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Adaptive distributed observer design for containment control of heterogeneous discrete-time swarm systems

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Abstract This paper develops both adaptive distributed dynamic state feedback control law and adaptive distributed measurement output feedback control law for heterogeneous discrete-time swarm systems with multiple leaders. The convex hull formed by the leaders and the system matrix of leaders is estimated via an adaptive distributed containment observer. Such estimations will feed the followers so that every follower can update the system matrix of the corresponding adaptive distributed containment observer and the system state of their neighbors. The followers cooperate with each other to achieve leader–follower consensus and thus solve the containment control problem over the network. Numerical results demonstrate the effectiveness and computational feasibility of the proposed control laws.

1. Introduction

Cooperative control of swarm systems (also known as multi-agent systems) has been a recent research hotspot for its extensive applications in engineering.[1] Generally speaking, cooperative control problems of multi-agent systems can be mainly classified into the consensus problems[2–5] and the containment control problems.[6–8] The consensus problem is to design the adaptive distributed control strategy regarding local information to fulfill the condition that the state errors between any two agents in the network tend to zero. On the other hand,

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the containment control problem is to design control laws so that all the followers converge to the convex hull formed by the leaders. In this paper, we investigate the containment control problem for heterogeneous discrete-time swarm systems with multiple leaders.

Extensive research effort has been dedicated to containment control problems for continuous-time and discrete-time swarm systems of first-order integrators and second-order integrators with stationary or dynamic leaders. Ref.[9] studied the distributed containment control of a group of mobile autonomous agents with multiple stationary or dynamic leaders under both fixed and switching network topologies using continuous-time formulation. Ref.[10] studied the containment control problem for multiple continuous-time and discrete-time single integrator systems. Control protocols were devised by exploiting the control input information of neighbors so that the leaders will converge to the desired convex formation while the followers converge to the convex hull of the leaders. Ref.[11] considered the containment control problems for both continuous-time and discrete-time multi-agent systems with general linear dynamics under general directed communication topologies. Distributed dynamic containment controllers based on the relative outputs of neighboring agents were constructed. Ref.[12] solved containment control problems for networked single integrator systems with multiple stationary or dynamic leaders over directed graphs. Ref.[13] proposed the containment control problem for a group of agents with heterogeneous dynamics modeled by both first-order integrators and second-order integrators. Due to the packet drop and node failure phenomena during the information transmission, the containment control of multi-agent systems over switching communication topologies is of importance. Ref.[14] considered the distributed containment control for second-order multi-agent systems guided by multiple leaders with random switching topologies. Ref.[15] investigated the containment control of linear multi-agent systems with input saturation on switching topologies. Both state feedback and output feedback containment control protocols were proposed. The containment control problems are achieved via local relative position and velocity measurements with constraints when the velocity and acceleration are difficult or impossible to measure in certain scenarios. Ref.[16] investigated the distributed containment control problem for a group of autonomous vehicles modeled by double-integrator dynamics with multiple dynamic leaders over a fixed network that the velocities and the accelerations of both leaders and followers are not available. Ref.[17] investigated the robust global containment control problem for continuous-time second-order multi-agent systems subject to input saturation wherein only local velocity measurements, relative positions, and velocity measurements are involved in the controller design. Distributed observers were constructed for followers to estimate the states of the leaders since they are not available for the followers. Ref.[18] addressed the design of distributed observers for agents with identical linear discrete-time dynamics over a directed graph interaction topology. Ref.[19] addressed the cooperative output regulation problem for discrete-time linear multi-agent systems with a new type of adaptive distributed observer for the leader systems. Ref.[20] studied the cooperative output regulation problem for the discrete-time linear time-delay multi-agent systems by a distributed observer approach. Refs.[21,22] considered the containment problem of heterogeneous linear multi-agent systems via the dynamic compensator technique wherein the containment control problem was converted into a cooperative output regulation problem.

A common assumption adopted in the literature addressing the containment control problem for discrete-time swarm systems is that every follower needs to know the system matrix of its leader. However, this assumption seems rather vulnerable. To tackle this challenge, we revisit the containment control problem for discrete-time swarm systems using an adaptive distributed containment observer inspired by the adaptive distributed observer design in Refs.[23,24]. In this paper, the discrete-time swarm systems under consideration consist of multiple leaders. The containment control problem is addressed by a new design of distributed control laws with the aid of an adaptive distributed containment observer for the followers to infer the system matrices of their leaders and the convex hull formed by the leaders. The proposed control laws circumvent the restrictive assumption that every follower needs to know the system matrix and signal of its leader as proposed in Refs.[20,21,22].

The rest of this paper is organized as follows: Section 2 formulates the problem. Section 3 presents some existing results from Refs.[23,24] and devises a new lemma for the design of an adaptive distributed containment observer. The main results are outlined in Section 4. A numerical example is conducted in Section 5. Companion materials are provided in the appendix.

Notation: For any matrix $A \in \mathrm{R}^{m\times n}$, $\mathrm{vec}\,(\boldsymbol{A}) = \mathrm{col}\,(\boldsymbol{A}_1, \boldsymbol{A}_2, \cdots, \boldsymbol{A}_n)$ where $\boldsymbol{A}_i \in \mathbf{R}^m$ is the ith column of $\boldsymbol{A}$. $\otimes$ denotes the Kronecker product of matrices. For $\boldsymbol{X}_1, \boldsymbol{X}_2 \in \mathbf{R}^n$, let $\mathrm{col}(\boldsymbol{X}_1, \boldsymbol{X}_2) = \left[\boldsymbol{X}_1^{\mathrm{T}}, \boldsymbol{X}_2^{\mathrm{T}}\right]^{\mathrm{T}}$. $\mathbf{1}_N = \mathrm{col}(1, 1, \cdots, 1)$, i.e., an N dimensional column vector whose components are all 1.

2. Problem formulation and assumptions

We consider the following discrete-time linear heterogeneous swarm system:

$$\boldsymbol{x}_i(t+1) = \boldsymbol{A}_i\boldsymbol{x}_i(t) + \boldsymbol{B}_i\boldsymbol{u}_i(t), t \in \mathbf{Z}^+ \tag{1a}$$

$$\boldsymbol{y}_{mi}(t) = \boldsymbol{C}_{mi}\boldsymbol{x}_i(t) + \boldsymbol{D}_{mi}\boldsymbol{u}_i(t) \tag{1b}$$

where $\boldsymbol{x}_i(t) \in \mathbf{R}^n; \boldsymbol{y}_{mi}(t) \in \mathbf{R}^{p_m}$ and $\boldsymbol{u}_i(t) \in \mathbf{R}^m$ are the state, measurement output, and control input of the ith subsystem, respectively. The M leaders are assumed to be exosystems of the following form:

$$\boldsymbol{w}_k(t+1) = \boldsymbol{S}\boldsymbol{w}_k(t), t \in \mathbf{Z}^+ \tag{2}$$

where $\boldsymbol{w}_k(t) \in \mathbf{R}^n$ is the state of the ith leader, for $k = N+1, N+2, \cdots, N+M$, and $\boldsymbol{S}$ represents the system matrix of all leaders.

Define a graph $\mathscr{G} = \{\mathscr{V}, \mathscr{E}, \mathscr{A}\}$ whose vertex set is $\mathscr{V} = \{1, 2, \cdots, N+M\}$, and the edge set is $\mathscr{E} \subseteq \mathscr{V} \times \mathscr{V}$. We use $\mathscr{A} = \left[a_{ij}\right]_{i,j=1}^{N+M} \in \mathbf{R}^{(N+M)\times(N+M)}$ to denote the adjacent matrix of graph $\mathscr{G}$, where a_{ij} is the weight of edge (j, i) with $a_{ij} > 0$ if $(i, j) \in \mathscr{E}$, and $a_{ij} = 0$ otherwise. And $\mathscr{L}$ is the Laplacian of $\mathscr{G}$ corresponding to $\mathscr{A}$, which is defined as $\mathscr{L} = \left[l_{ij}\right] \in \mathbf{R}^{n\times n}$ and $l_{ij} = -a_{ij}$ for any $j \neq i$, and $l_{ii} = \sum_{j=1, j\neq i}^{n} a_{ij}$. Agents 1 to N denote the followers while agents $N+1$ to $N+M$ are leaders. Each follower has at least one neighbor while the leaders have no neighbor. Accordingly, the Laplacian matrix of graph $\mathscr{G}$ can be partitioned as

$$\begin{bmatrix} \mathscr{L}_1 & \mathscr{L}_2 \\ \mathbf{0}_{M\times N} & \mathbf{0}_{M\times M} \end{bmatrix}$$

where $\mathscr{L}_1 \in \mathbf{R}^{N\times N}$ reflects the communication relation between each follower and other followers as well as leaders, and $\mathscr{L}_2 \in \mathbf{R}^{N\times M}$ reflects the communication relation between this follower and the leaders. More details of the graph theory can be found in Ref.[25].

