Weak Formulation of Finite Element Method Using Wavelet Basis Functions

S. L. Ho, Shiyou Yang, and H. C. Wong

***Abstract*—This paper details the development of the weak form formulations of finite element type methods using wavelets as basis functions. Such approaches are different from most wavelets based ones that are derived from the strong form. The advantages of the proposed formulation are that there is no need to enforce natural boundary conditions and that the lower order derivatives of the wavelet bases are involved in the connection coefficients. Various approaches to deal with essential boundary and interface conditions are investigated, and algorithms to compute the associated connection coefficients are derived. To validate the proposed method, two numerical examples are described.**

***Index Terms*—Connection coefficient, Galerkin approach, wavelet bases, wavelet-Galerkin method, weak form.**

I. Introduction

SUBSEQUENT to the success of the applied mathematicians to use Wavelet Basis Functions (WBFs) as a new tool for local time–frequency analysis, the electrical engineers have also reported success in using WBFs to solve electromagnetic problems [1]–[3]. However, most dedicated formulae are derived from the strong form and are unfortunately rather difficult to enforce the natural boundary conditions. Moreover, as the higher order differentials of the basis function are involved in the connection coefficients in the strong form, the connection coefficients do vary greatly because of the oscillatory nature of the differentials of the wavelet bases, thereby resulting in numerical instability.

As it is well known, there are two main obstacles in applying wavelet-Galerkin methods to solve differential and integral equations; namely (1) the difficulty to impose boundary conditions, since both scaling and wavelet functions do not satisfy the Kronecker delta criterion; (2) the complexity in computing the connection coefficients, because each equation requires a different type of connection coefficients that are dictated by the various terms in the equation. Moreover, the implicit representation of both scaling and wavelet functions as well as their high oscillatory natures can give rise to programming complexities. This paper will demonstrate how one could develop a weak form formulation of the finite element method by using the wavelets as basis functions, which will alleviate, at least partly, some of these difficulties. Algorithms to compute the corresponding connection coefficients are also described.

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II. Formulation

For general engineering problems, a Poisson equation in a two-dimensional region Ω enclosed by a boundary Γ is:

$$\Omega: \beta\frac{\partial^2 u}{\partial x^2}+\beta\frac{\partial^2 u}{\partial y^2}=-f. \quad (1)$$

$$\Gamma_2: \beta\frac{\partial u}{\partial n}=g. \quad (2a)$$

$$\Gamma_1: u=u_0. \quad (2b)$$

where $\Gamma=\Gamma_1\cup\Gamma_2$.

A. General Formulation

The residual of (1) is

$$R=\beta\frac{\partial^2 u}{\partial x^2}+\beta\frac{\partial^2 u}{\partial y^2}+f. \quad (3).$$

The general form of the weighted residual then becomes

$$\iint_\Omega wR\,dxdy=\iint w(x,y)\left(\beta\frac{\partial^2 u}{\partial x^2}+\beta\frac{\partial^2 u}{\partial y^2}+f\right)dxdy =0. \quad (4)$$

where $w(x,y)$ is the weighting function.

Selecting the scaling function $\phi(2^m x-i)\phi(2^m y-j)$ as both the shape and weighting functions, we approximate $u(x,y)$ by

$$u(x,y)=\sum_i\sum_j s_{i,j}\phi(2^m x-i)\phi(2^m y-j). \quad (5)$$

where

m	is the resolution,
i and j	are integers, and
$\phi_i^J(z)=2^{J/2}\phi(2^J z-i)$	is the one dimensional Daubechies' scaling function,

and it can be determined from the following two scale relation

$$\phi(x)=\sum_{k=0}^{L-1}p_k\phi(2x-k). \quad (6)$$

where L is a finite even integer.

One then reads

$$\iint_\Omega \phi(2^m x-p)\phi(2^m y-q)\left(\beta\frac{\partial^2 u}{\partial x^2}+\beta\frac{\partial^2 u}{\partial y^2}+f\right)dxdy =0. \quad (7)$$

where p, q are also integers.

