Wavelet-Galerkin Method for Computations of Electromagnetic Fields—Computation of Connection Coefficients

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***Abstract*—One of the key issues which makes the wavelet-Galerkin method unsuitable for solving general electromagnetic problems is a lack of exact representations of the connection coefficients. This paper presents the mathematical formulae and computer procedures for computing some common connection coefficients. The characteristic of the present formulae and procedures is that the arbitrary point values of the connection coefficients, rather than the dyadic point values, can be determined. A numerical example is also given to demonstrate the feasibility of using the wavelet-Galerkin method to solve engineering field problems.**

***Index Terms*—Connection coefficients, wavelet bases, wavelet-Galerkin method.**

I. Introduction

THE wavelet-Galerkin method is a powerful alternative for the numerical solution of both integral and differential equations by pure mathematicians and engineers alike [1]–[4]. As similar to the Galerkin discretization approach, the wavelet-Galerkin scheme involves the evaluation of connection coefficients. These connection coefficients are quadratures with integrands being products of wavelet bases and their derivatives. As the derivatives of compactly supported wavelets are highly oscillatory and also due to the implicit representation of wavelet bases, it is difficult to compute these connection coefficients by numerical methods. Thanks to the development in algorithms for computing these coefficients [5]–[7], the wavelet- Galerkin method has now been applied successfully to solve some typical benchmark problems in mathematics and in almost all branches of engineering. Since most algorithms developed are essentially based on unbounded domains, the general wavelet bases have to be periodized, and the applications of wavelet-Galerkin methods are limited to cases where the problem domain is unbounded or the boundary condition is periodic. To use the wavelet-Galerkin method to solve finite domain problems, Chen *et al.* have developed algorithms for computing some finite integrals of wavelets on a bounded interval [8]. However, the algorithms of Chen *et al.* are still restricted to the computation of dyadic point values. On the other hand, the authors believe that these algorithms can be improved by a more generalized definition of normalization conditions.

For practical engineering field problems, especially those in electromagnetics, one should note that:

1) The solution domains are normally bounded to some specific regions. For those wavelet bases whose supports lie partly on the domains, the computations of the connection coefficients depends on the integrals between some limits;
2) Many, possibly most, engineering field problems involve distributed sources and multiple physical media, the so called inhomogeneous sources and materials, which means that even for wavelets whose supports are entirely within the solution domain, the computation of connection coefficients must also be carried out region by region, i.e., the connection coefficients are the sums of different bounded integrals, because of the differences in the media parameters and sources.

In the application of general wavelet bases in the wavelet-Galerkin method for solving engineering field problems, it is necessary to have an exact determination of the arbitrary point values of the connection coefficients. In short, this paper is a continuation of the previous work of the authors to develop an improved wavelet-Galerkin method for the numerical computations of electromagnetic fields. The emphasis of the work being reported is the computation of the arbitrary point values of the connection coefficients.

II. Computation of Connection Coefficients

Since the present work is a continuation of [9], thus the mathematical notations used in what follows are as defined in [8] and [9] to allow easy cross referencing.

A. Computation of $\Gamma_k^n(x) = \int_0^x \phi^{(n)}(y-k)\phi(y)\,dy$

The computation of this connection coefficient at integer points is reported by Chen *et al.* although there were something wrong with the normalizing conditions [8]. In order to derive the correct coefficients, one needs to use the following equation

$$\sum_{k=-\infty}^{\infty} k^n \phi^{(n)}(x-k) = n! \tag{1}$$

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Multiplying both sides of (1) with $\phi(y)$, and taking integration from limit 0 to limit x, one has

$$\sum_{k=-\infty}^{\infty} k^n \int_0^x \phi(y)\phi^{(n)}(y-k)\,dy = n! \int_0^y \phi(y)\,dy \qquad (2)$$

Considering the facts

$$\begin{aligned}\Gamma_k^n(x) &= 0 \qquad (|k \geq |L-1, \text{ or } x \leq \min\{0, k\}) \\ \Gamma_{-k}^n(x) &= (-1)^n \Gamma_k^n(L-1) \qquad (x+k \geq L-1) \qquad (3)\end{aligned}$$

one obtains

$$\begin{aligned}&\sum_{m-L+2}^{m-1} k^n \Gamma_k^n(m) \\ &\quad = n!\theta_1(m) - (-1)^n \left[\sum_{2-L\leq k\leq m-L+1} k^n \Gamma_{-k}^n(L-1)\right] \qquad (4)\end{aligned}$$

where $\theta_1(x) = \int_0^x \phi(y)\,dy$, and its computation is reported in [8].

