A performance-based warranty for products subject to competing hard and soft failures

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Abstract

This article studies a performance-based warranty for products subject to competing hard and soft failures. The two failure modes are competing in the sense that either one, on a "whichever-comes-first" basis, can cause the product to fail. A performance-based warranty not only covers the repair or replacement of any defect, but also guarantees the minimum performance level throughout the warranty period. In this article, we propose three compensation policies—that is, free replacement, penalty, and full refund, when a product's performance fails to meet the guaranteed level. The expected warranty servicing costs for the three policies are derived, based on the competing risks concept. A warranty design problem is further formulated to simultaneously determine the optimal product price, warranty length, and performance guarantee level so as to maximize the manufacturer's total profit. Numerical studies are conducted to demonstrate and compare the three performance-based compensation policies. It is shown that the full refund policy always leads to the lowest total profit, whereas neither of the other two policies can dominate each other in all scenarios. In particular, the free replacement policy results in a higher total profit than the penalty policy when the replacement cost is low, the penalty cost coefficient is high, and/or the product reliability is high.

Keywords: Warranty, performance deterioration, competing risks, warranty policy design

1. Introduction

Many durable products have specific key performance characteristics (e.g., capacity of batteries, energy efficiency of refrigerators, rated power output of solar panels) that deteriorate with time and usage. Such performance-critical products are subject to two failure modes—hard and soft failures. Hard failures are usually caused by manufacturing defects,

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wearout/aging, or even external shocks, whereas soft failures occur when the product performance becomes unsatisfactory—precisely, when the performance degradation exceeds a pre-set threshold. In principle, hard and soft failure processes are competing, meaning that either of the two processes can cause the product to fail.

In today's highly competitive market, it is a common practice for manufacturers to provide attractive product warranties along with the sales of their products, in order to protect consumers against premature failures and signal product quality and reliability (Murthy and Djamaludin, 2002; Xie et al., 2017). In addition to product failures, consumers are increasingly concerned about product performance deterioration during the use period (Jin et al., 2015; Kim et al., 2007). Driven by consumer desires as well as technological advances, more and more manufacturers are offering performance warranties (Koschnick and Hartman, 2020). Unlike product warranties that provide protection against functional failures, performance warranties focus on the degradation of key performance characteristics and provide guarantee on the minimum performance level(s) over the warranty period. In general, performance warranties and product warranties are offered simultaneously, although the lengths of their protection periods might differ. In this article, we refer to the combination of product and performance warranties as *performance-based warranty*, and our aim is to study this new type of warranty for products subject to competing hard and soft failure processes.

1.1. Motivating examples

Performance-based warranties have received a few applications. Two typical examples are presented below.

Lithium-ion battery warranties: Lithium-ion batteries experience gradual energy or power loss with time and usage, which results in capacity reduction. Hard failures of lithium-ion batteries may take place because of manufacturing defects or wearout/aging, etc. A typical example of performance-based warranties is the mid-range battery warranty for Tesla Model 3 electric cars: It covers the repair or replacement of any malfunctioning or defective battery for 8 years or 100000 miles (whichever comes first), with minimum 70% retention of battery capacity over the warranty period.[1] In this case, the protection periods of the product and performance warranties coincide.

Solar panel warranties: The performance of photovoltaic panels is also subject to stochastic degradation which is dependent on operational and environmental conditions. Brand-new solar panels are usually protected by both performance and product warranties. Take Canadian Solar's warranty policy[2] as an example. The performance warranty guarantees that the solar panels' actual power output should be no less than 97.5% of the rated power output during the first year, and the actual annual power decline should be no more than 0.5% from year 2 to year 30. That is, the actual power output should be no less than 83.0% of the rated power output by the end of 30 years of operations. In addition, the

[1]Tesla's new vehicle limited warranty: https://www.tesla.com/sites/default/files/downloads/tesla-new-vehicle-limited-warranty-en-us.pdf.

[2]Canadian Solar's limited warranty statement–photovoltaic double glass module products: https://www.canadiansolar.com/wp-content/uploads/2019/12/PV_Double_Glass_Module_Warranty_en.pdf.

product warranty guarantees that the solar panels should be free from defects in materials and workmanship for 12 years. In this case, however, the protection lengths of the product and performance warranties are different.

1.2. Related literature

In the literature, there are two streams of research that come closely to our work. The first one is product warranty modeling and analysis, and the second one is reliability assessment and maintenance planning for systems subject to competing hard and soft failures.

Product warranty modeling and analysis have long been a vibrant topic in the warranty management field (Murthy and Djamaludin, 2002). In recent years, quite a few novel warranty concepts and policies (in terms of protection duration, compensation mechanisms, and maintenance strategies, etc.) have been investigated in the literature. Ye and Murthy (2016) study the design of a two-dimensional warranty menu that contains a number of rectangular regions. Luo and Wu (2018) collectively price the warranty policies for a portfolio of different products through the mean-variance optimization approach. Wang et al. (2019) develop cost model for a new piece-wise renewing free replacement warranty policy. Lu and Shang (2019) develop a new warranty mechanism for online pre-owned tech products to encourage product quality information disclosure between e-tailers and online warranty provider. Wang et al. (2020b) study the design and pricing of extended warranty menus which offer multiple options with differentiated lengths and prices. Liu et al. (2020) investigate the profit and pricing strategy for a complimentary extended warranty. Cha et al. (2021) propose a new renewing warranty policy with inspection for heterogeneous, stochastically degrading items. In addition, some studies attempt to develop novel maintenance strategies for better warranty servicing. Su and Wang (2016) investigate customized preventive-maintenance warranty policies, where preventive maintenance strategies are tailored for different consumer categories. Wang et al. (2020a) propose an unpunctual preventive maintenance policy for repairable products under two-dimensional warranties, where consumers are entitled to slightly advance or postpone maintenance executions. Peng et al. (2021) investigate a dynamic preventive maintenance problem under two-dimensional warranties, and show that the optimal policy is a control limit policy with usage-dependent failure rate thresholds. Furthermore, Shang et al. (2018) make the first attempt to study condition-based warranties for products suffering from stochastic degradation, but the warranty in their work is a traditional policy (failure-based, not performance-based).

To our knowledge, Su and Cheng (2018) and Koschnick and Hartman (2020) are the most closely related to our work. Su and Cheng (2018) propose an availability-based warranty under which the manufacturer not only provides free repair or replacement upon any failure, but also ensures that the operational availability over the warranty period meets a negotiated level. Our work differs from theirs in two main aspects: (i) The performance measure of interest in their work is the operational availability, whereas we focus on the stochastic deterioration of key performance characteristics; and (ii) the products in their work are subject to only one failure mode (i.e., hard failure), whereas the products in our work exhibit two competing failure modes—hard and soft failures. Recently, Koschnick and Hartman (2020) also introduce a performance-based warranty policy, where the manufacturer may offer to cap the amount of operating costs the consumer will pay each period for a certain amount of time. That is, if a consumer's operating cost exceeds the guaranteed level, then the manufacturer has to offer a compensation. As Koschnick and Hartman (2020) merely focus on performance warranties against excessive operational costs, their problem setting is clearly different from ours.

The second stream of research focuses on reliability analysis and maintenance planning for systems with competing hard and soft failures. Most existing studies in this stream consider (in)dependent competing degradation processes and random shocks. In general, they assume that hard failures (resp. soft failures) are induced by fatal shocks (resp. stochastic degradation of key performance characteristics), and nonfatal shocks contribute to instantaneous increments in degradation levels or rates. Peng et al. (2010) develop reliability models and preventive maintenance policies for complex systems with dependent competing failure processes. Wang and Pham (2011) propose a s-dependent competing risk model for systems subject to degradation processes and random shocks. Huynh et al. (2012) study maintenance strategies for single-unit repairable systems subject to dependent competing failures due to degradation and shocks. Rafiee et al. (2014) model the reliability of devices experiencing dependent competing failure processes of random shocks and degradation with a changing rate. Jiang et al. (2015) explicitly consider zoned shock effects on stochastic degradation when modeling dependent degradation and shock processes. Song et al. (2016) develop new reliability models for series systems subject to competing hard and soft failures with dependent shock effects. Qiu et al. (2018) propose a new maintenance model under power purchase agreements for complex energy generation systems that experience competing hard and soft failure processes. Hao and Yang (2018) conduct reliability assessment for dependent competing failure processes with changing degradation rate and hard failure thresholds. Yousefi et al. (2019) attempt to optimize on-condition failure thresholds and inspection intervals for multi-component systems with each component undergoing degradation and shock processes. Gao et al. (2019) develop reliability models for systems subject to competing risks with degradation-shock dependency.

Our work differs from the second stream of research in two main perspectives: (i) None of these studies takes product/performance warranties into consideration; and (ii) our work considers a slightly different setting of competing risks: Hard failures are triggered by manufacturing defects, wearout/aging, or external shocks and can be described by a lifetime-based reliability model; whereas soft failures are induced by the stochastic degradation of key performance characteristics and can be characterized by a degradation-based reliability model.

1.3. Overview of this article

This article makes an early attempt to model and optimize the performance-based warranty for products subject to competing hard and soft failure processes. Under this performance-based warranty, the manufacturer not only provides free replacement upon any failure but also guarantees the minimum performance level throughout the warranty period. We propose three types of compensation mechanisms when a unit's actual performance fails to meet the guaranteed level over the warranty period: *i*) the unit will be replaced with a new one; *ii*) a penalty cost will be induced; and *iii*) a full refund will be issued and

Nomenclature

W	Warranty period [decision variable]
P	Product price [decision variable]
$\mathcal{D}$	Warranty threshold [decision variable]
$\mathcal{L}$	Failure threshold
r_0	Nominal usage rate
R, r	Random usage rate and its realization
$G(r)$	Cumulative distribution function of R
$T_r^{(1)}$	Latent time to first hard failure under usage rate r
$T_r^{(2)}$	Latent first hitting time to $\mathcal{D}$ under usage rate r
$T_r^{(3)}$	Latent first hitting time to $\mathcal{L}$ under usage rate r
$F^{(i)}(t; r)$	Conditional distribution function of $T_r^{(i)}$, $i = 1, 2, 3$
T_r^{I}	$= \min\{T_r^{(1)}, T_r^{(2)}\}$
T_r^{II}	$= \min\{T_r^{(1)}, T_r^{(3)}\}$
$\Psi^i(t; r)$	Conditional distribution function of T_r^i, $i = \mathrm{I}, \mathrm{II}$
α, β	Scale and shape parameters of Weibull distribution
$\eta(t; \omega)$	Actual degradation path under usage rate r_0
ω	Degradation rate of the soft failure process
μ, σ	mean and standard deviation of normal distribution
γ_1, γ_2	Accelerating factors of hard and soft failure processes
c_f	Average replacement cost of a single unit
c_p	Penalty cost coefficient
C_0	Unit manufacturing cost
$E[\mathcal{C}_{\mathcal{T}}^{i}(W, \mathcal{D})]$	Total expected warranty cost under Policy j, $j = \mathrm{I}, \mathrm{II}$
$E[\mathcal{C}_{\mathcal{T}}^{\mathrm{III}}(P, W, \mathcal{D})]$	Total expected warranty cost under Policy III
$Q(P, W, \mathcal{D})$	Product demand function
$\Pi_j(P, W, \mathcal{D})$	Total expected profit under Policy j, $j = \mathrm{I}, \mathrm{II}, \mathrm{III}$

the warranty terminates. We then derive the expected warranty servicing expenses for the three compensation policies, based on the well-known competing risks model. An optimization problem is further formulated to simultaneously determine the optimal product price, warranty length, and performance guarantee level of the product so as to maximize the manufacturer's total profit. Numerical experiments are carried out to demonstrate and compare the three performance-based warranty policies, as well as answering the following questions: *In terms of total expected profit, does the performance-based warranty has a better outcome than traditional product warranties? If yes, which compensation policy is the most beneficial to the manufacturer?*

The rest of the article is organized as follows. Section 2 defines the three performance-based warranty policies, formulates hard and soft failure processes, and then develops the associated warranty cost models. On this basis, Section 3 develops and discusses a profit-maximization optimization problem. Section 4 illustrates and compares the proposed compensation policies through numerical experiments. Finally, Section 5 concludes the article with some suggestions for future research.