Transforming the second order differentials in (7) into first order ones by means of integration by parts, one gets

$$\begin{aligned}&2^{2m}\beta\sum_i\sum_j s_{i,j}\\&\times\left[\int\left(\phi_i^m(x)\right)^{(1)}\left(\phi_p^m(x)\right)^{(1)}dx\right.\\&\qquad\times\int\phi_j^m(y)\phi_q^m(y)dy+\int\phi_i^m(x)\phi_p^m(x)dx\\&\qquad\left.\times\int\left(\phi_j^m(y)\right)^{(1)}\left(\phi_q^m(y)\right)^{(1)}dy\right]\\&=\int_{\Gamma_2}\phi_p^m(x)\phi_q^m(y)gds+\int_{\Gamma_1}\beta\frac{\partial u}{\partial n}\left[\phi_p^m(x)\phi_q^m(y)\right]ds\\&\quad+\iint_\Omega f\phi_p^m(x)\phi_q^m(y)dxdy.\end{aligned}\tag{8}$$

Then (8) can be written as

$$\sum_{i,j}k_{p,q;i,j}s_{i,j}=p_{p,q}.\tag{9}$$

where

$$\begin{aligned}k_{p,q;i,j}=&\beta\left[\int\phi^{(1)}(X-i)\phi^{(1)}(X-p)dX\right.\\&\times\int\phi(Y-j)\phi(Y-q)dY\\&+\int\phi(X-i)\phi(X-p)dX\\&\left.\times\int\phi^{(1)}(Y-j)\phi^{(1)}(Y-q)dY\right].\end{aligned}\tag{10}$$

$$\begin{aligned}p_{p,q}=&\iint_{\Gamma_2}\phi_p^m(x)\phi_q^m(y)gds\\&+\int_{\Gamma_1}\beta\frac{\partial u}{\partial n}\left[\phi_p^m(x)\phi_q^m(y)\right]ds\\&+\iint_\Omega f\phi_p^m(x)\phi_q^m(y)dxdy.\end{aligned}\tag{11}$$

For a rectangular element $\boldsymbol{e}$ with sides parallel to the x- and y-axes with x varying between x_1 to x_2 and y within the limits of y_1 to y_2, the contributions of the element $\boldsymbol{e}$ are

$$\begin{aligned}(k_{p,q;i,j})_e=&\beta\left[\int_{X_1}^{X_2}\phi^{(1)}(X-i)\phi^{(1)}(X-p)dX\right.\\&\times\int_{Y_1}^{Y_2}\phi(Y-j)\phi(Y-q)dY\\&+\int_{X_1}^{X_2}\phi(X-i)\phi(X-p)dX\\&\left.\times\int_{Y_1}^{Y_2}\phi^{(1)}(Y-j)\phi^{(1)}(Y-q)dY\right]\\(Z_i=&2^mz_i;\quad z=x,y;\ \ i=1,2).\end{aligned}\tag{12}$$

$$\begin{aligned}p_{p,q}=&\int_{(\Gamma_2)_e}\phi_p^m(x)\phi_q^m(y)gds\\&+\int_{(\Gamma_1)_e}\beta\frac{\partial u}{\partial n}\left[\phi_p^m(x)\phi_q^m(y)\right]ds\\&+\int_{x_1}^{x_2}\phi_p^m(x)dx\int_{y_1}^{y_2}f\phi_q^m(y)dy.\end{aligned}\tag{13}$$

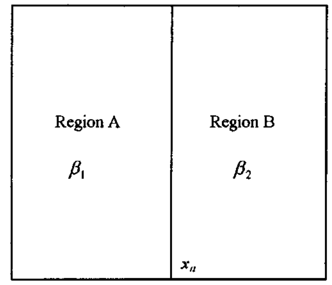

Fig. 1. Schematic diagram of interfaces.

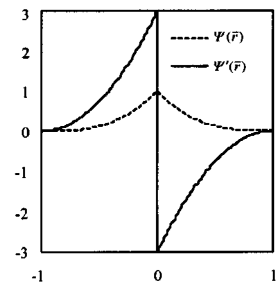

Fig. 2. Jump function and its derivative.

B. Interface Condition Approximation

While the high order continuity that the wavelet-Galerkin method inherits from the wavelet/scaling functions can be considered a blessing because it provides solutions with smooth derivatives, it can also be a disadvantage in engineering problems involving multiple materials (i.e., where the coefficients in the PDE are discontinuous along the interface of different materials), because this will result in solutions with discontinuous derivatives along the interface. This discontinuity in the derivatives introduces spurious oscillations, and a special technique is required to approximate this derivative discontinuity. Here the idea of a jump function approach, which is similar to that used in meshless methods [4], is introduced in the proposed method.