Definition 1 [26]. *A set $\mathscr{C} \subseteq \mathbf{R}^N$ is convex if $(1-\lambda)\boldsymbol{x} + \lambda\boldsymbol{y} \in \mathscr{C}$, for any $\boldsymbol{x}, \boldsymbol{y} \in \mathscr{C}$ and any $\lambda \in [0,1]$. The convex hull $\mathrm{Co}(\boldsymbol{X})$ of a finite set of points $\boldsymbol{X} = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_q\}$ is the minimal convex set containing all points in $\boldsymbol{X}$, that is,*

$$\mathrm{Co}(\boldsymbol{X}) = \left\{ \sum_{i=1}^{q} \alpha_i \boldsymbol{x}_i | \boldsymbol{x}_i \in \boldsymbol{X}, \alpha_i \geqslant 0, \sum_{i=1}^{q} \alpha_i = 1 \right\} \tag{3}$$

Now we describe the containment control problem for the system composed of Eqs. (1) and (2) as follows:

Problem 1 [9]. *The swarm system (1)--(2) achieves containment if the designed control law for each follower makes sure that all followers will converge to the convex hull spanned by the dynamic leaders as $t \to \infty$, that is, $\forall i = 1, 2\cdots, N$ and $\boldsymbol{X}(t) = \{\boldsymbol{w}_{N+1}(t), \boldsymbol{w}_{N+2}(t), \cdots, \boldsymbol{w}_{N+M}(t)\}$*

$$\lim_{t\to\infty} \| \boldsymbol{x}_i(t) - \mathrm{Co}(\boldsymbol{X}(t)) \| = 0 \tag{4}$$

where

$$\mathrm{Co}(\boldsymbol{X}(t)) = \left\{ \sum_{k=N+1}^{N+M} \alpha_{ik} \boldsymbol{w}_k(t) | \boldsymbol{w}_k(t) \in \boldsymbol{X}(t), \alpha_{ik} \geqslant 0, \sum_{k=N+1}^{N+M} \alpha_{ik} = 1 \right\} \tag{5}$$

To simplify the analysis, we introduce the error vector of the ith follower as

$$\boldsymbol{e}_i(t) = \sum_{j=1}^{N} a_{ij}(\boldsymbol{x}_j(t) - \boldsymbol{x}_i(t)) + \sum_{k=N+1}^{N+M} \delta_i^k (\boldsymbol{w}_k(t) - \boldsymbol{x}_i(t)) \tag{6}$$

where $\boldsymbol{x}_i(t)$ is the state of the ith agent and $\boldsymbol{w}_k(t)$ is the state of the kth leader. Eq. (6) can be rewritten in a compact form as

$$\boldsymbol{e}(t) = -(\mathscr{L}_1 \otimes \boldsymbol{I}_n)\boldsymbol{x}(t) - (\mathscr{L}_2 \otimes \boldsymbol{I}_n)\boldsymbol{w}(t) \tag{7}$$

where $\boldsymbol{e}(t)$, $\boldsymbol{x}(t)$ and $\boldsymbol{w}(t)$ are obtained by stacking the columns $\boldsymbol{e}_i$, $\boldsymbol{x}_i$ and $\boldsymbol{w}_k$, $i = 1, 2, \cdots, N$, $k = N+1, N+2, \cdots, N+M$.

The following standard assumptions are necessary for solving the containment control problem:

Assumption 1. *For each follower, there exists at least one leader that has a directed path to the follower.*

Assumption 2. *All the eigenvalues of $\boldsymbol{S}$ have modulus smaller than or equal to 1.*

Assumption 3. *$(\boldsymbol{A}_i, \boldsymbol{B}_i)$ is stabilizable, $i = 1, 2, \cdots, N$.*

Assumption 4. *$(\boldsymbol{A}_i, \boldsymbol{C}_{mi})$ is detectable, $i = 1, 2, \cdots, N$.*

Assumption 5. *The linear matrix equations $\boldsymbol{S} = \boldsymbol{A}_i + \boldsymbol{B}_i\boldsymbol{U}_i$ have solutions $\boldsymbol{U}_i$, for all $i = 1, 2, \cdots, N$.*

Remark 1. Under Assumption 5, there exist matrices $\boldsymbol{U}_i$ such that $\boldsymbol{S} = \boldsymbol{A}_i + \boldsymbol{B}_i\boldsymbol{U}_i$. This implies $\boldsymbol{BU} = \boldsymbol{A} - \boldsymbol{I}_N \otimes \boldsymbol{S}$, where $\boldsymbol{A} = \mathrm{blockdiag}\{\boldsymbol{A}_1, \boldsymbol{A}_2, \cdots, \boldsymbol{A}_N\}$, $\boldsymbol{B} = \mathrm{blockdiag}\{\boldsymbol{B}_1, \boldsymbol{B}_2, \cdots, \boldsymbol{B}_N\}$, $\boldsymbol{U} = \mathrm{blockdiag}\{\boldsymbol{U}_1, \boldsymbol{U}_2, \cdots, \boldsymbol{U}_N\}$.

3. Useful lemmas

In this section, we introduce and establish some useful lemmas on the design of an adaptive distributed containment observer.

Lemma 1 [9]. *Under Assumption 1, all the eigenvalues of $\mathscr{L}_1$ have positive real parts, each entry of $-\mathscr{L}_1^{-1}\mathscr{L}_2$ is non-negative, and each row of $-\mathscr{L}_1^{-1}\mathscr{L}_2$ has a sum equal to 1.*

Lemma 2 [24]. *Consider the following system:*

$$\boldsymbol{x}(t+1) = \boldsymbol{F}\boldsymbol{x}(t) + \boldsymbol{F}_1(t)\boldsymbol{x}(t) + \boldsymbol{F}_2(t) \tag{8}$$

where $\boldsymbol{x}(t) \in \mathbf{R}^n$, $\boldsymbol{F} \in \mathbf{R}^n$ is Schur, and $\boldsymbol{F}_1(t)$ and $\boldsymbol{F}_2(t)$ are well defined for all $t \in \mathbf{Z}^+$. If $\boldsymbol{F}_1(t), \boldsymbol{F}_2(t) \to \mathbf{0}$ (exponentially) as $t \to \infty$, then for any $\boldsymbol{x}(0) \in \mathbf{R}^n$, $\boldsymbol{x}(t) \to \mathbf{0}$ (exponentially) as $t \to \infty$.

We recall the concept of adaptive distributed observer proposed in Ref. [24] for solving the leader-following consensus problem of heterogeneous swarm systems. Suppose there are N followers and one leader. Use $\mathscr{A} = [a_{ij}]$ to denote the weighted adjacency matrix of $\mathscr{G}$. The adaptive distributed observer proposed in Ref. [24] is

$$\boldsymbol{S}_i(t+1) = \boldsymbol{S}_i(t) + \mu_1 (\sum_{j=1}^{N} a_{ij}(\boldsymbol{S}_j(t) - \boldsymbol{S}_i(t)) + a_{i0}(\boldsymbol{S} - \boldsymbol{S}_i(t))) \tag{9a}$$

$$\boldsymbol{\eta}_i(t+1) = \boldsymbol{S}_i(t)[\boldsymbol{\eta}_i(t) + \mu_2 \sum_{j=1}^{N} a_{ij}(\boldsymbol{\eta}_j(t) - \boldsymbol{\eta}_i(t)) + a_{i0}(\boldsymbol{\eta}_0(t) - \boldsymbol{\eta}_i(t))] \tag{9b}$$

where $i = 1, 2, \cdots, N$, and $\boldsymbol{S} \in \mathbf{R}^{n\times n}$ and $\boldsymbol{\eta}_0(t) \in \mathbf{R}^n$ are the leader's system matrix and signal, respectively. The ith follower can receive information about the leader if and only if $a_{i0} \neq 0$. It is assumed that only those followers within the prescribed neighbor of a leader can access the leader's system matrix and signal.

Remark 2. If there exists a spanning tree and the leader is the root, then $\boldsymbol{S}_i(t) \to \boldsymbol{S}$ and $\boldsymbol{\eta}_i(t) \to \boldsymbol{\eta}_0(t)$ with time $t \to \infty$. The adaptive distributed observer not only can estimate the leader's signal but also can infer the leader's system matrix.