From (3), it is clear that there were $(L-1)^2$ independent members, $\{\Gamma_k^n(i+x)|\, 0 \leq i \leq L-2,\, i-L+2 \leq k \leq i\}$, in the set $\Gamma_k^n(i+x)$ $(i = 0, 1, \cdots, L-2)$ $(|k| \leq L-2)$), that exist for some fixed $x \in (0, 1)$. Expressing these independent members in a vector form as

$$\overrightarrow{v}(t) = [v(t)\, v(1+t) \cdots v(L-2+t)]^T \in R^{(L-1)\times(L-1)} \qquad (5)$$

$$\begin{aligned}v(i+t) &= \Gamma^n(i+t) \\ &= [\Gamma_{i-L+2}^n(i+t)\Gamma_{i-L+3}^n(i+t) \cdots \Gamma_i^n(i+t)]^T \\ &\qquad (i = 0, 1, \cdots, L-2) \qquad (6)\end{aligned}$$

one then reads

$$\overrightarrow{v}(0) = [\Gamma^n(0)\,\Gamma^n(1) \cdots \Gamma^n(L-2)]^T \qquad (7)$$

According to the following two scale relationships

$$\Gamma_k^n(x) = 2^{n-1} \sum_{i=0}^{L-1}\sum_{j=0}^{L-1} p_i p_j \Gamma_{2k+i-j}^n(2x-j) \qquad (8)$$

one gets

$$\overrightarrow{v}\left(\frac{t}{2}\right) = T_0 \overrightarrow{v}(t) + g_0 \qquad (9)$$

where

$$\begin{aligned}(T_0)_{i,j,k,m} &= 2^{n-1} p_{2i-j} p_{L-1-2k+m} \\ &\quad \cdot (i, j = 0, 1, \cdots, L-2; \\ &\qquad k, m = 1, 2, \cdots, L-1)\end{aligned}$$

$$g_0 = [\,g_0(1) \quad g_0(2) \quad \cdots \quad g_0((L-2)(L-1))\,]^T$$

$$\begin{aligned}&g_0(i[L-1] + [k-(i-L+1)]) \\ &\quad = 2^{n-1}\left\{\sum_{p,\,q\in\mu_1(i,\,k,\,L)} p_p p_q \Gamma_{2k+p-q}^n(L-1) + (-1)^n \right. \\ &\qquad \left. \cdot \sum_{p,\,q\in\mu_2(i,\,k,\,L)} p_p p_q \Gamma_{-2k-p+q}^n(L-1)\right\} \\ &\qquad (i = 0, 1, \cdots, L-2; \\ &\qquad k = i-L+2, i-L+3, \cdots, i)\end{aligned}$$

$$\begin{aligned}&\mu_1(i, k, L) \\ &\quad = \{(p, q)|L-1 \leq 2i-q(0 \leq p, q \leq L-1)\} \\ &\mu_2(i, k, L) \\ &\quad = \{(p, q)|L-1 \leq 2i-2k-p(0 \leq p, q \leq L-1)\}\end{aligned}$$

Similarly

$$\overrightarrow{v}\left(\frac{t+1}{2}\right) = T_1\vec{v}(t) + g_1 \qquad (10)$$

where

$$\begin{aligned}(T_1)_{i,j,k,m} &= 2^{n-1} p_{2i-j+1} p_{L-2k+m} \\ &\quad (i, j = 0, 1, \cdots, L-2; \\ &\qquad k, m = 1, 2, \cdots, L-1)\end{aligned}$$

$$g_1 = [\,g_1(1) \quad g_1(2) \quad \cdots \quad g_1((L-2)(L-1))\,]^T$$

$$\begin{aligned}&g_1(i[L-1] + [k-(i-L+1)]) \\ &\quad = 2^{n-1}\left\{\sum_{p,\,q\in\mu_3(i,\,k,\,L)} p_p p_q \Gamma_{2k+p-q}^n(L-1) + (-1)^n \right. \\ &\qquad \left. \cdot \sum_{p,\,q\in\mu_4(i,\,k,\,L)} p_p p_q \Gamma_{-2k-p+q}^n(L-1)\right\} \\ &\qquad (i = 0, 1, \cdots, L-2; \\ &\qquad k = i-L+2, i-L+3, \cdots, i)\end{aligned}$$

$$\begin{aligned}\mu_3(i, k, L) &= \{(p, q)|L-1 \leq 2i-q+1 \\ &\qquad \cdot (0 \leq p, q \leq L-1)\} \\ \mu_4(i, k, L) &= \{(p, q)|L-1 \leq 2i-2k-p+1 \\ &\qquad \cdot (0 \leq p, q \leq L-1)\}.\end{aligned}$$

