IEEE Access
Multidisciplinary : Rapid Review : Open Access Journal

Received March 18, 2019, accepted April 15, 2019, date of publication April 22, 2019, date of current version May 1, 2019.

Digital Object Identifier 10.1109/ACCESS.2019.2912653

Optimal Continuous/Impulsive LQ Control With Quadratic Constraints

QINGYUAN QI[1], ZHENGHONG QIU[2], AND ZHIJIAN JI[1]
[1]College of Automation, Institute of Complexity Science, Qingdao University, Qingdao 266071, China
[2]Department of Applied Mathematics, The Hong Kong Polytechnic University, Hong Kong

Corresponding author: Zhijian Ji (jizhijian@pku.org.cn)

This work was supported by the National Natural Science Foundation of China under Grant 61573221, Grant 61633014, Grant 61873136, and Grant 61873179.

ABSTRACT In this paper, the optimal continuous/impulsive linear quadratic (LQ) control problem with quadratic constraints is thoroughly solved for the first time. The main contributions of this paper can be stated as in the following. First, the maximum principle is developed by using the variational method. Then, by using the Lagrange duality principle, the optimal continuous/impulsive control can thus be obtained via decoupling the forward and backward differential/difference equation (FBSDE). Finally, the optimal parameter can be calculated by using the gradient-type optimization algorithm.

INDEX TERMS Continuous/impulsive control, quadratic constraints, maximum principle, solution to FBSDE.

I. INTRODUCTION

Different to the traditional instantaneous stochastic control problem, the impulsive control problem has to choose the optimal actions at every chosen time instant, and the intervention times don't accumulate (see [1]–[4]). Generally, for stochastic continuous/impulsive control problem, the decision-maker has to choose both the instantaneous and the cumulative components. The stochastic impulsive control can be applied to solve problems raised in mathematical finance (see [4]–[6]). For example, the stochastic continuous/impulsive control can be used to solve the portfolio selection problem with transaction costs, the instantaneous control is used to indicate the consumption process, and the impulse control signifies the transactions cost at some stopping times (see [6]).

Due to the potential applications in many fields, especially in natural resource economics and in studies considering the optimal control of exchange and interest rates (see [7]–[9] and references therein), the study on continuous/impulsive control have attracted many researchers' interest in the last decade, and significant contributions have been made. In [3], some applications of stochastic impulsive control model were studied, including the optimal control of an exchange rate, the portfolio optimization under transaction cost and the cash management problem. On one hand, the stochastic impulse control problem can be solved by using the dynamic programming principle, and the value function satisfies some HJB quasi-variational inequalities (see [1], [6], [7], [10]–[12]); On the other hand, stochastic maximum principle can be also used to obtain the necessary conditions for the optimal impulse control problem (see [13]–[15]). For recent development on the impulse control model (see [16]–[18], [24]–[27]).

Besides, the study on control problems with constraint has been another hot research topic in recent years. Especially, the constrained LQR problem has been investigated in many literatures (see [19]–[21]). Usually, the unconstrained control problem can be viewed as single objective problem. While in practical applications, many objectives must be achieved simultaneously, then the multi-objective control problems should be under consideration. For example, the well-celebrated portfolio selection problem can be viewed as multi-objective optimization problem (see [21]). To handle with the multi-objective control problem, generally, the multi-objective problem is converted to single-objective problem subject to some constraints associated with other objectives, readers can refer to [28], [30]. This framework of LQ control problem with constraints is very useful in practical applications, and some progress has been made in the last several decades, [19]–[22]. Also, it is also possible to remove some special type constraints (if such constraint

The associate editor coordinating the review of this manuscript and approving it for publication was Bora Onat.

can be embedded into a revised functional) via the revised equivalent method transform as proposed in [23].

In this paper, we consider the LQ continuous/impulsive control problem with quadratic constraints. At the instantaneous time, the system dynamics is described with the Itô process; While at the impulsive time, the system state can be calculated via the given discrete-time multiplicative noise system. Meanwhile, the quadratic constraint should be satisfied for the investigated problem. In this paper, the problem can be solved in the following way. Firstly, by defining the Lagrangian, the original optimization problem with quadratic constraints is converted to the unconstrained one. Secondly, by using the convex variational method, the necessary condition for the optimality (maximum principle) will be developed. In what follows, the optimal control can be derived by decoupling the FBSDE both for the instantaneous time and the impulsive time. Finally, by using the Lagrangian duality theorem, the original LQ optimal continuous/impulsive control problem with quadratic constraints can be solved, and the optimal parameter can be calculated via the gradient-type algorithms.

It should be pointed out that previous works like [16]–[18] studied the impulsive control problem, while other works like [19]–[21] investigated the control problem with constraints. Furthermore, the continuous/impulsive LQ control problem with quadratic constraints has not be solved in previous literatures. While, both the impulsive control and the quadratic constraints are considered in this paper. It is stressed that the continuous/impulsive LQ control problem with quadratic constraints will be thoroughly solved in this paper. Moreover, the obtained results in this paper are innovative in the following aspects: Firstly, by using the variational method, the maximum principle for continuous/impulsive LQ control problem with quadratic constraint will be proposed for the first time; Secondly, by decoupling the FBSDE raised by the maximum principle, the optimal continuous/impulsive control will be derived; Finally, from the Lagrange duality principle, the optimal parameter selection problem will be solved by using the gradient-type optimization algorithm.

The remainder of this paper can be organized as below. In Section II, the investigated stochastic impulsive control problem is formulated; the main results are presented in Section III, also the maximum principle for the optimization is developed; some examples are provided in Section IV; finally, this paper is concluded in Section V.

The following notations will be used in this paper:

Notations: I_n means the unit matrix with rank n; Superscript $'$ denotes the transpose of a matrix. Real symmetric matrix $A > 0$ (or ≥ 0) implies that A is strictly positive definite (or positive semi-definite). $\mathbb{R}^n$ signifies the n-dimensional Euclidean space. B^{-1} is used to indicate the inverse of real matrix B. $\{\Omega, \mathcal{F}, \mathcal{P}, \{\mathcal{F}_t\}_{t\geq 0}\}$ represents a filtered complete probability space, with natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$ generated by $\{x_0, \{W_\theta\}_{\theta\geq 0}, \cdots, \{w_k\}_0^{\{k|\tau_k\leq t\}}\}$ augmented by all the $\mathcal{P}$-null sets. $E[\cdot|\mathcal{F}_k]$ means the conditional expectation with respect to $\mathcal{F}_k$ and $\mathcal{F}_{-1}$ is understood as $\{\emptyset, \Omega\}$. *a.s.* denotes the 'almost surely' sense.

II. PROBLEM FORMULATION

In this paper, for $t \in [0, T]$, we consider the following continuous/impulsive system:

$$\begin{cases} dx_t &= (Ax_t + Bu_t)dt + (Cx_t + Du_t)dW_t, \quad t \neq \tau_k, & (1)\\ x_{\tau_k^+} &= \bar{A}x_{\tau_k^-} + \bar{B}u_{\tau_k} + w_k(\bar{C}x_{\tau_k^-} + \bar{D}u_{\tau_k}), \quad t = \tau_k, & (2)\end{cases}$$

where $x_t \in \mathbb{R}^n$ is the system state, $u_t \in \mathbb{R}^m$ is the control input, $u_{\tau_k} \in \mathbb{R}^m$ denotes the impulsive control at time $\tau_k, k = 1, \cdots, N$, and $\tau_{N+1} = T$. The superscripts $^-, ^+$ means the instant immediately before and after the impulsive dynamics is applied. $\{w_k\}_{k=1}^N$ is the Gaussian white noise with zero mean and covariance 1, W_t indicates the 1-dimensional standard Brownian motion.

The associated cost function is given by:

$$J_T = E\Big\{\frac{1}{2}\sum_{k=0}^{N}\int_{\tau_k^+}^{\tau_{k+1}^-}[x_t'Qx_t + u_t'Ru_t]dt + \frac{1}{2}\sum_{k=1}^{N}[x_{\tau_k^-}'\bar{Q}x_{\tau_k^-} + u_{\tau_k}'\bar{R}u_{\tau_k}]\Big\}, \tag{3}$$

where the weighting matrices $Q \in \mathbb{R}^{n\times n}, R \in \mathbb{R}^{m\times m}$ and $\bar{Q} \in \mathbb{R}^{n\times n}, \bar{R} \in \mathbb{R}^{m\times m}$ are symmetric.

Moreover, we denote

$$\bar{J}_N \triangleq E\Big\{\mathcal{A}\sum_{i=1}^{N}u_{\tau_i}'\mathcal{R}u_{\tau_i} + \mathcal{B}'\sum_{i=1}^{N}\bar{\mathcal{R}}u_{\tau_i} + \mathcal{C}\sum_{i=1}^{N}x_{\tau_i^-}'\mathcal{Q}x_{\tau_i^-} + \mathcal{D}'\sum_{i=1}^{N}\bar{\mathcal{Q}}x_{\tau_i^-}\Big\}, \tag{4}$$

where $\mathcal{A} \in \mathbb{R}^1, \mathcal{B} \in \mathbb{R}^m, \mathcal{C} \in \mathbb{R}^1, \mathcal{D} \in \mathbb{R}^n$ are the given coefficients, and $\mathcal{R}$, $\bar{\mathcal{R}}$, $\mathcal{Q}$, $\bar{\mathcal{Q}}$ are given symmetric weighting matrices with appropriate dimensions.

