{ij}(\pi^*) = E_{ij}(\sigma^{**}) - E_{ij}(\pi^*) \geq 0,$$

which implies that

$$|E_{ij}(\sigma^*) - E_{ij}(\pi^*)| - |E_{ij}(\sigma^{**}) - E_{ij}(\pi^*)| = (E_{ij}(\sigma^*) - E_{ij}(\pi^*)) - (E_{ij}(\sigma^{**}) - E_{ij}(\pi^*)) = E_{ij}(\sigma^*) - E_{ij}(\sigma^{**}).$$

This, together with (A8), implies that

$$G(\sigma^*) - G(\sigma^{**}) = \gamma\big[(E_{ij}(\sigma^*) - E_{ij}(\sigma^{**})) + |E_{il}(\sigma^*) - E_{il}(\pi^*)| - |E_{il}(\sigma^{**}) - E_{il}(\pi^*)|\big].$$

If $E_{il}(\sigma^{**}) \leq E_{il}(\pi^*)$, then

$$\begin{aligned} G(\sigma^*) - G(\sigma^{**}) &= \gamma\big[(E_{ij}(\sigma^*) - E_{ij}(\sigma^{**})) + |E_{il}(\sigma^*) - E_{il}(\pi^*)| - (E_{il}(\pi^*) - E_{il}(\sigma^{**}))\big] \\ &\geq \gamma\big[(E_{ij}(\sigma^*) - E_{ij}(\sigma^{**})) + (E_{il}(\pi^*) - E_{il}(\sigma^*)) - (E_{il}(\pi^*) - E_{il}(\sigma^{**}))\big] \\ &= \gamma\big[E_{ij}(\sigma^*) - E_{ij}(\sigma^{**}) + E_{il}(\sigma^{**}) - E_{il}(\sigma^*)\big] \\ &\geq 0, \end{aligned}$$

where the last inequality follows from (A4) and (A5). If $E_{il}(\sigma^{**}) > E_{il}(\pi^*)$, then

$$\begin{aligned} G(\sigma^*) - G(\sigma^{**}) &= \gamma\big[(E_{ij}(\sigma^*) - E_{ij}(\sigma^{**})) + |E_{il}(\sigma^*) - E_{il}(\pi^*)| - (E_{il}(\sigma^{**}) - E_{il}(\pi^*))\big] \\ &\geq \gamma\big[(E_{ij}(\sigma^*) - E_{ij}(\sigma^{**})) + (E_{il}(\sigma^*) - E_{il}(\pi^*)) - (E_{il}(\sigma^{**}) - E_{il}(\pi^*))\big] \\ &= \gamma\big[E_{ij}(\sigma^*) - E_{ij}(\sigma^{**}) + E_{il}(\sigma^*) - E_{il}(\sigma^{**})\big] \\ &= 0, \end{aligned}$$

where the last equality follows from (A2) and (A3). Therefore, in Case 1, $G(\sigma^{**}) \leq G(\sigma^*)$.

Case 2: $E_{ij}(\sigma^{**}) < E_{ij}(\pi^*)$ and $E_{il}(\sigma^{**}) \geq E_{il}(\pi^*)$. In this case, by (A1) and (A3), we have

$$E_{il}(\pi^*) - E_{il}(\sigma^*) \geq E_{ij}(\pi^*) - E_{ij}(\sigma^{**}) > 0,$$

which implies that

$$|E_{il}(\sigma^*) - E_{il}(\pi^*)| \geq |E_{ij}(\sigma^{**}) - E_{ij}(\pi^*)|. \tag{A9}$$

By (A1) and (A2), we have

$$E_{ij}(\sigma^*) - E_{ij}(\pi^*) \geq E_{il}(\sigma^{**}) - E_{il}(\pi^*) \geq 0,$$

which implies that

$$|E_{ij}(\sigma^*) - E_{ij}(\pi^*)| \geq |E_{il}(\sigma^{**}) - E_{il}(\pi^*)|. \tag{A10}$$

From (A8), (A9), and (A10), we have $G(\sigma^*) - G(\sigma^{**}) \geq 0$. Therefore, in Case 2, $G(\sigma^{**}) \leq G(\sigma^*)$.

Case 3: $E_{ij}(\sigma^{**}) < E_{ij}(\pi^*)$ and $E_{il}(\sigma^{**}) < E_{il}(\pi^*)$. In this case, by (A1) and (A3), we have

$$E_{il}(\pi^*) - E_{il}(\sigma^*) \geq E_{ij}(\pi^*) - E_{ij}(\sigma^{**}) > 0.$$

If $E_{ij}(\sigma^*) \geq E_{ij}(\pi^*)$, then equation (A8) becomes

$$\begin{aligned}
&G(\sigma^*) - G(\sigma^{**}) \\
&= \gamma\big[(E_{ij}(\sigma^*) - E_{ij}(\pi^*)) - (E_{ij}(\pi^*) - E_{ij}(\sigma^{**})) + (E_{il}(\pi^*) - E_{il}(\sigma^*)) - (E_{il}(\pi^*) - E_{il}(\sigma^{**}))\big] \\
&= \gamma\big[E_{ij}(\sigma^*) + E_{ij}(\sigma^{**}) + E_{il}(\sigma^{**}) - 2E_{ij}(\pi^*) - E_{il}(\sigma^*)\big] \\
&= \gamma\big[2E_{ij}(\sigma^*) - 2E_{ij}(\pi^*)\big] \\
&\geq 0,
\end{aligned}$$

where the third equality follows from (A2) and (A3). If $E_{ij}(\sigma^*) < E_{ij}(\pi^*)$, then equation (A8) becomes

$$\begin{aligned}
&G(\sigma^*) - G(\sigma^{**}) \\
&= \gamma\big[(E_{ij}(\pi^*) - E_{ij}(\sigma^*)) - (E_{ij}(\pi^*) - E_{ij}(\sigma^{**})) + (E_{il}(\pi^*) - E_{il}(\sigma^*)) - (E_{il}(\pi^*) - E_{il}(\sigma^{**}))\big] \\
&= \gamma\big[E_{ij}(\sigma^{**}) + E_{il}(\sigma^{**}) - E_{ij}(\sigma^*) - E_{il}(\sigma^*)\big] \\
&= 0,
\end{aligned}$$

where the last equality follows from (A2) and (A3). Therefore, in Case 3, $G(\sigma^{**}) \leq G(\sigma^*)$.

In all three cases, $G(\sigma^{**}) \leq G(\sigma^*)$, which implies that $\tilde{\Gamma}_1(\sigma^{**}) \leq \tilde{\Gamma}_1(\sigma^*)$. By repeatedly applying this job interchange operation, we obtain an alternative optimal solution in which the jobs that belong to the same customer are processed by the machine in the same SPT sequence as in π^*. This alternative optimal solution of problem $\mathbf{P}_1$ also satisfies properties (C1)–(C3).

Next, we consider problem $\mathbf{P}_m$ for $m = 2, 3$. We will show that

$$\max\{|E_{ij}(\sigma^*) - E_{ij}(\pi^*)|, |E_{il}(\sigma^*) - E_{il}(\pi^*)|\} \geq \max\{|E_{ij}(\sigma^{**}) - E_{ij}(\pi^*)|, |E_{il}(\sigma^{**}) - E_{il}(\pi^*)|\}. \quad \text{(A11)}$$

We divide the analysis into two cases.