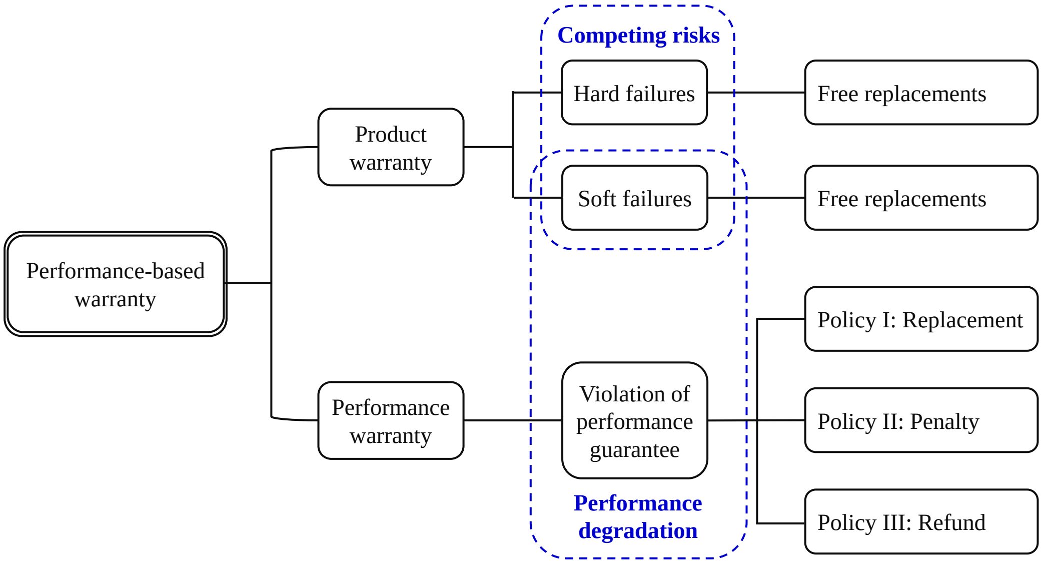

Figure 1: Framework of the proposed performance-based warranty.

2. Model formulation

Before formally defining the performance-based warranty, the following assumptions are made to facilitate policy definition.

(i) The performance characteristic of interest is *the-higher-the-better*, e.g., battery capacity and solar panel's power output, so that we normalize the initial performance level to 100%.
(ii) The performance characteristic is gradually deteriorating. Mathematically, the actual performance level is continuously decreasing.
(iii) The product performance is continuously monitored.

2.1. Performance-based warranty policies

The product is protected by a performance-based warranty of length W, which involves both product and performance warranties (see Fig. 1). The product warranty guarantees that if a hard failure or a soft failure (when the actual performance level reaches a predetermined *failure threshold* $\mathcal{L}$) occurs within the warranty period $[0, W]$, the failed unit will be *replaced* with a new identical one, at no cost to the consumer. In addition, the performance warranty specifies that the product's performance level over the warranty period $[0, W]$ should be no lower than the guaranteed level (referred to as the *warranty threshold* $\mathcal{D} \geq \mathcal{L}$).

We consider three compensation policies for the performance-based warranty. An underlying difference among them lies in the compensation mechanism when the performance guarantee level is not satisfied (see Fig. 2). Under Policy I, the manufacturer will provide a free replacement service immediately; under Policy II, the manufacturer will bear a penalty cost that is proportional to the additional amount of performance degradation compared

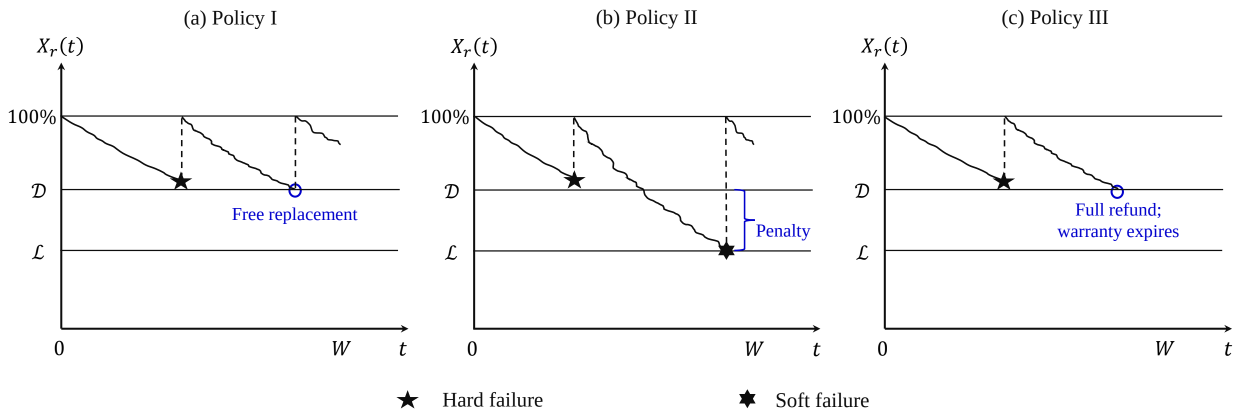

Figure 2: Schematic description of the three compensation policies.

with the warranty threshold; whereas Policy III specifies that a full refund will be offered by the manufacturer and the warranty thus terminates.

Several properties of the three compensation policies can be drawn:

1) Soft failures will never occur under Policies I and III because of the constraint $\mathcal{D} \geq \mathcal{L}$.
2) When $\mathcal{D} = \mathcal{L}$, all the three policies reduce to a pure product warranty policy. That is to say, the manufacturer only provides free replacement services for failed units, without any guarantee on their performance deterioration levels.
3) The warranty threshold $\mathcal{D}$ has different influences on the three compensation policies. Under Policy I, it affects the total replacement cost, as free replacement is the only strategy of servicing warranty claims; under Policy II, it solely impacts the penalty cost, without any influence on the associated replacement cost; whereas under Policy III, it has an influence on the time to full refund (if any) and thus on both the replacement cost and refund cost.

2.2. Modeling competing failure processes

Products deteriorate with both time and usage. Different products might have different usage measures, e.g., the number of charging/discharging cycles for batteries, working hours for solar panels, and driving miles for cars, among others. Suppose that the product of interest is designed for some nominal *usage rate* r_0 under which the design reliability well satisfies related requirements. Consumers, however, use the product in different rates. In reality, both hard and soft failures could be highly related to the usage rate (Jack et al., 2009). A higher usage rate could accelerate product wearout/aging which results in a shorter time to hard failure, as well as accelerating performance deterioration process which, in turn, leads to an earlier soft failure.

In this work, we assume that the usage rate remains constant over the warranty period for a specific consumer, but varies randomly across the consumer population. Let R represent the random usage rate with a cumulative distribution function (CDF) $G(r)$, $0 \leq r < \infty$. Further assume that the manufacturer has enough information on this distribution in advance, either through historical data on previous product generations or from a consumer survey (Su and Wang, 2016). Conditional on $R = r$, the hard and soft failure processes are assumed to be independent (Zhao et al., 2020). Such conditional independence assumption is reasonable for products such as lithium-ion batteries, of whom capacity reduction is attributed to the deterioration of lithium-ions and electrolyte, whereas hard failures (of, e.g., electrodes, casing, and separators) are mainly due to manufacturing defects, material fatigue/aging, and shocks (Hendricks et al., 2015). In this case, the two failure processes are largely independent.

Let $T_r^{(1)}$, $T_r^{(2)}$, and $T_r^{(3)}$ represent the *latent* time to first hard failure, first hitting time to $\mathcal{D}$, and first hitting time to $\mathcal{L}$, respectively, under usage rate r. Further define $F^{(i)}(t;r)$ as the conditional CDF associated with $T_r^{(i)}$, $i = 1, 2, 3$. Recall that under Policy I, a unit will be replaced when either a hard failure occurs or the performance level reaches $\mathcal{D}$, whichever happens first. According to the classical competing risks concept, conditional on $R = r$, the overall time to first replacement under Policy I is given by $T_r^{\mathrm{I}} = \min\{T_r^{(1)}, T_r^{(2)}\}$. The conditional CDF of T_r^{I} can be derived as

$$\begin{aligned}
\Psi^{\mathrm{I}}(t;r) &= \Pr\{\min\{T_r^{(1)}, T_r^{(2)}\} \le t \mid R = r\} \\
&= 1 - \Pr\{\min\{T_r^{(1)}, T_r^{(2)}\} > t \mid R = r\} \\
&= 1 - \Pr\{T_r^{(1)} > t, T_r^{(2)} > t \mid R = r\} \\
&= 1 - (1 - F^{(1)}(t;r))(1 - F^{(2)}(t;r)).
\end{aligned} \tag{1}$$

Under Policy II, however, product replacement will be triggered by either a hard failure or a soft failure (i.e., when the performance level hits $\mathcal{L}$). As a result, the overall time to first replacement is $T_r^{\mathrm{II}} = \min\{T_r^{(1)}, T_r^{(3)}\}$, and the associated conditional CDF becomes

$$\begin{aligned}
\Psi^{\mathrm{II}}(t;r) &= \Pr\{\min\{T_r^{(1)}, T_r^{(3)}\} \le t \mid R = r\} \\
&= 1 - (1 - F^{(1)}(t;r))(1 - F^{(3)}(t;r)).
\end{aligned} \tag{2}$$

Under Policy III, free replacement and full refund—caused by a hard failure and the violation of performance guarantee (i.e., when the performance level is lower than $\mathcal{D}$), respectively—are competing on a "whichever-occurs-first" basis. Under this policy, the overall time to first *event* (either replacement or refund) is also given by T_r^{I}, with conditional CDF $\Psi^{\mathrm{I}}(t;r)$.

In the next subsections, we will derive the conditional distribution functions $F^{(1)}(t;r)$, $F^{(2)}(t;r)$, and $F^{(3)}(t;r)$, respectively.

2.2.1. Hard failure process

The hard failure process is described by a lifetime-based reliability model. The effect of usage rate on the hard failure process is characterized by the well-known accelerated failure time (AFT) model (Jack et al., 2009). Denote by $T_0^{(1)}$ (resp. $T_r^{(1)}$) the time to first hard failure under usage rate r_0 (resp. r). Using the AFT formulation in Jack et al. (2009), we have

$$\frac{T_r^{(1)}}{T_0^{(1)}} = \left(\frac{r_0}{r}\right)^{\gamma_1}, \tag{3}$$

where $\gamma_1 > 0$ is the associated accelerating factor. As the usage rate r increases, the product wearout/aging accelerates, which, in turn, shortens the time to hard failure.

In this study, $T_0^{(1)}$ is assumed to follow a Weibull distribution with scale parameter α and shape parameter β, which is widely used to model product (hard) failures due to its flexibility in describing various failure rate properties (Murthy et al., 2004). Then, the CDF of $T_0^{(1)}$ is given by

$$F_0^{(1)}(t) = 1 - \exp\left\{-\left(\frac{t}{\alpha}\right)^{\beta}\right\}. \tag{4}$$

According to Eqs. (3) and (4), the conditional CDF of the time to first hard failure $T_r^{(1)}$ can be derived as

$$\begin{aligned} F^{(1)}(t;r) &= F_0^{(1)}(t(r/r_0)^{\gamma_1}) \\ &= 1 - \exp\left\{-\left(\frac{\Lambda_1(t)}{\alpha}\right)^{\beta}\right\}, \end{aligned} \tag{5}$$

where $\Lambda_1(t) = t(r/r_0)^{\gamma_1}$ is a time-scale transformation. It is clear that $\Lambda_1(T_r^{(1)}) \sim F_0^{(1)}(t)$.