In this paper, we consider the following *quadratic* constraint imposed on the stochastic impulsive control problem:

$$\bar{J}_N \leq K, \quad K \in \mathbb{R}^1. \tag{5}$$

The admissible control set $\mathcal{U}_{ad}$ is given by:

$$\mathcal{U}_{ad} = \{u_t | u_t \text{ is } \mathcal{F}_t - \text{adapted, and } \int_0^T E(u_t'u_t)dt < \infty, \sum_{k=1}^{N}u_{\tau_k}'u_{\tau_k} < \infty\}. \tag{6}$$

In this paper, we make the following standard assumption:

Assumption 1: 1) The weighting matrices in (3) *satisfy:* $Q \geq 0, R > 0$, *and* $\bar{Q} \geq 0, \bar{R} > 0$*; 2) The coefficients in* (4) *satisfy* $\mathcal{A} \geq 0, \mathcal{C} \geq 0, \mathcal{Q} \geq 0$ *and* $\mathcal{R} \geq 0$.

Assumption 2: The impulsive time $\tau_1, \cdots, \tau_N$ *are fixed, and satisfy:* $0 = \tau_0 \leq \tau_1 < \cdots < \tau_N \leq \tau_{N+1} = T$.

The main problem to be solved in this paper can be described as:

Problem 1: Under Assumption 1, find the continuous/impulsive control $u_t \in \mathcal{U}_{ad}$ for $t \in [0, T]$ to minimize the cost function (3) J_T *subject to the constraint* (5)*:* $\bar{J}_N \le K$.

Remark 1: Assumption 1 indicates the cost function (3) *is strictly convex on the admissible set $\mathcal{U}_{ad}$. Furthermore, if R and $\bar{R}$ are assumed to be positive semi-definite, the associated impulsive control problem can be derived by introducing generalized Riccati equations, which will not be discussed in this paper, see [19], [20].*

III. CONTINUOUS/IMPULSIVE CONTROL DESIGN

In this section, the closed-loop continuous/impulsive control $\{u_{\tau_k}\}_{k=1}^N$ will be derived via the duality theory.

To guarantee the solvability of the optimal continuous/impulsive control problem (Problem 1), we assume:

Assumption 3: There exists $u \in \mathcal{U}_{ad}$ which is feasible for Problem 1.

Theorem 1: Suppose Assumption 3 holds, then there exists a unique ***optimal*** *solution (x^*, u^*) for Problem 1.*

Proof: Firstly, we define the following set:

$$\mathcal{U}_\sigma = \{u_t | u_t \in \mathcal{U}_{ad}, J_T \le \sigma\}.$$

We consider the problem as below:

$$\begin{aligned} \text{To find} \quad & u \in \mathcal{U}_\sigma \text{ to minimize } J_T \text{ in (3)}, \\ & \text{subjecting to } J_T \le \sigma, \text{ and } \bar{J}_N \le K. \end{aligned} \tag{7}$$

It is noted from Remark 1 that J_T is strictly convex under Assumption 1. Moreover, the constraint (5) and $J_T \le \sigma$ define a convex subset (closed and bounded) on the admissible set $\mathcal{U}_{ad}$. Thus, problem (7) admits a unique solution, and the uniqueness can be implied from the strictly convexity of J_T.

Now we will show Problem 1 has a unique optimal solution. To show this, it is supposed that $u^* \in \mathcal{U}_{ad}$ is feasible for problem (7). In this case, if u^* is not optimal for Problem 1, then there exists $\bar{u}$ which is feasible satisfying $J_T(\bar{u}) \le J_T(u^*) \le \sigma$. This is contracted with u^* is feasible for problem (7). This ends the proof. □

A. DUALITY THEORY

In this section, we adopt the duality theory to calculate the optimal impulsive control.

For the given parameter $\lambda \ge 0$, from (4) and (5), we define the Lagrangian as:

$$\begin{aligned} L(u, \lambda) = E \sum_{k=1}^{N} \Big\{ & x'_{\tau_k^-} (\frac{1}{2}\bar{Q} + \lambda \mathcal{C}\mathcal{Q}) x_{\tau_k^-} \\ & + u'_{\tau_k} (\frac{1}{2}\bar{R} + \lambda \mathcal{A}\mathcal{R}) u_{\tau_k} + \lambda \mathcal{B}'\bar{\mathcal{R}} u_{\tau_k} + \lambda \mathcal{D}'\bar{\mathcal{Q}} x_{\tau_k^-} \Big\} \\ & + E\Big\{ \frac{1}{2} \sum_{k=0}^{N} \int_{\tau_k^+}^{\tau_{k+1}^-} [x'_t Q x_t + u'_t R u_t] dt \Big\}, \end{aligned} \tag{8}$$

where λ is called the Lagrange multiplier.

Under Assumption 1, from [29] we know that for any $\lambda \ge 0$, the following relationship holds:

$$L^*(u, \lambda) = \inf_{u \in \mathcal{U}_{ad}} \sup_{\lambda} L(u, \lambda) = \sup_{\lambda} \inf_{u \in \mathcal{U}_{ad}} L(u, \lambda). \tag{9}$$

Next we will establish the Maximum Principle for system (1)-(1) and cost function (8) as follows, which is divided into two parts: the instantaneous part and the impulsive part.

Lemma 1: For regular control ($t \ne \tau_k$), the adjoint equation (for $t \in [\tau_k^+, \tau_{k+1}^-]$) is given by:

$$\begin{cases} dp_t = -(A'p_t + C'q_t + Qx_t)dt + q_t dW_t, \\ p_{\tau_{k+1}^-}, \end{cases} \tag{10}$$

where p_t is the costate for regular control, and $p_{\tau_{N+1}^-} = p_T = 0$.

For impulsive control ($t = \tau_k, k = 1, \cdots, N$), the adjoint equation is given by:

$$\begin{cases} p_{\tau_k^-} = E[(\bar{A} + w_k \bar{C})' p_{\tau_k^+} + \lambda \bar{\mathcal{Q}} \mathcal{D} | \mathcal{F}_{k-1}] \\ \qquad + (\bar{Q} + 2\lambda \mathcal{C}\mathcal{Q}) x_{\tau_k^-}, \\ p_{\tau_k^+}. \end{cases} \tag{11}$$

It is remarkable that the final condition $p_{\tau_{k+1}^-}$ can be obtained from (11)*, and $p_{\tau_k^+}$ can be calculated from* (10).

Furthermore, the optimal instantaneous control ($t \ne \tau_k$) satisfies the stationary condition:

$$Ru_t + B'p_t + D'q_t = 0. \tag{12}$$

While the stationary condition at $t = \tau_k$ is given as:

$$(\bar{R} + 2\lambda\ \mathcal{A}\mathcal{R}) u_{\tau_k} + E[(\bar{B} + w_k \bar{D})' p_{\tau_k^+} | \mathcal{F}_{k-1}] + \lambda \mathcal{B}' \bar{\mathcal{R}} = 0. \tag{13}$$

Proof: For the convenience of discussion, we define the following notations.

$$\begin{aligned} L_{\tau_l}(u, \lambda) = E \sum_{k=l}^{N} \Big\{ & x'_{\tau_k^-} (\frac{1}{2}\bar{Q} + \lambda \mathcal{C}\mathcal{Q}) x_{\tau_k^-} + \lambda \mathcal{D}'\bar{\mathcal{Q}} x_{\tau_k^-} \\ & + u'_{\tau_k} (\frac{1}{2}\bar{R} + \lambda \mathcal{A}\mathcal{R}) u_{\tau_k} + \lambda \mathcal{B}'\bar{\mathcal{R}} u_{\tau_k} \Big\} \\ & + E\Big\{ \frac{1}{2} \sum_{k=l}^{N} \int_{\tau_k^+}^{\tau_{k+1}^-} [x'_t Q x_t + u'_t R u_t] dt \Big\}, \end{aligned} \tag{14}$$

and

$$\begin{aligned} L_{l}(u, \lambda) = E \sum_{k=l+1}^{N} \Big\{ & x'_{\tau_k^-} (\frac{1}{2}\bar{Q} + \lambda \mathcal{C}\mathcal{Q}) x_{\tau_k^-} + \lambda \mathcal{D}'\bar{\mathcal{Q}} x_{\tau_k^-} \\ & + u'_{\tau_k} (\frac{1}{2}\bar{R} + \lambda \mathcal{A}\mathcal{R}) u_{\tau_k} + \lambda \mathcal{B}'\bar{\mathcal{R}} u_{\tau_k} \Big\} \\ & + E\Big\{ \frac{1}{2} \sum_{k=l}^{N} \int_{\tau_k^+}^{\tau_{k+1}^-} [x'_t Q x_t + u'_t R u_t] dt \Big\}. \end{aligned} \tag{15}$$

By using the Bellman optimality principle, we know that the optimal control u_t^*, $t \in [0, T]$ is still optimal of minimizing $L_{\tau_l}(u, \lambda)$, $L_l(u, \lambda)$ in (14), (15), respectively.