Case 1: $E_{ij}(\pi^*) \leq E_{il}(\sigma^*)$. From (A3) and (A4), we have

$$E_{ij}(\sigma^*) - E_{ij}(\pi^*) \geq E_{ij}(\sigma^{**}) - E_{ij}(\pi^*) = E_{il}(\sigma^*) - E_{ij}(\pi^*) \geq 0,$$

which implies that $|E_{ij}(\sigma^*) - E_{ij}(\pi^*)| \geq |E_{ij}(\sigma^{**}) - E_{ij}(\pi^*)|$. Thus,

$$\max\{|E_{ij}(\sigma^*) - E_{ij}(\pi^*)|, |E_{il}(\sigma^*) - E_{il}(\pi^*)|\} \geq |E_{ij}(\sigma^{**}) - E_{ij}(\pi^*)|. \tag{A12}$$

From (A1), (A2), and (A5), we have

$$E_{ij}(\sigma^*) - E_{ij}(\pi^*) \geq E_{ij}(\sigma^*) - E_{il}(\pi^*) = E_{il}(\sigma^{**}) - E_{il}(\pi^*)$$

and

$$E_{il}(\pi^*) - E_{il}(\sigma^*) \geq E_{il}(\pi^*) - E_{il}(\sigma^{**}).$$

If $E_{il}(\sigma^{**}) - E_{il}(\pi^*) \geq 0$, then $|E_{ij}(\sigma^*) - E_{ij}(\pi^*)| \geq |E_{il}(\sigma^{**}) - E_{il}(\pi^*)|$; otherwise, $|E_{il}(\sigma^*) - E_{il}(\pi^*)| \geq |E_{il}(\sigma^{**}) - E_{il}(\pi^*)|$. Hence,

$$\max\{|E_{ij}(\sigma^*) - E_{ij}(\pi^*)|, |E_{il}(\sigma^*) - E_{il}(\pi^*)|\} \geq |E_{il}(\sigma^{**}) - E_{il}(\pi^*)|. \tag{A13}$$

By (A12) and (A13), inequality (A11) holds.

Case 2: $E_{ij}(\pi^*) > E_{il}(\sigma^*)$. From (A1), we have $E_{il}(\pi^*) - E_{il}(\sigma^*) \geq E_{ij}(\pi^*) - E_{il}(\sigma^*) > 0$. From (A3), we have $E_{ij}(\pi^*) - E_{il}(\sigma^*) = E_{ij}(\pi^*) - E_{ij}(\sigma^{**}) > 0$. Thus, $|E_{il}(\sigma^*) - E_{il}(\pi^*)| \geq |E_{ij}(\sigma^{**}) - E_{ij}(\pi^*)|$. This implies that

$$\max\{|E_{ij}(\sigma^*) - E_{ij}(\pi^*)|, |E_{il}(\sigma^*) - E_{il}(\pi^*)|\} \geq |E_{ij}(\sigma^{**}) - E_{ij}(\pi^*)|. \tag{A14}$$

From (A1), (A2), and (A5), we have

$$E_{il}(\pi^*) - E_{il}(\sigma^*) \geq E_{il}(\pi^*) - E_{il}(\sigma^{**})$$

and

$$E_{ij}(\sigma^*) - E_{ij}(\pi^*) = E_{il}(\sigma^{**}) - E_{ij}(\pi^*) \geq E_{il}(\sigma^{**}) - E_{il}(\pi^*).$$

If $E_{il}(\pi^*) - E_{il}(\sigma^{**}) \geq 0$, then $|E_{il}(\pi^*) - E_{il}(\sigma^*)| \geq |E_{il}(\pi^*) - E_{il}(\sigma^{**})|$; otherwise, $|E_{ij}(\sigma^*) - E_{ij}(\pi^*)| \geq |E_{il}(\sigma^{**}) - E_{il}(\pi^*)|$. Hence,

$$\max\{|E_{ij}(\sigma^*) - E_{ij}(\pi^*)|, |E_{il}(\sigma^*) - E_{il}(\pi^*)|\} \geq |E_{il}(\sigma^{**}) - E_{il}(\pi^*)|. \tag{A15}$$

By (A14) and (A15), inequality (A11) holds.

In both cases, inequality (A11) holds. Thus, the job interchange operation described in steps (i)–(iii) does not increase the maximum delivery time disruption. Clearly, this job interchange operation does not affect the total cost of delivery $\sum_{i=1}^{k} r_i(\sigma)\phi_i$. By (A2) and (A3),

$$E_{ij}(\sigma^*) + E_{il}(\sigma^*) = E_{ij}(\sigma^{**}) + E_{il}(\sigma^{**}).$$

Thus, this job interchange operation does not affect the value of $\sum_{i=1}^{k}\sum_{j=1}^{n_i} E_{ij}(\sigma)$. Hence, for problem $\mathbf{P}_2$, σ^{**} satisfies constraint (2), and $\tilde{\Gamma}_2(\sigma^{**}) = \tilde{\Gamma}_2(\sigma^*)$. For problem $\mathbf{P}_3$, we have $\tilde{\Gamma}_3(\sigma^{**}) \leq \tilde{\Gamma}_3(\sigma^*)$. By repeatedly applying this job interchange operation, we obtain an alternative optimal solution in which the jobs that belong to the same customer are processed by the machine in the same SPT sequence as in π^*. This alternative optimal solution of problem $\mathbf{P}_m$ also satisfies properties (C1)–(C3). ∎

Proof of Lemma 3: We refer to a group of ℓ delivery shipments as "consecutive shipments" if their departure times are $t, t+\delta, \ldots, t+(\ell-1)\delta$ for some $t > 0$. We say that this group of consecutive shipments is a group of "maximal consecutive shipments" if neither $t-\delta$ nor $t+\ell\delta$ is a departure time of one of the shipments in the solution.

Suppose there exists an optimal solution σ^* of $\mathbf{P}_1$ that satisfies the properties in Lemma 2 but contains a delivery shipment, say the sth shipment, with a departure time not equal to any of the following values:

(i) T;

(ii) one of the values in $\mathcal{E}_0$ minus $y\delta$, for some $y = 0, 1, \ldots, n-1$; or

(iii) departure time of the $(s-1)$st shipment plus δ (if $s > 1$).

Let r be the number of delivery shipments in solution σ^*. For $\ell = 1, 2, \ldots, r$, let B_ℓ be the job subset associated with the ℓth delivery shipment, C_ℓ be the completion time of production of job subset B_ℓ, and E_ℓ be the departure time of the ℓth delivery shipment. Let $E_0 = -\infty$ and $E_{r+1} = +\infty$. Then,

- $E_s > T$;
- $E_s \neq t - y\delta$ for all $t \in \mathcal{E}_0$ and $y = 0, 1, \ldots, n-1$; and

- $E_s > E_{s-1} + \delta$.