This approach introduces additional shape functions, the jump functions, in sub-regions where discontinuities appear. For example, if the interface of two different materials coincides with line $x=x_a$, as shown in Fig. 1, the approximation of (5) is:

$$\begin{aligned}u(x,y)=&\sum_i\sum_j s_{i,j}\phi_i^m(x)\phi_j^m(y)\\&+\sum_j b_j\phi_j^m(y)\Psi(x-x_a).\end{aligned}\tag{14}$$

where $\Psi(x-x_a)$ is the jump function which will generate the discontinuous (normal) derivatives along the interface, and b_j will be determined in the same way as that for $s_{i,j}$. The jump function used in this paper is of the form

$$\Psi(\bar{r})=\begin{cases}-\bar{r}^3+3\bar{r}^2-3\bar{r}+1 & (\bar{r}\le 1)\\0 & \text{(otherwise)}.\end{cases}\tag{15}$$

where $\bar{r}=|x-x_a|/r_{\max}$, $r_{\max}$ is a measure to control the support of the jump function.

Fig. 2 shows the typical shapes of the jump function and its derivatives for the case of $r_{\max}=1$ and $x_a=0$. From this figure it can be seen that the jump function generates discontinuous derivatives while ensuring the continuity of the approximated functions.

C. Enforcement of Essential Boundary Conditions

Due to the fact that both the scaling and wavelet functions do not satisfy the Kronecker delta criterion, the enforcement of the boundary conditions for wavelet-Galerkin type methods is quite awkward when compared with other types of finite element methods. Fortunately, the natural boundary conditions are included in the discretized formulations for the proposed method, thus only a special treatment is needed for the essential boundary conditions. For the solution $u(x,y)$ to satisfy the essential boundary condition (2b), the coefficients $s_{i,j}$ and b_j must also satisfy the following relations

$$\sum_i \sum_j s_{i,j}\phi_i^m(x_k)\phi_j^m(y_k) + \sum_j b_j \phi_j^m(y_k)\Psi(x_k - x_a) = (u_0)_k \quad (k = 1,2,\ldots,N_{\Gamma_1}). \tag{16}$$

where N_{Γ_1} is the node number in boundary Γ_1, and (x_k, y_k) is the coordinates of the kth node.

As a result, the number of the equations of (9) and (16) is greater than the freedom of the coefficients. The least squares technique is used to determining the coefficients uniquely.

III. Computation of Connection Coefficients

As described earlier in this paper, if one applies wavelet-Galerkin methods to solve both differential and integral equations, one requires different types of connection coefficients, such as those formulated in (10)–(13). In addition to the connection coefficients reported in [5], the following ones are also required for the proposed method, i.e.,

$$M_k^m(x) = \int_0^x y^m \phi(y-k)dy. \tag{17}$$

$$\Theta_k^{m,n}(x) = \int_0^x \phi^{(m)}(y-k)\phi^{(n)}(y)dy. \tag{18}$$

A. Algorithm

This section will develop algorithms and procedures for computing the values of (17) and (18) at arbitrary points for the reasons as were described in [5].

1) $M_k^m(x) = \int_0^x y^m \phi(y-k)dy$

The algorithm for computing $M_k^m(x)$ has being divided into two steps.

Step 1) Compute $M_0^m(x)$

The values of $M_0^m(x)$ can be determined by [6]

$$M_0^m(x) = \sum_{i=0}^m (-1)^i \frac{m!}{(m-i)!} x^{m-i}\theta_{i+1}(x). \tag{19}$$

where

$$\theta_n(x) = \overbrace{\int_0^x \int_0^{y_n} \cdots \int_0^{y_2}}^{n\text{-tuple}} \phi(y_1)dy_1 \cdots dy_{n-1}dy_n. \tag{20}$$

Step 2) Compute $M_k^m(x)$ for $k \neq 0$

For any $k \neq 0$, the values of $M_k^m(x)$ can be determined using $M_k^m(x)$ according to the following relationship

$$M_k^m(x) = \int_0^x y^m \phi(y-k)dy = \int_{-k}^{x-k} (y+k)^m \phi(y)dy = \sum_{i=0}^m C_m^i k^{m-i}\left[M_0^i(x-k) - M_0^i(-k)\right]. \tag{21}$$

where $C_m^i = m(m-1)\cdots(m-i+1)/i!$.

Thus the computation of $M_k^m(x)$ at arbitrary point values is dominated by that of $\theta_n(x)$. The values of $\theta_n(x)$ at dyadic points and points $x > L-1$ are given in [6]. The following only describes the computations of $\theta_n(x)$ at arbitrary points $(t+j-1)$ $(t \in [0,1], j = 1,2,\ldots,L-1)$. In doing that, one can define a vector $\bar{v}(t) \in R^{L-1}$ $(t \in (0,1)$ such that

$$\bar{v}(t) = [\theta_n(t)\theta_n(t+1)\cdots\theta_n(t+L-2)]^{\mathrm{T}}. \tag{22}$$

$$v_j(t) = \theta_n(t+j-1) \quad (j = 1,2,\ldots,L-1). \tag{23}$$

Then one has $\vec{v}(0) = [\theta_n(0)\theta_n(1)\cdots\theta_n(L-2)]^{\mathrm{T}}$.