Thus, with $l = N$, it follows from (15) that

$$L_N(u,\lambda) = E\Big\{\tfrac{1}{2}\int_{\tau_N^+}^{T}[x_t'Qx_t + u_t'Ru_t]dt\Big\}. \quad (16)$$

At time interval $[\tau_N^+, T]$, for system (1)-(1), the task is to find optimal control of minimizing $L_N(u,\lambda)$ in (16).

At time interval $[\tau_N^+, T]$, set $u_t^\varepsilon = u_t + \varepsilon\delta u_t$, in which $\varepsilon \in (0,1)$, δu_t denotes the variation of the optimal control. Moreover, we use $L_N^\varepsilon(u,\lambda), L_N(u,\lambda)$ represent the Lagrangian associated with u_t^ε and u_t, respectively.

Through calculation, the variation of $\delta L_N = L_N^\varepsilon(u,\lambda) - L_N(u,\lambda)$ can be calculated as:

$$\delta L_k = E\int_{\tau_k^+}^{\tau_{k+1}^-} E[H_s'|\mathcal{F}_s]\varepsilon\delta u_s ds + o(\varepsilon),$$

where $o(\cdot)$ means the infinitesimal of higher order, and

$$H_s = \Big[\int_s^{\tau_{k+1}^-} x_t'Q\Phi_t dt + x_T'P_T\Phi_T\Big]\Phi_s^{-1}(B - CD) + u_s'R + \eta_s'\Phi_s^{-1}\bar{B},$$

while Φ_t is the unique solution of the following SDE

$$\begin{cases} d\Phi_t = A\Phi_t dt + C\Phi_t dW_t, \\ \Phi_0 = I_n, \end{cases}$$

and $\Phi_t^{-1} = \Psi_t$ exists, satisfying

$$\begin{cases} d\Psi_t = \Psi_t(-A + C^2)dt - \Psi_t\bar{A}dW_t, \\ \Psi_0 = I_n. \end{cases}$$

Apparently from the above derivation, we know that the optimality condition of minimizing $L_N(u,\lambda)$ at time interval $t \in [\tau_N^+, T]$ is:

$$E[H_s'|\mathcal{F}_s] \equiv 0, \quad s \in [\tau_k^+, \tau_{k+1}^-].$$

Next, if p_t, q_t are set to satisfy the backward stochastic differential equation as (10), and via applying Itô's formula to $\langle p_t, \delta x_t\rangle$, then it can be verified that the optimality condition of $E[H_s'|\mathcal{F}_s] \equiv 0$ can be rewritten as equation (12).

Next, by applying Itô's formula to $x_t'p_t$, $t \in [\tau_N^+, T]$, there holds

$$\begin{aligned} d(x_t'p_t) &= (dx_t)'p_t + x_t'dp_t + (dx_t)'dp_t \\ &= [(Ax_t + Bu_t)dt + (Cx_t + Du_t)dW_t]'p_t \\ &\quad + x_t'[-(A'p_t + C'q_t + Qx_t)dt + q_t dW_t] \\ &\quad + (Cx_t + Du_t)'q_t dt \\ &= [u_t'(B'p_t + D'q_t) - x_t'Qx_t]dt + \{\cdots\}dW_t \\ &= -(x_t'Qx_t + u_t'Ru_t)dt + \{\cdots\}dW_t. \end{aligned} \quad (17)$$

Then, by taking expectation of (17), and taking integral from τ_N^+ to T, the optimal $L_N(u,\lambda)$ can be given as:

$$L_N^*(u,\lambda) = \tfrac{1}{2}E(x_{\tau_N^+}'p_{\tau_N^+}) \quad (18)$$

where $p_T = 0$ has been used.

In what follows, there holds from (14) and (18) that

$$\begin{aligned} L_{\tau_N}(u,\lambda) = E\Big\{& x_{\tau_N^-}'(\frac{1}{2}\bar{Q} + \lambda\mathcal{C}\mathcal{Q})x_{\tau_N^-} + \lambda\mathcal{D}'\bar{\mathcal{Q}}x_{\tau_N^-} \\ & + u_{\tau_N}'(\frac{1}{2}\bar{R} + \lambda\mathcal{A}\mathcal{R})u_{\tau_N} + \lambda\mathcal{B}'\bar{\mathcal{R}}u_{\tau_N}\Big\} \\ & + \frac{1}{2}E(x_{\tau_N^+}'p_{\tau_N^+}). \end{aligned} \quad (19)$$

To minimize $L_{\tau_N}(u,\lambda)$, by using the results in [32],

At time τ_N, we set $u_{\tau_N}^\varepsilon = u_{\tau_N} + \varepsilon\delta u_{\tau_N}$, and $\varepsilon \in (0,1)$, δu_{τ_N} denotes the variation of the optimal control. $L_{\tau_N}^\varepsilon(u,\lambda), L_{\tau_N}(u,\lambda)$ are used to represent the Lagrangian associated with $u_{\tau_N}^\varepsilon$ and u_{τ_N}, respectively.

The variation of $\delta L_{\tau_N} = L_{\tau_N}^\varepsilon(u,\lambda) - L_{\tau_N}(u,\lambda)$ can be calculated as:

$$\begin{aligned} L_{\tau_N}(u,\lambda) = E\Big\{& x_{\tau_N^-}'(\frac{1}{2}\bar{Q} + \lambda\mathcal{C}\mathcal{Q})x_{\tau_N^-} \\ & + u_{\tau_N}'(\frac{1}{2}\bar{R} + \lambda\mathcal{A}\mathcal{R})u_{\tau_N} + \lambda\mathcal{B}'\bar{\mathcal{R}}u_{\tau_N} \\ & + \lambda\mathcal{D}'\bar{\mathcal{Q}}x_{\tau_N^-}\frac{1}{2}E(x_{\tau_N^+}'p_{\tau_N^+})\Big\}, \end{aligned}$$

and

$$\begin{aligned} L_{\tau_N}^\varepsilon(u,\lambda) = E\Big\{& x_{\tau_N^-}'(\frac{1}{2}\bar{Q} + \lambda\mathcal{C}\mathcal{Q})x_{\tau_N^-} \\ & + (u_{\tau_N}^\varepsilon)'(\frac{1}{2}\bar{R} + \lambda\mathcal{A}\mathcal{R})u_{\tau_N}^\varepsilon + \lambda\mathcal{B}'\bar{\mathcal{R}}u_{\tau_N}^\varepsilon + \lambda\mathcal{D}'\bar{\mathcal{Q}}x_{\tau_N^-} \\ & + \frac{1}{2}[\bar{A}x_{\tau_N^-} + \bar{B}u_{\tau_N} + w_N(\bar{C}x_{\tau_N^-} + \bar{D}u_{\tau_N})]'p_{\tau_N^+}\Big\}. \end{aligned}$$

In what follows, we have

$$\begin{aligned} \delta L_{\tau_N} = E\Big\{& [u_{\tau_N}'(\bar{R} + 2\lambda\mathcal{A}\mathcal{R}) + \lambda\mathcal{B}'\bar{\mathcal{R}}]\varepsilon\delta u_{\tau_N} \\ & + \frac{1}{2}E([(\bar{B} + w_N\bar{D})\varepsilon u_{\tau_N}]'p_{\tau_N^+})\Big\} + o(\varepsilon), \end{aligned}$$

where $o(\cdot)$ means the infinitesimal of higher order.

Therefore, at time τ_k, the optimality condition of minimizing $L_{\tau_N}(u,\lambda)$ is as equation (13), i.e.,

$$(\bar{R} + 2\lambda\ \mathcal{A}\mathcal{R})u_{\tau_k} + E[(\bar{B} + w_k\bar{D})'p_{\tau_k^+}|\mathcal{F}_{k-1}] + \lambda\mathcal{B}'\bar{\mathcal{R}} = 0.$$

If costate $p_{\tau_k^-}$ is defined to satisfy (11), there holds

$$\begin{aligned} L_{\tau_N}^*(u,\lambda) &= E\Big\{x_{\tau_N^-}'(\frac{1}{2}\bar{Q} + \lambda\mathcal{C}\mathcal{Q})x_{\tau_N^-} + \lambda\mathcal{D}'\bar{\mathcal{Q}}x_{\tau_N^-} \\ &\quad + u_{\tau_N}'(\frac{1}{2}\bar{R} + \lambda\mathcal{A}\mathcal{R})u_{\tau_N} + \lambda\mathcal{B}'\bar{\mathcal{R}}u_{\tau_N}\Big\} \\ &\quad + \frac{1}{2}E\{[(\bar{A} + w_N\bar{C})x_{\tau_N^-} + (\bar{B} + w_N\bar{D})u_{\tau_N}]'p_{\tau_N^+}\} \\ &= \frac{1}{2}E(x_{\tau_N^-}'p_{\tau_N^-}). \end{aligned} \quad (20)$$

Finally, by repeating the above procedures backwardly, the relationships (10)-(13) can be verified to be the necessary conditions of minimizing $L(u,\lambda)$ in (8). This ends the proof. □

Remark 2: For the first time, the maximum principle for the LQ continuous/impulsive control problem with quadratic constraint has been explored in Theorem 1. By decoupling the FBSDE composed by (1)-(1) associated with (10)-(13), the optimal continuous/impulsive control can be derived.