Consider the group of maximal consecutive shipments that the sth shipment belongs to. Let $B_s, B_{s+1}, \ldots, B_{\hat{s}}$ be the job subsets associated with these consecutive shipments, where $1 \le s \le \hat{s} \le r$ (note: the sth shipment is the first shipment in this group because $E_s > E_{s-1} + \delta$). Let Φ denote the total delivery cost of these $\hat{s} - s + 1$ shipments. Let $H(\sigma^*)$ denote the total contribution of these $\hat{s} - s + 1$ shipments to the objective value $\tilde{\Gamma}_1(\sigma^*)$ of this solution. That is,

$$\begin{aligned} H(\sigma^*) &= \alpha \sum_{\ell=s}^{\hat{s}} \sum_{J_{ij} \in B_\ell} E_{ij}(\sigma^*) + \beta\Phi + \gamma \sum_{\ell=s}^{\hat{s}} \sum_{J_{ij} \in B_\ell} |E_{ij}(\sigma^*) - E_{ij}(\pi^*)| \\ &= \alpha \sum_{\ell=s}^{\hat{s}} |B_\ell| \cdot E_\ell + \beta\Phi + \gamma \sum_{\ell=s}^{\hat{s}} \sum_{J_{ij} \in B_\ell} |E_\ell - E_{ij}(\pi^*)|. \end{aligned}$$

Because $E_s \ne t - y\delta$ for all $t \in \mathcal{E}_0$ and $y = 0, 1, \ldots, n-1$, we have $E_s, E_s + \delta, \ldots, E_s + (\hat{s} - s)\delta \notin \mathcal{E}_0$. Note that $E_{ij}(\pi^*) \in \mathcal{E}_0$ for all $J_{ij} \in \mathcal{J}$. Thus, $E_\ell \ne E_{ij}(\pi^*)$ for all $\ell = s, s+1, \ldots, \hat{s}$ and all $J_{ij} \in B_\ell$. For $\ell = s, s+1, \ldots, \hat{s}$, we partition set B_ℓ into subsets B'_ℓ and B''_ℓ, where B'_ℓ contains jobs with departure times in solution π^* being less than E_ℓ, and B''_ℓ contains jobs with departure times in solution π^* being greater than E_ℓ. Then,

$$H(\sigma^*) = \alpha \sum_{\ell=s}^{\hat{s}} |B_\ell| \cdot E_\ell + \beta\Phi + \gamma \sum_{\ell=s}^{\hat{s}} \sum_{J_{ij} \in B'_\ell} (E_\ell - E_{ij}(\pi^*)) + \gamma \sum_{\ell=s}^{\hat{s}} \sum_{J_{ij} \in B''_\ell} (E_{ij}(\pi^*) - E_\ell).$$

Let

$$\zeta' = \begin{cases} \min_{\ell=s,s+1,\ldots,\hat{s} \text{ s.t. } B'_\ell \ne \emptyset} \left\{ \min_{J_{ij} \in B'_\ell} \{E_\ell - \max\{E_{ij}(\pi^*), C_\ell\}\} \right\}, & \text{if } B'_s \cup B'_{s+1} \cup \cdots \cup B'_{\hat{s}} \ne \emptyset; \\ +\infty, & \text{if } B'_s \cup B'_{s+1} \cup \cdots \cup B'_{\hat{s}} = \emptyset; \end{cases}$$

and

$$\hat{\zeta}' = \min\{\zeta', E_s - T, E_s - E_{s-1} - \delta\}.$$

For $\ell = s, s+1, \ldots, \hat{s}$, since $C_\ell \in \mathcal{E}_0$ and $E_\ell \notin \mathcal{E}_0$, we have $E_\ell > C_\ell$. Hence, $E_\ell - \max\{E_{ij}(\pi^*), C_\ell\} > 0$ for all $\ell = s, s+1, \ldots, \hat{s}$ and all $J_{ij} \in B'_\ell$. Therefore, $\zeta' > 0$. In addition, because $E_s > T$ and $E_s > E_{s-1} + \delta$, we have $\hat{\zeta}' > 0$. Let

$$\zeta'' = \begin{cases} \min_{\ell=s,s+1,\ldots,\hat{s} \text{ s.t. } B''_\ell \ne \emptyset} \left\{ \min_{J_{ij} \in B''_\ell} \{E_{ij}(\pi^*) - E_\ell\} \right\}, & \text{if } B''_s \cup B''_{s+1} \cup \cdots \cup B''_{\hat{s}} \ne \emptyset; \\ +\infty, & \text{if } B''_s \cup B''_{s+1} \cup \cdots \cup B''_{\hat{s}} = \emptyset; \end{cases}$$

and

$$\hat{\zeta}'' = \min\{\zeta'', E_{\hat{s}+1} - E_{\hat{s}} - \delta\}.$$

It is easy to see that $\zeta'' > 0$. Since $E_{\hat{s}+1} > E_{\hat{s}} + \delta$, we have $\hat{\zeta}'' > 0$.

Suppose, to the contrary, that $\gamma \sum_{\ell=s}^{\hat{s}}(|B''_\ell| - |B'_\ell|) > \alpha \sum_{\ell=s}^{\hat{s}} |B_\ell|$. Consider an alternative solution σ of $\mathbf{P}_1$ obtained by increasing E_ℓ by $\hat{\zeta}''$ for $\ell = s, s+1, \ldots, \hat{s}$. Clearly, σ is a feasible solution of $\mathbf{P}_1$. The total contribution of the shipments of $B_s, B_{s+1}, \ldots, B_{\hat{s}}$ to the objective value $\tilde{\Gamma}_1(\sigma)$ of this alternative solution is

$$H(\sigma) = \alpha \sum_{\ell=s}^{\hat{s}} |B_\ell| \cdot (E_\ell + \hat{\zeta}'') + \beta\Phi + \gamma \sum_{\ell=s}^{\hat{s}} \sum_{J_{ij} \in B'_\ell} ((E_\ell + \hat{\zeta}'') - E_{ij}(\pi^*)) + \gamma \sum_{\ell=s}^{\hat{s}} \sum_{J_{ij} \in B''_\ell} (E_{ij}(\pi^*) - (E_\ell + \hat{\zeta}'')).$$

Thus,

$$H(\sigma^*) - H(\sigma) = \hat{\zeta}'' \left[\gamma \sum_{\ell=s}^{\hat{s}} (|B''_\ell| - |B'_\ell|) - \alpha \sum_{\ell=s}^{\hat{s}} |B_\ell| \right] > 0.$$

This contradicts the optimality of solution σ^*. Hence, $\gamma \sum_{\ell=s}^{\hat{s}}(|B''_\ell| - |B'_\ell|) \leq \alpha \sum_{\ell=s}^{\hat{s}} |B_\ell|$.

Consider an alternative solution σ^{**} of $\mathbf{P}_1$ obtained by decreasing E_ℓ by $\hat{\zeta}'$ for $\ell = s, s+1, \ldots, \hat{s}$. It is easy to check that σ^{**} is a feasible solution of $\mathbf{P}_1$ that satisfies the properties in Lemma 2. The total contribution of the shipments of $B_s, B_{s+1}, \ldots, B_{\hat{s}}$ to the objective value $\tilde{\Gamma}_1(\sigma^{**})$ of this alternative solution is

$$H(\sigma^{**}) = \alpha \sum_{\ell=s}^{\hat{s}} |B_\ell| \cdot (E_\ell - \hat{\zeta}') + \beta\Phi + \gamma \sum_{\ell=s}^{\hat{s}} \sum_{J_{ij} \in B'_\ell} ((E_\ell - \hat{\zeta}') - E_{ij}(\pi^*)) + \gamma \sum_{\ell=s}^{\hat{s}} \sum_{J_{ij} \in B''_\ell} (E_{ij}(\pi^*) - (E_\ell - \hat{\zeta}')).$$

Thus,

$$H(\sigma^*) - H(\sigma^{**}) = \hat{\zeta}' \left[\alpha \sum_{\ell=s}^{\hat{s}} |B_\ell| + \gamma \sum_{\ell=s}^{\hat{s}} (|B'_\ell| - |B''_\ell|) \right] \geq 0.$$

Hence, σ^{**} is also an optimal solution of $\mathbf{P}_1$. In solution σ^{**}, the group of maximal consecutive shipments that contains $B_s, B_{s+1}, \ldots, B_{\hat{s}}$ are merged with the group of maximal consecutive shipments that contains B_{s-1} into a single group of maximal consecutive shipments (i.e., when $\hat{\zeta}' = E_s - E_{s-1} - \delta$), or the departure time of job subset B_s is equal to T (i.e., when $\hat{\zeta}' = E_s - T$), or the departure time of one of $B_s, B_{s+1}, \ldots, B_{\hat{s}}$ is equal to a value in $\mathcal{E}_0$. By repeatedly applying this argument to different groups of maximal consecutive shipments, we obtain an optimal solution of problem $\mathbf{P}_1$ that satisfies the properties stated in Lemma 3. ∎

Proof of Lemma 4: Suppose there exists an optimal solution σ^* of $\mathbf{P}_2$ that satisfies the properties in Lemma 2 but contains a delivery shipment, say the sth shipment, with a departure time not equal to any of the following values:

(i) T;

(ii) the completion time of production of the sth shipment's last job;

(iii) the departure time of the sth shipment's last job in solution π^* minus θ; or

(iv) the departure time of the $(s-1)$st shipment plus δ (if $s \neq 1$).