In our case, there are N followers and M leaders. We need to extend the adaptive distributed containment observer for all followers ($i = 1, 2, \cdots, N$) as follows:

$$\boldsymbol{\eta}_i(t+1) = \boldsymbol{S}_i(t)[\boldsymbol{\eta}_i(t) + \mu_1 \sum_{j=1}^{N} a_{ij}(\boldsymbol{\eta}_j(t) - \boldsymbol{\eta}_i(t)) + \mu_1 \sum_{k=N+1}^{N+M} \delta_i^k (\boldsymbol{w}_k(t) - \boldsymbol{\eta}_i(t))] \tag{10a}$$

$$\boldsymbol{S}_i(t+1) = \boldsymbol{S}_i(t) + \mu_2 \sum_{j=1}^{N} a_{ij}(\boldsymbol{S}_j(t) - \boldsymbol{S}_i(t)) + \mu_2 \sum_{k=N+1}^{N+M} \delta_i^k (\boldsymbol{S} - \boldsymbol{S}_i(t)) \tag{10b}$$

where δ_i^k represents the communication between subsystem i and leader k, and the subsystem i can access the k leader, i.e., the i subsystem is of the first group, if and only if $\delta_i^k > 0$.

Remark 3. For each follower, if all leaders have a directed path to the follower, the containment control problem will be the cooperative output regulation problem. Otherwise, one cannot design the adaptive distributed observer (9) proposed in Ref. [24]. For example, the communication topology $\bar{\mathscr{G}}_1$ depicted in Fig. 1 has a directed path to every follower for each leader $j \in \{6,7,8,9\}$. We can design the adaptive distributed observer (9) proposed in Ref. [24] for each follower to estimate each leader's state and to solve the containment control problem. For another example, the communication topology $\bar{\mathscr{G}}_2$ depicted in Fig. 2 has no directed path to every follower for each leader $j \in \{6,7,8,9\}$. One cannot design the adaptive distributed observer (9) proposed in Ref. [24] for this swarm system to solve the containment control problem. On the other hand, we can design the adaptive containment distributed observer via Eqs. (10a) and (10b), which is summarized in the following lemma.

Let $\boldsymbol{w}(t) = \mathrm{col}(\boldsymbol{w}_{N+1}(t), \boldsymbol{w}_{N+2}(t), \cdots, \boldsymbol{w}_{N+M}(t))$, $\boldsymbol{\eta}(t) = \mathrm{col}(\boldsymbol{\eta}_1(t), \boldsymbol{\eta}_2(t), \cdots, \boldsymbol{\eta}_N(t))$, $\bar{\boldsymbol{S}} = \boldsymbol{1}_M \otimes \boldsymbol{S}$, $\tilde{\boldsymbol{S}}(t) = \mathrm{col}(\boldsymbol{S}_1(t), \boldsymbol{S}_2(t), \cdots, \boldsymbol{S}_N(t))$ and $\boldsymbol{S}_{\mathrm{d}}(t) = \mathrm{blockdiag}\{\boldsymbol{S}_1(t), \boldsymbol{S}_2(t), \cdots, \boldsymbol{S}_N(t)\}$, then

$$\begin{aligned}\boldsymbol{\eta}(t+1) = (\boldsymbol{S}_{\mathrm{d}}(t) - \mu_1 \boldsymbol{S}_{\mathrm{d}}(t)(\mathscr{L}_1 \otimes \boldsymbol{I}_n))\boldsymbol{\eta}(t) \\ - \mu_1 \boldsymbol{S}_{\mathrm{d}}(t)(\mathscr{L}_2 \otimes \boldsymbol{I}_n)\boldsymbol{w}(t)\end{aligned} \tag{11a}$$

$$\tilde{\boldsymbol{S}}(t+1) = \tilde{\boldsymbol{S}}(t) - \mu_2(\mathscr{L}_1 \otimes \boldsymbol{I}_n)\tilde{\boldsymbol{S}}(t) - \mu_2(\mathscr{L}_2 \otimes \boldsymbol{I}_n)\bar{\boldsymbol{S}}(t) \tag{11b}$$

$$\boldsymbol{w}(t+1) = (\boldsymbol{I}_M \otimes \boldsymbol{S})\boldsymbol{w}(t) \tag{11c}$$

Lemma 3. *Given the systems (2), (10a) and (10b), then for any initial condition $\boldsymbol{S}_i(0)$ and $\boldsymbol{\eta}_i(0)$, $i = 1, 2, \cdots, N$, we have*

A) under Assumption 1, for any μ_2 satisfying $0 < \mu_2 < \frac{2}{\rho(\mathscr{L}_1)}$,

$$\lim_{t\to\infty} \hat{\hat{\boldsymbol{S}}}(t) = \boldsymbol{0} \tag{12}$$

exponentially fast, and

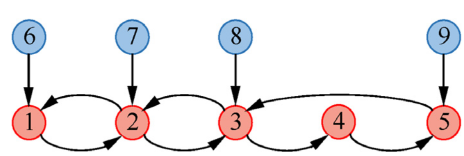

Fig. 1 Communication network graph $\bar{\mathscr{G}}_1$.

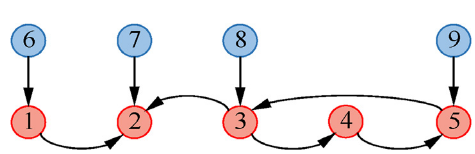

Fig. 2 Communication network graph $\bar{\mathscr{G}}_2$.

B) under Assumptions 1 and 2, with μ_1 satisfying $0 < \mu_1 < \frac{2}{\rho(\mathscr{L}_1)}$ and μ being such that the matrix $\boldsymbol{I}_{Nn} - \mu_1(\mathscr{L}_1 \otimes \boldsymbol{I}_n)$ is Schur,

$$\lim_{t\to\infty} \tilde{\boldsymbol{\eta}}(t) = \boldsymbol{0} \tag{13}$$

exponentially fast.

Proof. Part (A). Under Assumption 1, by Lemma 1, $\mathscr{L}_1$ is invertible. Let $\hat{\hat{\boldsymbol{S}}}(t) = \tilde{\boldsymbol{S}}(t) + (\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n)\bar{\boldsymbol{S}}$. Based on graph structure in Section 2, under Assumption 1, Eq. (11b) can be rewritten as

$$\begin{aligned}&\hat{\hat{\boldsymbol{S}}}(t+1) = \tilde{\boldsymbol{S}}(t+1) + (\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n)\bar{\boldsymbol{S}} \\ &= \tilde{\boldsymbol{S}}(t) - \mu_2(\mathscr{L}_1 \otimes \boldsymbol{I}_n)\tilde{\boldsymbol{S}}(t) - \mu_2(\mathscr{L}_2 \otimes \boldsymbol{I}_n)\bar{\boldsymbol{S}} + (\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n)\bar{\boldsymbol{S}} \\ &= (\boldsymbol{I}_{Nn} - \mu_2(\mathscr{L}_1 \otimes \boldsymbol{I}_n))\hat{\hat{\boldsymbol{S}}}(t)\end{aligned} \tag{14}$$

Under Assumption 1, by Lemma 1, all the eigenvalues of $\mathscr{L}_1$ have positive real parts. Thus, for any μ_2 satisfying $0 < \mu_2 < \frac{2}{\rho(\mathscr{L}_1)}$, the matrix $\boldsymbol{I}_{Nn} - \mu_2(\mathscr{L}_1 \otimes \boldsymbol{I}_n)$ is Schur. Therefore

$$\lim_{t\to\infty} \hat{\hat{\boldsymbol{S}}}(t) = \boldsymbol{0} \tag{15}$$

exponentially, and

$$\lim_{t\to\infty} \left[\tilde{\boldsymbol{S}}(t) + (\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n)\bar{\boldsymbol{S}}\right] = \boldsymbol{0} \tag{16}$$

exponentially fast, respectively

$$-(\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n)\bar{\boldsymbol{S}} = -(\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n)(\boldsymbol{1}_M \otimes \boldsymbol{S}) = \boldsymbol{1}_N \otimes \boldsymbol{S} \tag{17}$$

which implies $\lim_{t\to\infty}\left(\tilde{\boldsymbol{S}}(t) - \boldsymbol{1}_N \otimes \boldsymbol{S}\right) = \boldsymbol{0}$ exponentially.