According to the following two scale relationship

$$\theta_n(x) = 2^{-n}\sum_{k=0}^{L-1} p_k \theta_n(2x-k). \tag{24}$$

one obtains

$$\vec{v}\left(\frac{t}{2}\right) = T_0\vec{v}(t) + f_0(t). \tag{25}$$

$$\vec{v}\left(\frac{t+1}{2}\right) = T_1\vec{v}(t) + f_1(t). \tag{26}$$

where,

$$(T_0)_{i,j} = 2^{-n}p_{2i-j-1}, \quad (T_1)_{i,j} = 2^{-n}p_{2i-j} \quad (i,j = 1,2,\ldots,L-1),$$

$$[f_0(t)]_i = \sum_{\substack{2i-k-2\geq L-1 \\ i=1,2,\ldots,L-1 \\ k=0,1,\ldots,L-1}} 2^{-n}p_k\theta_n(2i-2-k+t), \text{ and}$$

$$[f_1(t)]_i = \sum_{\substack{2i-k-1\geq L-1 \\ i=1,2,\ldots,L-1 \\ k=0,1,\ldots,L-1}} 2^{-n}p_k\theta_n(2i-1-k+t).$$

Thus for an arbitrary $t \in (0,1)$, one could assume its approximation of the dyadic expansion be

$$t = \sum_{j=1}^{\infty} d_j 2^{-j}. \tag{27}$$

where $d_j = 0$ or 1.

If one defines a shift operator

$$\tau t = \sum_{j=2}^{\infty} d_j 2^{-j+1}. \tag{28}$$

Application of the two scale relation of (24)–(27) yields

$$\bar{v}(t) = T_{d_1}\bar{v}(\tau\, t) + f_{d_1}(\tau\, t). \tag{29}$$

If $t = 0, d_1, d_2, \ldots, d_m$, then the application of (29) m times yields

$$\bar{v}(t) = T_{d_1}\left(T_{d_2}\left(\cdots\left(T_{d_m}\bar{v}(0) + f_{d_m}\left(t_{d_m}\right)\right)\cdots + f_{d_3}\left(t_{d_3}\right)\right) + f_{d_2}\left(t_{d_2}\right)\right) + f_{d_1}\left(t_{d_1}\right). \tag{30}$$

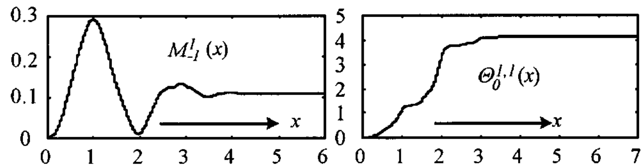

Fig. 3. The computed results of some typical connection coefficients.

where

$$[f_{d_n}(t_{d_n})]_i = \sum_{\substack{2i-2-k+d_i \geq L-1 \\ i=1,2,\ldots,L-1 \\ j=0,1,\ldots,L-1}} 2^{-n} p_k \theta_n \times \left(2i-2-k+d_n+\sum_{l=n+1}^{m} 2^{n-l} d_l\right).$$

According to (30), the procedure for computing $\theta_n(x)$ at arbitrary points $(t+j-1)$ $(t \in [0,1], j=1,2,\ldots,L-1)$ is

1) compute T_0, T_1;
2) for $t \in (0,1)$, determine the value of m to approximate t for $t \approx \sum_{j=1}^{m} d_j 2^{-j}$ with suitable precision;
3) compute

$$\bar{v}(t) = T_{d_1}\left(T_{d_2}\left(\cdots(T_{d_m}\bar{v}(0) + f_{d_m}(t_{d_m}))\cdots + f_{d_3}(t_{d_3})\right) + f_{d_2}(t_{d_2})\right) + f_{d_1}(t_{d_1}).$$

Then the value of $\theta_n(x)$ at an arbitrary point $(t+j-1)$ can be determined from $\vec{v}(t)$ according to (22)–(23).

$$2)\ \Theta_k^{m,n}(x) = \int_0^x \phi^{(m)}(y-k)\phi^{(n)}(y)dy$$

Performing integration by parts successively for n times on (18) yields

$$\Theta_k^{m,n}(x) = \sum_{i=0}^{n-1}(-1)^i \phi^{(m+i)}(x-k)\phi^{(n-i-1)}(x) + (-1)^n \Gamma_k^{m+n}. \tag{31}$$

where

$$\Gamma_k^n(x) = \int_0^x \phi^{(n)}(y-k)\phi(y)dy. \tag{32}$$

So the computation of $\Theta_k^{m,n}(x)$ is replaced by that of $\Gamma_k^n(x)$ and $\phi^{(n)}(x)$ as reported in [5].