Let C_s and E_s be the production completion time and departure time, respectively, of the sth shipment. Let E_{s-1} be the departure time of the $(s-1)$st shipment if $s \neq 1$, and let $E_0 = -\infty$. Let K_i be the customer that the sth shipment belongs to. Let $B_s = \{J_{ij_1}, J_{ij_2}, \ldots, J_{ij_u}\}$ be the job subset associated with the sth shipment, where jobs $J_{ij_1}, J_{ij_2}, \ldots, J_{ij_u}$ are indexed in increasing order of their completion times of production in solution π^*. Since solution π^* satisfies property (C3), we have $E_{ij_1}(\pi^*) \le E_{ij_2}(\pi^*) \le \cdots \le E_{ij_u}(\pi^*)$. Constraint (2) implies that $E_{ij_l}(\pi^*) - \theta \le E_s \le E_{ij_l}(\pi^*) + \theta$, for $l = 1, 2, \ldots, u$. Then,

- $E_s > T$;
- $E_s > C_s$;
- $E_s > E_{ij_u}(\pi^*) - \theta$; and
- $E_s > E_{s-1} + \delta$.

Consider an alternative solution σ^{**} of $\mathbf{P}_2$ obtained by decreasing E_s to E_s', where $E_s' = \max\{T, C_s, E_{ij_u}(\pi^*) - \theta, E_{s-1} + \delta\}$. Clearly, σ^{**} satisfies the minimum headway constraint. Because $E_s' \ge E_{ij_u}(\pi^*) - \theta$, we have

$$E_s' \ge E_{ij_l}(\pi^*) - \theta,$$

for $l = 1, 2, \ldots, u$. Because $E_s' < E_s$ and $E_s \le E_{ij_l}(\pi^*) + \theta$, we have

$$E_s' < E_{ij_l}(\pi^*) + \theta,$$

for $l = 1, 2, \ldots, u$. Thus, σ^{**} also satisfies constraint (2). The objective value of σ^{**} is less than the objective value of σ^* by $\alpha u(E_s - E_s') \ge 0$. Hence, σ^{**} is also an optimal solution of $\mathbf{P}_2$. In the solution σ^{**}, the departure time of the sth shipment is equal to T, C_s, $E_{ij_u}(\pi^*) - \theta$, or $E_{s-1} + \delta$. By repeatedly applying this argument to different shipments, we obtain an optimal solution of problem $\mathbf{P}_2$ in which the departure time of each shipment is equal to one of the following values: (i) T; (ii) the completion time of production of the current shipment's last job; (iii) the departure time of the current shipment's last job in solution π^* minus θ; or (iv) the departure time of the previous shipment plus δ (if the current shipment is not the first shipment). Note that the completion time of production of the current shipment's last job is an element of $\mathcal{E}_0$, and the departure time of the current shipment's last job in solution π^* is also an element of $\mathcal{E}_0$. Therefore, this optimal solution satisfies all the properties stated in Lemma 4. ∎

Proof of Lemma 5: Suppose, to the contrary, that $\Delta_{\max}(\sigma^*) \notin \mathcal{S}$ for every optimal solution σ^* of problem $\mathbf{P}_3$ that satisfies the properties in Lemma 2. Let σ^{**} be such an optimal solution where the sum of departure times of the shipments is the smallest. Let r be the number of delivery shipments in solution σ^{**}, and let $B_1, B_2, \ldots, B_r$ be the job subsets associated with these shipments. Let $\mathcal{B} = \{B_1, B_2, \ldots, B_r\}$. For $\ell = 1, 2, \ldots, r$, let

$$\varepsilon_\ell = \min_{s \in \mathcal{S}} \left\{ \left| \max_{J_{ij} \in B_\ell} \{\Delta_{ij}(\sigma^{**})\} - s \right| \right\},$$

which is the amount that the ℓth shipment's maximum delivery time disruption deviates from an element of $\mathcal{S}$. Since $\Delta_{\max}(\sigma^{**}) \notin \mathcal{S}$, at least one of $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r$ is strictly positive. Let

$$\varepsilon = \min \left\{ \varepsilon_\ell \;\middle|\; \ell = 1, \ldots, r \text{ such that } \varepsilon_\ell > 0 \right\} > 0.$$

Then, $\max_{J_{ij} \in B_\ell}\{\Delta_{ij}(\sigma^{**})\}$ either equals an element of $\mathcal{S}$ or deviates from an element of $\mathcal{S}$ by at least ε, for each $\ell = 1, 2, \ldots, r$. Note that $E_{11}(\pi^*) = \frac{E_{11}(\pi^*)+E_{11}(\pi^*)-0\delta}{2} \in \mathcal{E}_3$ and $|E_{11}(\pi^*) - E_{11}(\pi^*)| = 0$. Thus, $0 \in \mathcal{S}$. Hence, if $\max_{J_{ij} \in B_\ell}\{\Delta_{ij}(\sigma^{**})\} \notin \mathcal{S}$, then $\max_{J_{ij} \in B_\ell}\{\Delta_{ij}(\sigma^{**})\} \geq \varepsilon$.

For $\ell = 1, 2, \ldots, r$, let $B_\ell = \{J_{i_\ell u_\ell}, J_{i_\ell, u_\ell+1}, \ldots, J_{i_\ell v_\ell}\}$, where $J_{i_\ell j}$ is processed by the machine before $J_{i_\ell, j+1}$ for $j = u_\ell, u_\ell+1, \ldots, v_\ell - 1$. Let E_ℓ be the departure time of job subset B_ℓ; that is, $E_\ell = E_{i_\ell u_\ell}(\sigma^{**}) = E_{i_\ell, u_\ell+1}(\sigma^{**}) = \cdots = E_{i_\ell v_\ell}(\sigma^{**})$. Then,

$$\max_{J_{ij} \in B_\ell} \{\Delta_{ij}(\sigma^{**})\} = \begin{cases} E_{i_\ell v_\ell}(\pi^*) - E_\ell, & \text{if } E_\ell \leq \frac{1}{2}[E_{i_\ell u_\ell}(\pi^*) + E_{i_\ell v_\ell}(\pi^*)]; \\ E_\ell - E_{i_\ell u_\ell}(\pi^*), & \text{if } E_\ell > \frac{1}{2}[E_{i_\ell u_\ell}(\pi^*) + E_{i_\ell v_\ell}(\pi^*)]. \end{cases}$$