B. OPTIMAL CONTROL

In this section, we will derive the optimal control by decoupling the FBSDE in Lemma 1. We have the following results.

Theorem 2: Under Assumptions 1-3, for the given $\lambda \geq 0$, the optimal control of minimizing $L(u, \lambda)$ in (8) can be given as:

$$\begin{cases} u_t = -\Upsilon_t^{-1}M_t x_t - \Upsilon_t^{-1}B'f_t, t \neq \tau_k, & (21)\\ u_{\tau_k} = -\Upsilon_{\tau_k}^{-1}M_{\tau_k}x_{\tau_k^-} - \Upsilon_{\tau_k}^{-1}(B'f_{\tau_k^+} + \lambda\bar{\mathcal{R}}\mathcal{B}), t = \tau_k, & (22)\end{cases}$$

where $\Upsilon_t, \Upsilon_{\tau_k}, M_t, M_{\tau_k}$ can be presented as:

$$\begin{cases} \Upsilon_t = R + D'P_tD,\ t \neq \tau_k, & (23)\\ \Upsilon_{\tau_k} = \bar{R} + 2\lambda\mathcal{A}\mathcal{R} + \bar{B}'P_{\tau_k^+}\bar{B} + \bar{D}'P_{\tau_k^+}\bar{D}, & (24)\\ M_t = B'P_t + D'P_tC,\ t \neq \tau_k, & (25)\\ M_{\tau_k} = \bar{B}'P_{\tau_k^+}\bar{A} + \bar{D}'P_{\tau_k^+}\bar{C}, & (26)\end{cases}$$

while P_t, f_t satisfy the following differential/difference equations:

$$\begin{cases} -\dot{P}_t = Q + A'P_t + P_tA + C'P_tC - M_t'\Upsilon_t^{-1}M_t, & (27)\\ -\dot{f}_t = A'f_t - M_t'\Upsilon_t^{-1}B'f_t, & (28)\\ P_{\tau_k^-} = \bar{Q} + 2\lambda\mathcal{C}\mathcal{Q} + \bar{A}'P_{\tau_k^+}\bar{A} + \bar{C}'P_{\tau_k^+}\bar{C} - M_{\tau_k}'\Upsilon_{\tau_k}^{-1}M_{\tau_k}, & (29)\\ f_{\tau_k^-} = M_{\tau_k}'\Upsilon_{\tau_k}^{-1}B'f_{\tau_k^+} + \lambda\bar{\mathcal{Q}}\mathcal{D} + \bar{A}'f_{\tau_k^+}, & (30)\end{cases}$$

where $P_{\tau_k^+}, f_{\tau_k^+}$ can be obtained from (21)-(21), and $P_{\tau_k^-}, f_{\tau_k^-}$ serve as the final condition in (21)-(21) at $t \in [\tau_{k-1}^+, \tau_k^-]$, $k = 2, \cdots, N$, which can be induced from (21)-(21). In addition, $P_T = f_T = 0$.

Moreover, the relationship between the costate and the state is presented as:

$$p_t = P_t x_t + f_t, \tag{31}$$

where P_t, f_t can be calculated through (21)-(21) for $t \in [0, T]$. With the optimal control (21)-(21), the optimal $L^*(u, \lambda)$ is given by:

$$\begin{aligned} L^*(u, \lambda) = &\frac{1}{2}E(x_0'P_0x_0 + 2x_0'f_0)\\ &-E\sum_{k=0}^{N}\int_{\tau_k^+}^{\tau_{k+1}^-} f_t'B\Upsilon_t^{-1}B'f_t dt\\ &-\sum_{k=1}^{N}(B'f_{\tau_k^+} + \lambda\bar{\mathcal{R}}\mathcal{B})'\Upsilon_{\tau_k}^{-1}(B'f_{\tau_k^+} + \lambda\bar{\mathcal{R}}\mathcal{B}). \end{aligned} \tag{32}$$

Proof: The backward recursive method is adopted here to derive the main results in the theorem.

Firstly, from Assumption 2 and using Lemma 1, we know that for $t \in [\tau_N^+, T]$, the solution to (10) is assumed to be $p_t = P_t x_t + f_t$, then by using the Itô's formula, there holds:

$$\begin{aligned} dp_t &= [\dot{P}_t + P_t(Ax_t + Bu_t)]dt + P_t(Cx_t + Du_t)dW_t\\ &= -(A'p_t + C'q_t + Qx_t)dt + q_t dW_t. \end{aligned} \tag{33}$$

Combining with (10) yields $q_t = P_t(Cx_t + Du_t)$, and then (12) can be rewritten as:

$$0 = Ru_t + B'(P_t x_t + f_t) + D'P_t(Cx_t + Du_t).$$

Thus, we have

$$u_t = -\Upsilon_t^{-1}(M_t x_t + B'f_t). \tag{34}$$

Combining (33) and (34), for $t \in [\tau_N^+, T]$, the following relationship can be easily derived:

$$\begin{cases} -\dot{P}_t = Q + A'P_t + P_tA + C'P_tC - M_t'\Upsilon_t^{-1}M_t,\\ -\dot{f}_t = A'f_t - M_t'(R + D'P_tD)^{-1}B'f_t, \end{cases}$$

where the final condition is given by $P_T = 0, f_T = 0$, i.e., relationships (21)-(21) have been verified for $t \in [\tau_N^+, T]$.

On the other hand, at time τ_N, it can be induced from above that $p_{\tau_N^+} = P_{\tau_N^+}x_{\tau_N^+} + f_{\tau_N^+}$, then it follows from (13) that

$$\begin{aligned} 0 &= E[(\bar{B} + w_N\bar{D})'p_{\tau_N^+}|\mathcal{F}_{N-1}]\\ &\quad + (\bar{R} + 2\lambda\mathcal{A}\mathcal{R})u_{\tau_N} + \lambda\bar{\mathcal{R}}\mathcal{B}\\ &= [\bar{R} + \lambda 2\mathcal{A}\mathcal{R} + \bar{B}'P_{\tau_N^+}\bar{B} + \bar{D}'P_{\tau_N^+}\bar{D}]u_{\tau_N}\\ &\quad + (\bar{B}'P_{\tau_N^+}\bar{A} + \bar{D}'P_{\tau_N^+}\bar{C})x_{\tau_N^-} + B'f_{\tau_N^+} + \lambda\bar{\mathcal{R}}\mathcal{B}. \end{aligned} \tag{35}$$

Then (21) can be derived for $k = N$.

Next using (11), the relationship (21) for $k = N$ can be obtained.

By repeating the above steps backwardly, then (21)-(21) can be derived for $t \in [0, T]$.

What remains to show is the optimal cost function.

By applying Itô's formula to $x_t'(P_t x_t + 2f_t)$, where P_t, f_t satisfy (21)-(21), thus we have

$$\begin{aligned} d[x_t'(P_tx_t + 2f_t)] &= dx_t'(P_tx_t + 2f_t) + x_t'\dot{P}_tx_t dt\\ &\quad + x_t'P_t dx_t + 2x_t'\dot{f}_t dt + dx_t' d(P_tx_t + 2f_t)\\ &= (Ax_t + Bu_t)'(P_tx_t + 2f_t)dt + (Cx_t + Du_t)'(P_tx_t + 2f_t)dW_t\\ &\quad + x_t'(-Q - A'P_t - P_tA - C'P_tC + M_t'\Upsilon_t^{-1}M_t)x_t dt\\ &\quad + x_t'P_t(Ax_t + Bu_t)dt + x_t'P_t(Cx_t + Du_t)dW_t\\ &\quad + 2x_t'(-A'f_t + M_t'\Upsilon_t^{-1}B'f_t)dt\\ &\quad + (Cx_t + Du_t)'P_t(Cx_t + Du_t)dt\\ &= [2x_t'A'f_t + 2u_t'B'P_tx_t - 2x_t'f_t + 2u_t'Bf_t]dt\\ &\quad + [x_t'(-Q + M_t'\Upsilon_t^{-1}M_t)x_t + 2x_t'M_t\Upsilon_t^{-1}B'f_t]dt\\ &\quad + (2x_t'C'P_tDu_t + u_t'D'P_tDu_t)dt + \{\ldots\}dW_t\\ &= -(x_t'Qx_t + u_t'Ru_t)dt + (u_t + \Upsilon_t^{-1}M_tx_t + \Upsilon_t^{-1}B'f_t)'\Upsilon_t\\ &\quad \times(u_t + \Upsilon_t^{-1}M_tx_t + \Upsilon_t^{-1}B'f_t)dt + \{\ldots\}dW_t\\ &\quad - f_t'B\Upsilon_t^{-1}B'f_t dt, \end{aligned} \tag{36}$$

where the equations (21)-(21) are inserted in the above.