2.2.2. Soft failure process

In the literature, three categories of degradation models have been proposed (Ye and Xie, 2015)—that is, general path models, stochastic process models, and other models beyond these two (such as machine learning models). In this work, we adopt the well-known general path model in Lu and Meeker (1993) for degradation modeling purposes. It should be noted that any feasible degradation model, e.g., Wiener process (Zhang et al., 2018), gamma process (van Noortwijk, 2009), and inverse Gaussian process (Ye and Chen, 2014), can be used in our problem, as long as the real degradation data justify the model.

Suppose that the product has only one key performance characteristic that is of primary concern to consumers. Under nominal usage rate r_0, the *observed* degradation path $X_0(t)$ of the performance characteristic is given by

$$X_0(t) = \eta(t;\omega) + \varepsilon, \tag{6}$$

where $\eta(t;\omega)$ is the *actual* degradation path which is a continuously decreasing function, and ε is a normally distributed random error with mean zero. Recall that the initial performance level is normalized to 100%, i.e., $\eta(0;\omega) = 1$. Possible formulations of $\eta(t;\omega)$ include the linear form $\eta(t;\omega) = 1 - \omega t$, $0 \le t \le 1/\omega$, and the exponential form $\eta(t;\omega) = \exp\{-\omega t\}$, $t \ge 0$, where $\omega > 0$ represents the degradation rate. In this work, the *exponential* form is adopted for degradation modeling. In practice, ω might vary randomly across the consumer population due to certain heterogeneity beyond the usage rate (e.g., working temperature of batteries). Such heterogeneous factors can result in random degradation rates, even for the same usage rate.

Likewise, the usage rate also has a significant impact on the stochastic degradation of the key performance characteristic. Conditional on $R = r$, the observed degradation path under usage rate r is modeled by

$$X_r(t) = \eta(\Lambda_2(t); \omega) + \varepsilon, \tag{7}$$

where $\Lambda_2(t) = t(r/r_0)^{\gamma_2}$ is a time-scale transformation with accelerating factor $\gamma_2 > 0$. Note that the factor γ_2 is not necessarily equal to γ_1.

Under Policy I, a unit will be replaced immediately when its actual performance level is lower than the warranty threshold $\mathcal{D}$, provided that there is no hard failure. Mathematically, conditional on $R = r$, the latent first hitting time to $\mathcal{D}$ is defined by

$$T_r^{(2)} = \inf\{t; \eta(\Lambda_2(t); \omega) \leq \mathcal{D} \mid R = r\}. \tag{8}$$

The conditional CDF of $T_r^{(2)}$ for the exponential degradation path can then be derived as

$$\begin{aligned} F^{(2)}(t; r) &= \Pr\{\eta(\Lambda_2(t); \omega) \leq \mathcal{D} \mid R = r\} \\ &= \Pr\{\exp\{-\omega\Lambda_2(t)\} \leq \mathcal{D} \mid R = r\} \\ &= \Pr\{\omega \geq -\ln \mathcal{D}/\Lambda_2(t) \mid R = r\}. \end{aligned} \tag{9}$$

We further assume that ω follows a normal distribution $N(\mu, \sigma^2)$ with $\sigma \ll \mu$ so that $\Pr\{\omega \leq 0\}$ is negligible (Lu and Meeker, 1993). Then, Eq. (9) can be rewritten as

$$F^{(2)}(t; r) = 1 - \Phi\left(\frac{-\ln \mathcal{D}/\mu - \Lambda_2(t)}{\sigma\Lambda_2(t)/\mu}\right), \tag{10}$$

where $\Phi(\cdot)$ is the standard normal distribution function. Note that if the aforementioned linear form of $\eta(\Lambda_2(t); \omega)$ is applied, then $F^{(2)}(t; r)$ is given by Eq. (10), with $1-\mathcal{D}$ replacing $-\ln \mathcal{D}$.

Under Policy II, a soft failure (resulting in product replacement) occurs when the actual performance degradation path hits the failure threshold $\mathcal{L}$. Similarly, conditional on $R = r$, the latent time to first soft failure, $T_r^{(3)} = \inf\{t; \eta(\Lambda_2(t); \omega) \leq \mathcal{L} \mid R = r\}$, has a conditional CDF $F^{(3)}(t; r)$ given by Eq. (10), with $\mathcal{L}$ replacing $\mathcal{D}$.

2.3. Modeling warranty costs

In this subsection, we derive the expected warranty servicing costs for the three compensation policies of performance-based warranty, respectively.

2.3.1. Policy I: Free replacement

Under this policy, a unit will be replaced with a new identical one once a hard failure occurs or its actual performance level is lower than $\mathcal{D}$, whichever comes first. The associated conditional CDF, $\Psi^{\mathrm{I}}(t; r)$, of the overall time to first replacement T_r^{I} can be derived by substituting (5) and (10) into (1). That is,

$$\Psi^{\mathrm{I}}(t; r) = 1 - \exp\left\{-\left(\frac{\Lambda_1(t)}{\alpha}\right)^{\beta}\right\} \times \Phi\left(\frac{-\ln \mathcal{D}/\mu - \Lambda_2(t)}{\sigma\Lambda_2(t)/\mu}\right). \tag{11}$$

Since offering free replacement is the only way of servicing warranty claims, the total warranty servicing cost for an individual unit is simply equal to the replacement cost over the warranty period. According to the renewal process theory (Ross, 2014), the conditional expected warranty servicing cost under Policy I can be determined by

$$\begin{aligned} E[C^{\mathrm{I}}_{\mathcal{T}}(W,\mathcal{D}) \mid R=r] &= c_f\left[\Psi^{\mathrm{I}}(W;r)+\int_0^W M^{\mathrm{I}}(W-t;r)\mathrm{d}\Psi^{\mathrm{I}}(t;r)\right] \\ &= c_f\left[\Psi^{\mathrm{I}}(W;r)+\int_0^W \Psi^{\mathrm{I}}(W-t;r)\mathrm{d}M^{\mathrm{I}}(t;r)\right], \end{aligned} \tag{12}$$

where c_f is the average cost of replacing an individual unit, and $M^{\mathrm{I}}(t;r)$ is the conditional renewal function associated with $\Psi^{\mathrm{I}}(t;r)$. In essence, $M^{\mathrm{I}}(W;r) = \Psi^{\mathrm{I}}(W;r) + \int_0^W \Psi^{\mathrm{I}}(W-t;r)\mathrm{d}M^{\mathrm{I}}(t;r)$, which exactly represents the conditional expected number of replacements over the warranty period. The renewal equation $M^{\mathrm{I}}(t;r)$ can be efficiently evaluated by the so-called Riemann-Stieltjes method in Xie (1989), which is briefed in Appendix A.

By removing the conditioning on R, the total expected warranty servicing cost under Policy I is thus given by

$$E[C^{\mathrm{I}}_{\mathcal{T}}(W,\mathcal{D})] = \int_0^\infty E[C^{\mathrm{I}}_{\mathcal{T}}(W,\mathcal{D}) \mid R=r]\mathrm{d}G(r), \tag{13}$$

where $E[C^{\mathrm{I}}_{\mathcal{T}}(W,\mathcal{D}) \mid R=r]$ is given by (12).

2.3.2. Policy II: Penalty

Under this policy, a unit will be replaced with a new identical one once a hard or soft failure occurs, whichever comes first. In addition, if the performance guarantee over the warranty period is violated, then a penalty cost would be incurred to the manufacturer. Hence, the total warranty servicing cost can be decomposed into two parts: the replacement cost $C_{\mathcal{F}}(W)$ due to hard and/or soft failures, and the penalty cost $C_{\mathcal{P}}(W,\mathcal{D})$ due to the violation of performance guarantee. That is, $C^{\mathrm{II}}_{\mathcal{T}}(W,\mathcal{D}) = C_{\mathcal{F}}(W) + C_{\mathcal{P}}(W,\mathcal{D})$. Notice that the warranty threshold $\mathcal{D}$ has no influence on the replacement cost, but it does impact the penalty cost.

The conditional CDF, $\Psi^{\mathrm{II}}(t;r)$, of the overall time to first replacement T^{II}_r is given by Eq. (11), with $\mathcal{L}$ replacing $\mathcal{D}$. Similarly, the conditional expected replacement cost within the warranty period, $E[C_{\mathcal{F}}(W) \mid R=r]$, can be determined by Eq. (12), with $\Psi^{\mathrm{II}}(t;r)$ and $M^{\mathrm{II}}(t;r)$ replacing $\Psi^{\mathrm{I}}(t;r)$ and $M^{\mathrm{I}}(t;r)$, respectively, where $M^{\mathrm{II}}(t;r)$ is the conditional renewal function associated with $\Psi^{\mathrm{II}}(t;r)$.

On the other hand, the expected penalty cost is not easy to derive. Suppose that the penalty cost is evaluated upon a failure (if any) and at the expiry of the warranty period. If the corresponding performance level is below $\mathcal{D}$, then a penalty cost proportional to the additional amount of performance deterioration is induced (see Fig. 2(b)). For instance, if a unit's performance level upon a failure is d, then the associated penalty cost can be expressed as $c_p[\mathcal{D}-d]^+$, where c_p is the penalty cost coefficient and $[x]^+ = \max\{x,0\}$. We further define $T^{\mathrm{II}}_{r,i}$ as the conditional inter-replacement time between the $(i-1)$th and ith replacements within the warranty period. Clearly, $T^{\mathrm{II}}_{r,i}$'s are independent and identically distributed, with conditional CDF $\Psi^{\mathrm{II}}(t;r)$. Upon a product failure at $T^{\mathrm{II}}_{r,i} = t_i$, the probability that this failure is caused by a hard failure is equal to

$$
\begin{aligned}
z(t_i;r) &= \Pr\{T_r^{(1)} < T_r^{(3)} \mid T^{\mathrm{II}}_{r,i} = t_i\} \\
&= \frac{\Pr\{\{T_r^{(1)} < T_r^{(3)}\} \cap \{\min\{T_r^{(1)}, T_r^{(3)}\} = t_i\}\}}{\Pr\{\min\{T_r^{(1)}, T_r^{(3)}\} = t_i\}} \\
&= \frac{\Pr\{T_r^{(1)} = t_i < T_r^{(3)}\}}{\mathrm{d}\Psi^{\mathrm{II}}(t_i;r)/\mathrm{d}t_i} \\
&= \frac{f^{(1)}(t_i;r)(1 - F^{(3)}(t_i;r))}{f^{(1)}(t_i;r)(1 - F^{(3)}(t_i;r)) + f^{(3)}(t_i;r)(1 - F^{(1)}(t_i;r))},
\end{aligned}
\tag{14}
$$

where $f^{(1)}(t;r) = \mathrm{d}F^{(1)}(t;r)/\mathrm{d}t$ and $f^{(3)}(t;r) = \mathrm{d}F^{(3)}(t;r)/\mathrm{d}t$; the probability that this failure is due to a soft failure or both of the two triggers is thus $1 - z(t_i;r)$.