We partition $\mathcal{B}$ into three subsets $\mathcal{B}^0$, $\mathcal{B}^1$, and $\mathcal{B}^2$, where

$$\mathcal{B}^0 = \Big\{ B_\ell \;\Big|\; \max_{J_{ij}\in B_\ell} \{\Delta_{ij}(\sigma^{**})\} \in \mathcal{S} \Big\},$$

$$\mathcal{B}^1 = \Big\{ B_\ell \;\Big|\; \max_{J_{ij}\in B_\ell} \{\Delta_{ij}(\sigma^{**})\} \notin \mathcal{S} \text{ and } E_\ell < \frac{1}{2}\big[E_{i_\ell u_\ell}(\pi^*) + E_{i_\ell v_\ell}(\pi^*)\big] \Big\},$$

and

$$\mathcal{B}^2 = \Big\{ B_\ell \;\Big|\; \max_{J_{ij}\in B_\ell} \{\Delta_{ij}(\sigma^{**})\} \notin \mathcal{S} \text{ and } E_\ell > \frac{1}{2}\big[E_{i_\ell u_\ell}(\pi^*) + E_{i_\ell v_\ell}(\pi^*)\big] \Big\}.$$

Then, $\max_{J_{ij}\in B_\ell}\{\Delta_{ij}(\sigma^{**})\} = E_{i_\ell v_\ell}(\pi^*) - E_\ell \geq \varepsilon$ if $B_\ell \in \mathcal{B}^1$, and $\max_{J_{ij}\in B_\ell}\{\Delta_{ij}(\sigma^{**})\} = E_\ell - E_{i_\ell u_\ell}(\pi^*) \geq \varepsilon$ if $B_\ell \in \mathcal{B}^2$.

Using the same terminology as in the proof of Lemma 3, we refer to a group of ℓ delivery shipments as "consecutive shipments" if their departure times are $t, t+\delta, \ldots, t+(\ell-1)\delta$ for some $t > 0$, and we say that this is a group of "maximal consecutive shipments" if neither $t-\delta$ nor $t+\ell\delta$ is a departure time of one of the shipments in the solution. We consider another way of partitioning set $\mathcal{B}$, which partitions $\mathcal{B}$ according to the grouping of maximal consecutive shipments. Specifically, we let μ be the number of groups of maximal consecutive shipments in solution σ^{**}. For $s = 1, 2, \ldots, \mu$, let $\mathcal{A}^s = \{B_1^s, B_2^s, \ldots, B_{\lambda_s}^s\}$, where $\lambda_s = |\mathcal{A}^s|$, and B_h^s is the job subset associated with the hth shipment in the sth group of maximal consecutive shipments. Then, job subsets $\mathcal{A}^1, \mathcal{A}^2, \ldots, \mathcal{A}^\mu$ form a partition of $\mathcal{B}$. Note that the minimum departure time of the job subsets in $\mathcal{A}^{s+1}$ is greater than the maximum departure time of the job subsets in $\mathcal{A}^s$ plus δ, for $s = 1, \ldots, \mu-1$. For $s = 1, 2, \ldots, \mu$ and $h = 1, 2, \ldots, \lambda_s$, let C_h^s be the completion time of production of the last job in B_h^s, and E_h^s be the departure time of job subset B_h^s. Let $\lambda_0 = 0$, $E_0^0 = -\infty$, and $E_1^{\mu+1} = +\infty$. Then, for $s = 1, 2, \ldots, \mu+1$,

$$E_1^s > E_{\lambda_{s-1}}^{s-1} + \delta. \tag{A16}$$

In the following, we show that solution σ^{**} possesses the following four properties:

Property Π_1: For $s = 1, 2, \ldots, \mu$, if $\mathcal{A}^s \cap (\mathcal{B}^1 \cup \mathcal{B}^2) \neq \emptyset$, then $E_h^s > \max\{T, C_h^s\}$ for $h = 1, 2, \ldots, \lambda_s$.

Property Π_2: For $s = 1, 2, \ldots, \mu$ and $h = 1, 2, \ldots, \lambda_s$, if $B_1^s, B_2^s, \ldots, B_{h-1}^s \in \mathcal{B}^0$ and $B_h^s \in \mathcal{B}^1 \cup \mathcal{B}^2$ then $B_h^s \in \mathcal{B}^1$.

Property Π_3: For $s = 1, 2, \ldots, \mu$ and $h = 1, 2, \ldots, \lambda_s$, if $B_1^s, B_2^s, \ldots, B_{h-1}^s \in \mathcal{B}^0$ and $B_h^s \in \mathcal{B}^1$, then $\max_{J_{ij} \in B_h^s}\{\Delta_{ij}(\sigma^{**})\} = \Delta_{\max}(\sigma^{**})$.

Property Π_4: For $s = 1, 2, \ldots, \mu$ and $h = 1, 2, \ldots, \lambda_s$, if $\mathcal{A}^s \cap \mathcal{B}^1 \neq \emptyset$ and $B_h^s \in \mathcal{B}^2$, then $\max_{J_{ij} \in B_h^s}\{\Delta_{ij}(\sigma^{**})\} < \Delta_{\max}(\sigma^{**})$.

To prove Property Π_1, suppose, to the contrary, that there exist $s = 1, 2, \ldots, \mu$ and $h = 1, 2, \ldots, \lambda_s$ such that $\mathcal{A}^s \cap (\mathcal{B}^1 \cup \mathcal{B}^2) \neq \emptyset$ and $E_h^s = \max\{T, C_h^s\}$. Since $\mathcal{A}^s \cap (\mathcal{B}^1 \cup \mathcal{B}^2) \neq \emptyset$, there exists $\omega \in \{1, 2, \ldots, \lambda_s\}$ such that $B_\omega^s \in \mathcal{B}^1 \cup \mathcal{B}^2$. Because the shipments in $\mathcal{A}^s$ are consecutive shipments and the difference in departure times between any two consecutive shipments is δ, we have $E_\omega^s = E_h^s + (\omega - h)\delta = \max\{T, C_h^s\} + (\omega - h)\delta$. Since $E_\omega^s \geq T$, we have

$$E_\omega^s = \max\Big\{T,\, \max\{T, C_h^s\} + (\omega - h)\delta\Big\} = \max\Big\{T + \max\{\omega - h, 0\}\delta,\, C_h^s + (\omega - h)\delta\Big\}. \quad \text{(A17)}$$

Note that $|\omega - h| \leq n - 1$ and that $C_h^s = \sum_{i=1}^{k}\sum_{j=1}^{q_i} p_{ij}$ for some $q_i = 0, 1, \ldots, n_i$; $i = 1, 2, \ldots, k$. Thus, $C_h^s + (\omega - h)\delta \in \mathcal{E}_3$. In addition, $T + \max\{\omega - h, 0\}\delta \in \mathcal{E}_3$. Hence, from (A17), $E_\omega^s \in \mathcal{E}_3$. This implies that $\max_{J_{ij} \in B_\omega^s}\{\Delta_{ij}(\sigma^{**})\} \in \mathcal{S}$, which contradicts that $B_\omega^s \in \mathcal{B}^1 \cup \mathcal{B}^2$. Therefore, σ^{**} satisfies Property Π_1.