Actually, for $t \in [\tau_N^+, T]$, by taking expectation, then integrating from τ_N^+ to T, we have

$$\begin{aligned} &\int_{\tau_N^+}^{T} E(x_t'Qx_t + u_t'Ru_t)dt + E[x_T'(P_Tx_T + 2f_T)]\\ &= E[x_{\tau_N^+}'(P_{\tau_N^+}x_{\tau_N^+} + 2f_{\tau_N^+})] - \int_{\tau_N^+}^{T} f_t'B\Upsilon_t^{-1}B'f_t dt\\ &\quad + E\int_{\tau_N^+}^{T}(u_t + \Upsilon_t^{-1}M_tx_t + \Upsilon_t^{-1}B'f_t) \end{aligned}$$

$$\times \Upsilon_t(u_t+\Upsilon_t^{-1}M_tx_t+\Upsilon_t^{-1}B'f_t)dt$$
$$= E[x'_{\tau_N^+}P_{\tau_N^+}x_{\tau_N^+}+2x'_{\tau_N^+}f_{\tau_N^+}]-\int_{\tau_N^+}^T f_t'B\Upsilon_t^{-1}B'f_tdt, \quad (37)$$

where the optimal control (21) has been used in the last equality.

Next, by using u_{τ_N} in (21), it can be obtained

$$\begin{aligned}
&E\{x'_{\tau_N^-}(\frac{1}{2}\bar{Q}+\lambda\mathcal{C}\mathcal{Q})x_{\tau_N^-}+\lambda\mathcal{D}'\bar{\mathcal{Q}}x_{\tau_N^-}\\
&\quad+u'_{\tau_N}(\frac{1}{2}\bar{R}+\lambda\mathcal{A}\mathcal{R})u_{\tau_N}+\lambda\mathcal{B}'\bar{\mathcal{R}}u_{\tau_N}\\
&\quad+\frac{1}{2}\int_{\tau_N^+}^T E(x_t'Qx_t+u_t'Ru_t)dt+E[x_T'(P_Tx_T+2f_T)]\\
&=E\{x'_{\tau_N^-}(\frac{1}{2}\bar{Q}+\lambda\mathcal{C}\mathcal{Q})x_{\tau_N^-}+\lambda\mathcal{D}'\bar{\mathcal{Q}}x_{\tau_N^-}\\
&\quad+u'_{\tau_N}(\frac{1}{2}\bar{R}+\lambda\mathcal{A}\mathcal{R})u_{\tau_N}+\lambda\mathcal{B}'\bar{\mathcal{R}}u_{\tau_N}\\
&\quad+\frac{1}{2}E[x'_{\tau_N^+}P_{\tau_N^+}x_{\tau_N^+}+2x'_{\tau_N^+}f_{\tau_N^+}]-\frac{1}{2}\int_{\tau_N^+}^T f_t'B\Upsilon_t^{-1}B'f_tdt\\
&=\frac{1}{2}E(x'_{\tau_N^-}P_{\tau_N^-}x_{\tau_N^-}+2x'_{\tau_N^-}f_{\tau_N^-})-\frac{1}{2}\int_{\tau_N^+}^T f_t'B\Upsilon_t^{-1}B'f_tdt\\
&\quad-\frac{1}{2}(B'f_{\tau_k^+}+\lambda\bar{\mathcal{R}}\mathcal{B})'\Upsilon_{\tau_k}^{-1}(B'f_{\tau_k^+}+\lambda\bar{\mathcal{R}}\mathcal{B}) \qquad (38)
\end{aligned}$$

where $P_{\tau_N^-}, f_{\tau_N^-}$ satisfy (21)-(21).

Finally, by repeating the above steps, the optimal $L(u,\lambda)$ can be calculated as below:

$$\begin{aligned}
L^*(u,\lambda)&=E\sum_{k=1}^N\Big\{x'_{\tau_k^-}(\frac{1}{2}\bar{Q}+\lambda\mathcal{C}\mathcal{Q})x_{\tau_k^-}+\lambda\mathcal{D}'\bar{\mathcal{Q}}x_{\tau_k^-}\\
&\quad+u'_{\tau_k}(\frac{1}{2}\bar{R}+\lambda\mathcal{A}\mathcal{R})u_{\tau_k}+\lambda\mathcal{B}'\bar{\mathcal{R}}u_{\tau_k}\Big\}\\
&\quad+E\Big\{\frac{1}{2}\sum_{k=1}^N\int_{\tau_k^+}^{\tau_{k+1}^-}[x_t'Qx_t+u_t'Ru_t]dt\Big\}\\
&=\frac{1}{2}E(x_0'P_0x_0+2x_0'f_0)-\sum_{k=0}^N\int_{\tau_k^+}^{\tau_{k+1}^-}f_t'B\Upsilon_t^{-1}B'f_tdt\\
&\quad-\sum_{k=1}^N(B'f_{\tau_k^+}+\lambda\bar{\mathcal{R}}\mathcal{B})'\Upsilon_{\tau_k}^{-1}(B'f_{\tau_k^+}+\lambda\bar{\mathcal{R}}\mathcal{B}) \qquad (39)
\end{aligned}$$

where P_0, f_0 is calculated from (21)-(21).

Thus (32) has been verified, and this ends the proof. □

Induced from Theorem 2, the following results can be immediately obtained.

Corollary 1: For the given $\lambda \geq 0$, from (5), we use $u_t(\lambda)$ to denote the optimal continuous/impulsive control, which make the parameter λ explicit. The optimal control of minimizing $L(u,\lambda)-\lambda K$ can be presented as:

$$\begin{cases}
u_t(\lambda)=-\Upsilon_t^{-1}(\lambda)M_t(\lambda)x_t\\
\qquad -\Upsilon_t^{-1}(\lambda)B'f_t(\lambda),\ t\neq\tau_k, & (40)\\
u_{\tau_k}(\lambda)=-\Upsilon_{\tau_k}^{-1}(\lambda)M_{\tau_k}(\lambda)x_{\tau_k^-}\\
\qquad -\Upsilon_{\tau_k}^{-1}(\lambda)(B'f_{\tau_k^+}(\lambda)+\lambda\bar{\mathcal{R}}\mathcal{B}),\ t=\tau_k, & (41)
\end{cases}$$

where $\Upsilon_t(\lambda)$, $\Upsilon_{\tau_k}(\lambda)$, $M_t(\lambda)$, $M_{\tau_k}(\lambda)$ satisfy:

$$\begin{cases}
\Upsilon_t(\lambda)=R+D'P_t(\lambda)D,\ t\neq\tau_k, & (42)\\
\Upsilon_{\tau_k}(\lambda)=\bar{R}+2\lambda\mathcal{A}\mathcal{R}+\bar{B}'P_{\tau_k^+}(\lambda)\bar{B}+\bar{D}'P_{\tau_k^+}(\lambda)\bar{D}, & (43)\\
M_t(\lambda)=B'P_t(\lambda)+D'P_t(\lambda)C,\ t\neq\tau_k, & (44)\\
M_{\tau_k}(\lambda)=\bar{B}'P_{\tau_k^+}(\lambda)\bar{A}+\bar{D}'P_{\tau_k^+}(\lambda)\bar{C}, & (45)
\end{cases}$$

and $P_t(\lambda), f_t(\lambda)$ are given by:

$$\begin{cases}
-\dot{P}_t(\lambda)=Q+A'P_t(\lambda)+P_t(\lambda)A+C'P_t(\lambda)C\\
\qquad -M_t'(\lambda)\Upsilon_t^{-1}(\lambda)M_t(\lambda), & (46)\\
-\dot{f}_t(\lambda)=A'f_t(\lambda)-M_t'(\lambda)\Upsilon_t^{-1}(\lambda)B'f_t(\lambda), & (47)\\
P_{\tau_k^-}(\lambda)=\bar{Q}+2\lambda\mathcal{C}\mathcal{Q}+\bar{A}'P_{\tau_k^+}(\lambda)\bar{A}+\bar{C}'P_{\tau_k^+}(\lambda)\bar{C}\\
\qquad -M'_{\tau_k}(\lambda)\Upsilon_{\tau_k}^{-1}(\lambda)M_{\tau_k}(\lambda), & (48)\\
f_{\tau_k^-}(\lambda)=M'_{\tau_k}(\lambda)\Upsilon_{\tau_k}^{-1}(\lambda)B'f_{\tau_k^+}(\lambda)+\lambda\bar{\mathcal{Q}}\mathcal{D}\\
\qquad +\bar{A}'f_{\tau_k^+}(\lambda), & (49)
\end{cases}$$

with final condition $P_T(\lambda)=f_T(\lambda)=0$.