For arbitrary $t \in (0, 1)$, let

$$t = \sum_{j=1}^{\infty} d_j 2^{-j} (d_j = 0, \text{ or } 1) \qquad (11)$$

Define the shift operator τ as

$$\tau t = \sum_{j=2}^{\infty} d_j 2^{-j+1} \tag{12}$$

From (8), one obtains

$$\vec{v}(t) = T_{d_1} \vec{v}(\tau t) + g_{d_1} \tag{13}$$

If $t = 0.d_1 \, d_2 \, \cdots \, d_m$, then by applying (13) repeatedly one gets

$$\vec{v}(t) = T_{d_1}(T_{d_2}(\cdots (T_{d_m} \vec{v}(0) + g_{d_m}) \cdots + g_{d_3}) + g_{d_2}) + g_{d_1} \tag{14}$$

Now the procedure for computing $\Gamma_k^n(x)$at an arbitrary point $(t+i)$ can be described as

1) Compute T_0, T_1, g_0 and g_1;
2) For $t \in (0, 1)$, evaluate m to approximate t as $\sum_{j=1}^{m} d_j 2^{-j}$;
3) Compute $\vec{v}(t) = T_{d_1}(T_{d_2}(\cdots (T_{d_m} \vec{v}(0) + g_{d_m}) \cdots + g_{d_3}) + g_{d_2}) + g_{d_1}$.

The value of $\Gamma_k^n(x)$ at the arbitrary point $(t+i)$ can be determined according to (5) and (6).

B. Computation of $\Lambda_k^{m,n}(x) = \int_0^x y^m \phi^{(n)}(y-k)\phi(y)\,dy$

The determination of this connection coefficient at integer points including points $L-1$ is also reported in [8] with the exception that the normalizing conditions are replaced by

$$\sum_{k=i-L+2}^{i-1} k^n \Lambda_k^{m,n}(i) = n! M_0^m(i) - \sum_{k=2-L}^{i-L+1} k^n \Lambda_k^{m,n}(L-1) \tag{15}$$

Equation (15) is derived in the same way as for (4).

In the derivation of the formulae for computing $\Lambda_k^{m,n}(x) = \int_0^x y^m \phi^{(n)}(y-k)\phi(y)\,dy$ at arbitrary points, one must note that

$$\Lambda_k^{m,n}(x) = \Lambda_k^{m,n}(L-1) \qquad (x \geq L-1, \text{ or } x-k \geq L-1) \tag{16}$$

$$\Lambda_k^{m,n}(x) = 0 \qquad (x \leq 0, \text{ or } |k| \geq L-1, \text{ or } x \leq k) \tag{17}$$

It is very obvious that there are only $(L-1)^2$ independent members $\{\Lambda_k^{m,n}(i+x) | \, 0 \leq i \leq L-2, \, i-L+2 \leq k \leq i\}$ in the set $\Lambda_k^{m,n}(i+x)$ $(i = 0, 1, \cdots, L-2)$ $(|k| \leq L-2))$ for some fixed $x \in (0, 1)$. Let the independent members be expressed in the following compact form

$$\vec{v}(t) = [v(t)\, v(1+t) \, \cdots \, v(L-2+t)]^T \in R^{(L-1)\times(L-1)} \tag{18}$$

$$\begin{aligned} v(i+t) &= \Lambda^{m,n}(i+t) \\ &= [\Lambda_{i-L+2}^{m,n}(i+t) \Lambda_{i-L+3}^{m,n}(i+t) \, \cdots \, \Lambda_i^{m,n}(i+t)] \\ &\quad (i = 0, 1, \cdots, L-2) \end{aligned} \tag{19}$$

then one has

$$\vec{v}(0) = [\Lambda^{m,n}(0)\, \Lambda^{m,n}(1) \, \cdots \, \Lambda^{m,n}(L-2)]^T. \tag{20}$$