Part (B). Let $\tilde{\boldsymbol{\eta}}(t) = \boldsymbol{\eta}(t) + (\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n)\boldsymbol{w}(t)$, from system (11a), we have

$$\begin{aligned}&\tilde{\boldsymbol{\eta}}(t+1) = \boldsymbol{\eta}(t+1) + (\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n)\boldsymbol{w}(t+1) \\ &= \boldsymbol{S}_{\mathrm{d}}(t)\boldsymbol{\eta}(t) - \mu_1\boldsymbol{S}_{\mathrm{d}}(t)((\mathscr{L}_1 \otimes \boldsymbol{I}_n)\boldsymbol{\eta}(t) + (\mathscr{L}_2 \otimes \boldsymbol{I}_n)\boldsymbol{w}(t)) \\ &+(\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n)(\boldsymbol{I}_M \otimes \boldsymbol{S})\boldsymbol{w}(t) \\ &= \boldsymbol{S}_{\mathrm{d}}(t)\tilde{\boldsymbol{\eta}}(t) - \boldsymbol{S}_{\mathrm{d}}(t)(\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n)\boldsymbol{w}(t) - \mu_1\boldsymbol{S}_{\mathrm{d}}(t)((\mathscr{L}_1 \otimes \boldsymbol{I}_n)\tilde{\boldsymbol{\eta}}(t)) \\ &+(\boldsymbol{I}_N \otimes \boldsymbol{S})(\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n)\boldsymbol{w}(t) \\ &= ((\boldsymbol{I}_N \otimes \boldsymbol{S}) - \mu_1(\mathscr{L}_1 \otimes \boldsymbol{S})\tilde{\boldsymbol{\eta}}(t) + (\boldsymbol{S}_{\mathrm{d}}(t) - \boldsymbol{I}_N \otimes \boldsymbol{S})\tilde{\boldsymbol{\eta}}(t) - \mu_1(\boldsymbol{S}_{\mathrm{d}}(t) \\ &-(\boldsymbol{I}_N \otimes \boldsymbol{S}))(\mathscr{L}_1 \otimes \boldsymbol{I}_n)\tilde{\boldsymbol{\eta}}(t) - (\boldsymbol{S}_{\mathrm{d}}(t) - \boldsymbol{I}_N \otimes \boldsymbol{S})(\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n)\boldsymbol{w}(t)\end{aligned} \tag{18}$$

By assumption, the matrix $((\boldsymbol{I}_n \otimes \boldsymbol{S}) - \mu_1(\mathscr{L}_1 \otimes \boldsymbol{S}))$ is Schur. Since

$$\lim_{t\to\infty}\left(\tilde{\boldsymbol{S}}(t) - (\boldsymbol{1}_N \otimes \boldsymbol{S})\right) = \boldsymbol{0} \tag{19}$$

exponentially fast, we have

$$\lim_{t\to\infty}(\boldsymbol{S}_{\mathrm{d}}(t) - \boldsymbol{I}_N \otimes \boldsymbol{S}) = \boldsymbol{0} \tag{20}$$

exponentially fast. Therefore, with Lemma 2, $\lim_{t\to\infty}\tilde{\boldsymbol{\eta}}(t) = \boldsymbol{0}$ exponentially fast.

Remark 4. $\boldsymbol{\eta}_i$ is the state of the ith observer of the convex hull spanned by the leaders. Let $\boldsymbol{\eta}(t) = \text{col}(\boldsymbol{\eta}_1(t), \boldsymbol{\eta}_2(t), \cdots, \boldsymbol{\eta}_N(t))$, the proof of Lemma 3 indicates that $\boldsymbol{\eta}(t) \to -(\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n)\boldsymbol{w}(t)$ as $t \to \infty$. $\boldsymbol{S}_i(t)$ is the estimation of the leaders' system matrix $\boldsymbol{S}$ with $\boldsymbol{S}_i(t)$ converging to $\boldsymbol{S}$ as $t \to \infty$.

4. Main results

Based on the adaptive distributed containment observer devised in the previous section, two adaptive distributed control laws for the followers are developed in this section. We first define the distributed control laws as follows:

(1) Distributed dynamic state feedback control

$$\boldsymbol{u}_i(t) = \boldsymbol{K}_{1i}\boldsymbol{x}_i(t) + (\boldsymbol{U}_i(t) - \boldsymbol{K}_{1i})\boldsymbol{\eta}_i(t) \tag{21}$$

(2) Distributed dynamic measurement output feedback

$$\boldsymbol{u}_i(t) = \boldsymbol{K}_{1i}\boldsymbol{\xi}_i(t) + (\boldsymbol{U}_i(t) - \boldsymbol{K}_{1i})\boldsymbol{\eta}_i(t) \tag{22a}$$

$$\xi_i(t+1) = \boldsymbol{A}_i\xi_i(t) + \boldsymbol{B}_i\boldsymbol{u}_i(t) + \boldsymbol{L}_i(\boldsymbol{y}_{mi}(t) - \boldsymbol{C}_{mi}\xi_i(t) - \boldsymbol{D}_{mi}\boldsymbol{u}_i(t)) \tag{22b}$$

in which $\boldsymbol{K}_{1i}$ is chosen in such a way that $\boldsymbol{A}_i + \boldsymbol{B}_i\boldsymbol{K}_{1i}$ is Schur, and $\boldsymbol{L}_i$ is chosen in such a way that $\boldsymbol{A}_i - \boldsymbol{L}_i\boldsymbol{C}_{mi}$ is Schur for $i = 1, 2, \cdots, N$. $\boldsymbol{U}_i(t)$ is defined in Remark 5.

Remark 5. Note that $\boldsymbol{\eta}_i$ is the state of the ith observer of the convex hull spanned by the leader, Lemma 3 showed that $\boldsymbol{\eta}(t) \to -(\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n)\boldsymbol{w}(t)$ as $t \to \infty$. Assumption5 guarantees that the matrix equations $\boldsymbol{S} = \boldsymbol{A}_i + \boldsymbol{B}_i\boldsymbol{U}_i$ have solutions $\boldsymbol{U}_i(t)$. For example, if $\boldsymbol{B}_i$ is of full column rank, $\boldsymbol{U}_i(t) = (\boldsymbol{B}_i^{\text{T}}\boldsymbol{B}_i)^{-1}\boldsymbol{B}_i^{\text{T}}\boldsymbol{S}_i(t) - (\boldsymbol{B}_i^{\text{T}}\boldsymbol{B}_i)^{-1}\boldsymbol{B}_i^{\text{T}}\boldsymbol{A}_i$, $\forall i = 1, 2, \cdots, N$.

Let $\boldsymbol{x}(t) = \text{col}(\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \cdots, \boldsymbol{x}_N(t))$, $\boldsymbol{e}(t) = \text{col}(\boldsymbol{e}_1(t), \boldsymbol{e}_2(t), \cdots, \boldsymbol{e}_N(t))$, $\boldsymbol{u}(t) = \text{col}(\boldsymbol{u}_1(t), \boldsymbol{u}_2(t), \cdots, \boldsymbol{u}_N(t))$, $\boldsymbol{A} = \text{blockdiag}(\boldsymbol{A}_1, \boldsymbol{A}_2, \cdots, \boldsymbol{A}_N)$ and $\boldsymbol{B} = \text{blockdiag}(\boldsymbol{B}_1, \boldsymbol{B}_2, \cdots, \boldsymbol{B}_N)$. Then we can put Eq. (1) into the following compact form:

$$\boldsymbol{x}(t+1) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \tag{23a}$$

$$\begin{aligned}\boldsymbol{\eta}(t+1) = {} & (\boldsymbol{S}_\text{d}(t) - \mu_1\boldsymbol{S}_\text{d}(t)(\mathscr{L}_1 \otimes \boldsymbol{I}_n))\boldsymbol{\eta}(t) \\ & - \mu_1\boldsymbol{S}_\text{d}(t)(\mathscr{L}_2 \otimes \boldsymbol{I}_n)\boldsymbol{w}(t)\end{aligned} \tag{23b}$$

$$\tilde{\boldsymbol{S}}(t+1) = \tilde{\boldsymbol{S}}(t) - \mu_1(\mathscr{L}_1 \otimes \boldsymbol{I}_n)\,\tilde{\boldsymbol{S}}(t) - \mu_2(\mathscr{L}_2 \otimes \boldsymbol{I}_n)\,\bar{\boldsymbol{S}} \tag{23c}$$

$$\boldsymbol{w}(t+1) = (\boldsymbol{I}_M \otimes \boldsymbol{S})\boldsymbol{w}(t) \tag{23d}$$

$$\boldsymbol{e} = -(\mathscr{L}_1 \otimes \boldsymbol{I}_n)\boldsymbol{x}(t) - (\mathscr{L}_2 \otimes \boldsymbol{I}_n)\boldsymbol{w}(t) \tag{23e}$$