Remark 3: It is noted that $P_t(\lambda), f_t(\lambda), M_t(\lambda), \Upsilon_t(\lambda)$ in (40)-(40) are dependent on the given parameter λ. In what follows, the optimal parameter λ^* will be explored through the parameter selection methods.

Before adopting the Lagrangian duality theorem, we introduce the following **slater condition**:

Assumption 4: For every $\lambda \geq 0$, there exists an admissible control $u \in \mathcal{U}_{ad}$ such that:

$$\lambda(\bar{J}_N-K)<0. \qquad (50)$$

Remark 4: From the slater condition in Assumption 4, we know if there exists $u \in \mathcal{U}_{ad}$ such that $\bar{J}_N < K$, then Assumption 4 holds.

C. OPTIMAL PARAMETER SELECTION

By using the Lagrange Duality theorem, the following result can be obtained.

Theorem 3: Under Assumptions 3 and 4, there exists λ^* which is optimal for Problem 1, which can also be stated as:

$$\sup_{\lambda\geq 0}\{L^*(u,\lambda)-\lambda K\}=\sup_{\lambda\geq 0}E\Big\{x_0'P_0(\lambda)x_0+2x_0'f_0(\lambda)-\lambda K\Big\},$$

Subject to:

$$\begin{cases}
-\dot{P}_t(\lambda)=Q+A'P_t(\lambda)+P_t(\lambda)A+C'P_t(\lambda)C\\
\qquad -M_t'(\lambda)\Upsilon_t^{-1}(\lambda)M_t(\lambda),\\
-\dot{f}_t(\lambda)=A'f_t(\lambda)-M_t'(\lambda)\Upsilon_t^{-1}(\lambda)B'f_t(\lambda),\\
P_{\tau_k^-}(\lambda)=\bar{Q}+2\lambda\mathcal{C}\mathcal{Q}+\bar{A}'P_{\tau_k^+}(\lambda)\bar{A}+\bar{C}'P_{\tau_k^+}(\lambda)\bar{C}\\
\qquad -M'_{\tau_k}(\lambda)\Upsilon_{\tau_k}^{-1}(\lambda)M_{\tau_k}(\lambda),\\
f_{\tau_k^-}(\lambda)=M'_{\tau_k}(\lambda)\Upsilon_{\tau_k}^{-1}(\lambda)B'f_{\tau_k^+}(\lambda)+\lambda\bar{\mathcal{Q}}\mathcal{D}+\bar{A}'f_{\tau_k^+}(\lambda),\\
\lambda\geq 0, P_T(\lambda)=f_T(\lambda)=0.
\end{cases}$$

In this case, the optimal continuous/impulsive control can be calculated as below:

$$\begin{cases} u_t(\lambda^*) = -\Upsilon_t^{-1}(\lambda^*)M_t(\lambda^*)x_t \\ \qquad\qquad - \Upsilon_t^{-1}(\lambda^*)B'f_t(\lambda^*), t \neq \tau_k, & (51) \\ u_{\tau_k}(\lambda^*) = -\Upsilon_{\tau_k}^{-1}(\lambda^*)M_{\tau_k}(\lambda^*)x_{\tau_k^-} \\ \qquad\qquad - \Upsilon_{\tau_k}^{-1}(\lambda^*)(B'f_{\tau_k^+}(\lambda^*)+\lambda\bar{\mathcal{R}}\mathcal{B}), t = \tau_k, & (52) \end{cases}$$

where $\Upsilon_t(\lambda^*)$, $\Upsilon_{\tau_k}(\lambda^*)$, $M_t(\lambda^*)$, $M_{\tau_k}(\lambda^*)$ *satisfy*:

$$\begin{cases} \Upsilon_t(\lambda^*) = R + D'P_t(\lambda^*)D, \ t \neq \tau_k, & (53) \\ \Upsilon_{\tau_k}(\lambda^*) = \bar{R} + 2\lambda^*\mathcal{A}\mathcal{R} + \bar{B}'P_{\tau_k^+}(\lambda^*)\bar{B} \\ \qquad\qquad + \bar{D}'P_{\tau_k^+}(\lambda^*)\bar{D}, & (54) \\ M_t(\lambda^*) = B'P_t(\lambda^*) + D'P_t(\lambda^*)C, \ t \neq \tau_k, & (55) \\ M_{\tau_k}(\lambda^*) = \bar{B}'P_{\tau_k^+}(\lambda^*)\bar{A} + \bar{D}'P_{\tau_k^+}(\lambda^*)\bar{C}. & (56) \end{cases}$$

Proof: The above theorem can be obtained by using Lagrangian duality theorem, the detailed proof can be found in [29] . □

Remark 5: The optimal control for the considered Problem 1 is proposed in Theorem 2 and Corollary 1 for the first time. The main approach is the solution to the FBSDE from maximum principle (Lemma 1). In what follows, we will explore the calculation method of parameter λ *in Theorem 2.*

The optimal λ^* can be calculated via the gradient-type optimization algorithms, and the gradient is given as the following theorem.

Lemma 2: Suppose Assumptions 3 and 4 hold. Let $\lambda \geq 0$ *be given, then*

$$\frac{d(L(u,\lambda) - \lambda K)}{d\lambda} = E\Big\{\sum_{i=1}^N x'_{\tau_i^-}G_{\tau_k}x'_{\tau_i^-} + \sum_{i=1}^N x'_{\tau_i^-}H_{\tau_k} + \sum_{i=1}^N I_{\tau_k}\Big\} - K, \tag{57}$$

where $\Lambda_k = E(x_{\tau_k^-}x'_{\tau_k^-})$ *satisfies the following relationship*:

$$\begin{cases} G_{\tau_k} = \mathcal{A}M'_{\tau_k}(\lambda)\Upsilon_{\tau_k}^{-1}(\lambda)\mathcal{R}\Upsilon_{\tau_k}^{-1}(\lambda)M_{\tau_k}(\lambda) + \mathcal{C}\mathcal{Q}, & (58) \\ H_{\tau_k} = 2\mathcal{A}M'_{\tau_k}(\lambda)\Upsilon_{\tau_k}^{-1}(\lambda)\mathcal{R}\Upsilon_{\tau_k}^{-1}(\lambda)(B'f_{\tau_k^+}(\lambda)+\lambda\bar{\mathcal{R}}\mathcal{B}) \\ \qquad + \bar{\mathcal{Q}}\mathcal{D} - M'_{\tau_k}(\lambda)\Upsilon_{\tau_k}^{-1}(\lambda)\bar{\mathcal{R}}\mathcal{B}, & (59) \\ I_{\tau_k} = (B'f_{\tau_k^+}(\lambda) + \lambda\bar{\mathcal{R}}\mathcal{B})'\Upsilon_{\tau_k}^{-1}(\lambda)\mathcal{R}\Upsilon_{\tau_k}^{-1}(\lambda) \\ \qquad \times (B'f_{\tau_k^+}(\lambda) + \lambda\bar{\mathcal{R}}\mathcal{B}) \\ \qquad - (B'f_{\tau_k^+}(\lambda) + \lambda\bar{\mathcal{R}}\mathcal{B})'\Upsilon_{\tau_k}^{-1}(\lambda)\bar{\mathcal{R}}\mathcal{B}, & (60) \end{cases}$$

where $\Upsilon_{\tau_k}(\lambda)$, $M'_{\tau_k}(\lambda)$ *satisfy* (40), (40), *respectively.*

Proof: Obviously, through simple calculation, we have

$$\frac{d(L(u,\lambda) - \lambda K)}{d\lambda} = \bar{J}_N - K. \tag{61}$$

Following from Theorem 2 and Corollary 1, the optimal control at $t = \tau_k$ for the given $\lambda \geq 0$ can be given by:

$$\begin{cases} u_t(\lambda) = -\Upsilon_t^{-1}(\lambda)M_t(\lambda)x_t \\ \qquad\qquad - \Upsilon_t^{-1}(\lambda)B'f_t(\lambda), t \neq \tau_k, \\ u_{\tau_k}(\lambda) = -\Upsilon_{\tau_k}^{-1}(\lambda)M_{\tau_k}(\lambda)x_{\tau_k^-} \\ \qquad\qquad - \Upsilon_{\tau_k}^{-1}(\lambda)(B'f_{\tau_k^+}(\lambda) + \lambda\bar{\mathcal{R}}\mathcal{B}), t = \tau_k, \end{cases}$$

In this case, (1) can be rewritten as:

$$\begin{aligned} x_{\tau_k^+} &= [\bar{A} - \bar{B}\Upsilon_{\tau_k}^{-1}(\lambda)M_{\tau_k}(\lambda)]x_{\tau_k^-} \\ &\quad + w_k[\bar{C} - \bar{D}\Upsilon_{\tau_k}^{-1}(\lambda)M_{\tau_k}(\lambda)]x_{\tau_k^-} \\ &\quad + (\bar{B} + w_k\bar{D})\Upsilon_{\tau_k}^{-1}(\lambda)(B'f_{\tau_k^+}(\lambda) + \lambda\bar{\mathcal{R}}\mathcal{B}). \end{aligned} \tag{62}$$

Thus, we have

$$\begin{aligned} &\frac{d(L(u,\lambda) - \lambda K)}{d\lambda} = \bar{J}_N - K \\ &= E\Big\{\mathcal{A}\sum_{i=1}^N u'_{\tau_i}\mathcal{R}u_{\tau_i} + \mathcal{B}'\sum_{i=1}^N \bar{\mathcal{R}}u_{\tau_i} + \mathcal{C}\sum_{i=1}^N x'_{\tau_i^-}\mathcal{Q}x_{\tau_i^-} \\ &\quad + \mathcal{D}'\sum_{i=1}^N \bar{\mathcal{Q}}x_{\tau_i^-}\Big\} - K \\ &= E\Big\{\sum_{i=1}^N x'_{\tau_i^-}G_{\tau_k}x'_{\tau_i^-} + \sum_{i=1}^N x'_{\tau_i^-}H_{\tau_k} + \sum_{i=1}^N I_{\tau_k}\Big\} - K, \end{aligned}$$

where $G_{\tau_k}, H_{\tau_k}, I_{\tau_k}$ are as (58)-(58).

