Refinement Computations of Electromagnetic Fields Using FE and Meshless Methods

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A refinement algorithm for electromagnetic field computations using a combination of finite element and meshless methods is introduced. Bridging scales are used to separate the finite element and meshless shape functions to make the refinement hierarchical and to uphold the mathematical properties such as consistency and linear independence for all the bases. To facilitate the application of the proposed algorithm, details about the node addition, requirements for the node distribution, and relationships between the finite element and meshless shape functions, as well as the determination of the stop criterion are also fully addressed. Primary numerical results are reported to demonstrate and validate the applicability and advantages of the proposed algorithm over traditional ones.

Index Terms—**Bridge scales, meshless method, moving least squares approximation, refinement computation.**

I. Introduction

ALTHOUGH the finite element (FE) method is well developed, widely trusted, and used as the most powerful numerical tool for solving boundary value problems in engineering, it is not equally successful in every electromagnetic field problem. In optimization and nondestructive evaluation simulation studies with geometrical deformations, for example, an adaptive remeshing producer is usually required in order to uphold the accuracy and numerical convergence of the solutions. However, the mesh generation is an extremely time consuming and expensive task when compared to the assembly and solution of FE equations [1]. Therefore, it is highly desirable to explore methods which will alleviate, at least partly, the onerous mesh generation or adaptive updating process. Accordingly, many meshless methods, all of which originate from computational mechanics, have been proposed and proved to be very promising in the study of electromagnetic field problems [2]–[5]. Since the interpolation of meshless methods is based on a set of nodes, and a connectivity of elements is not required, they offer the flexibility of additions and deletions of a set of nodes, which may be distributed in the solution domain irregularly, with relative ease. Even meshless methods were originally designed to alleviate the burdensome meshing or adaptive meshing requirements, they are now being used increasingly in the development of efficient adaptive strategies [6]–[8]. In this paper, the ideas proposed by Wagner *et al.* for enforcing boundary conditions of meshless methods using FE methods are generalized in the development of a simple yet efficient hierarchical refinement procedure of FE solutions. Compared with the original procedure of [6], the proposed algorithm requires no *a priori* knowledge about the localized behaviors of the solution variable. In contrary to other refinement approaches, the most salient characteristic of the proposed method is that no further meshing process is required, making the algorithm ideal for producing very accurate solutions from a coarse mesh and such characteristics are particularly useful for three-dimensional (3-D) problems. Primary numerical results are used to test and validate the proposed algorithm.

II. Combined Meshless and FE Methods

Meshless methods are notorious when it comes to enforcing boundary and interface conditions when solving engineering boundary value problems. On the other hand, FE methods can implement boundary and interface conditions very readily. To make full use of the FE and meshless methods, a combined FE and element-free Galerkin (EFG) method is proposed.

A. A Brief Introduction of Meshless Methods

Although the EFG method is well documented in literature, a brief introduction about this method, which is based on the moving least squares approximation, is firstly reviewed to make the paper self-contained.

For any function $u(X) \in \Omega$, its local approximation using the moving least-squares approximation can be given as

$$u_L^h(X) = \sum_{j=0}^{m} p_j(X) a_j(X) = p^T(X) a(X) \tag{1}$$

where the unknown parameters $a_j(X)$ will vary with X and $p(X)$ is the basis of a complete polynomial of order m.

For two-dimensional (2-D) problems, $X^T = [x, y]$ and a quadratic basis $p^T(X) = [1, x, y, xy, x^2, y^2]$ is used.

By minimizing some weighted discrete L_2 norms, one can determine the unknown parameters, $a(X)$, as

$$a(X) = A^{-1}(X) B(X) c \tag{2}$$

where $A(x) = \sum_{I=1}^{n} w(X - X_I) p(X_I) p^T(X_I)$, n is the number of nodes (particles) in Ω, w is a compactly supported weight function, $[B(X)]_I = w(X - X_I) p(X_I)$, $c^T = [c_1\ c_2\ \cdots\ c_n]$, and c_I refers to the nodal parameter of $u(X)$ at $X = X_I$.