Theorem 1. *Under Assumptions 1–3 and 5. Let μ_1 and μ_2 belong to $(0, \frac{2}{\rho(\mathscr{L}_1)})$. Then the swarm system can achieve containment control by the state feedback control law (21).*

Proof. Let $\boldsymbol{K}_1 = \text{blockdiag}(\boldsymbol{K}_{11}, \boldsymbol{K}_{12}, \cdots, \boldsymbol{K}_{1N})$ and $\boldsymbol{U}(t) = \text{blockdiag}(\boldsymbol{U}_1(t), \boldsymbol{U}_2(t), \cdots, \boldsymbol{U}_N(t))$, then Eq. (21) can be rewritten into an augmented system as

$$\boldsymbol{u}(t) = \boldsymbol{K}_1\boldsymbol{x}(t) + (\boldsymbol{U}(t) - \boldsymbol{K}_1)\boldsymbol{\eta}(t) \tag{24}$$

Let $\tilde{\boldsymbol{u}}(t) = \boldsymbol{u}(t) + \boldsymbol{U}(\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n)\boldsymbol{w}(t)$, $\tilde{\boldsymbol{\eta}}(t) = \boldsymbol{\eta}(t) + (\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n)\boldsymbol{w}(t)$, $\zeta(t) = \boldsymbol{x}(t) + (\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n)\boldsymbol{w}(t)$ and $\tilde{\boldsymbol{U}}(t) = (\boldsymbol{U} - \boldsymbol{U}(t))$, and we have

$$\begin{aligned}\tilde{\boldsymbol{u}}(t) &= \boldsymbol{K}_1\boldsymbol{x}(t) + (\boldsymbol{U}(t) - \boldsymbol{K}_1)\left(\tilde{\boldsymbol{\eta}}(t) - (\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n)\boldsymbol{w}(t)\right) \\ &+ \boldsymbol{U}(\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n)\boldsymbol{w}(t) \\ &= \boldsymbol{K}_1\left(\boldsymbol{x}(t) + (\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n)\boldsymbol{w}(t)\right) + (\boldsymbol{U}(t) - \boldsymbol{K}_1) \\ &\left(\boldsymbol{\eta}(t) + (\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n)\boldsymbol{w}(t)\right) \\ &+ (\boldsymbol{U} - \boldsymbol{U}(t))(\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n)\boldsymbol{w}(t) \\ &= \boldsymbol{K}_1\zeta(t) + (\boldsymbol{U}(t) - \boldsymbol{K}_1)\,\tilde{\boldsymbol{\eta}}(t) + \tilde{\boldsymbol{U}}(t)(\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n)\boldsymbol{w}(t)\end{aligned} \tag{25}$$

Evaluating $\zeta(t+1)$ along the trajectory (22), we have

$$\begin{aligned}\zeta(t+1) &= \boldsymbol{x}(t+1) + (\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n)\boldsymbol{w}(t+1) \\ &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) + (\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n)(\boldsymbol{I}_M \otimes \boldsymbol{S})\boldsymbol{w}(t) \\ &= \boldsymbol{A}\left(\zeta(t) - (\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n)\boldsymbol{w}(t)\right) + \boldsymbol{B}\left(\tilde{\boldsymbol{u}}(t) - \boldsymbol{U}(\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n)\boldsymbol{w}(t)\right) \\ &+ (\boldsymbol{I}_N \otimes \boldsymbol{S})(\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n)\boldsymbol{w}(t) \\ &= (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}_1)\zeta(t) + \boldsymbol{B}\left(\tilde{\boldsymbol{u}}(t) - \boldsymbol{K}_1\zeta(t)\right) \\ &+ ((\boldsymbol{I}_N \otimes \boldsymbol{S}) - \boldsymbol{A} - \boldsymbol{B}\boldsymbol{U})(\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n)\boldsymbol{w}(t) \\ &= (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}_1)\zeta(t) + \boldsymbol{B}\left(\tilde{\boldsymbol{u}}(t) - \boldsymbol{K}_1\zeta(t)\right) \\ &= (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}_1)\zeta(t) + \boldsymbol{B}(\boldsymbol{U}(t) - \boldsymbol{K}_1)\,\tilde{\eta}(t) + \boldsymbol{B}\,\tilde{\boldsymbol{U}}(t)(\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n)\boldsymbol{w}(t)\end{aligned} \tag{26}$$

$$\begin{aligned}\tilde{\boldsymbol{\eta}}(t+1) = {} & ((\boldsymbol{I}_N \otimes \boldsymbol{S}) - \mu_1(\mathscr{L}_1 \otimes \boldsymbol{S}))\,\tilde{\boldsymbol{\eta}}(t) + (\boldsymbol{S}_\text{d}(t) - \boldsymbol{I}_N \otimes \boldsymbol{S})\,\tilde{\boldsymbol{\eta}}(t) \\ & - \mu_1(\boldsymbol{S}_\text{d}(t) - (\boldsymbol{I}_N \otimes \boldsymbol{S}))(\mathscr{L}_1 \otimes \boldsymbol{I}_n)\,\tilde{\boldsymbol{\eta}}(t) - (\boldsymbol{S}_\text{d}(t) - \boldsymbol{I}_N \otimes \boldsymbol{S})(\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n)\boldsymbol{w}(t)\end{aligned} \tag{27}$$

$$\hat{\tilde{\boldsymbol{S}}}(t+1) = (\boldsymbol{I}_{Nn} - \mu_2(\mathscr{L}_1 \otimes \boldsymbol{I}_n))\hat{\tilde{\boldsymbol{S}}}(t) \tag{28}$$

$$\begin{aligned}\boldsymbol{e}(t) = {} & (\mathscr{L}_1 \otimes \boldsymbol{I}_n)(\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n)\boldsymbol{w}(t) - (\mathscr{L}_2 \otimes \boldsymbol{I}_n)\boldsymbol{w}(t) \\ & - (\mathscr{L}_1 \otimes \boldsymbol{I}_n)\zeta(t)\end{aligned} \tag{29}$$

Stacking ζ and $\tilde{\boldsymbol{\eta}}$, and by Remark 1, we have

$$\begin{bmatrix}\zeta(t+1) \\ \tilde{\boldsymbol{\eta}}(t+1)\end{bmatrix} = \begin{bmatrix}(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}_1) & \boldsymbol{0} \\ \boldsymbol{0} & \hat{\mathscr{L}}_1\end{bmatrix}\begin{bmatrix}\zeta(t) \\ \tilde{\boldsymbol{\eta}}(t)\end{bmatrix} + \begin{bmatrix}\boldsymbol{f}_1(t) \\ \boldsymbol{f}_2(t)\end{bmatrix} \tag{30a}$$

$$\hat{\tilde{\boldsymbol{S}}}(t+1) = (\boldsymbol{I}_{Nn} - \mu_2(\mathscr{L}_1 \otimes \boldsymbol{I}_n))\hat{\tilde{\boldsymbol{S}}}(t) \tag{30b}$$

$$\boldsymbol{e}(t) = -(\mathscr{L}_1 \otimes \boldsymbol{I}_n)\zeta(t) \tag{30c}$$

where

$$\hat{\mathscr{L}}_1 = ((\boldsymbol{I}_N \otimes \boldsymbol{S}) - \mu_2(\mathscr{L}_1 \otimes \boldsymbol{S})) \tag{31}$$

$$\boldsymbol{f}_1(t) = \boldsymbol{B}(\boldsymbol{U}(t) - \boldsymbol{K}_1)\,\tilde{\boldsymbol{\eta}}(t) + \boldsymbol{B}\,\tilde{\boldsymbol{U}}(t)(\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n)\boldsymbol{w}(t) \tag{32}$$

$$\begin{aligned}\boldsymbol{f}_2(t) = {} & (\boldsymbol{S}_\text{d}(t) - \boldsymbol{I}_N \otimes \boldsymbol{S})\,\tilde{\boldsymbol{\eta}}(t) - \mu_1(\boldsymbol{S}_\text{d}(t) - (\boldsymbol{I}_N \otimes \boldsymbol{S}))(\mathscr{L}_1 \otimes \boldsymbol{I}_n)\,\tilde{\boldsymbol{\eta}}(t) \\ & - (\boldsymbol{S}_\text{d}(t) - \boldsymbol{I}_N \otimes \boldsymbol{S})(\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n)\boldsymbol{w}(t)\end{aligned} \tag{33}$$