It is obvious that if the failure of a unit is caused by a soft failure, then the unit's performance level upon this failure is $\mathcal{L}$ for sure; otherwise, its performance level is random. Denote by $\mathcal{N}(W)$ the total number of replacements within the warranty period, under Policy II. Then, conditional on $R = r$, $\mathcal{N}(W) = n$, and $T^{\mathrm{II}}_{r,1} = t_1$, $T^{\mathrm{II}}_{r,2} = t_2$, $\ldots$, $T^{\mathrm{II}}_{r,n} = t_n$, the expected penalty cost over the warranty period can be derived as

$$
\begin{aligned}
&E[C_{\mathcal{P}}(W,\mathcal{D}) \mid T^{\mathrm{II}}_{r,1} = t_1, \ldots, T^{\mathrm{II}}_{r,n} = t_n, \mathcal{N}(W) = n, R = r] \\
&= \sum_{i=1}^{n} \left[c_p z(t_i;r) \int_{\frac{-\ln \mathcal{D}}{\Lambda_2(t_i)}}^{\frac{-\ln \mathcal{L}}{\Lambda_2(t_i)}} (\mathcal{D} - \eta(\Lambda_2(t_i);u)) \mathrm{d}F_\omega(u) + c_p(1 - z(t_i;r))(\mathcal{D} - \mathcal{L}) \right] \\
&\quad + c_p \int_{\frac{-\ln \mathcal{D}}{\Lambda_2(t_{n+1})}}^{\frac{-\ln \mathcal{L}}{\Lambda_2(t_{n+1})}} (\mathcal{D} - \eta(\Lambda_2(t_{n+1});u)) \mathrm{d}F_\omega(u),
\end{aligned}
\tag{15}
$$

where $F_\omega(\cdot)$ is the CDF of ω, a normally distributed random variable with mean μ and standard deviation $\sigma \ll \mu$, and $t_{n+1} = W - \sum_{i=1}^{n} t_i$ is the time interval between the last replacement action and the end of the warranty period. The upper and lower limits in the integrals are the values of ω under which the performance levels are between $\mathcal{L}$ and $\mathcal{D}$. This specific interval is concerned because any performance level below $\mathcal{L}$ will result in a soft failure (so that the product failure would be caused by a soft failure, rather than a hard failure), whereas any performance level above $\mathcal{D}$ will not be penalized.

Removing the conditioning on $T^{\mathrm{II}}_{r,i}$, $i = 1, 2, \ldots, n$, yields

$$
\begin{aligned}
&E[C_{\mathcal{P}}(W,\mathcal{D}) \mid \mathcal{N}(W) = n, R = r] \\
&= \int_0^W \int_0^{W-t_1} \cdots \int_0^{W - \sum_{i=1}^{n-1} t_i} E\Big[C_{\mathcal{P}}(W,\mathcal{D}) \mid T^{\mathrm{II}}_{r,1} = t_1, \ldots, T^{\mathrm{II}}_{r,n} = t_n, \\
&\qquad \mathcal{N}^{\mathrm{II}}(W) = n, R = r\Big] \mathrm{d}\Psi^{\mathrm{II}}(t_n;r) \cdots \mathrm{d}\Psi^{\mathrm{II}}(t_2;r) \mathrm{d}\Psi^{\mathrm{II}}(t_1;r).
\end{aligned}
\tag{16}
$$

According to the renewal process theory, we have $\Pr\{\mathcal{N}(W) = n \mid R = r\} = \Psi_n^{\mathrm{II}}(W;r) - \Psi_{n+1}^{\mathrm{II}}(W;r)$, where $\Psi_n^{\mathrm{II}}(t;r)$ is the n-fold convolution of $\Psi^{\mathrm{II}}(t;r)$ with itself. Then, we further have

$$\begin{aligned} E[C_{\mathcal{P}}(W,\mathcal{D}) \mid R = r] = \sum_{n=0}^{\infty} \Big(& E[C_{\mathcal{P}}(W,\mathcal{D}) \mid \mathcal{N}(W) = n, R = r] \\ & \times \Pr\{\mathcal{N}(W) = n \mid R = r\}\Big). \end{aligned} \tag{17}$$

In general, the closed-form determination of $E[C_{\mathcal{P}}(W,\mathcal{D}) \mid R = r]$ appears to be an onerous task; numerical integration or simulation offers an alternative. The latter method is adopted in this work, which is elaborated in Appendix B.

Finally, by removing the conditioning on $R = r$, the total expected warranty servicing cost under Policy II is given by

$$E[C_{\mathcal{T}}^{\mathrm{II}}(W,\mathcal{D})] = \int_0^{\infty} \Big(E[C_{\mathcal{F}}(W) \mid R = r] + E[C_{\mathcal{P}}(W,\mathcal{D}) \mid R = r] \Big) \mathrm{d}G(r), \tag{18}$$

where $E[C_{\mathcal{F}}(W) \mid R = r]$ is given by (12), with $\Psi^{\mathrm{II}}(t;r)$ and $M^{\mathrm{II}}(t;r)$ replacing $\Psi^{\mathrm{I}}(t;r)$ and $M^{\mathrm{I}}(t;r)$, respectively, and $E[C_{\mathcal{P}}(W,\mathcal{D}) \mid R = r]$ is given by (17).

2.3.3. Policy III: Full refund

Under this policy, if a unit's actual performance fails to meet the guaranteed level, then a full refund would be offered to the consumer, and the warranty period terminates immediately. On the other hand, if any hard failure occurs before a full refund (if any), then the unit would be replaced with a new identical one. In this case, the product price P, in addition to the warranty length W and performance guarantee level $\mathcal{D}$, will have an impact on the total expected warranty servicing cost.

We derive the total expected warranty servicing cost by conditioning on the time to first event T_r^{I}. Conditional on $T_r^{\mathrm{I}} = t$, the probability that this event is caused by a hard failure is equal to

$$\begin{aligned} q(t;r) &= \Pr\{T_r^{(1)} < T_r^{(2)} \mid T_r^{\mathrm{I}} = t\} \\ &= \frac{f^{(1)}(t;r)(1 - F^{(2)}(t;r))}{f^{(1)}(t;r)(1 - F^{(2)}(t;r)) + f^{(2)}(t;r)(1 - F^{(1)}(t;r))}, \end{aligned} \tag{19}$$

where $f^{(2)}(t;r) = \mathrm{d}F^{(2)}(t;r)/\mathrm{d}t$; the probability that this event is induced by the violation of performance guarantee or both is thus $1 - q(t;r)$.

Under this policy, if the event is caused by the violation of performance guarantee, then a full refund is issued and the warranty cost is thus equal to the product price P; otherwise, the warranty cost should be c_f plus that in the remaining warranty period. Therefore, conditional on $R = r$ and $T_r^{\mathrm{I}} = t$, the conditional expected warranty cost over the warranty period is given by

$$\begin{aligned} & E[C_{\mathcal{T}}^{\mathrm{III}}(P,W,\mathcal{D}) \mid T_r^{\mathrm{I}} = t, R = r] \\ &= \begin{cases} P, & T_r^{(2)} < T_r^{(1)} < W, \\ c_f + E[C_{\mathcal{T}}^{\mathrm{III}}(P, W - t, \mathcal{D}) \mid R = r], & T_r^{(1)} \le T_r^{(2)} < W. \end{cases} \end{aligned} \tag{20}$$

Removing the conditioning on $T_r^{\mathrm{I}} = t$ yields

$$\begin{aligned} E[C_{\mathcal{T}}^{\mathrm{III}}(P, W, \mathcal{D}) \mid R = r] &= \int_0^W E[C_{\mathcal{T}}^{\mathrm{III}}(P, W, \mathcal{D}) \mid T_r^{\mathrm{I}} = t, R = r] \mathrm{d}\Psi^{\mathrm{I}}(t; r) \\ &= P \int_0^W (1 - q(t; r)) \mathrm{d}\Psi^{\mathrm{I}}(t; r) + c_f \int_0^W q(t; r) \mathrm{d}\Psi^{\mathrm{I}}(t; r) \\ &\quad + \int_0^W q(t; r) E[C_{\mathcal{T}}^{\mathrm{III}}(P, W - t, \mathcal{D}) \mid R = r] \mathrm{d}\Psi^{\mathrm{I}}(t; r) \\ &= P\Psi^{\mathrm{I}}(W; r) + (c_f - P) \int_0^W q(t; r) \mathrm{d}\Psi^{\mathrm{I}}(t; r) \\ &\quad + \int_0^W q(t; r) E[C_{\mathcal{T}}^{\mathrm{III}}(P, W - t, \mathcal{D}) \mid R = r] \mathrm{d}\Psi^{\mathrm{I}}(t; r). \end{aligned} \tag{21}$$

The expression above is a generalized renewal-type equation and can be solved by the Riemann-Stieltjes method as well, with a slight modification.

By further removing the conditioning on $R = r$, the total expected warranty cost under Policy III becomes

$$E[C_{\mathcal{T}}^{\mathrm{III}}(P, W, \mathcal{D})] = \int_0^\infty E[C_{\mathcal{T}}^{\mathrm{III}}(P, W, \mathcal{D}) \mid R = r] \mathrm{d}G(r), \tag{22}$$

where $E[C_{\mathcal{T}}^{\mathrm{III}}(P, W, \mathcal{D}) \mid R = r]$ is given by (21).

3. A profit-maximization optimization problem

In this section, an optimization problem is developed to simultaneously determine the optimal product price, warranty length, and performance guarantee level, so as to maximize the manufacturer's total profit. The optimization of this problem is briefly discussed, although there is no closed-form solutions to most decision variables.

Product price and warranty length are two typical factors that have a significant effect on the product demand or sales volume (Glickman and Berger, 1976; Xie et al., 2014). In addition, the performance guarantee level shall also influence the product sales volume, especially when it is treated as an advertising weapon. One would intuitively expect that the sales volume increases with the performance guarantee level $\mathcal{D}$. By generalizing the conventional wisdom in Glickman and Berger (1976), we model the product demand as a log-linear function of price P, warranty length W, and performance guarantee level $\mathcal{D}$, as follows:

$$Q(P, W, \mathcal{D}) = \kappa_1 P^{-\varphi_1} (W + \kappa_2)^{\varphi_2} (\mathcal{D} + \kappa_3)^{\varphi_3}, \tag{23}$$

where $\kappa_1 > 0$, $\kappa_2 \geq 0$, $\kappa_3 \geq 0$, $\varphi_1 > 1$, $0 < \varphi_2 < 1$, $0 < \varphi_3 < 1$. The constant κ_1 is an amplitude factor, and κ_2, κ_3 are constants for time and performance displacements that allow for nonzero demand when W or $\mathcal{D}$ is zero. Parameters φ_1, φ_2, and φ_3 can be interpreted as the price elasticity, displaced warranty period elasticity, and displaced performance guarantee elasticity, respectively. According to Eq. (23), it is clear that the product sales volume decreases exponentially with respect to product price, and increases exponentially with warranty length and performance guarantee level.