Substituting (2) into (1) yields

$$u^h(x) = \sum_{I=1}^{n} \Phi_I(x) c_I = \Phi(x) c \tag{3}$$

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where the shape function $\Phi_I(x)$ of the EFG method is given as

$$\Phi_I(x) = p^T A^{-1} B_I. \tag{4}$$

B. Combined Model of FE and EFG Using Bridge Scales

For elements of the FE solutions where refinement computations are required, some "nodes" (particles) are firstly added to the elements and the general form of the approximations of the solution variable $u(x,y)$ becomes

$$u(x,y) = \sum_i u_i N_i^{\mathrm{FEM}}(x,y) + \sum_j \Phi_j^{\mathrm{EFG}}(x,y) c_j \tag{5}$$

where $\sum_i u_i N_i^{\mathrm{FEM}}(x,y)$ and $\sum_j \Phi_j^{\mathrm{EFG}}(x,y)c_j$ are, respectively, the FE and EFG refinement solutions of the solution variable; Φ_j^{EFG} is the EFG shape function related to particle j.

To uphold the mathematical properties of the entire bases regarding consistency and linear independence, the bridge scales as proposed in [6] are used. The basic concept of the bridging scales is based on a hierarchical decomposition of a function u which is dependent on some projection operator P to represent, for example, the projection of u onto the span of the FE shape functions. To add a refinement function w to this space, one employs the property of a projection operator such that multiple projections of the function will leave the function unchanged, i.e., $PPu = Pu$. Using this concept, the total function u of (5) can now be rewritten as

$$u = Pu + w - Pw \tag{6}$$

where Pw is the bridging scale term.

The objective to include the bridging scale term is to make the refinement term, $w - Pw$, to contain only the parts of u which are not included in Pu, thereby ensuring a hierarchical decomposition of u such that it can be easily demonstrated by taking the projection operator P to both sides of (6) to give

$$Pu = PPu + Pw - PPw = Pu. \tag{7}$$

For the case study, the FE solutions are being refined in some specific regions, Pu, w, and Pw become, respectively, as

$$Pu = \sum_i u_i N_i^{\mathrm{FEM}}(x,y) \tag{8}$$

$$w = \sum_j \Phi_j^{\mathrm{EFG}}(x,y) c_j \tag{9}$$

$$Pw = \sum_j \sum_i N_i^{\mathrm{FEM}}(x,y) \Phi_j^{\mathrm{EFG}}(x_i,y_i) c_j. \tag{10}$$

Substituting (8)–(10) into (6), one obtains

$$u(x,y) = \sum_i u_i N_i^{\mathrm{FEM}}(x,y) + \sum_j \overline{\Phi}_j^{\mathrm{EFG}}(x,y) c_j \tag{11}$$

where $\overline{\Phi}_j^{\mathrm{EFG}}$ is the modified EFG shape function based on bridging scale, and is defined by

$$\overline{\Phi}_j^{\mathrm{EFG}}(x,y) = \Phi_j^{\mathrm{EFG}}(x,y) - \sum_i N_i^{\mathrm{FEM}}(x,y) \Phi_j^{\mathrm{EFG}}(x_i,y_i). \tag{12}$$

C. Convergence Analysis

A general convergence analysis of the combined FE and EFG methods is referred to [9]. In order to avoid problems of possible poor matrix conditioning which results from significant differences in the ratio between the values of the quantities in the submatrix K and those in the submatrix L of the stiffness matrix in (14) (see the next section), the iterative solution procedure that decouples the FE and the wavelet systems of [10] is used in the proposed algorithm. However, to guarantee the regularity of matrix A of (2), it is recommended that, for the proposed method in which the FE solution is refined by EFG method, the order of $p(X)$, i.e., m, in (1) must be greater than that of the FE shape function [7].