By Lemma 3, $\lim_{t\to\infty}\left(\tilde{\boldsymbol{S}}(t) - \boldsymbol{1}_N \otimes \boldsymbol{S}\right) = \boldsymbol{0}$ and $\lim_{t\to 0}\tilde{\boldsymbol{\eta}}(t) = \boldsymbol{0}$, we have

$$\lim_{t\to\infty} \boldsymbol{B}(\boldsymbol{S}_{\mathrm{d}}(t) - \boldsymbol{I}_N \otimes \boldsymbol{S}) = \boldsymbol{0}. \tag{34}$$

Hence, $\lim_{t\to\infty} \boldsymbol{B}(\boldsymbol{U}(t) - \boldsymbol{U}) = \lim_{t\to\infty} \boldsymbol{B}(\boldsymbol{S}_{\mathrm{d}}(t) - \boldsymbol{I}_N \otimes \boldsymbol{S}) = \boldsymbol{0}$. Then we have $\boldsymbol{f}_1(t) \to \boldsymbol{0}$ and $\boldsymbol{f}_2(t) \to \boldsymbol{0}$ as $t \to \infty$. Since $\mu_k,\ k = 1, 2$ is chosen, $\boldsymbol{I}_{Nn} - \mu_k(\mathscr{L}_1 \otimes \boldsymbol{I}_n)$ is Schur, while note that $\boldsymbol{A}_i + \boldsymbol{B}_i\boldsymbol{K}_{1i}$ is Schur for $i = 1, 2, \cdots, N$, we have $\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}$ and $\hat{\mathscr{L}}_1$ are Schur. By Lemma 2, $\lim_{t\to\infty} \left(\zeta(t), \tilde{\boldsymbol{\eta}}(t)\right) = \boldsymbol{0}$, for arbitrary $\boldsymbol{x}(0)$, $\boldsymbol{\eta}(0)$, and $\boldsymbol{w}(0)$. Therefore $\lim_{t\to\infty} \boldsymbol{e}(t) = \boldsymbol{0}$.

Theorem 2. *Under Assumptions 1–5. Let μ_1 and μ_2 belong to $(0, \frac{2}{\rho(\mathscr{L}_1)})$. Then the swarm system can achieve containment control by the dynamic measurement output feedback control law* Eq. (22*a*), Eq. ((22)*b*)

Proof. *Let* $\xi(t) = \mathrm{col}(\xi_1(t), \xi_2(t), \cdots, \xi_N(t))$, $\boldsymbol{y}_m(t) = \mathrm{col}(\boldsymbol{y}_{m1}(t), \boldsymbol{y}_{m2}(t), \cdots, \boldsymbol{y}_{mN}(t))$, $\boldsymbol{L} = \mathrm{blockdiag}(\boldsymbol{L}_1, \boldsymbol{L}_2, \cdots, \boldsymbol{L}_N)$, $\boldsymbol{C}_m = \mathrm{col}(\boldsymbol{C}_{m1}, \boldsymbol{C}_{m2}, \cdots, \boldsymbol{C}_{mN})$, $\boldsymbol{D}_m = \mathrm{col}(\boldsymbol{D}_{m1}, \boldsymbol{D}_{m2}, \cdots, \boldsymbol{D}_{mN})$, $\boldsymbol{K}_1 = \mathrm{blockdiag}(\boldsymbol{K}_{11}, \boldsymbol{K}_{12}, \cdots, \boldsymbol{K}_{1N})$ and $\boldsymbol{U}(t) = \mathrm{blockdiag}(\boldsymbol{U}_1(t), \boldsymbol{U}_2(t), \cdots, \boldsymbol{U}_N(t))$, and we rewrite Eqs. (22a, 22b) into an augmented system as

$$\boldsymbol{u}(t) = \boldsymbol{K}_1\xi(t) + (\boldsymbol{U}(t) - \boldsymbol{K}_1)\boldsymbol{\eta}(t) \tag{35a}$$

$$\xi(t+1) = \boldsymbol{A}\xi(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{L}(\boldsymbol{y}_m(t) - \boldsymbol{C}_m\xi(t) - \boldsymbol{D}_m\boldsymbol{u}(t)) \tag{35b}$$

Let $\tilde{\boldsymbol{x}}(t) = \xi(t) - \boldsymbol{x}(t)$, $\tilde{\boldsymbol{U}}(t) = \boldsymbol{U} - \boldsymbol{U}(t)$, $\tilde{\boldsymbol{u}}(t) = \boldsymbol{u}(t) + \boldsymbol{U}\left(\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n\right)\boldsymbol{w}(t)$, $\tilde{\boldsymbol{\eta}}(t) = \boldsymbol{\eta}(t) + \left(\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n\right)\boldsymbol{w}(t)$ and $\zeta(t) = \boldsymbol{x}(t) + \left(\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n\right)\boldsymbol{w}(t)$, and we have

$$\begin{aligned}
\tilde{\boldsymbol{u}}(t) &= \boldsymbol{K}_1\xi(t) + (\boldsymbol{U}(t) - \boldsymbol{K}_1)\left(\tilde{\boldsymbol{\eta}}(t) - \left(\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n\right)\boldsymbol{w}(t)\right) \\
&\quad + \boldsymbol{U}\left(\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n\right)\boldsymbol{w}(t) \\
&= \boldsymbol{K}_1(\xi(t) - \boldsymbol{x}(t)) + \boldsymbol{K}_1(\boldsymbol{x}(t) + (\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n)\boldsymbol{w}(t)) \\
&\quad + (\boldsymbol{U}(t) - \boldsymbol{K}_1)\left(\boldsymbol{\eta}(t) + \left(\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n\right)\boldsymbol{w}(t)\right) \\
&\quad + (\boldsymbol{U} - \boldsymbol{U}(t))\left(\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n\right)\boldsymbol{w}(t) \\
&= \boldsymbol{K}_1\tilde{\boldsymbol{x}}(t) + \boldsymbol{K}_1\zeta(t) + (\boldsymbol{U}(t) - \boldsymbol{K}_1)\tilde{\boldsymbol{\eta}}(t) + \tilde{\boldsymbol{U}}(t)\left(\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n\right)\boldsymbol{w}(t)
\end{aligned} \tag{36a}$$

$$\tilde{\boldsymbol{x}}(t+1) = (\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C}_m)\tilde{\boldsymbol{x}}(t) \tag{36b}$$

Evaluating $\zeta(t+1)$ along the trajectory (22), we have

$$\begin{aligned}
\zeta(t+1) &= \boldsymbol{x}(t+1) + \left(\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n\right)\boldsymbol{w}(t+1) \\
&= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \left(\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n\right)(\boldsymbol{I}_M \otimes \boldsymbol{S})\boldsymbol{w}(t) \\
&= \boldsymbol{A}\left(\zeta(t) - \left(\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n\right)\boldsymbol{w}(t)\right) + \boldsymbol{B}\left(\tilde{\boldsymbol{u}}(t) - \boldsymbol{U}\left(\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n\right)\boldsymbol{w}(t)\right) \\
&\quad + (\boldsymbol{I}_N \otimes \boldsymbol{S})\left(\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n\right)\boldsymbol{w}(t) \\
&= (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}_1)\zeta(t) + \boldsymbol{B}(\tilde{\boldsymbol{u}}(t) - \boldsymbol{K}_1\zeta(t)) \\
&\quad + ((\boldsymbol{I}_N \otimes \boldsymbol{S}) - \boldsymbol{A} - \boldsymbol{B}\boldsymbol{U})\left(\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n\right)\boldsymbol{w}(t) \\
&= (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}_1)\zeta(t) + \boldsymbol{B}\boldsymbol{K}_1\tilde{\boldsymbol{x}}(t) + \boldsymbol{B}(\boldsymbol{U}(t) - \boldsymbol{K}_1)\tilde{\boldsymbol{\eta}}(t) \\
&\quad + \boldsymbol{B}(\boldsymbol{U} - \boldsymbol{U}(t))\left(\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n\right)\boldsymbol{w}(t)
\end{aligned} \tag{37}$$

$$\begin{aligned}
\tilde{\boldsymbol{\eta}}(t+1) &= ((\boldsymbol{I}_N \otimes \boldsymbol{S}) - \mu_1(\mathscr{L}_1 \otimes \boldsymbol{S}))\tilde{\boldsymbol{\eta}}(t) + (\boldsymbol{S}_{\mathrm{d}}(t) - \boldsymbol{I}_N \otimes \boldsymbol{S})\tilde{\boldsymbol{\eta}}(t) \\
&\quad - \mu_1(\boldsymbol{S}_{\mathrm{d}}(t) - (\boldsymbol{I}_N \otimes \boldsymbol{S}))(\mathscr{L}_1 \otimes \boldsymbol{I}_n)\tilde{\boldsymbol{\eta}}(t) - (\boldsymbol{S}_{\mathrm{d}}(t) - \boldsymbol{I}_N \otimes \boldsymbol{S})\left(\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n\right)\boldsymbol{w}(t)
\end{aligned} \tag{38}$$

$$\hat{\tilde{\boldsymbol{S}}}(t+1) = (\boldsymbol{I}_{Nn} - \mu_2(\mathscr{L}_1 \otimes \boldsymbol{I}_n))\hat{\tilde{\boldsymbol{S}}}(t) \tag{39}$$

$$\begin{aligned}
\boldsymbol{e}(t) &= (\mathscr{L}_1 \otimes \boldsymbol{I}_n)\left(\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n\right)\boldsymbol{w}(t) - (\mathscr{L}_2 \otimes \boldsymbol{I}_n)\boldsymbol{w}(t) \\
&\quad - (\mathscr{L}_1 \otimes \boldsymbol{I}_n)\zeta(t)
\end{aligned} \tag{40}$$