Suppose that the unit manufacturing cost C_0 contains all variable costs related to the manufacturing of an individual unit, and is independent of the number of units produced. Then, the expected profit of selling an individual unit is $\pi_j = P - C_0 - E[C_{\mathcal{T}}^j]$, where, for notation brevity, $E[C_{\mathcal{T}}^j]$ represents the total expected warranty servicing cost of Policy j, $j = \mathrm{I}, \mathrm{II}, \mathrm{III}$, given by Eqs. (13), (18), and (22), respectively. This way, the manufacturer's total expected profit can be obtained by multiplying the expected profit extracted from the sales of each unit by the total number of units sold. That is,

$$\begin{aligned}\Pi_j(P, W, \mathcal{D}) &= Q(P, W, \mathcal{D})\pi_j \\ &= \kappa_1 P^{-\varphi_1}(W + \kappa_2)^{\varphi_2}(\mathcal{D} + \kappa_3)^{\varphi_3} \\ &\quad \times (P - C_0 - E[C_{\mathcal{T}}^j]), j = \mathrm{I}, \mathrm{II}, \mathrm{III}.\end{aligned} \tag{24}$$

The manufacturer's optimization problem is to determine the optimal product price, warranty length, and performance guarantee level to maximize the total expected profit:

$$\begin{aligned}&\max_{P,W,\mathcal{D}} \Pi_j(P, W, \mathcal{D}) \\ &s.t.\ P > 0, W \geq 0, \mathcal{L} \leq \mathcal{D} \leq 100\%.\end{aligned} \tag{25}$$

Analogous to Glickman and Berger (1976), we have the following result for optimization problem (25):

Proposition 1. *Given any $W \geq 0$ and $\mathcal{L} \leq \mathcal{D} \leq 100\%$, the optimal product price for Policies I and II can be obtained by*

$$P_j^* = \frac{\varphi_1}{\varphi_1 - 1}(C_0 + E[C_{\mathcal{T}}^j]), j = \mathrm{I}, \mathrm{II}. \tag{26}$$

Proof. By taking the first derivative of $\Pi_j(P, W, \mathcal{D})$, $j = \mathrm{I}, \mathrm{II}$, with respect to P, we have

$$\begin{aligned}\frac{\partial \Pi_j(P, W, \mathcal{D})}{\partial P} &= \kappa_1 P^{-\varphi_1 - 1}(W + \kappa_2)^{\varphi_2}(\mathcal{D} + \kappa_3)^{\varphi_3} \\ &\quad \times \Big(P - \varphi_1\big(P - C_0 - E[C_{\mathcal{T}}^j]\big)\Big), j = \mathrm{I}, \mathrm{II}.\end{aligned}$$

Setting $\partial \Pi_j(P, W, \mathcal{D})/\partial P = 0$ yields $P_j^* = \varphi_1(C_0 + E[C_{\mathcal{T}}^j])/(\varphi_1 - 1)$, $j = \mathrm{I}, \mathrm{II}$. Further take the second derivative of $\Pi_j(P, W, \mathcal{D})$ with respect to P:

$$\begin{aligned}\frac{\partial^2 \Pi_j(P, W, \mathcal{D})}{\partial P^2} &= \kappa_1 \varphi_1 P^{-\varphi_1 - 2}(W + \kappa_2)^{\varphi_2}(\mathcal{D} + \kappa_3)^{\varphi_3} \\ &\quad \times \Big((\varphi_1 + 1)\big(P - C_0 - E[C_{\mathcal{T}}^j]\big) - 2P\Big), j = \mathrm{I}, \mathrm{II}.\end{aligned}$$

Due to the facts that $\partial^2 \Pi_j(P, W, \mathcal{D})/\partial P^2|_{P_j^*} = -(C_0 + E[C_{\mathcal{T}}^j]) < 0$ and P_j^* is the unique solution to $\partial \Pi_j(P, W, \mathcal{D})/\partial P = 0$, we know that P_j^* is indeed the optimal solution to problem (25). This completes the proof. □

In essence, Proposition 1 analytically presents the optimal product price for Policies I and II, when the warranty period W and performance guarantee level $\mathcal{D}$ are fixed. It is straightforward that the optimal product price decreases as the price elasticity φ_1 increases. This is because the sales volume would decrease with respect to product price in a faster manner when the price elasticity is higher, which, in turn, negatively affect the total profit. As a result, the manufacturer has to set a lower product price to mitigate this effect.

Nevertheless, the optimal product price for Policy III is not easy to derive in an analytical way, as $E[C_T^{\mathrm{III}}]$ is a complex function of product price. Likewise, the optimal values of the warranty period W and performance guarantee level $\mathcal{D}$ for all the three policies are also difficult to obtain, given the complicated forms of expected warranty costs in Eqs. (13), (18), and (22), so that we resort to numerical search methods. In this study, the Nelder-Mead method proposed in Nelder and Mead (1965) is employed to solve the optimization problem. This method does not require the derivatives of the objective function and is known to be effective for such a complex nonlinear optimization problem.

4. Numerical experiments

In this section, numerical examples are presented to illustrate and compare the three compensation policies of performance-based warranty. Comprehensive sensitivity analyses and policy comparisons are conducted with respect to key model parameters. The managerial insights would be of importance to manufacturers who seek the maximum profit to be generated from new durable products.

Consider that a firm produces and sells a hypothetical lithium-ion battery model, with unit manufacturing cost $C_0 = \$2500$. The firm intends to design an optimal performance-based warranty policy for the battery to maximize its total profit. For this purpose, lab reliability tests are conducted to obtain necessary information on the battery's hard and soft failure processes. Reliability and usage data show that the battery's hard failures are Weibull distributed, with scale parameter $\alpha = 15$ and shape parameter $\beta = 2$. On the other hand, the battery's actual performance degradation path is $\eta(t;\omega) = \exp\{-\omega t\}$, where ω follows a normal distribution $N(\mu, \sigma^2)$ with $\mu = 0.02$ and $\sigma = 0.005$. In principle, the battery is said to have failed when its capacity reduces to $\mathcal{L} = 50\%$ of its original capacity. Moreover, a market survey shows that consumer usage rates (in 100 cycles/year) obey a Gamma distribution $g(r) = \mathrm{d}G(r)/\mathrm{d}r = \frac{1}{\phi^\rho \Gamma(\rho)} r^{\rho-1} e^{-r/\phi}$ with shape parameter $\rho = 5.88$ and scale parameter $\phi = 0.35$, where $\Gamma(\rho) = \int_0^\infty x^{\rho-1} e^{-x} \mathrm{d}x$. Also, the minimal and maximal values of consumer usage rates are $r_{min} = 0.5 \times 100$ cycles/year and $r_{max} = 6.5 \times 100$ cycles/year, respectively. Further assume that the nominal usage rate is designed to $r_0 = 3 \times 100$ cycles/year, and the accelerating factors for hard and soft failure processes are $\gamma_1 = 0.8$ and $\gamma_2 = 1.0$, respectively. The parameters in the product demand model are set to $\kappa_1 = 5 \times 10^{11}$, $\kappa_2 = 1.5$, $\kappa_3 = 0.55$, $\varphi_1 = 1.75$, $\varphi_2 = 0.30$, and $\varphi_3 = 0.75$. Furthermore, the average replacement cost for an individual unit is $c_f = \$2000$; the penalty cost coefficient is $c_p = \$100000$, which implies that if the actual performance level is lower than the guaranteed level by 1%, then the penalty cost would be \$1000.

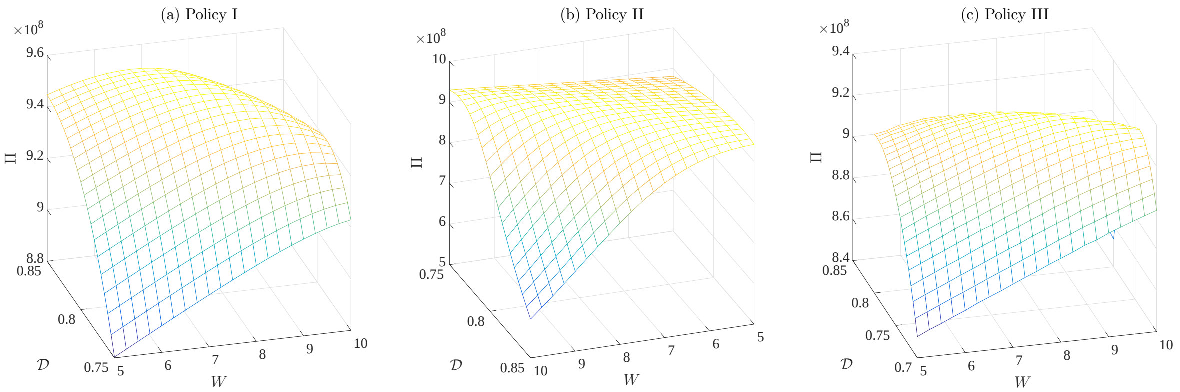

Figure 3: Total expected profits versus W and $\mathcal{D}$ for the three policies, with optimal P^* fixed. The profit functions are concave in W and $\mathcal{D}$, ensuring the existence of optimality.

Based on the arbitrarily set parameter values, Fig. 3 illustrates the profit functions for the three policies, with optimal P^* fixed. As can be observed, the profit functions are concave in W and $\mathcal{D}$, which ensures the existence of optimality. The optimal decision for Policy I is to sell the product at price $P^* = \$6885.9$, offer a performance-based warranty of length $W^* = 8.8$ years, and guarantee a minimum performance level $\mathcal{D}^* = 79.9\%$, of its original capacity, over the warranty period. The resultant sales volume is $Q^* = 242046$, unit profit is $\pi^* = \$3934.8$, and total expected profit is $\Pi_{\mathrm{I}}^* = \$952400973.1$. Likewise, the optimal results for Policy II are $P^* = \$6580.5$, $W^* = 7.8$, $\mathcal{D}^* = 79.2\%$, $Q^* = 253349$, $\pi^* = \$3774.7$, and $\Pi_{\mathrm{II}}^* = \$956312258.9$; the optimal results for Policy III are $P^* = \$6643.6$, $W^* = 8.1$, $\mathcal{D}^* = 76.3\%$, $Q^* = 247254$, $\pi^* = \$3758.9$, and $\Pi_{\mathrm{III}}^* = \$929408859.4$.

As can be seen, under the current parameter setting, Policies II and III specify shorter warranty lengths, lower performance guarantees, and lower product prices, when compared with Policy I. This is because the warranty costs in Policies II and III would be higher than that in their counterpart if the same W and $\mathcal{D}$ are applied to all of them (Notice that the penalty cost and full refund, especially the latter one, are much higher than the replacement cost). Hence, the two policies have to set smaller W^* and $\mathcal{D}^*$ to control the warranty servicing costs, which, in turn, would negatively affect the sales volumes. As a consequence, they have to set lower prices to help attract consumers, although in Policy III a lower product price is beneficial to warranty cost reduction as well. This is reflected in the fact that the sales volumes of Policies II and III are larger, and the associated unit profits are lower.

On the other hand, when comparing the penalty and full refund policies, one can see that the latter policy has a lower performance guarantee level to control the warranty servicing cost; the profit margin of the full refund policy is slightly lower, although its optimal price is higher. This is not surprising because of the high expense of full refund. Nevertheless, in this single case, we find that the free replacement policy results in a higher total profit than the other two policies, thanks to its lower warranty cost. However, it may be not always the case when the values of model parameters vary.