D. Discrete Equation

For illustrative purposes, one considers the following 2-D Poisson problem:

$$\begin{aligned} &\Omega : \beta \frac{\partial^2 u}{\partial x^2} + \beta \frac{\partial^2 u}{\partial y^2} = -f \\ &\Gamma_D : u = u_0, \\ &\Gamma_N : \beta \frac{\partial u}{\partial n} = q. \end{aligned} \tag{13}$$

Based on the weak form of (13), by using the approximation of (11) and a Galerkin approach, one can obtain the final discrete equations as

$$\begin{bmatrix} K & M \\ M^T & L \end{bmatrix} \begin{Bmatrix} U \\ C \end{Bmatrix} = \begin{Bmatrix} F \\ G \end{Bmatrix} \tag{14}$$

where

$$k_{ij} = \int_\Omega \int \left\{ \beta \left[\left(N_j^{\mathrm{FEM}}\right)_x \left(N_i^{\mathrm{FEM}}\right)_x + \left(N_j^{\mathrm{FEM}}\right)_y \left(N_i^{\mathrm{FEM}}\right)_y \right\} dxdy \tag{15}$$

$$f_i = \int_\Omega \int f N_i^{\mathrm{FEM}} dxdy + \int_{\Gamma_N} q N_i^{\mathrm{FEM}} ds \tag{16}$$

$$m_{ji} = \int_\Omega \int \beta \left[\left(\overline{\Phi}_j^{\mathrm{EFG}}\right)_x \left(N_i^{\mathrm{FEM}}\right)_x + \left(\overline{\Phi}_j^{\mathrm{EFG}}\right)_y \left(N_i^{\mathrm{FEM}}\right)_y \right] dxdy \tag{17}$$

$$l_{ij} = \int_\Omega \int \beta \left[\left(\overline{\Phi}_i^{\mathrm{FEM}}\right)_x \left(\overline{\Phi}_j^{\mathrm{FEM}}\right)_x + \left(\overline{\Phi}_i^{\mathrm{FEM}}\right)_y \left(\overline{\Phi}_j^{\mathrm{FEM}}\right)_y \right] dxdy \tag{18}$$

$$f_i = \int_\Omega \int f N_i^{\mathrm{FEM}} dxdy + \int_{\Gamma_N} q N_i^{\mathrm{FEM}} ds \tag{19}$$

$$g_i = \int_{\Gamma_N} \overline{\Phi}_i^{\mathrm{EFG}} q ds + \int_\Omega \int f \overline{\Phi}_i^{\mathrm{EFG}} dxdy. \tag{20}$$

III. Adaptive Strategy

To take full advantages of both FE and meshless methods, the refinement computation using the EFG method is activated only in regions where high and sharp gradients may occur. Thus the proposed algorithm allows one to deal with boundary and interface conditions in a manner which is as simple as that in pure FE methods. In other words, the proposed method does not require any special techniques to deal with essential and interface conditions normally required by other meshless methods.

A. Error Estimation

In the proposed algorithm, one uses an error estimation for each element of the FE solution in order to identify the elements that require refinements. Thus a post local error estimation approach, the field difference method [11], is used because of its simplicity in numerical implementations.

B. Strategy of Refinement

Once the elements are identified, a refinement computation procedure as described below will be activated.

Step 1: Activate the particle addition procedure to add nodes in the identified elements.
Step 2: Implement the EFG refinement computation.
Step 3: Termination test. If the test is passed, stop; otherwise, go to Step 1 for a finer refinement computation.

Although the distribution of the particles of the EFG method may have some irregularities, for each point in the domain where the EFG has influence, the particles whose influences cover this point must form at least one nondegenerated n-simplex (n being the dimension size) [7]. Thus, it is worth pointing out that when there is only one element in a subregion where the FE solution is required for refinement computations, the particles that are added in the first iterative step should be large enough to satisfy this admissible condition.

C. Stop Criterion

To stop the refinement process automatically, a criterion based on that for the wavelet solutions proposed in [12] is extended and used. Assume that the refinement solutions at some subregions at steps J and $J+1$ are, respectively,

$$u^J(x,y) = \sum_i u_i^J N_i^{\mathrm{FEM}}(x,y) + \sum_j \overline{\Phi}_j^{\mathrm{EFG}}(x,y) c_j^J \tag{21}$$

$$u^{J+1}(x,y) = \sum_i u_i^{J+1} N_i^{\mathrm{FEM}}(x,y) + \sum_j \overline{\Phi}_j^{\mathrm{EFG}}(x,y) c_j^{J+1}. \tag{22}$$

The corresponding wavelet solutions are defined as

$$\varphi^J(x,y) = u^{J+1}(x,y) - u^J(x,y). \tag{23}$$

Then the ratio of the rms value of $\varphi^J(x,y)$ to that of the averaged $u(x,y)$ is used as an indicator to stop the refinement procedure. Once this value reaches a threshold as specified by the user, the refinement procedure will stop the refinement process in the specific element.