By Remark 1, while stacking $\zeta(t)$, $\tilde{\boldsymbol{x}}(t)$ and $\tilde{\boldsymbol{\eta}}(t)$ into a vector, we have

$$\begin{bmatrix} \zeta(t+1) \\ \tilde{\boldsymbol{x}}(t+1) \\ \tilde{\boldsymbol{\eta}}(t+1) \end{bmatrix} = \boldsymbol{M}\begin{bmatrix} \zeta(t) \\ \tilde{\boldsymbol{x}}(t) \\ \tilde{\boldsymbol{\eta}}(t) \end{bmatrix} + \begin{bmatrix} \boldsymbol{g}_1(t) \\ \boldsymbol{0} \\ \boldsymbol{g}_2(t) \end{bmatrix} \tag{41}$$

$$\hat{\tilde{\boldsymbol{S}}}(t+1) = (\boldsymbol{I}_{Nn} - \mu_2(\mathscr{L}_1 \otimes \boldsymbol{I}_n))\hat{\tilde{\boldsymbol{S}}}(t) \tag{42}$$

$$\boldsymbol{e}(t) = (\mathscr{L}_1 \otimes \boldsymbol{I}_n)\zeta(t) \tag{43}$$

where

$$\boldsymbol{M} = \begin{bmatrix} (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}_1) & \boldsymbol{B}\boldsymbol{K}_1 & \boldsymbol{0} \\ \boldsymbol{0} & (\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C}_m) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \hat{\mathscr{L}}_1 \end{bmatrix} \tag{44}$$

$$\hat{\mathscr{L}}_1 = \boldsymbol{I}_N \otimes \boldsymbol{S} - \mu_1(\mathscr{L}_1 \otimes \boldsymbol{S}) \tag{45}$$

$$\boldsymbol{g}_1(t) = \boldsymbol{B}(\boldsymbol{U}(t) - \boldsymbol{K}_1)\tilde{\boldsymbol{\eta}} + \boldsymbol{B}(\boldsymbol{U} - \boldsymbol{U}(t))\left(\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n\right)\boldsymbol{w}(t) \tag{46}$$

$$\begin{aligned}
\boldsymbol{g}_2(t) &= (\boldsymbol{S}_{\mathrm{d}}(t) - \boldsymbol{I}_N \otimes \boldsymbol{S})\tilde{\boldsymbol{\eta}}(t) - \mu_1(\boldsymbol{S}_{\mathrm{d}}(t) - (\boldsymbol{I}_N \otimes \boldsymbol{S}))(\mathscr{L}_1 \otimes \boldsymbol{I}_n)\tilde{\boldsymbol{\eta}}(t) \\
&\quad - (\boldsymbol{S}_{\mathrm{d}} - \boldsymbol{I}_N \otimes \boldsymbol{S})\left(\mathscr{L}_1^{-1}\mathscr{L}_2 \otimes \boldsymbol{I}_n\right)\boldsymbol{w}(t)
\end{aligned} \tag{47}$$

By Lemma 3, $\lim_{t\to\infty} \left(\tilde{\boldsymbol{S}}(t) - \boldsymbol{1}_N \otimes \boldsymbol{S}\right) = \boldsymbol{0}$ and $\lim_{t\to 0} \tilde{\boldsymbol{\eta}}(t) = \boldsymbol{0}$, we have

$$\lim_{t\to\infty} \boldsymbol{B}(\boldsymbol{S}_{\mathrm{d}}(t) - \boldsymbol{I}_N \otimes \boldsymbol{S}) = \boldsymbol{0} \tag{48}$$

Hence, $\lim_{t\to\infty} \boldsymbol{B}(\boldsymbol{U}(t) - \boldsymbol{U}) = \lim_{t\to\infty} \boldsymbol{B}(\boldsymbol{S}_{\mathrm{d}}(t) - \boldsymbol{I}_N \otimes \boldsymbol{S}) = \boldsymbol{0}$. Then we have $\boldsymbol{g}_1(t) \to \boldsymbol{0}$ and $\boldsymbol{g}_2(t) \to \boldsymbol{0}$ as $t \to \infty$. For $\mu_k,\ k = 1, 2$ is chosen so that $\boldsymbol{I}_{Nn} - \mu_k(\mathscr{L}_1 \otimes \boldsymbol{I}_n)$ is Schur. Furthermore, $\boldsymbol{A}_i + \boldsymbol{B}_i\boldsymbol{K}_{1i}$ and $\boldsymbol{A}_i - \boldsymbol{L}_i\boldsymbol{C}_{mi}$ are Schur, for $i = 1, 2, \cdots, N$. Therefore, $\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C}_m$ and $\hat{\mathscr{L}}_1$ are Schur, and thus $\boldsymbol{M}$ is Schur by Lemma 2 while $\lim_{t\to\infty} \mathrm{col}\left(\zeta(t), \tilde{\boldsymbol{x}}(t), \tilde{\boldsymbol{\eta}}(t)\right) = \boldsymbol{0}$, for arbitrary initial conditions. This concludes $\lim_{t\to\infty} \boldsymbol{e}(t) = \boldsymbol{0}$.

5. Numerical example

Consider the heterogeneous swarm system with five followers and three leaders in $\mathbf{R}^2$. The communication topology of the followers and leaders is shown in Fig. 3. The dynamics of the followers and leaders are given by Eqs. (1) and (2), respectively, with

$$\boldsymbol{A}_i = \begin{bmatrix} 0 & \alpha_i \\ 0 & 0 \end{bmatrix}, \boldsymbol{B}_i = \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix}$$

$$\boldsymbol{C}_{mi} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}^{\mathrm{T}}, \boldsymbol{S} = \begin{bmatrix} \cos\frac{\pi}{6} & \sin\frac{\pi}{6} \\ -\sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{bmatrix}$$

where $\alpha_1 = 0.4587$, $\alpha_2 = 0.7703$, $\alpha_3 = 0.6619$, $\alpha_4 = 0.3502$, $\alpha_5 = 0.6620$. The initial states are $\boldsymbol{w}_1(0) = \mathrm{col}(0.8819, 0.6692)$, $\boldsymbol{w}_2(0) = \mathrm{col}(0.1904, 0.3689)$, $\boldsymbol{w}_3(0) = \mathrm{col}(0.4607, 0.9816)$,

$\boldsymbol{\eta}_i(0) = \mathbf{0}$, $\boldsymbol{S}_i(0) = \mathbf{0}$, and $\boldsymbol{x}_i(0)$ is randomly chosen in [0, 1], for $i = 1, 2, \cdots, 5$.

Assumptions 1–4 can be easily verified. Since $\boldsymbol{B}_i$ is of full column rank, we have

$$\boldsymbol{U}_i(t) = \left(\boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{B}_i\right)^{-1}\boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{S}_i(t) - \left(\boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{B}_i\right)^{-1}\boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{A}_i \tag{49}$$

where, $i = 1, 2, \cdots, 5$, from Remark 5. Thus, Assumption 5 is satisfied. By choosing

$$\boldsymbol{K}_{1i} = \begin{bmatrix} 0.15 & 0 \\ 0 & 0 \end{bmatrix}, \ \boldsymbol{L}_i = \mathrm{col}(0.15, 0.3)$$

$\mu_1 = 0.5$ and $\mu_2 = 0.5$, so conditions of Theorems 1–2 hold.

Since the 7th leader has no directed path to every follower, one cannot design the distributed observer proposed in Refs. [20,24] for this heterogeneous swarm system to solve the containment control problem. We design the adaptive containment distributed observer in Eqs. (10a)–(10b) according to Lemma 3.