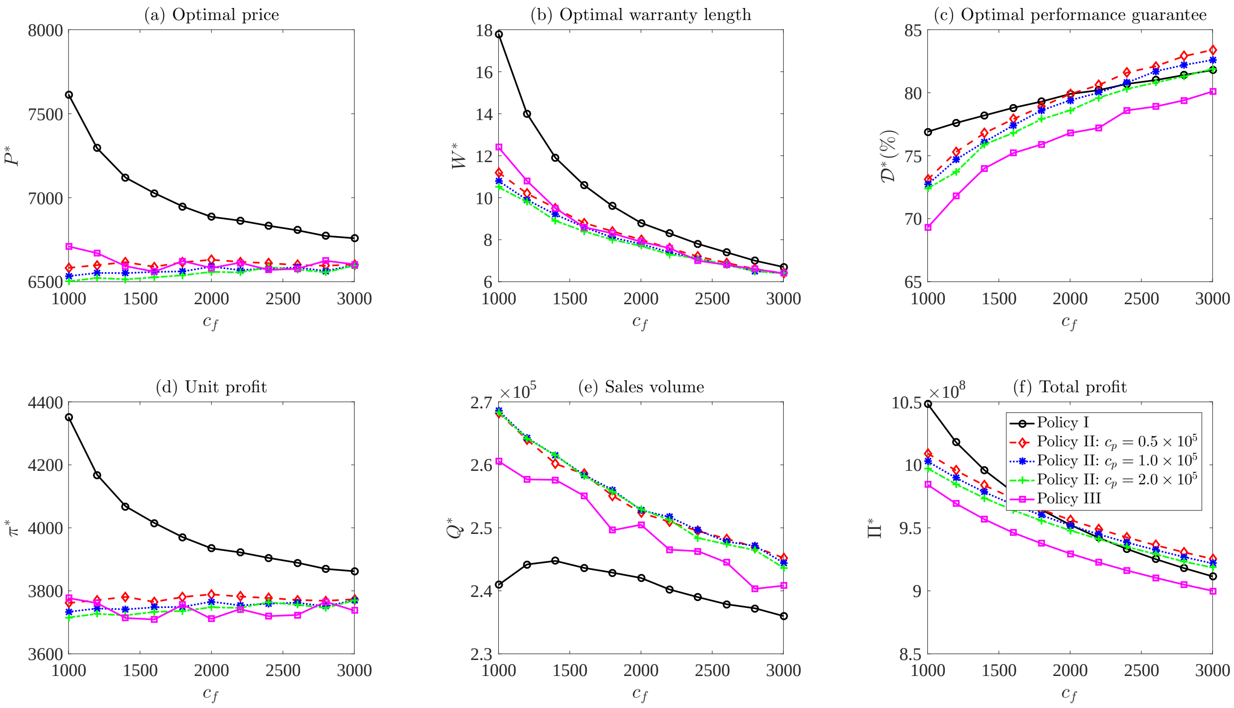

Figure 4: Comparison of the three performance-based policies under various combinations of c_p and c_f.

4.1. Sensitivity analyses and policy comparisons

In what follows, we conduct thorough sensitivity analyses and policy comparisons of the three compensation policies under various parameter settings. One thing noteworthy is that the penalty cost coefficient c_p is unique for Policy II so that its value shall have an impact on the policy comparisons. Below we investigate the effects of model parameters c_f, α, and μ, in combination with c_p, by varying two parameters at a time while keeping the others unchanged.

We first focus on the combination of penalty cost coefficient c_p and replacement cost c_f. In this scenario, three values of c_p, i.e., $c_p = \$50000$, $c_p = \$100000$, and $c_p = \$200000$, are considered, with c_f increasing from \$1000 to \$3000. The corresponding results are summarized in Table 1 and displayed in Fig. 4. The following observations can be obtained accordingly:

1. Under Policy I, as the replacement cost c_f increases from \$1000 to \$3000, the optimal warranty length declines rapidly from 17.8 years to 6.7 years, in order to reduce the warranty servicing cost. In the meantime, the optimal product price decreases from \$7615.0 to \$6758.8, and the optimal performance guarantee level increases slightly from 76.9% to 81.8%. As a result, the total expected profit shows a downward trend. To provide a more comprehensive view, Fig. 4 also shows the unit profit and total sales volume at optimality. One can observe that the unit profit is consistently decreasing in c_f, which is driven by the reduction in product price and the growth in warranty cost. Interestingly, the total sales volume exhibits an increasing-then-decreasing pattern. It

Table 1: Optimal results for various combinations of c_p and c_f (note: Π^* is in million dollars).

c_f	Policy I				Policy II												Policy III			
					$c_p = 50000$				$c_p = 100000$				$c_p = 200000$							
	P^*	W^*	$\mathcal{D}^*$	Π^*	P^*	W^*	$\mathcal{D}^*$	Π^*	P^*	W^*	$\mathcal{D}^*$	Π^*	P^*	W^*	$\mathcal{D}^*$	Π^*	P^*	W^*	$\mathcal{D}^*$	Π^*
1000	7615.0	17.8	76.9%	1048.4	6582.4	11.2	73.1%	1009.1	6533.3	10.8	72.7%	1002.8	6502.2	10.5	72.4%	997.2	6710.1	12.4	69.3%	984.2
1200	7294.6	14.0	77.6%	1017.9	6598.4	10.2	75.3%	995.4	6551.9	9.9	74.7%	989.5	6522.0	9.8	73.7%	984.6	6669.8	10.8	71.8%	969.1
1400	7119.2	11.9	78.2%	995.7	6616.3	9.5	76.8%	983.6	6551.0	9.2	76.1%	978.3	6514.0	8.9	75.9%	973.6	6592.6	9.5	74.0%	956.7
1600	7027.4	10.6	78.8%	978.4	6588.8	8.8	77.9%	973.5	6558.2	8.6	77.4%	968.4	6525.8	8.4	76.8%	964.0	6559.6	8.6	75.2%	946.2
1800	6947.7	9.6	79.3%	964.3	6615.8	8.4	78.9%	964.4	6561.1	8.1	78.6%	959.9	6538.3	8.0	77.9%	955.6	6622.5	8.3	75.9%	937.5
2000	6885.9	8.8	79.9%	952.4	6606.4	7.9	80.0%	956.3	6580.5	7.8	79.2%	951.9	6568.7	7.7	78.8%	948.0	6581.3	7.9	76.8%	929.6
2200	6863.6	8.3	80.2%	942.1	6618.2	7.6	80.6%	949.0	6568.9	7.4	80.0%	944.9	6554.9	7.3	79.6%	941.1	6613.3	7.6	77.2%	922.5
2400	6832.9	7.8	80.7%	933.2	6610.0	7.2	81.6%	942.4	6578.6	7.1	80.8%	938.3	6586.5	7.1	80.3%	934.9	6569.7	7.0	78.6%	916.1
2600	6806.3	7.4	81.0%	925.2	6599.2	6.9	82.1%	936.2	6584.2	6.8	81.7%	932.4	6572.3	6.8	80.8%	929.1	6576.6	6.8	78.9%	910.3
2800	6771.9	7.0	81.4%	917.9	6594.7	6.6	82.9%	930.6	6563.6	6.5	82.2%	926.9	6555.3	6.5	81.3%	923.2	6624.6	6.6	79.4%	904.9
3000	6758.8	6.7	81.8%	911.5	6602.9	6.4	83.4%	925.3	6598.4	6.4	82.6%	921.7	6596.1	6.4	81.9%	918.4	6603.1	6.4	80.1%	899.9

is increasing at the very beginning because of the rapid decrease in product price; while it becomes decreasing later, as the sales reduction effect due to the rapid decline in warranty length dominates the sales growth effect caused by the decrease in product price and the increase in performance guarantee.

2. Under Policy II, as c_f rises, the patterns of optimal warranty length, performance guarantee level, and total expected profit are the same as those in Policy I; however, the optimal product price and associated unit profit are relatively stable, although the optimal price shows a slightly increasing trend. The sales volume under Policy II, at optimality, exhibits a decreasing trend as c_f increases. A decreasing sales volume, combining a relatively stable unit profit, leads to a decreasing total profit. Under Policy III, the patterns of all the variables are analogous to those in Policy II.
3. In terms of the impact of penalty cost coefficient c_p on the optimal results under Policy II, one can find that almost all the reported quantities (except the sales volume) decreases as c_p increases. This is consistent with our intuition: When c_p becomes larger, the manufacturer needs to pay more for the violation of the performance guarantee. Therefore, the manufacturer has to reduce the warranty length and performance guarantee level to avoid excessive penalty cost, and also set a lower product price to mitigate sales volume reduction. Eventually, it results in a lower total profit for the manufacturer. Nevertheless, compared with c_f, the impact of c_p on the optimal results is quite insignificant.
4. More importantly, we compare the reported quantities of the three compensation policies. One can clearly see that the total profit of Policy III is the lowest, whereas the total profit of Policy I is higher than those of Policy II when c_f is small while becomes lower than those when c_f is large; however, the turning point comes later when c_p is larger. This implies that the free replacement policy is superior to the penalty policy, in terms of total expected profit, when c_f is small and/or c_p is large. Moreover, the optimal product price, warranty length, and unit profit of Policy I are consistently higher than those in Policies II and III, respectively, whereas the total sales volumes exhibit an inverse relationship. This shows that Policy I pursues a high profit by extracting more money from each sold unit, whereas Policy II adopts a different strategy, i.e., selling more units at a lower profit margin. Furthermore, the optimal performance guarantee levels of Policies I and II exhibit a similar pattern to the corresponding total profits, although the performance guarantee levels are increasing, rather than decreasing, in c_f. The optimal performance guarantee level of Policy III, however, is obviously lower than those of the other two policies, due to the high expense of full refund. In addition, for the optimal product price, warranty length, unit profit and total sales volume, the gap between that of Policy I and those of Policies II and III becomes narrower as c_f increases.

We then examine the combination of penalty cost coefficient c_p and scale parameter α of the Weibull distribution. Basically, a larger (resp. smaller) value of α corresponds to a higher (resp. lower) product reliability, in terms of the hard failure process. In this scenario, we consider α increasing from 10 to 20, in combination with $c_p = \$50000$, $c_p = \$100000$,

Table 2: Optimal results for various combinations of c_p and α (note: Π^* is in million dollars).

α	Policy I				Policy II												Policy III			
					$c_p = 50000$				$c_p = 100000$				$c_p = 200000$							
	P^*	W^*	$\mathcal{D}^*$	Π^*	P^*	W^*	$\mathcal{D}^*$	Π^*	P^*	W^*	$\mathcal{D}^*$	Π^*	P^*	W^*	$\mathcal{D}^*$	Π^*	P^*	W^*	$\mathcal{D}^*$	Π^*
10	6865.0	6.0	84.9%	891.9	6654.0	5.5	85.7%	898.2	6617.4	5.4	85.1%	895.2	6585.5	5.3	84.7%	892.6	6653.4	5.5	82.4%	877.1
11	6878.4	6.6	83.8%	906.1	6661.9	6.1	84.1%	911.9	6626.4	6.0	83.4%	908.5	6605.1	5.9	83.1%	905.6	6642.8	5.9	81.1%	889.5
12	6889.9	7.2	82.7%	919.0	6634.2	6.5	83.2%	924.5	6607.0	6.4	82.6%	920.8	6579.3	6.3	82.1%	917.7	6625.4	6.6	79.9%	900.1
13	6879.8	7.7	81.8%	931.0	6601.8	6.9	82.1%	935.9	6587.6	6.8	81.7%	932.0	6564.6	6.7	81.2%	928.5	6609.2	7.0	78.8%	911.1
14	6894.5	8.3	80.8%	942.1	6616.9	7.5	80.7%	946.6	6586.7	7.3	80.5%	942.4	6566.9	7.2	80.0%	938.7	6606.9	7.4	78.1%	920.9
15	6885.9	8.8	79.9%	952.4	6606.4	7.9	80.0%	956.3	6580.5	7.8	79.2%	951.9	6568.7	7.7	78.8%	948.0	6581.3	7.9	76.8%	929.6
16	6898.6	9.4	78.9%	962.0	6602.1	8.4	78.8%	965.4	6553.5	8.1	78.6%	960.7	6535.1	8.0	78.0%	956.5	6551.9	8.3	76.0%	937.8
17	6895.6	9.9	78.1%	970.9	6580.9	8.8	77.8%	973.8	6554.1	8.6	77.4%	968.9	6499.7	8.3	77.1%	964.5	6571.7	8.8	75.1%	945.2
18	6893.3	10.4	77.3%	979.4	6546.0	9.0	77.5%	981.7	6522.5	8.9	76.6%	976.5	6493.2	8.7	76.2%	971.9	6538.8	9.1	74.2%	952.3
19	6896.5	10.9	76.6%	987.3	6567.6	9.6	76.3%	988.9	6546.9	9.5	75.4%	983.5	6511.3	9.3	74.8%	978.7	6548.8	9.6	73.0%	958.8
20	6894.7	11.4	75.8%	994.7	6542.0	9.9	75.6%	995.8	6509.6	9.7	74.9%	990.2	6467.9	9.4	74.6%	985.2	6495.2	9.7	72.7%	965.1