According to the following two scale relationships

$$\Lambda_k^{m,n}(x) = 2^{n-m-1} \sum_{i=0}^{L-1} \sum_{j=0}^{L-1} \sum_{l=0}^{m} p_i p_j \cdot \binom{m}{l} j^l \Lambda_{2k-j+i}^{m-l,n}(2x-j) \tag{21}$$

one obtains

$$\vec{v}\left(\frac{t}{2}\right) = T_0 \vec{v}(t) + g_0 + f_0(t) \tag{22}$$

where

$$\begin{aligned} (T_0)_{i,j,k,m} = {} & 2^{n-m-1} p_{2i-j} p_{L-1-2k+m} \\ & (i, j = 0, 1, \cdots, L-2; \\ & k, m = 1, 2, \cdots, L-1) \end{aligned}$$

$$g_0 = [\, g_0(1) \quad g_0(2) \quad \cdots \quad g_0((L-1)(L-1)) \,]^T$$

$$\begin{aligned} & g_0(i[L-1] + [k-(i-L+1)]) \\ & = 2^{n-m-1} \Bigg\{ \sum_{p,q \in \mu_0(i,k,L)} p_p p_q \Lambda_{2k+p-q}^{m,n}(L-1) + \sum_{l=1}^{m} \\ & \qquad \cdot \sum_{p,q \in \mu_0(i,k,L)} p_p p_q \binom{m}{l} q^l \Lambda_{2k+p-q}^{m-l,n}(L-1) \Bigg\} \\ & \cdot (i = 0, 1, \cdots, L-2; \\ & \quad k = i-L+2, i-L+3, \cdots, i) \end{aligned}$$

$$\begin{aligned} & [f_0(t)]_{i[L-1]+[k-(i-L+1)]} \\ & \quad = 2^{n-m-1} \sum_{p=0}^{L-1} \sum_{q=0}^{L-1} \sum_{l=1}^{m} p_p p_q \\ & \qquad \cdot \binom{m}{l} q^l \Lambda_{2k+p-q}^{m-l,n}(2i-q+t) \\ & \qquad (p, q \notin \mu_0(i,k,L)) \, (i = 0, 1, \cdots, L-2; \\ & \qquad k = i-L+2, i-L+3, \cdots, i) \end{aligned}$$

$$\mu_0(i,k,L) = \{(p,q) | L-1 \leq 2i-q \text{ or } L-1 \leq 2i-2k - p (0 \leq p, q \leq L-1)$$

Similarly

$$\vec{v}\left(\frac{t+1}{2}\right) = T_1 \vec{v}(t) + g_1 + f_1(t) \tag{23}$$

where

$$(T_1)_{i,\,j,\,k,\,m} = 2^{n-m-1} p_{2i-j+1} p_{L-2k+m} \quad (i,\, j = 0,\, 1,\, \cdots, L-2;\ k,\, m = 1,\, 2,\, \cdots,\, L-1)$$

$$g_1 = [g_1(1) \quad g_1(2) \quad \cdots \quad g_1((L-1)(L-1))]^T$$

$$\begin{aligned} g_1(i[L-1] + [k-(i-L+1)]) &= 2^{n-m-1} \Bigg\{ \sum_{p,\,q \in \mu_1(i,\,k,\,L)} p_p p_q \Lambda^{m,\,n}_{2k+p-q}(L-1) + \sum_{l=1}^{m} \\ &\quad \cdot \sum_{p,\,q \in \mu_1(i,\,k,\,L)} p_p p_q \binom{m}{l} q^l \Lambda^{m-l,\,n}_{2k+p-q}(L-1) \Bigg\} \\ &\quad (i = 0,\, 1,\, \cdots,\, L-2; \\ &\quad k = i-L+2,\, i-L+3,\, \cdots,\, i) \end{aligned}$$

$$\mu_1(i,\,k,\,L) = \{(p,\,q) | L-1 \le 2i-q+1 \text{ or } 2i-2k-p + 1 (0 \le p,\, q \le L-1)\}$$

$$\begin{aligned} [f_1(t)]_{i[L-1]+[k-(i-L+1)]} &= 2^{n-m-1} \sum_{p=0}^{L-1} \sum_{q=0}^{L-1} \sum_{l=1}^{m} p_p p_q \binom{m}{l} q^l \Lambda^{m-l,\,n}_{2k+p-q} \\ &\quad \cdot (2i-q+1+t) \qquad (p,\, q \notin \mu_1(i,\,k,\,L)) \\ &\quad (i = 0,\, 1,\, \cdots,\, L-2; \\ &\quad k = i-L+2,\, i-L+3,\, \cdots,\, i) \end{aligned}$$

Arbitrarily $t \in (0,\, 1)$ may be expressed as (11). According to (21) one obtains

$$\overrightarrow{v}(t) = T_{d_1} \overrightarrow{v}(\tau t) + g_{d_1} + f_{d_1}(\tau\, t) \tag{24}$$

here the definition of $\tau t$is as given in (12).