IV. Numerical Examples

The computation of the end fields of a power transformer as shown in Fig. 1 is selected as the numerical example to demonstrate the applicability and advantages of the proposed procedure over the traditional approaches. The corresponding boundary value problem is formulated as

$$\varepsilon \frac{\partial^2 \varphi}{\partial x^2} + \varepsilon \frac{\partial^2 \varphi}{\partial y^2} = 0 \tag{24}$$

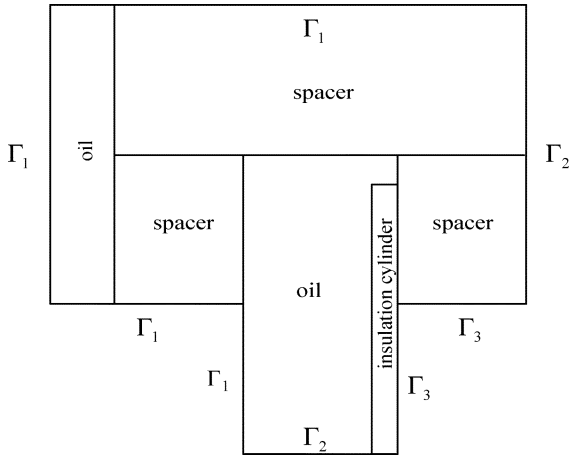

Fig. 1. Schematic diagram of the end region of a power transformer.

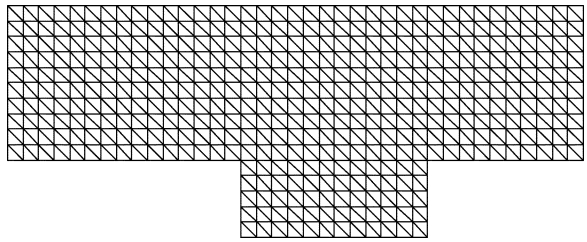

Fig. 2. Finite element mesh used for the proposed FE+EFG method.

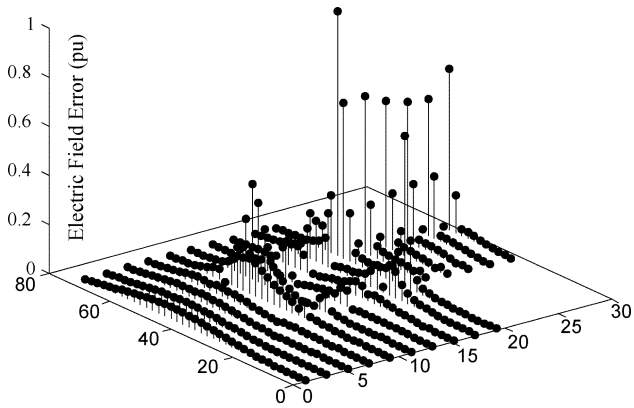

Fig. 3. Error distribution for each element using the field difference method.

$$\varphi|_{\Gamma_1} = 0 \quad \varphi|_{\Gamma_3} = 1 \quad \frac{\partial \varphi}{\partial n}|_{\Gamma_2} = 0. \tag{25}$$

In the numerical implementation, the fields are first determined using only the FE method with a coarse mesh as shown in Fig. 2. The corresponding computed field error distributions under the definition of the field difference method for every element are demonstrated in Fig. 3, in which the per unit value is used. From this error distribution, an error threshold value of 0.15 is then used to select the elements where the refinement computations are required, and the corresponding identified elements are marked in Fig. 4. Thirdly, by means of the aforementioned node addition procedure, some particles are added to the identified elements and the program repeats the refinement computations by using the EFG method, in which a quadratic basis and a cubic spline weight function are used, until the stop criterion is satisfied. The wavelet solution as defined by (23)