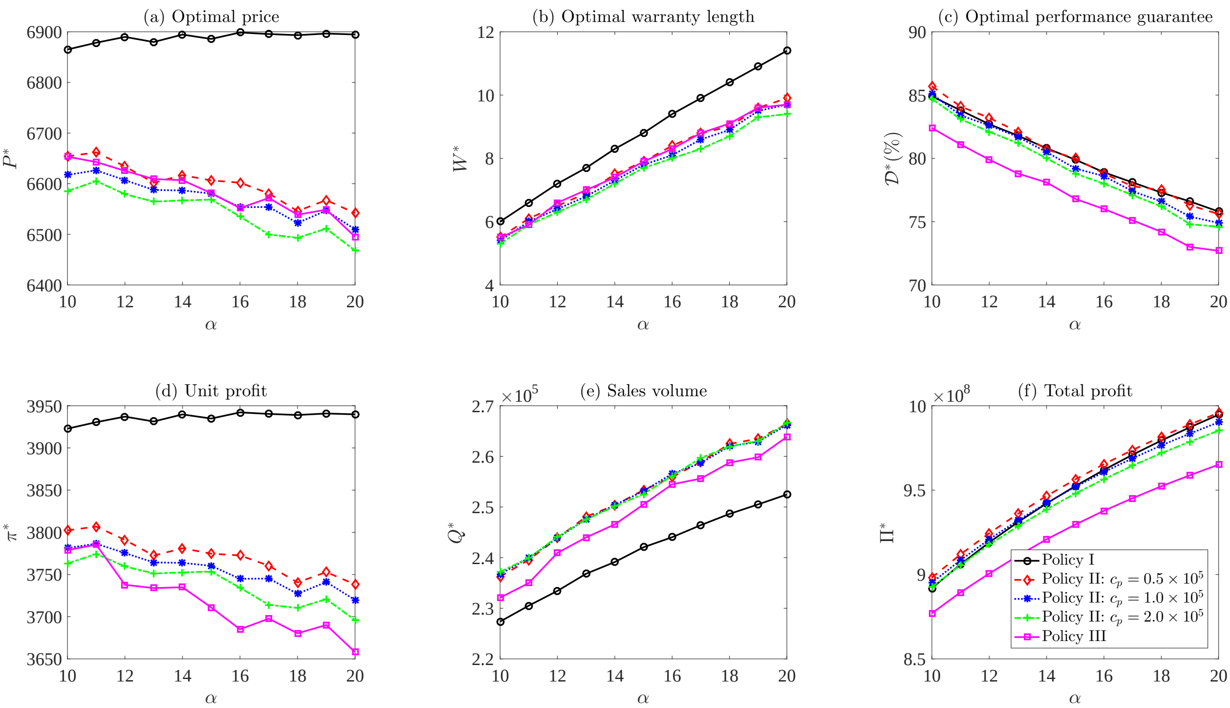

Figure 5: Comparison of the three performance-based policies under various combinations of c_p and α.

and $c_p = \$200000$, respectively. The corresponding results are summarized in Table 2 and illustrated in Fig. 5. The following findings can be drawn:

1. Under Policy I, as the scale parameter α increases from 10 to 20, the optimal product price remains relatively stable, and the optimal performance guarantee level decreases from 84.9% to 75.8%; accordingly, the optimal warranty length increases from 6.0 years to 11.4 years so as to boost the product sales. On the other hand, the unit profit is relatively stable, as the warranty cost growth effect induced by the increase in warranty length is largely offset by the cost reduction effect caused by the decline in performance guarantee level. As a result, the total sales volume shows an upward trend, which, in combination with a relatively constant unit profit, results in an increasing total profit.
2. Under Policy II, as α becomes large, the patterns of the optimal warranty length, performance guarantee, sales volume, and total expected profit are the same as those in Policy I; however, the optimal product price and associated unit profit show a slightly decreasing tendency. An increasing sales volume and a slightly decreasing unit profit lead to an increasing total profit. As before, the patterns of all the variables under Policy III is analogous to those in Policy II.
3. In terms of the impact of penalty cost coefficient c_p on the optimal results under Policy II, one can still observe that almost all the reported quantities (except the sales volume) become smaller as c_p becomes larger. Again, compared with α, the impact of c_p on most quantities is insignificant.
4. When comparing the reported quantities of the three compensation policies, one can

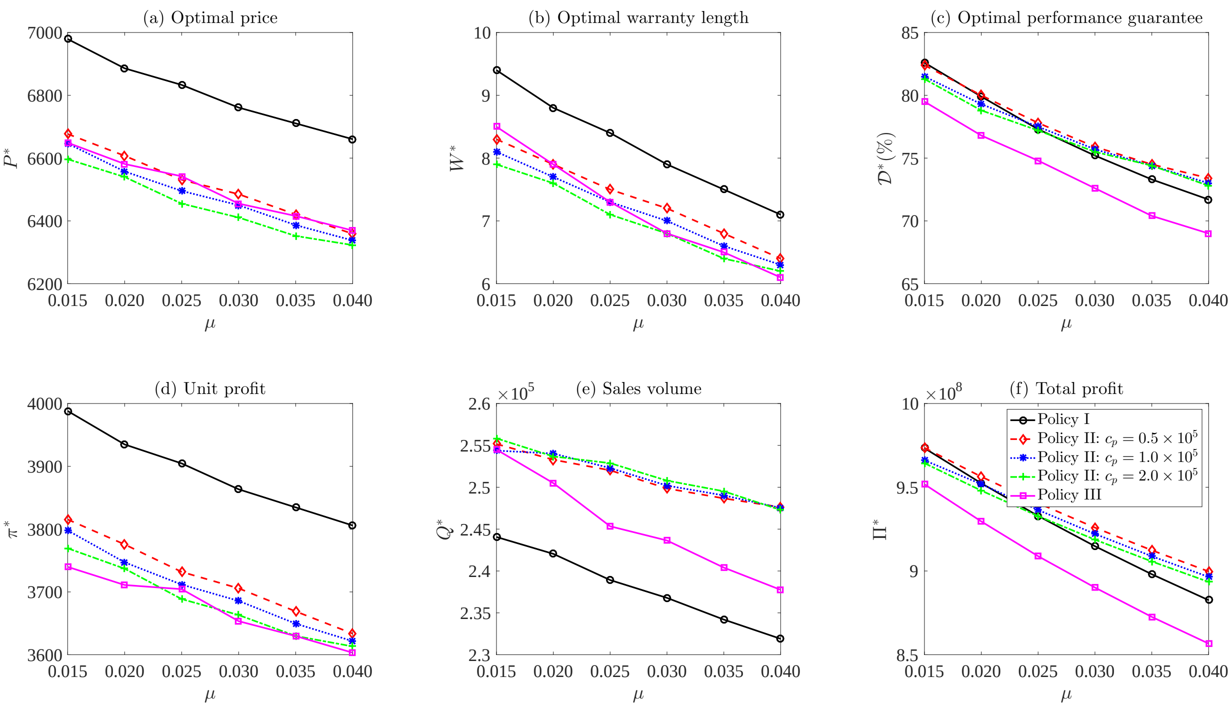

Figure 6: Comparison of the three performance-based policies under various combinations of c_p and μ.

find that Policy III is always inferior to the other two, in terms of total expected profit. Policy I is dominated by Policy II, when α is small, while it becomes dominant when α becomes large; however, the turning point comes earlier when c_p is larger. This implies that Policy I is inferior to Policy II, when α is small and/or c_p is small. In addition, the relative patterns of optimal product prices, warranty lengths, unit profits and sales volumes for the three policies are consistent with those in Fig. 4. This comes to the same finding that Policy I seeks a high total profit by extracting more money from each sold unit, whereas Policy II sells more units at a lower unit profit. Moreover, the optimal guarantee level in Policy I is lower than those in Policy II when α is small and then becomes higher than those when α is large; however, the optimal performance guarantee of Policy III is always lower than those of the other two policies. Furthermore, for the optimal product price, warranty length, unit profit and total sales volume, the gap between that of Policy I and those of Policies II and III grows up as α increases.

We further look at the combination of penalty cost coefficient c_p and mean parameter μ of the normal distribution. In essence, a larger (resp. smaller) value of μ implies a higher (resp. lower) degradation rate, with respect to the soft failure process. We consider μ growing from 0.015 to 0.040 with a step size of 0.005, in combination with $c_p = \$50000$, $c_p = \$100000$, and $c_p = \$200000$ as well. The corresponding results are listed in Table 3 and illustrated in Fig. 6. The following observations can be summarized:

1. As the mean parameter μ increases from 0.015 to 0.040, all the reported quantities

Table 3: Optimal results for various combinations of c_p and μ (note: Π^* is in million dollars).

μ	Policy I				Policy II												Policy III			
					$c_p = 50000$				$c_p = 100000$				$c_p = 200000$							
	P^*	W^*	$\mathcal{D}^*$	Π^*	P^*	W^*	$\mathcal{D}^*$	Π^*	P^*	W^*	$\mathcal{D}^*$	Π^*	P^*	W^*	$\mathcal{D}^*$	Π^*	P^*	W^*	$\mathcal{D}^*$	Π^*
0.015	6979.2	9.4	82.6%	973.4	6676.9	8.3	82.4%	973.5	6646.4	8.1	81.5%	966.1	6596.6	7.9	81.3%	964.4	6649.0	8.5	79.5%	951.8
0.020	6885.9	8.8	79.9%	952.4	6606.4	7.9	80.0%	956.3	6580.5	7.8	79.2%	951.9	6568.7	7.7	78.8%	948.0	6581.3	7.9	76.8%	929.6
0.025	6833.1	8.4	77.3%	932.9	6530.9	7.5	77.8%	940.5	6495.5	7.3	77.5%	936.4	6455.1	7.1	77.2%	932.8	6542.1	7.3	74.8%	908.8
0.030	6760.9	7.9	75.2%	914.8	6485.0	7.2	75.9%	925.9	6449.9	7.0	75.7%	922.1	6410.7	6.8	75.5%	918.7	6455.1	6.8	72.6%	890.1
0.035	6710.6	7.5	73.3%	898.1	6420.7	6.8	74.5%	912.4	6386.6	6.6	74.4%	908.8	6352.1	6.4	74.4%	905.7	6415.7	6.5	70.4%	872.6
0.040	6660.2	7.1	71.7%	882.6	6358.8	6.4	73.4%	899.7	6337.7	6.3	73.0%	896.4	6323.0	6.2	72.8%	893.5	6369.3	6.1	69.0%	856.7

(including the optimal product price, warranty length, performance guarantee level, unit profit, sales volume, and total expected profit) of the three policies exhibit a decreasing pattern. This can be explained by the fact that when μ rises, the performance degradation process will accelerate in an exponential manner. Hence, the manufacturer has to reduce the warranty length and performance guarantee level to cut down the warranty servicing cost, and also set a lower product price to mitigate sales volume decline. On the other hand, the unit profit decreases as μ increases, driven by the reduction in product price and the growth in warranty cost. As a consequence, the declines in both sales volume and unit profit generate a decreasing pattern of the manufacturer's total profit.

2. Likewise, the impact of penalty cost coefficient c_p on the optimal results under Policy II is not significant, although almost all the reported quantities (except the sales volume) become smaller as c_p rises. This observation is consistent with those in Figs. 4 and 5.
3. One can see that the total profit of Policy I is higher than those of Policy II when μ is small while becomes lower than them when μ is large; however, the turning point comes later when c_p is larger. This indicates that Policy I is superior to its counterpart, in terms of total expected profit, when μ is small and/or c_p is large. In contrast, Policy III has the lowest total profit among the three policies, as before. Furthermore, the relative patterns of optimal product prices, warranty lengths, unit profits and sales volumes for the three policies are the same as those in Figs. 4 and 5. In addition, the optimal performance guarantee level in Policy III is the lowest among the three policies, whereas the optimal guarantee level in Policy I is higher than those in Policy II when μ is small and then becomes lower than those when μ is large.