If $t = 0.d_1\, d_2, \cdots,\, d_m$, then applying (24) repeatedly yields

$$\begin{aligned} \overrightarrow{v}(t) = & T_{d_1}(T_{d_2}(\cdots (T_{d_m} \overrightarrow{v}(0) + g_{d_m} + f_{d_m}(t_{d_m})) \cdots + g_{d_3} \\ & + f_{d_3}(t_{d_3})) + g_{d_2} + f_{d_2}(t_{d_2})) + g_{d_1} + f_{d_1}(t_{d_1}) \end{aligned} \tag{25}$$

where

$$\begin{aligned} [f_{d_n}(t_{d_n})]_{i[L-1]+[k-(i-L+1)} &= 2^{n-m-1} \sum_{p=0}^{L-1} \sum_{q=0}^{L-1} \sum_{l=1}^{m} p_p p_q \binom{m}{l} q^l \Lambda^{m-l,\,n}_{2k+p-q} \\ &\quad \cdot \left(2i - q + d_n + \sum_{l=n+1}^{m} 2^{n-l}\, d_l \right) \\ &\quad (p,\, q \notin \mu_{d_i}(i,\,k,\,L),\, i = 0,\, 1,\, \cdots,\, L-2; \\ &\quad k = i-L+2,\, i-L+3,\, \cdots,\, i) \end{aligned}$$

Special attention should be paid to the value of t_{d_n} here.

The procedure for computing the values of $\Lambda^{m,\,n}_k(x)$at an arbitrary point is then summarized as

1) Compute $T_0,\, T_1,\, g_0$ and g_1;
2) For $t \in (0,\, 1)$, determine m to approximate t as $\sum_{j=1}^{m} d_j 2^{-j}$;
3) Compute

$$\begin{aligned} \overrightarrow{v}(t) = & T_{d_1}(T_{d_2}(\cdots (T_{d_m} \overrightarrow{v}(0) + g_{d_m} + f_{d_m}(t_{d_m})) \cdots + g_{d_3} \\ & + f_{d_3}(t_{d_3})) + g_{d_2} + f_{d_2}(t_{d_2})) + g_{d_1} + f_{d_1}(t_{d_1}). \end{aligned}$$

The value of $\Lambda^{m,\,n}_k(x)$at an arbitrary point $(t+i)$ can be determined according to (18) and (19).

TABLE I
COMPUTED RESULTS BY USING DIFFERENT NORMALIZATION CONDITIONS FOR THE DAUBECHIES SCALING FUNCTION WITH $L = 6$

k	$\Lambda^{1,1}_k(5)$	
	Proposed method	Reference 8
-4	-0.16773451E-05	-0.39622225E-05
-3	0.68129507E-03	-0.67621931E-03
-2	0.39894768E-01	0.19212883E-03
-1	-0.33984185E+00	-0.12104326E+00
0	-0.50000014E+00	0.10224223E+01
1	0.10850474E+01	-0.12104326E+00
2	-0.33030568E+00	0.19212883E-01
3	0.43154324E-01	-0.67621931E-03
4	0.13716132E-02	-0.39622225E-05

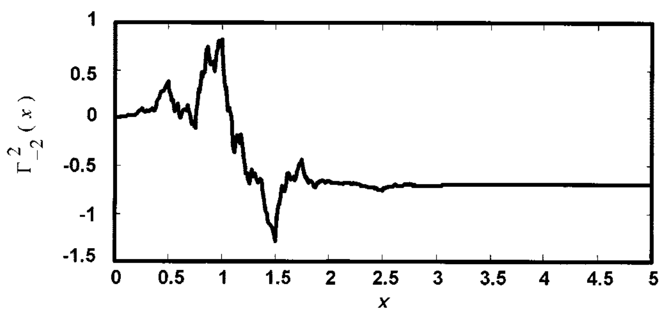


Fig. 1. Computed $\Gamma^2_{-2}(x)$for the Daubechies scaling function with $L = 8$

III. NUMERICAL RESULTS

A. Connection Coefficients

According to the previous formulae and procedures, a software program has been developed to calculate all the aforementioned connection coefficients. Due to the limitation of space, only parts of the results are presented in this paper. Table I presents the computed results of $\Lambda^{1,\,1}_k(5)$ for the Daubechies scaling function with $L = 6$ as well as those given by [8]. Figs. 1 and 2 show some other computed results of connection coefficients. The smoothness of these curves is a good indicator of the validity of the proposed formulae and procedure, especially the new normalization conditions developed in this paper.