Thus relationship (57) has been verified, and the proof is complete. □

Remark 6: The optimal parameter λ^* *can be calculated from the algorithm in Lemma 2. It is noted that the calculation of* λ^* *can be viewed as an optimal parameter selection problem, which is actually a finite dimensional optimization problem.*

IV. NUMERICAL EXAMPLE

In this section, for $n = 2$, we will investigate the 2-dimensional case to illustrate the main results in this paper.

For (1)-(4), we choose the following coefficients:

$$A = \begin{pmatrix} 0.5895 & 0 \\ 0 & 0.2262 \end{pmatrix}, \bar{A} = \begin{pmatrix} 0.3846 & 0 \\ 0 & 0.5830 \end{pmatrix}, B = \begin{pmatrix} 0.2518 & 0 \\ 0 & 0.2904 \end{pmatrix},$$
$$\bar{B} = \begin{pmatrix} 0.6171 & 0 \\ 0 & 0.2653 \end{pmatrix}, C = \begin{pmatrix} 0.8244 & 0 \\ 0 & 0.9827 \end{pmatrix}, \bar{C} = \begin{pmatrix} 0.7302 & 0 \\ 0 & 0.3439 \end{pmatrix},$$
$$D = \begin{pmatrix} 0.5841 & 0 \\ 0 & 0.1078 \end{pmatrix}, \bar{D} = \begin{pmatrix} 0.9063 & 0 \\ 0 & 0.8787 \end{pmatrix}, Q = \begin{pmatrix} 0.8178 & 0 \\ 0 & 0.2607 \end{pmatrix},$$
$$\bar{Q} = \begin{pmatrix} 0.5944 & 0 \\ 0 & 0.0225 \end{pmatrix}, R = \begin{pmatrix} 0.4253 & 0 \\ 0 & 0.3127 \end{pmatrix}, \bar{R} = \begin{pmatrix} 0.1615 & 0 \\ 0 & 0.1788 \end{pmatrix},$$
$$\mathcal{A} = 0.4229, \ \mathcal{B} = \begin{pmatrix} 0.0942 \\ 0.5985 \end{pmatrix}, \ \mathcal{C} = 0.4709, \ \mathcal{D} = \begin{pmatrix} 0.6959 \\ 0.6999 \end{pmatrix},$$
$$\mathcal{Q} = \begin{pmatrix} 0.6385 & 0 \\ 0 & 0.0336 \end{pmatrix}, \mathcal{R} = \begin{pmatrix} 0.0688 & 0 \\ 0 & 0.3196 \end{pmatrix}, \bar{\mathcal{Q}} = \begin{pmatrix} 0.5309 & 0 \\ 0 & 0.6544 \end{pmatrix},$$
$$\bar{\mathcal{R}} = \begin{pmatrix} 0.4076 & 0 \\ 0 & 0.8200 \end{pmatrix}, K = 0.8060, x_0 = \begin{pmatrix} 0.7184 \\ 0.9686 \end{pmatrix}.$$

The time horizon is $T = 1$ and the impulse time is $\tau_1 = 0.3, \tau_2 = 0.6$. We can calculate the optimal λ, which is approximately equal to 0.2205, via the gradient method. Viewing **FIGURE 1**, We can see that the optimal λ^* to maximize $L(u,\lambda) - \lambda K$ is $\lambda^* \approx 0.2205$. The maximum value of $L(u,\lambda) - \lambda K$ is approximately equal to 0.5090. Moreover, the minimum cost is $J_T^* \approx 0.4822$ and the constraint

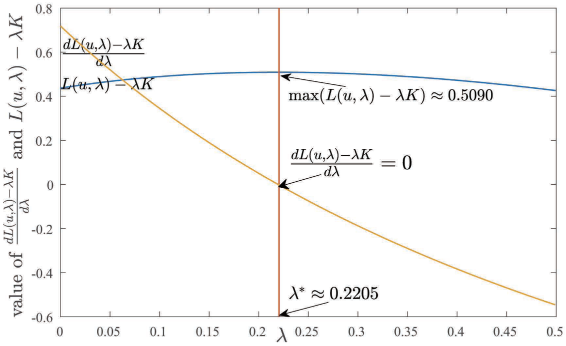


FIGURE 1. The relation among λ, $\frac{dL(u,\lambda)-\lambda K}{d\lambda}$, $L(u,\lambda)-\lambda K$.

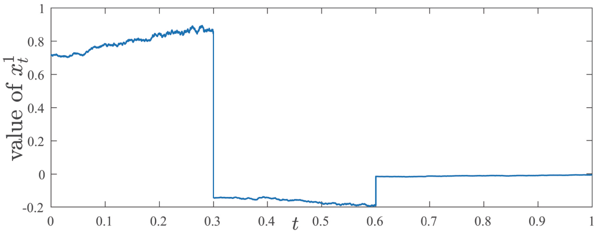


FIGURE 2. x_t^1: the first coordinate of x_t^*.

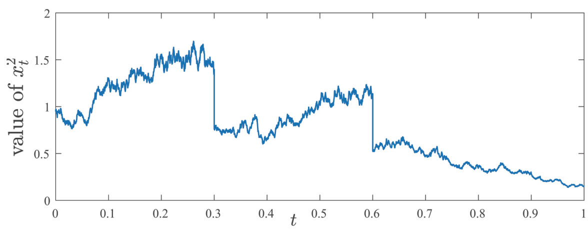


FIGURE 3. x_t^2: the second coordinate of x_t^*.

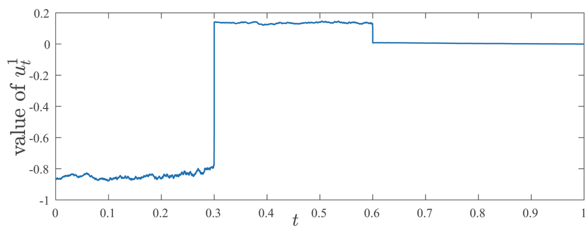


FIGURE 4. u_t^1: the first coordinate of u_t^*.

$\bar{J}_N \approx 0.7549 < K = 0.8060$. Correspondingly, the optimal control u_t^* and the corresponding optimal trajectory x_t^* are as **FIGURE 2 - FIGURE 5**. The impulse control at time τ_1 and τ_2 can be represented as the following closed-loop form

$$\begin{cases} u_{\tau_1} \approx \begin{pmatrix} -0.6785 & 0 \\ 0 & -0.2364 \end{pmatrix} x_{\tau_1} - \begin{pmatrix} 0.0166 \\ 0.3283 \end{pmatrix}, \\ u_{\tau_2} \approx \begin{pmatrix} -0.5676 & 0 \\ 0 & -0.1787 \end{pmatrix} x_{\tau_1} - \begin{pmatrix} 0.0117 \\ 0.3016 \end{pmatrix}, \end{cases}$$

respectively. Moreover, at $t \neq \tau_1, \tau_2$, the optimal continuous control is given as

$$u_t = -\Upsilon_t^{-1} M_t x_t - \Upsilon_t^{-1} B' f_t,$$

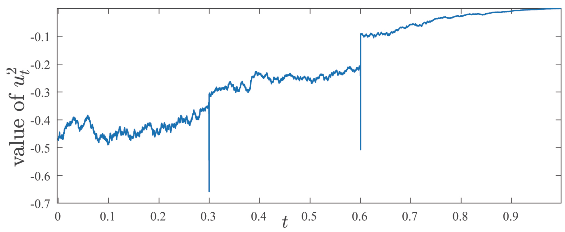


FIGURE 5. u_t^2: the second coordinate of u_t^*.

where

$$\begin{cases} \Upsilon_t = \begin{pmatrix} 0.4253 & 0 \\ 0 & 0.3127 \end{pmatrix} + \begin{pmatrix} 0.5841 & 0 \\ 0 & 0.1078 \end{pmatrix} P_t \begin{pmatrix} 0.5841 & 0 \\ 0 & 0.1078 \end{pmatrix} \\ M_t = \begin{pmatrix} 0.2518 & 0 \\ 0 & 0.2904 \end{pmatrix} P_t + \begin{pmatrix} 0.5841 & 0 \\ 0 & 0.1078 \end{pmatrix} P_t \begin{pmatrix} 0.8244 & 0 \\ 0 & 0.9827 \end{pmatrix}, \end{cases}$$

and P_t, f_t can be obtained by (40)-(40) via Runge-Kutta numerical methods.