B. Numerical Results

Some typical numerical results of the aforementioned connection coefficients for the Daubechies scaling function with $L = 8$ computed by using the proposed algorithms and procedures are given in Fig. 3.

IV. Numerical Examples

Two numerical examples are investigated to validate the proposed method.

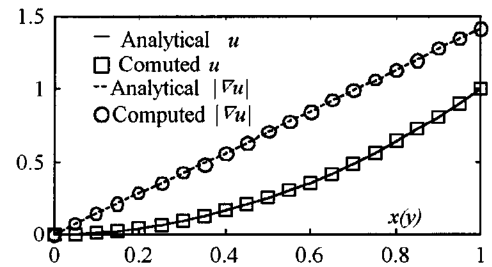

Fig. 4. The comparison of analytical and numerical solutions in line $x = y$.

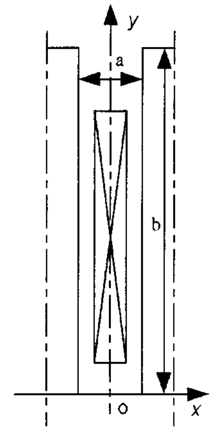

Fig. 5. Electrical machine rotor, filled by current-carrying conductor.

A. A 2-D Test Problem with Analytical Solutions

This simple example presents the comparison of the numerical results obtained using the proposed method with analytical ones. The problem is to find the solution of

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0 \quad (0 \leq x, y \leq 1) \\ u|_{x=0} &= u|_{y=0} = 0, \quad u|_{x=y=1} = 1.\end{aligned} \tag{33}$$

The exact solution of (33) is $u = xy$. Fig. 4 gives the comparison of the numerical and the analytical solutions on the line $x = y$. In the numerical computations, one uses the Daubeiches scaling function with $L = 6$; $m = 0$ and i, j both varying from -4 to 0, resulting total 25 basis functions to represent the shape and weighting functions. From these results, it can be seen that only 25 functions can approximate the solution with acceptable precisions.

B. A 2-D Magnetostatic Problem

The second example is to investigate the feasibility of the proposed method in solving electromagnetic field problems. The problem is to calculate the magnetic fields of an electrical conductor lying in a rotor slot, as shown in Fig. 5. Because of symmetry, only half of the region is analyzed. The boundary value problem is then formulated as

$$\begin{aligned}\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} &= -\mu J_z \\ \left.\frac{\partial A_z}{\partial x}\right|_{x=0} &= \left.\frac{\partial A_z}{\partial x}\right|_{x=a/2} = \left.\frac{\partial A_z}{\partial y}\right|_{y=0} = 0, \\ A_z|_{y=b} &= 0.\end{aligned} \tag{34}$$

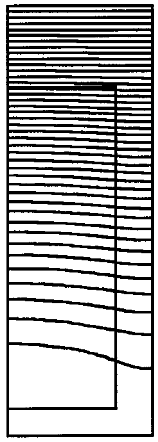

Fig. 6. Equipotential lines of the electrical machine rotor problem.

The solution of this problem by using the proposed wavelet-Galerkin method with Daubechies scaling function of $L = 6$ is given in Fig. 6. Only four rectangular elements are needed for this specific problem. In the numerical calculation, the resolution parameter is set to 2, and only ten equi-distance sample points along the boundary $y = b$ are used to enforce the zero essential boundary condition. The total linear equation set is of the order of 1126, and the CPU time needed to compute the solution is about 38 seconds on a Pentium III 600 MHz Clock. These computed results show the advantages of the proposed method over those wavelet formulations based on strong form, in that the proposed method requires no special treatment when imposing the natural boundary conditions, thus resulting in simplicity in both computation and programming.

V. Conclusion

A weak form formulation, including the technique for dealing with discontinuous derivatives and the approach for enforcing essential boundary conditions, of wavelet-Galerkin methods is derived in this paper. The computations of the associated connection coefficients at arbitrary point values are also investigated. Numerical results on both the test and a practical field problem demonstrate the feasibility of the proposed method to study practical field problems.

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