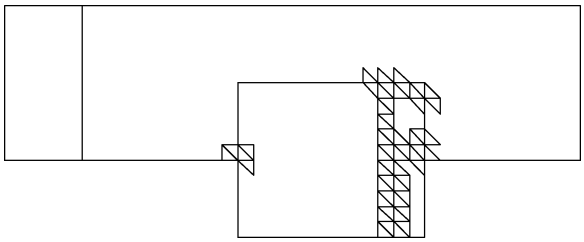

Fig. 4. Elements where refinements are required.

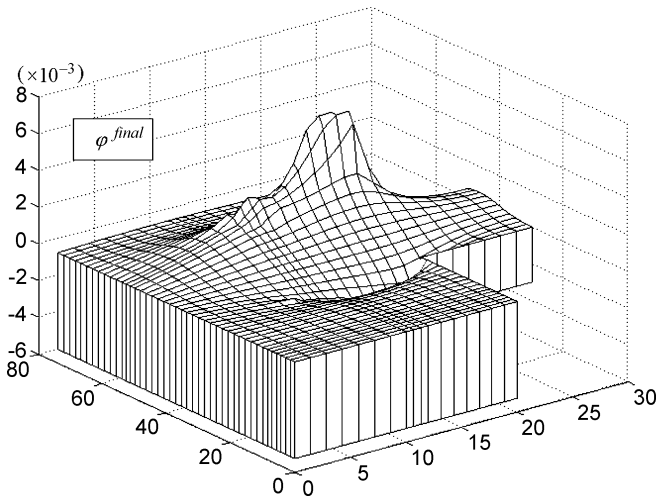

Fig. 5. Wavelet solution under the final and the penultimate refinements.

TABLE I
PERFORMANCE COMPARISON OF AN ADAPTIVE FE AND THE PROPOSED METHODS HAVING THE SAME SOLUTION ACCURACY

	DoFs	CPU time (pu)
Adaptive FEM	483*+355**	4.4
Proposed	483+282***	4.2

*483 is the node number of the original coarse finite element mesh of Fig. 2. **355 is the number of nodes added in the adaptive process of FE method, **282 is the number of the additional nodes of the EFG method in the final step refinements.

under the final and the penultimate refinement computations are shown in Fig. 5. To compare the performances of an adaptive FE strategy and the proposed method, this problem is solved using a pure adaptive FE, other things being equal, by successively densifying the mesh of the refinement regions until the same accuracy as that of the proposed method is obtained. The corresponding performance comparison results are given in Table I. To present an overview about the solution times used by different methods, the CPU time is given in Table I in per unit values using the FE solution time, with the mesh of Fig. 2, as the base value. From these numerical results one can see that: 1) for the same numerical accuracy, both the degree of freedoms (DoFs) and the solution time for the proposed method are smaller than those in a FE algorithm; 2) compared with traditional FE adaptive computation procedures, the most salient characteristics of the proposed one is that the refinement computation can be carried out by only adding particles, irrespective of their connectivities, rather than remeshing the solution domain, hence making the numerical implementation of the proposed algorithm simple and easy; 3) since the situation with respect to the storage technique of the resulting stiffness matrix and the numerical solver for EFG methods is not as satisfactory as that for FE method, the CPU time reduction used by the proposed algorithm is not proportional to that of the DoFs.

V. CONCLUSION

A combined FE and EFG method for the refinement computation of electromagnetic fields is proposed. Primary simulation results show that: 1) compared with the traditional adaptive meshing based refinement procedures, the most salient feature of the proposed one is that no cumbersome remeshing procedure is required in the refinement process, making it possible to use a coarse mesh to produce high precision numerical results; 2) compared with the pure meshless refinement procedures, the proposed one needs no special techniques to deal with the essential and interface conditions. In addition, to make the EFG based method as widely applicable as FE ones, one needs to synchronize the development of the storage technique of the resulting stiffness and the numerical solver in addition to making significant advancements in the study of the new methods as proposed.

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