B. Numerical Example of Wavelet-Galerkin Method

One of the numerical examples of the wavelet-Galerkin method is reported [9]. This section presents another example on the computation of magnetic fields of a typical U-magnet as shown in Fig. 3(a). Due to the geometrical symmetry, only

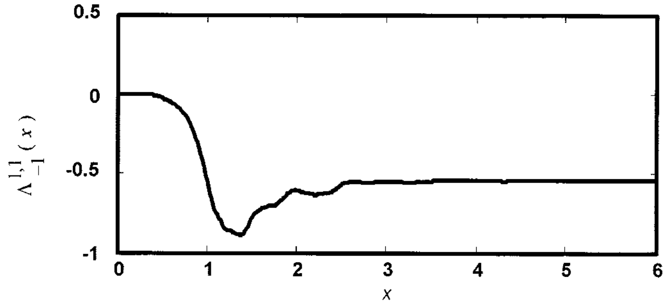
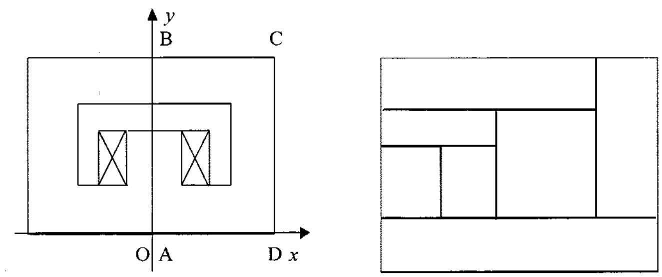


Fig. 2. Computed $\Lambda_{-1}^{1,1}(x)$ for the Daubechies scaling function with $L = 8$



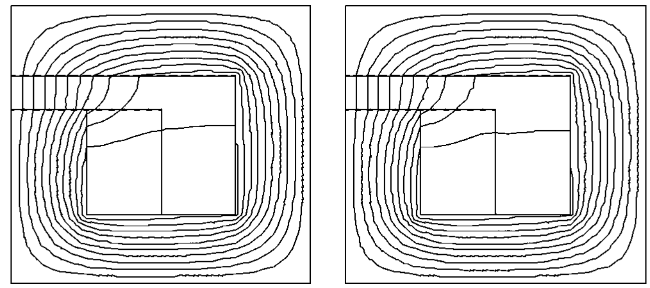
(a) (b)

Fig. 3. (a) The U-Magnet and (b) the wavelet-Galerkin method mesh

half of the actual region, i.e., the region enclosed by ABCD in Fig. 3(a), needs to be analyzed. This fields are governed by the vector form of Poisson equation after introducing the magnetic vector potential A. By assuming an infinite length in the longitudinal direction, this vector Poisson equation degenerates to its scalar counterpart, and the boundary value problem is formulated as

$$\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} = -\mu J \qquad A|_{ABCD} = 0 \tag{26}$$

The solutions of this U-Magnet problem using both the wavelet-Galerkin method with the mesh of Fig. 3(b) and $L = 6$, and that by FEM are given in Fig. 4. The orders of the linear equation sets for the proposed method and that of FEM are, respectively, 1000 and 1106. The CPU time required by both methods are almost the same, i.e., about 20 seconds on an Acer 586 computer. Please note that a very simple "mesh" is needed for the present method. These computed results reveal again that: 1) the Wavelet-Galerkin method is virtually a meshless method, and this is very promising for 3D problems and 2) wavelet-Galerkin method is a strong contender to conventional FEM for case where boundaries are oblique at the present stage.



(a) (b)

Fig. 4. Solution of U-Magnet. The equipotentials of A corresponding to magnetic flux lines, (a) by wavelet-Galerkin method, (b) by FEM

IV. Conclusion

This paper details the mathematical formulae and the corresponding computer procedures for the exact determination of arbitrary point values of some typical connection coefficients encountered in wavelet-Galerkin method. These results play an essential role in extending the wavelet-Galerkin method to solve general field problems in electromagnetics. One should note that the computation of these coefficients is done once and for all. The numerical example given in this paper has also demonstrated the potential of the wavelet-Galerkin method in solving electromagnetic field problems.

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