4.2. Comparison with the pure product warranty policy

As discussed earlier, the performance-based warranty policies reduce to the pure product warranty policy when $\mathcal{D} = \mathcal{L}$. It is thus of interest to compare the proposed performance-based warranties with the pure product warranty, so as to demonstrate the superiority of the performance-based warranties. For this purpose, Table 4 shows the optimal results for the pure product warranty policy under various settings of parameters c_f, α, and μ, the same as those in Section 4.1.

It is clear to observe that the optimal product price and warranty length in the pure product warranty policy are higher, whereas the resulting total profit is lower than those in the performance-based warranty policies, respectively. The explanation of this observation is intuitive: Without the performance guarantee as an advertising weapon, the pure product warranty policy has to offer a longer protection period so as to mitigate sales volume decrease. On the other hand, the product price is lifted to cover the growing warranty servicing cost. As a result, the pure warranty policy has a lower sales volume and a higher unit profit, although the detailed results are not shown in Table 4. The ability of increasing total profit shows the superiority of the proposed performance-based warranty, in comparison with traditional product warranties. In addition, the profit difference between the performance-based and product warranties becomes larger when the penalty cost coefficient c_p is larger, the scale parameter α is smaller, and/or the mean parameter μ is smaller.

Table 4: Optimal results for the pure product warranty policy (note: Π^* is in million dollars).

c_f	P^*	W^*	Π^*	α	P^*	W^*	Π^*	μ	P^*	W^*	Π^*
1000	7975.6	23.4	921.2	10	7209.8	7.6	734.7	0.015	7359.5	12.1	816.1
1200	7748.1	19.1	889.4	11	7250.2	8.5	752.6	0.020	7359.6	12.1	816.1
1400	7599.5	16.4	865.1	12	7284.1	9.4	769.6	0.025	7338.6	12.0	816.0
1600	7488.7	14.5	845.8	13	7313.0	10.3	785.8	0.030	7341.7	12.0	815.8
1800	7406.1	13.1	829.7	14	7337.9	11.2	801.3	0.035	7284.7	11.7	815.1
2000	7359.6	12.1	816.1	15	7359.6	12.1	816.1	0.040	7236.3	11.4	813.5
2200	7301.3	11.2	804.3	16	7358.4	12.9	830.3				
2400	7262.5	10.5	793.9	17	7376.7	13.8	844.0				
2600	7227.1	9.9	784.6	18	7393.3	14.7	857.1				
2800	7200.3	9.4	776.3	19	7408.6	15.6	869.8				
3000	7186.5	9.0	768.8	20	7406.4	16.4	882.0				

5. Conclusions and future research topics

This article proposes the concept of performance-based warranty for products subject to competing hard and soft failure modes. Unlike product warranties that protect consumers solely from premature failures, the performance-based warranty not only covers the repair or replacement of any failure, but also guarantees the minimum performance level over the warranty period. This warranty policy would compensate consumers if the product's actual performance throughout the warranty period fails to meet the guaranteed level, which offers a feasible mechanism to address consumers' concern on the deterioration of key product performance characteristics. In this work, three types of compensation mechanisms are studied, namely, free replacement, penalty, and full refund. We first formulate the warranty cost models for the three compensation policies, and then develop an optimization problem to determine the optimal product price, warranty length, and performance guarantee level so as to maximize the manufacturer's total profit. Numerical experiments are carried out to demonstrate and compare the three compensation policies.

Numerical results show that the full refund policy always leads to the lowest total profit, whereas neither of the other two policies can dominate each other in all scenarios. In particular, the free replacement policy is superior to the penalty policy, in terms of total expected profit, when the replacement cost is low, the penalty cost coefficient is high, and/or the product reliability is high. More interestingly, we find that under different policies, manufacturers seek profit maximization through distinct strategies: Under the free replacement policy, the manufacturers would extract more money from each sold unit but have a lower sales volume, whereas under the penalty policy they would sell more product units with a lower profit margin for each unit. Moreover, comparing the performance-based warranty policies with a pure product warranty policy shows that introducing an additional performance guarantee mechanism is beneficial for manufacturers to increase sales profit. Nevertheless, the new type of performance-based warranty has shown a significant potential, and more manufacturers could consider this direction for capital-intensive and performance-critical products whose performance characteristics can be clearly defined and measured.

This work makes an early attempt to study the performance-based warranty. There are several ways in which the work can be extended. Practical performance-based warranties might involve multiple performance guarantee levels at different time points within the entire warranty period; see, e.g., the warranty policies for solar panels mentioned earlier. This will complicate the warranty modeling and analysis, and Monte Carlo simulation methods shall be effective for warranty cost evaluation. Moreover, preventive maintenance activities can be carried out to improve product reliability and mitigate performance deterioration, especially for capital-intensive products. How to optimally schedule preventive maintenance activities for performance-critical products is an open problem. Finally, it is also interesting to treat the penalty cost coefficient c_p as an additional decision variable, so that the manufacturer could seek a tradeoff between the product demand growth and the penalty cost incurred.

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Appendix A. Riemann-Stieltjes method

Under policy I, when computing the conditional total expected warranty servicing cost, i.e., Eq. (12), we need to solve a renewal-type integral equation. It is not easy to derive an analytical solution for this equation. Based on the Riemann-Stieltjes sums, Xie (1989) proposes a simple yet accurate numerical method to evaluate this type of equations. To implement this method, the warranty period is (uniformly) divided into n intervals with endpoints $0 = t_0 < t_1 < \cdots < t_n = W$. Then, using the well-known midpoint method in numerical analysis, the renewal equation can be rewritten as

$$M^{\mathrm{I}}(t_j; r) \approx \Psi^{\mathrm{I}}(t_j; r) + \sum_{k=1}^{j} \Psi^{\mathrm{I}}(t_j - t_{k-1/2}; r)\big(M^{\mathrm{I}}(t_k; r) - M^{\mathrm{I}}(t_{k-1}; r)\big), \tag{A.1}$$

where $t_{j-1/2} = (t_{j-1} + t_j)/2$ is the midpoint of the jth interval.

Therefore, the numerical approximation of $M^{\mathrm{I}}(t_j; r)$ can be calculated recursively through

$$\tilde{M}^{\mathrm{I}}(t_j; r) = \frac{\Psi^{\mathrm{I}}(t_j; r) + S_j - \Psi^{\mathrm{I}}(t_j - t_{j-1/2}; r)\tilde{M}^{\mathrm{I}}(t_{j-1}; r)}{1 - \Psi^{\mathrm{I}}(t_j - t_{j-1/2}; r)}, \tag{A.2}$$

where $\tilde{M}^{\mathrm{I}}(t_0; r) = 0$ and $S_j = \sum_{k=1}^{j-1} \Psi^{\mathrm{I}}(t_j - t_{k-1/2}; r)(\tilde{M}^{\mathrm{I}}(t_k; r) - \tilde{M}^{\mathrm{I}}(t_{k-1}; r))$. We use $\tilde{M}^{\mathrm{I}}(t_j; r)$ to indicate that it is a numerical approximation of $M^{\mathrm{I}}(t_j; r)$.

One result noteworthy is that Sun et al. (2019) prove that under some mild conditions, the approximation error, $M^{\mathrm{I}}(t_n; r) - \tilde{M}^{\mathrm{I}}(t_n; r)$, is $\mathcal{O}(n^{-1})$.

Appendix B. Simulation algorithm for evaluating Eq. (17)

Given any value of r, we start the simulation by drawing $T_r^{(1)}$ and $T_r^{(3)}$ from $F^{(1)}(\cdot;r)$ and $F^{(3)}(\cdot;r)$, respectively. Then, the time to first replacement can be determined by $T_r^{\mathrm{II}} = \min\{T_r^{(1)}, T_r^{(3)}\}$. Define $\zeta(t) = \int_{-\ln \mathcal{D}/\Lambda_2(t)}^{-\ln \mathcal{L}/\Lambda_2(t)} (\mathcal{D} - \eta(\Lambda_2(t);u))\mathrm{d}F_\omega(u)$. If $T_r^{\mathrm{II}} \geq W$, then the penalty cost is $C_{\mathcal{P}}(W,\mathcal{D};r) = c_p\zeta(W)$; otherwise, calculate the penalty cost as follows: If $T_r^{(1)} \geq T_r^{(3)}$, i.e., a soft failure induces the product failure, then $C_{\mathcal{P}}(W,\mathcal{D};r) = c_p(\mathcal{D}-\mathcal{L})$; otherwise, $C_{\mathcal{P}}(W,\mathcal{D};r) = c_p\zeta(T_r^{(1)})$. Repeat this procedure until the end of warranty period is reached.

After a large number of simulation runs, the conditional expected penalty cost in Eq. (17) can be well approximated by the average of $C_{\mathcal{P}}(W,\mathcal{D};r)$, i.e., $\overline{C}_{\mathcal{P}}(W,\mathcal{D};r) = \sum_{i=1}^{Sim} C_{\mathcal{P}}^{(i)}(W,\mathcal{D};r)/Sim$, where $C_{\mathcal{P}}^{(i)}(W,\mathcal{D};r)$ is the penalty cost in the ith simulation run and Sim is the total number of simulation runs for each single case. The Monte Carlo simulation procedure is detailed in Algorithm 1.

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Algorithm 1

Input: r, $F^{(1)}(\cdot; r)$, $F^{(3)}(\cdot; r)$, $\zeta(\cdot)$, c_p

Output: $\overline{C}_{\mathcal{P}}(W, \mathcal{D}; r)$

for $i = 1$ to Sim **do**
  $k \leftarrow 0$
  $T^C_{r,k} \leftarrow 0$
  $C^{(i)}_{\mathcal{P}}(W, \mathcal{D}; r) \leftarrow 0$
  while $T^C_{r,k} < W$ **do**
    $k \leftarrow k + 1$
    Simulate $T^{(1)}_{r,k} \sim F^{(1)}(\cdot; r)$
    Simulate $T^{(3)}_{r,k} \sim F^{(3)}(\cdot; r)$
    $T^{\mathrm{II}}_{r,k} = \min\{T^{(1)}_{r,k}, T^{(3)}_{r,k}\}$
    $T^C_{r,k} \leftarrow T^C_{r,k-1} + T^{\mathrm{II}}_{r,k}$
    if $T^C_{r,k} \geq W$ **then**
      $C^{(i)}_{\mathcal{P}}(W, \mathcal{D}; r) \leftarrow C^{(i)}_{\mathcal{P}}(W, \mathcal{D}; r) + c_p\zeta(W - T^C_{r,k-1})$
      Break;
    else
      if $T^{(1)}_{r,k} < T^{(3)}_{r,k}$ **then**
        $C^{(i)}_{\mathcal{P}}(W, \mathcal{D}; r) \leftarrow C^{(i)}_{\mathcal{P}}(W, \mathcal{D}; r) + c_p\zeta(T^{(1)}_{r,k})$
      else
        $C^{(i)}_{\mathcal{P}}(W, \mathcal{D}; r) \leftarrow C^{(i)}_{\mathcal{P}}(W, \mathcal{D}; r) + c_p(\mathcal{D} - \mathcal{L})$
      end if
    end if
  end while
end for
return $\overline{C}_{\mathcal{P}}(W, \mathcal{D}; r) = \sum_{i=1}^{Sim} C^{(i)}_{\mathcal{P}}(W, \mathcal{D}; r)/Sim$

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