To prove Property Π_2, consider any $s = 1, 2, \ldots, \mu$ such that $\mathcal{A}^s \cap (\mathcal{B}^1 \cup \mathcal{B}^2) \neq \emptyset$. Let B_h^s be the first job subset in $\mathcal{A}^s$ that belongs to $\mathcal{B}^1 \cup \mathcal{B}^2$; that is, $B_1^s, B_2^s, \ldots, B_{h-1}^s \in \mathcal{B}^0$ and $B_h^s \in \mathcal{B}^1 \cup \mathcal{B}^2$. Suppose, to the contrary, that $B_h^s \in \mathcal{B}^2$. Then,

$$\max_{J_{ij} \in B_h^s}\{\Delta_{ij}(\sigma^{**})\} = E_h^s - \min_{J_{ij} \in B_h^s}\{E_{ij}(\pi^*)\} \geq \varepsilon. \quad \text{(A18)}$$

Let

$$\varphi_1 = \min\Big\{\varepsilon,\, E_1^s - E_{\lambda_{s-1}}^{s-1} - \delta,\, \min_{\omega=1,\ldots,h}\big\{E_\omega^s - \max\{T, C_\omega^s\}\big\}\Big\}.$$

By (A16), $E_1^s - E_{\lambda_{s-1}}^{s-1} - \delta > 0$. By Property Π_1, $E_\omega^s > \max\{T, C_\omega^s\}$ for $\omega = 1, 2, \ldots, h$. Hence, $\varphi_1 > 0$. Consider a new solution σ_1 obtained by decreasing the departure times of the job subsets $B_1^s, B_2^s, \ldots, B_h^s$ by φ_1. Since the new departure time of job subset B_1^s is $E_1^s - \varphi_1 \geq E_1^s - (E_1^s - E_{\lambda_{s-1}}^{s-1} - \delta) = E_{\lambda_{s-1}}^{s-1} + \delta$, the new departure time of job subset B_1^s differs from the departure time of job subset $B_{\lambda_{s-1}}^{s-1}$ by at least δ. Thus, the new solution σ_1 is feasible. In addition, σ_1 satisfies the properties in Lemma 2.

Decreasing the departure times of $B_1^s, B_2^s, \ldots, B_h^s$ reduces the value of $\sum_{i=1}^{k}\sum_{j=1}^{n_i} E_{ij}(\sigma^{**})$ by $\sum_{\omega=1}^{h} |B_\omega^s| \cdot \varphi_1$. In the following, we show that decreasing the departure times of $B_1^s, B_2^s, \ldots, B_h^s$ does not increase the maximum delivery time disruption $\Delta_{\max}(\sigma^{**})$. First, consider the job subsets $B_1^s, B_2^s, \ldots, B_{h-1}^s$. Since $B_1^s, B_2^s, \ldots, B_{h-1}^s \in \mathcal{B}^0$, we have $\max_{J_{ij}\in B_\omega^s}\{\Delta_{ij}(\sigma^{**})\} \in \mathcal{S}$ for $\omega = 1, 2, \ldots, h-1$. Since $\Delta_{\max}(\sigma^{**})$ deviates from each element of $\mathcal{S}$ by at least ε, we have $\max_{J_{ij}\in B_\omega^s}\{\Delta_{ij}(\sigma^{**})\} \le \Delta_{\max}(\sigma^{**}) - \varepsilon$ for $\omega = 1, 2, \ldots, h-1$. Hence, in the new solution σ_1, the maximum delivery time disruption of the job subsets $B_1^s, B_2^s, \ldots, B_{h-1}^s$ is at most $\Delta_{\max}(\sigma^{**}) - \varepsilon + \varphi_1 \le \Delta_{\max}(\sigma^{**})$. Next, consider the job subset B_h^s. By (A18), when E_h^s decreases by φ_1, the maximum delivery time disruption of the job subset B_h^s also decreases by φ_1. Since the departure times of the other job subsets are unaffected by the decrease in departure times of $B_1^s, B_2^s, \ldots, B_h^s$, we conclude that the maximum delivery time disruption of solution σ_1 (i.e., $\Delta_{\max}(\sigma_1)$) is no greater than that of solution σ^{**} (i.e., $\Delta_{\max}(\sigma^{**})$). Summarizing the above discussion, we have

$$\tilde{\Gamma}_3(\sigma_1) \le \tilde{\Gamma}_3(\sigma^{**}) - \alpha\sum_{\omega=1}^{h} |B_\omega^s| \cdot \varphi_1 \le \tilde{\Gamma}_3(\sigma^{**}).$$

Thus, σ_1 is also an optimal solution. However, the sum of departure times of the shipments in solution σ_1 is smaller than that in solution σ^{**}. This contradicts that the fact that σ^{**} is an optimal solution which satisfies the properties in Lemma 2 with the smallest total shipment departure times. Hence, $B_h^s \in \mathcal{B}^1$. Therefore, σ^{**} satisfies Property Π_2.

To prove Property Π_3, consider any $s = 1, 2, \ldots, \mu$ and $h = 1, 2, \ldots, \lambda_s$ such that $B_1^s, B_2^s, \ldots, B_{h-1}^s \in \mathcal{B}^0$ and $B_h^s \in \mathcal{B}^1$. Since $B_h^s \in \mathcal{B}^1$, we have

$$\max_{J_{ij}\in B_h^s}\{\Delta_{ij}(\sigma^{**})\} = \max_{J_{ij}\in B_h^s}\{E_{ij}(\pi^*)\} - E_h^s \ge \varepsilon. \tag{A19}$$

Suppose, to the contrary, that $\max_{J_{ij}\in B_h^s}\{\Delta_{ij}(\sigma^{**})\} < \Delta_{\max}(\sigma^{**})$. Then, let

$$\varphi_2 = \min\Big\{\varepsilon,\, \Delta_{\max}(\sigma^{**}) - \max_{J_{ij}\in B_h^s}\{\Delta_{ij}(\sigma^{**})\},\, E_1^s - E_{\lambda_{s-1}}^{s-1} - \delta,\, \min_{\omega=1,\ldots,h}\big\{E_\omega^s - \max\{T, C_\omega^s\}\big\}\Big\}.$$

By (A16), $E_1^s - E_{\lambda_{s-1}}^{s-1} - \delta > 0$. By Property Π_1, $E_\omega^s > \max\{T, C_\omega^s\}$ for $\omega = 1, 2, \ldots, h$. Hence, $\varphi_2 > 0$. Consider a new solution σ_2 obtained by decreasing the departure times of the job subsets $B_1^s, B_2^s, \ldots, B_h^s$ by φ_2. Since the new departure time of job subset B_1^s is $E_1^s - \varphi_2 \ge E_1^s - (E_1^s - E_{\lambda_{s-1}}^{s-1} - \delta) = E_{\lambda_{s-1}}^{s-1} + \delta$, the new departure time of job subset B_1^s differs from the departure time of job subset $B^{s-1}_{\lambda_{s-1}}$ by at least δ. Thus, the new solution σ_2 is feasible. In addition, σ_2 satisfies the properties in Lemma 2.