V. CONCLUSION

In this paper, the stochastic LQ continuous/impulsive control problem with quadratic constraint has been investigated. The necessary conditions for the optimization has been developed for the first time; Under standard assumptions, the optimal continuous/impulsive control have been derived through decoupling the FBSDE; Finally, by using the Lagrangian duality theorem, the stochastic continuous/impulsive control problem has been solved, and the optimal parameter can be calculated by the gradient type algorithm. For future research, we will extend the obtained results to the random impulsive time case.

REFERENCES

[1] A. Bensoussan and J. L. Lions, *Impulse Control and Quasi-Variational Inequalities*. Paris, France: Gauthier-Villars, 1984.

[2] M. Jeanblanc-Piqué, "Impulse control method and exchange rate," *Math. Finance*, vol. 3, no. 2, pp. 161–177, 1993.

[3] L. H. R. Alvarez, "A class of solvable impulse control problems," *Appl. Math. Optim.*, vol. 49, no. 3, pp. 265–295, 2004.

[4] R. Korn, "Some applications of impulse control in mathematical finance," *Math. Methods Oper. Res.*, vol. 50, no. 3, pp. 493–518, 1999.

[5] Z. Yu, "The stochastic maximum principle for optimal control problems of delay systems involving continuous and impulse controls," *Automatica*, vol. 48, no. 10, pp. 2420–2432, 2012.

[6] M. H. A. Davis and A. R. Norman, "Portfolio selection with transaction costs," *Math. Oper. Res.*, vol. 15, no. 4, pp. 676–713, 1990.

[7] L. H. R. Alvarez, "Singular stochastic control, linear diffusions, and optimal stopping: A class of solvable problems," *SIAM J. Control Optim.*, vol. 39, no. 6, pp. 1697–1710, 2001.

[8] L. A. Sobiesiak and C. J. Damaren, "Linear quadratic optimal control for systems with continuous and impulsive inputs," in *Proc. IEEE 53rd Conf. Decision Control*, Los Angeles, CA, USA, Dec. 2014, pp. 5071–5076.

[9] S. M. McClure, D. I. Laibson, G. Loewenstein, and J. D. Cohen, "Separate neural systems value immediate and delayed monetary rewards," *Science*, vol. 306, no. 5695, pp. 503–507, 2004.

[10] S. T. Zavalishchin and A. N. Sesekin, *Dynamic Impulse Systems: Theory and Applications*. Dordrecht, The Netherlands: Kluwer, 1997.

[11] G. Mundaca and B. Øksendal, "Optimal stochastic intervention control with application to the exchange rate," *J. Math. Econ.*, vol. 29, no. 2, pp. 225–243, 1998.

[12] K. A. Brekke and B. Øksendal, "A verificiation theorem for combined stochastic control and impulse control," in *Stochastic Analysis and Related Topics VI*. Boston, MA, USA: Birkhäuser, 1996, pp. 211–220.

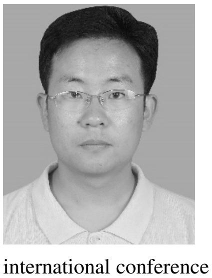
[13] A. Cadenillas and U. G. Haussmann, "The stochastic maximum principle for a singular control problem," *Stochastics Stochastic Rep.*, vol. 49, nos. 3–4, pp. 211–237, 1994.
[14] F. Dufour and B. Miller, "Maximum principle for singular stochastic control problems," *SIAM J. Control Optim.*, vol. 45, no. 2, pp. 668–698, 2006.
[15] J. Huang and D. Zhang, "The near-optimal maximum principle of impulse control for stochastic recursive system," *Sci. China Inf. Sci.*, vol. 59, no. 11, pp. 112206:1–112206:13, 2016.
[16] S. Baccarin, "Optimal impulse control for a multidimensional cash management system with generalized cost functions," *Eur. J. Oper. Res.*, vol. 196, no. 1, pp. 198–206, 2009.
[17] L. A. Sobiesiak and J. D. Christopher, "Lorentz-augmented spacecraft formation reconfiguration," *IEEE Trans. Control Syst. Technol.*, vol. 24, no. 2, pp. 514–524, Mar. 2016.
[18] B. Vatankhahghadim and C. J. Damaren, "Optimal hybrid magnetic attitude control: Disturbance accommodation and impulse timing," *IEEE Trans. Control Syst. Technol.*, vol. 25, no. 4, pp. 1512–1520, Jul. 2017.
[19] A. E. B. Lim and J. B. Moore, "A quasi-separation theorem for LQG optimal control with IQ constraints," *Syst. Control Lett.*, vol. 32, no. 1, pp. 21–33, 1997.
[20] A. E. B. Lim and X. Y. Zhou, "Stochastic optimal LQR control with integral quadratic constraints and indefinite control weights," *IEEE Trans. Autom. Control*, vol. 44, no. 7, pp. 1359–1369, Jul. 1999.
[21] X. Li, X. Y. Zhou, and A. E. B. Lim, "Dynamic mean-variance portfolio selection with no-shorting constraints," *SIAM J. Control Optim.*, vol. 40, no. 5, pp. 1540–1555, 2002.
[22] J. Qu, Z. Ji, C. Lin, and H. Yu, "Fast consensus seeking on networks with antagonistic interactions," *Complexity*, vol. 2018, Dec. 2018, Art. no. 7831317.
[23] J. Huang and Z. Yu, "Solvability of indefinite stochastic Riccati equations and linear quadratic optimal control problems," *Syst. Control Lett.*, vol. 68, pp. 68–75, Jun. 2014.
[24] D. Fudenberg and D. K. Levine, "A dual-self model of impulse control," *Amer. Econ. Rev.*, vol. 96, no. 5, pp. 1449–1476, 2006.
[25] J. Wu, "Optimal exchange rates management using stochastic impulse control for geometric Lévy processes," *J. Math. Methods Oper. Res.*, to be published. doi: 10.1007/s00186-018-0648-y.
[26] Y. Hu, Z. Liu, and J. Wu, "Optimal impulse control of a mean-reverting inventory with quadratic costs," *J. Ind. Manage. Optim.*, vol. 14, no. 4, pp. 1685–1700, 2018.
[27] Y. Guo and J. Cao, "Stability and stabilization for stochastic Cohen-Grossberg neural networks with impulse control and noise-induced control," *Int. J. Robust Nonlinear Control*, vol. 29, no. 1, pp. 153–165, 2019.
[28] H. T. Toivonen and P. M. Mäkilä, "Computer-aided design procedure for multiobjective LQG control problems," *Int. J. Control*, vol. 49, no. 2, pp. 655–666, 1989.
[29] D. G. Luenberger, *Optimization by Vector Space Methods*. Hoboken, NJ, USA: Wiley, 1997.
[30] A. E. B. Lim, X. Y. Zhou, and J. B. Moore, "Multiple-objective risk-sensitive control and its small noise limit," *Automatica*, vol. 39, no. 3, pp. 533–541, 2003.
[31] J. Yong and X. Y. Zhou, *Stochastic Controls: Hamiltonian Systems and HJB Equations*. New York, NY, USA: Springer, 1999.
[32] H. Zhang, L. Li, J. Xu, and M. Fu, "Linear quadratic regulation and stabilization of discrete-time systems with delay and multiplicative noise," *IEEE Trans. Autom. Control*, vol. 60, no. 10, pp. 2599–2613, Oct. 2015.

QINGYUAN QI received the B.S. degree in mathematics and the Ph.D. degree in control theory and engineering from Shandong University, Jinan, Shandong, China, in 2012 and 2018, respectively.

He is currently a Lecturer with the Institute of Complexity Science, School of Automation, Qingdao University. His research interests include optimal control, optimal estimation, stabilization, and stochastic systems.

ZHENGHONG QIU received the B.S. degree in applied mathematics from Wuhan University, Wuhan, China, in 2014. He is currently pursuing the Ph.D. degree with the Department of Applied Mathematics, The Hong Kong Polytechnic University.

His current research interests include optimal control, mean-field games, and stochastic systems.



ZHIJIAN JI was born in Qingdao, China, in 1973. He received the M.S. degree in applied mathematics from the Ocean University of China, Qingdao, in 1998, and the Ph.D. degree in control theory and control engineering from Peking University, Beijing, China, in 2015.

He is currently a Full-Time Professor with the Institute of Complexity Science, School of Automation, Qingdao University. He has authored or coauthored 81 refereed papers in journals and international conference proceedings. His current research interests include multiagent systems, switched control systems, networked control systems, intelligent control, and robot systems.