Fig. 4 shows the tracking errors under the state feedback control law (21). As shown in Fig. 4, it is clear that the tracking errors converge to zero as time tends to infinity. The state trajectories of the followers converge to the convex hull formed by the leaders as demonstrated in Fig. 6. The tracking errors and the state trajectories under the dynamic measurement output feedback control law Eq. (22a), Eq. (22b) are shown in Figs. 5 and 7, respectively. Similar to the results of state feedback control law (21), the tracking errors converge to zero asymptotically under the dynamic measurement output feedback control law Eq. (22a), Eq. (22b) while the state trajectories of the followers converge to the convex hull formed by the leaders. Fig. 8 shows the convergence of the estimation of the leader's system matrix S by the adaptive distributed observer. Satisfactory tracking performance confirms the analytical results of Theorems 1–2.

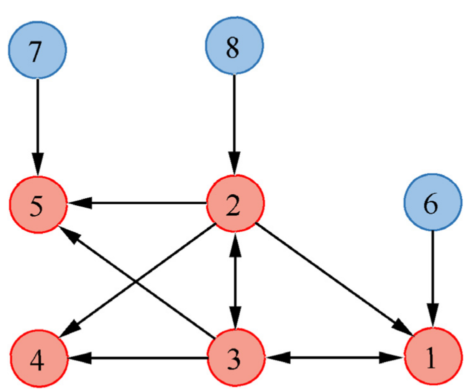

Fig. 3 Network topology $\bar{\mathscr{G}}$.

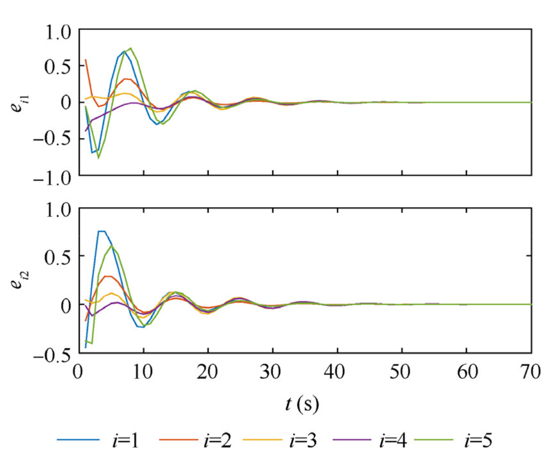

Fig. 4 Tracking errors under state feedback control law (21).

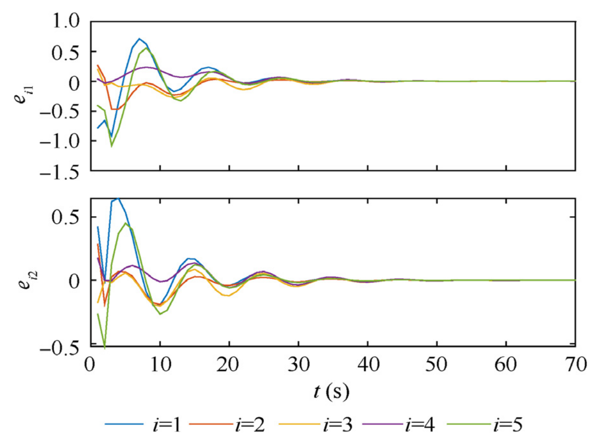

Fig. 5 Tracking errors under dynamic measurement output feedback control law (22).

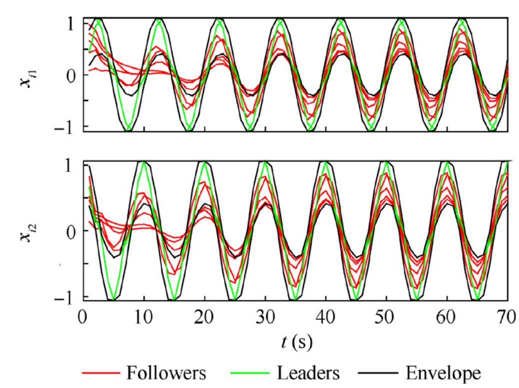

Fig. 6 State trajectories under state feedback control law (21).

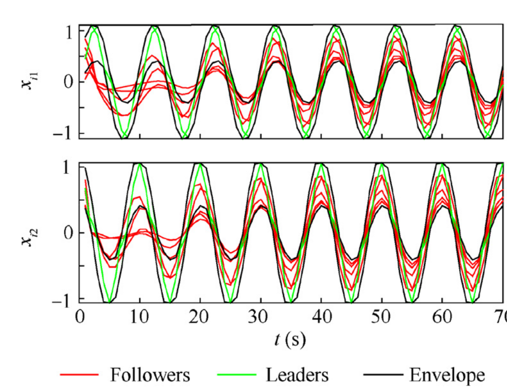

Fig. 7 State trajectories under dynamic measurement output feedback control law (22).

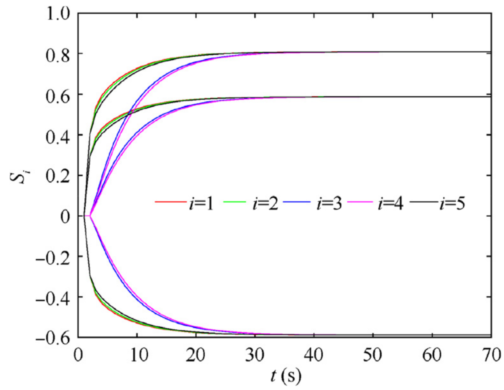

Fig. 8 Estimation of the leader's system matrix by adaptive distributed observer.

6. Conclusions

In this paper, the containment control problem for discrete-time swam system has been investigated. To circumvent a restrictive assumption in the literature that every follower needs to know the system matrix and signal of its leader, we extended the design of an adaptive distributed observer in Ref. [24] to estimate the leader's system matrix and signal. Both analytical and numerical results confirm that the followers cooperate with each other to achieve consensus and convergence to the convex hull spanned by the leaders under both the distributed adaptive dynamic state feedback and distributed adaptive measurement output feedback control laws.

Three issues deserve further research effort. First, the eigenvalues of $\boldsymbol{S}$ are assumed to have modulus smaller than or equal to 1 in Assumption 2. Removing this restricted assumption would yield more general and practical results. Second, it is interesting to look into the containment control problem of singular discrete-time linear heterogeneous swarm systems. Third, the topology of the communication network is confined to be static in the containment control problem. Extending the results for containment control problems, wherein the topology of the communication network is time-varying, is of importance.

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Appendix A. This appendix presents a brief introduction of graph theory in line with Ref. [25].

A digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ consists of a finite set of nodes $\mathscr{V} = \{1, 2, \cdots, N\}$ and an edge set $\mathscr{E} = \{(i,j), i,j \in \mathscr{V}, i \neq j\}$. A node i is called a neighbor of a node j if the edge $(i,j) \in \mathscr{E}$. $\mathscr{N}_i$ denotes the subset of $\mathscr{V}$ that consists of all the neighbors of the node i. If the graph $\mathscr{G}$ contains a sequence of edges of the form (i_1, i_2), (i_2, i_3), $\cdots$, (i_k, i_{k+1}), then the set $\{(i_1, i_2), (i_2, i_3), \cdots, (i_k, i_{k+1})\}$ is called a path of $\mathscr{G}$ from i_1 to i_{k+1}, and node i_{k+1} is said to be reachable from node i_1. A digraph $\mathscr{G}_s = (\mathscr{V}_s, \mathscr{E}_s)$ is a subgraph of $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ if $\mathscr{V}_s \subseteq \mathscr{V}$ and $\mathscr{E}_s \subseteq \mathscr{E} \cap (\mathscr{V}_s \times \mathscr{V}_s)$. Given a set of r digraphs$\{\mathscr{G}_i = (\mathscr{V}_i, \mathscr{E}_i, i = 1, 2, \cdots, r)\}$, the digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ where $\mathscr{E} = \cup_{i=1}^{r} \mathscr{E}_i$ is called the union of digraphs $\mathscr{G}_i$, denoted by $\mathscr{G} = \cup_{i=1}^{r} \mathscr{G}_i$. The weighted adjacency matrix of a digraph $\mathscr{G}$ is nonnegative matrix$\mathscr{A} = [a_{ij}] \in \mathbf{R}^{N\times N}$, where $\mathscr{A}_{ii} = 0$ and$a_{ij} > 0 \Longleftrightarrow (j,i) \in \mathscr{E}$. The Laplacian of a digraph $\mathscr{G}$ is denoted by $\mathscr{L} = [l_{ij}] \in \mathbf{R}^{N\times N}$, where $l_{ii} = \sum_{j=1}^{N} a_{ij}$ and $l_{ij} = -a_{ij}$ if $i \neq j$.

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