Decreasing the departure times of $B^s_1, B^s_2, \ldots, B^s_h$ reduces the value of $\sum_{i=1}^{k}\sum_{j=1}^{n_i} E_{ij}(\sigma^{**})$ by $\sum_{\omega=1}^{h} |B^s_\omega| \cdot \varphi_2$. In the following, we show that decreasing the departure times of $B^s_1, B^s_2, \ldots, B^s_h$ does not increase the maximum delivery time disruption $\Delta_{\max}(\sigma^{**})$. First, consider the job subsets $B^s_1, B^s_2, \ldots, B^s_{h-1}$. Since $B^s_1, B^s_2, \ldots, B^s_{h-1} \in \mathcal{B}^0$, we have $\max_{J_{ij}\in B^s_\omega}\{\Delta_{ij}(\sigma^{**})\} \in \mathcal{S}$ for $\omega = 1, 2, \ldots, h-1$. Since $\Delta_{\max}(\sigma^{**})$ deviates from each element of $\mathcal{S}$ by at least ε, we have $\max_{J_{ij}\in B^s_\omega}\{\Delta_{ij}(\sigma^{**})\} \leq \Delta_{\max}(\sigma^{**}) - \varepsilon$ for $\omega = 1, 2, \ldots, h-1$. Hence, in the new solution σ_2, the maximum delivery time disruption of the job subsets $B^s_1, B^s_2, \ldots, B^s_{h-1}$ is at most $\Delta_{\max}(\sigma^{**}) - \varepsilon + \varphi_2 \leq \Delta_{\max}(\sigma^{**})$. Next, consider the job subset B^s_h. By (A19), when E^s_h decreases by φ_2, the maximum delivery time disruption of the job subset B^s_h increases by φ_2. Because $\varphi_2 \leq \Delta_{\max}(\sigma^{**}) - \max_{J_{ij}\in B^s_h}\{\Delta_{ij}(\sigma^{**})\}$, the maximum delivery time disruption of B^s_h in solution σ_2 is no greater than $\Delta_{\max}(\sigma^{**})$. Since the departure times of the other job subsets are unaffected by the decrease in departure times of $B^s_1, B^s_2, \ldots, B^s_h$, we conclude that the maximum delivery time disruption of solution σ_2 (i.e., $\Delta_{\max}(\sigma_2)$) is no greater than that of solution σ^{**} (i.e., $\Delta_{\max}(\sigma^{**})$). Summarizing the above discussion, we have

$$\tilde{\Gamma}_3(\sigma_2) \leq \tilde{\Gamma}_3(\sigma^{**}) - \alpha\sum_{\omega=1}^{h} |B^s_\omega| \cdot \varphi_2 \leq \tilde{\Gamma}_3(\sigma^{**}).$$

Thus, σ_2 is also an optimal solution. However, the sum of departure times of the shipments in solution σ_2 is smaller than that in solution σ^{**}. This contradicts that the fact that σ^{**} is an optimal solution which satisfies the properties in Lemma 2 with the smallest total shipment departure times. Hence, $\max_{J_{ij}\in B^s_h}\{\Delta_{ij}(\sigma^{**})\} = \Delta_{\max}(\sigma^{**})$. Therefore, σ^{**} satisfies Property Π_3.

To prove Property Π_4, consider any $s = 1, 2, \ldots, \mu$ and $h = 1, 2, \ldots, \lambda_s$ such that $\mathcal{A}^s \cap \mathcal{B}^1 \neq \emptyset$ and $B^s_h \in \mathcal{B}^2$. Since $B^s_h \in \mathcal{B}^2$, we have

$$\max_{J_{ij}\in B^s_h}\{\Delta_{ij}(\sigma^{**})\} = E^s_h - \min_{J_{ij}\in B^s_h}\{E_{ij}(\pi^*)\}. \tag{A20}$$

Let B^s_ω be the first job subset in $\mathcal{A}^s$ which is not in $\mathcal{B}^0$. By Property Π_2, $B^s_\omega \in \mathcal{B}^1$. Hence,

$$\max_{J_{ij}\in B^s_\omega}\{\Delta_{ij}(\sigma^{**})\} = \max_{J_{ij}\in B^s_\omega}\{E_{ij}(\pi^*)\} - E^s_\omega.$$

By Property Π_3, $\max_{J_{ij}\in B^s_\omega}\{\Delta_{ij}(\sigma^{**})\} = \Delta_{\max}(\sigma^{**})$. Hence,

$$\Delta_{\max}(\sigma^{**}) = \max_{J_{ij}\in B^s_\omega}\{E_{ij}(\pi^*)\} - E^s_\omega. \tag{A21}$$

Note that $\omega < h$ and that $B^s_\omega, B^s_{\omega+1}, \ldots, B^s_h$ are job subsets associated with consecutive shipments. Thus, $E^s_h = E^s_\omega + (h-\omega)\delta$, and equation (A20) can be rewritten as

$$\max_{J_{ij}\in B^s_h}\{\Delta_{ij}(\sigma^{**})\} = E^s_\omega + (h-\omega)\delta - \min_{J_{ij}\in B^s_h}\{E_{ij}(\pi^*)\}. \tag{A22}$$

Suppose, to the contrary, that $\max_{J_{ij}\in B^s_h}\{\Delta_{ij}(\sigma^{**})\} = \Delta_{\max}(\sigma^{**})$. Then, from (A21),

$$\max_{J_{ij}\in B^s_h}\{\Delta_{ij}(\sigma^{**})\} = \max_{J_{ij}\in B^s_\omega}\{E_{ij}(\pi^*)\} - E^s_\omega. \tag{A23}$$

From (A22) and (A23),

$$E^s_\omega + (h-\omega)\delta - \min_{J_{ij}\in B^s_h}\{E_{ij}(\pi^*)\} = \max_{J_{ij}\in B^s_\omega}\{E_{ij}(\pi^*)\} - E^s_\omega,$$

which implies that

$$E^s_\omega = \frac{\min_{J_{ij}\in B^s_\omega}\{E_{ij}(\pi^*)\} + \max_{J_{ij}\in B^s_h}\{E_{ij}(\pi^*)\} - (h-\omega)\delta}{2} \in \mathcal{E}_3.$$

Hence, $\max_{J_{ij}\in B^s_\omega}\{\Delta_{ij}(\sigma^{**})\} \in \mathcal{S}$, which contradicts that B^s_ω is not in $\mathcal{B}^0$. Therefore, σ^{**} satisfies Property Π_4.

Denote

$$\mathcal{C} = \bigcup_{s\in\{1,\ldots,\mu\}\ \text{s.t.}\ \mathcal{A}^s\cap\mathcal{B}^1\neq\emptyset} \mathcal{A}^s.$$

Note that any job subset in $\mathcal{B}^1\cup\mathcal{B}^2$ must belong to $\mathcal{A}^s$ for some $s = 1, 2, \ldots, \mu$. By Property Π_2, the first job subset in $\mathcal{A}^s$ that belongs to $\mathcal{B}^1\cup\mathcal{B}^2$ must be an element of $\mathcal{B}^1$. Thus, any job subset in $\mathcal{B}^1\cup\mathcal{B}^2$ must belong to $\mathcal{A}^s$ for some $s = 1, 2, \ldots, \mu$ such that $\mathcal{A}^s\cap\mathcal{B}^1 \neq \emptyset$. Because $\Delta_{\max}(\sigma^{**}) \notin \mathcal{S}$, we have $\mathcal{B}^1\cup\mathcal{B}^2 \neq \emptyset$. Hence, there exists $s \in \{1, 2, \ldots, \mu\}$ such that $\mathcal{A}^s\cap\mathcal{B}^1 \neq \emptyset$. This implies that $\mathcal{C} \neq \emptyset$. We divide the analysis into two cases.

Case 1: $\alpha\sum_{B_\ell\in\mathcal{C}}|B_\ell| \geq \gamma$. Let

$$\eta_1 = \min_{s\in\{1,\ldots,\mu\}\ \text{s.t.}\ \mathcal{A}^s\cap\mathcal{B}^1\neq\emptyset} \min\Big\{\varepsilon,\ E^s_1 - E^{s-1}_{\lambda_{s-1}} - \delta,\ \min_{h=1,\ldots,\lambda_s}\{E^s_h - \max\{C^s_h, T\}\}\Big\}.$$

Consider any $s \in \{1, 2, \ldots, \mu\}$ such that $\mathcal{A}^s \cap \mathcal{B}^1 \neq \emptyset$. By (A16), $E_1^s - E_{\lambda_{s-1}}^{s-1} - \delta > 0$. By Property Π_1, $E_h^s - \max\{C_h^s, T\} > 0$ for any $h = 1, 2, \ldots, \lambda_s$. Hence, $\eta_1 > 0$.

Consider a solution σ_3 obtained by decreasing the departure times of the job subsets in $\mathcal{C}$ by η_1. Note that in solution σ^{**} the departure time of a job subset B_1^s that belongs to $\mathcal{C}$ is at least $\delta + \eta_1$ time units larger than the departure time of job subset $B_{\lambda_{s-1}}^{s-1}$ (if $s \neq 1$). Thus, any two consecutive shipments in the new solution σ_3 are at least δ time units apart. Note also that the departure time of each job subset B_h^s in σ_3 is at least $\max\{C_h^s, T\}$. Hence, σ_3 is a feasible solution of problem $\mathbf{P}_3$. In addition, σ_3 satisfies the properties in Lemma 2. Decreasing the departure times of the job subsets in $\mathcal{C}$ by η_1 reduces the value of $\sum_{i=1}^{k} \sum_{j=1}^{n_i} E_{ij}(\sigma^{**})$ by $\sum_{B_\ell \in \mathcal{C}} |B_\ell| \cdot \eta_1$ and increases the maximum delivery time disruption $\Delta_{\max}(\sigma^{**})$ by at most η_1. Thus,

$$\tilde{\Gamma}_3(\sigma_3) \leq \tilde{\Gamma}_3(\sigma^{**}) - \alpha \sum_{B_\ell \in \mathcal{C}} |B_\ell| \cdot \eta_1 + \gamma \eta_1 \leq \tilde{\Gamma}_3(\sigma^{**}).$$

Therefore, σ_3 is also an optimal solution. However, the sum of departure times of the shipments in solution σ_3 is smaller than that in solution σ^{**}. This contradicts that the fact that σ^{**} is an optimal solution which satisfies the properties in Lemma 2 with the smallest total shipment departure times.

Case 2: $\alpha \sum_{B_\ell \in \mathcal{C}} |B_\ell| < \gamma$. Let

$$\eta_2 = \min_{s \in \{1,\ldots,\mu\} \text{ s.t. } \mathcal{A}^s \cap \mathcal{B}^1 \neq \emptyset} \min\left\{\frac{\varepsilon}{2},\, E_1^{s+1} - E_{\lambda_s}^s - \delta,\, \frac{1}{2} \min_{h \in \{1,\ldots,\lambda_s\} \text{ s.t. } B_h^s \in \mathcal{B}^2} \left\{\Delta_{\max}(\sigma^{**}) - \max_{J_{ij} \in B_h^s}\{\Delta_{ij}(\sigma^{**})\}\right\}\right\},$$

where $\min_{h \in \{1,\ldots,\lambda_s\} \text{ s.t. } B_h^s \in \mathcal{B}^2}\{\Delta_{\max}(\sigma^{**}) - \max_{J_{ij} \in B_h^s}\{\Delta_{ij}(\sigma^{**})\}\} = +\infty$ when $\mathcal{A}^s \cap \mathcal{B}^2 = \emptyset$. Consider any $s \in \{1, 2, \ldots, \mu\}$ such that $\mathcal{A}^s \cap \mathcal{B}^1 \neq \emptyset$. By (A16), $E_1^{s+1} - E_{\lambda_s}^s - \delta > 0$. By Property Π_4, $\Delta_{\max}(\sigma^{**}) - \max_{J_{ij} \in B_h^s}\{\Delta_{ij}(\sigma^{**})\} > 0$ for any $h \in \{1, 2, \ldots, \lambda_s\}$ such that $B_h^s \in \mathcal{B}^2$. Hence, $\eta_2 > 0$.

Consider a solution σ_4 obtained by increasing the departure times of the job subsets in $\mathcal{C}$ by η_2. Note that in solution σ^{**} the departure time of a job subset $B_{\lambda_s}^s$ that belongs to $\mathcal{C}$ is at least $\delta + \eta_2$ time units smaller than the departure time of job subset B_1^{s+1} (if $s \neq \mu$). Thus, any two consecutive shipments in the new solution σ_4 are at least δ time units apart. Hence, σ_4 is a feasible solution of problem $\mathbf{P}_3$. In the following, we show that the maximum delivery time disruption of solution σ_4 is at most $\Delta_{\max}(\sigma^{**}) - \eta_2$. First, we consider shipments of those job subsets in $\mathcal{B}^0$. Note that in solution σ^{**} the maximum delivery time disruption of each job subset

B_ℓ in $\mathcal{B}^0$ is at most $\Delta_{\max}(\sigma^{**}) - \varepsilon$ (because $\max_{J_{ij} \in B_\ell}\{\Delta_{ij}(\sigma^{**})\} \in \mathcal{S}$ and $\Delta_{\max}(\sigma^{**})$ deviates from an element of $\mathcal{S}$ by at least ε). Thus, in solution σ_4, the maximum delivery time disruption of each of the job subsets in $\mathcal{B}^0$ is at most $\Delta_{\max}(\sigma^{**}) - \varepsilon + \eta_2 \leq \Delta_{\max}(\sigma^{**}) - \eta_2$ (because $\eta_2 \leq \frac{\varepsilon}{2}$). Next, we consider shipments of those job subsets in $\mathcal{B}^1$. Since $\mathcal{B}^1 \subseteq \mathcal{C}$, the departure time of each job subset in $\mathcal{B}^1$ is increased by η_2. Since $\max_{J_{ij} \in B_\ell}\{\Delta_{ij}(\sigma^{**})\} = E_{i_\ell v_\ell}(\pi^*) - E_\ell \geq \varepsilon$ for each $B_\ell \in \mathcal{B}^1$, the maximum delivery time disruption of each of the job subsets in $\mathcal{B}^1$ decreases by η_2 when we increase their departure times by η_2. Hence, in solution σ_4, the maximum delivery time disruption of each of the job subsets in $\mathcal{B}^1$ is at most $\Delta_{\max}(\sigma^{**}) - \eta_2$. Finally, we consider shipments of those job subsets in $\mathcal{B}^2$. Because $\mathcal{B}^2 \subseteq \mathcal{C}$, the departure time of each job subset in $\mathcal{B}^2$ is increased by η_2. The maximum delivery time disruption of each of the job subsets in $\mathcal{B}^2$ increases by no more than η_2 when we increase their departure times by η_2. Thus, in solution σ_4, the maximum delivery time disruption of each job subset $B_\ell \in \mathcal{B}^2$ is at most $\max_{J_{ij} \in B_\ell}\{\Delta_{ij}(\sigma^{**})\} + \eta_2 = \Delta_{\max}(\sigma^{**}) + \eta_2 - [\Delta_{\max}(\sigma^{**}) - \max_{J_{ij} \in B_\ell}\{\Delta_{ij}(\sigma^{**})\}] \leq \Delta_{\max}(\sigma^{**}) + \eta_2 - 2\eta_2 \leq \Delta_{\max}(\sigma^{**}) - \eta_2$. Summarizing the above discussion, we have $\Delta_{\max}(\sigma_4) \leq \Delta_{\max}(\sigma^{**}) - \eta_2$. Therefore,

$$\tilde{\Gamma}_3(\sigma_4) \leq \tilde{\Gamma}_3(\sigma^{**}) + \alpha \sum_{B_\ell \in \mathcal{C}} |B_\ell| \cdot \eta_2 - \gamma \eta_2 < \tilde{\Gamma}_3(\sigma^{**}).$$

This contradicts that σ^{**} is an optimal solution.

Combining Cases 1 and 2, we conclude that $\Delta_{\max}(\sigma^*) \in \mathcal{S}$ for some optimal solution σ^*. ∎