# Design and Analysis of Practical Induction Motors

W. N. Fu, S. L. Ho, and H. C. Wong

***Abstract*—A method to estimate the flux-linkages and parameters of induction motors with skewed rotor bars is presented. The proposed method is based directly on a multi-slice time stepping finite element model, hence the effects of saturation, eddy-current, skewed rotor bars and high-order harmonic fields, can all be included readily in the solution. Special formulas for calculating rotor currents, rotor flux-linkages and air-gap flux in quadrature-phase winding system are deduced. The computed currents and flux-linkages from the finite element model are used directly to compute the parameters of the lumped circuit model with the help of the least squares method. The advantages of the proposed method are its simple physical concept, high accuracy and versatility. The method is suitable to study virtually all modes of operating conditions, including dynamic operations, of induction motors.**

***Index Terms*—Finite element, induction motor, parameter, time stepping.**

## I. Introduction

CIRCUIT model of induction machines is simple and its equations can be solved quickly because there are only a few variables. Indeed the lumped parameters have been widely applied in real-time modeling and off-line simulation of motor control systems. However, such parameters of induction machines will vary during dynamic process due to magnetic saturation and skin effect. Moreover, the flux-linkages that change considerably during dynamic process are also interesting variables to the motor control designers.

Nowadays finite element method (FEM) has been a practical method to compute the parameters from magnetic fields [1]–[3]. However there are inherent disadvantages of such approaches since: 1) The motor must operate in some specific exciting conditions and this requires additional computing time. 2) The skin effects exhibited during dynamic operation cannot be precisely included in the computation of rotor parameters. 3) The skewing feature of the rotor bars can be considered only approximately. 4) In virtually all published methods a complex FEM model is used. Hence many co-researchers found that it is difficult to determine the reluctivity of the iron materials precisely and the effect of the high-order harmonics in the time domain cannot be included. The dynamic changes of the motor parameters cannot be computed either.

The field-circuit-torque coupled multi-slice time stepping FEM (TS-FEM), which is a perfect model in a 2-D domain, has been used to simulate the dynamic operation of induction motors [4]–[6]. The feature of the skewed rotor bars can be directly modeled using the multi-slice technique. The current waveform and the actual flux distribution in space against time can be found from the given terminal voltage waveforms using the circuit-field coupled technique. Hence the exact saturation of different parts can be readily accounted for. The skin effect in the rotor bars, which is essential in analyzing the dynamic behavior, can also be dealt with as eddy-current effect in the field equations.

In other words, the fact that the TS-FEM can precisely simulate all modes of operations of the motors lends itself conveniently for one to use the computed results to estimate the circuit model parameters, because both models describe exactly the same system. In this paper a method which is directly based on simulating the actual motor operations in order to estimate the flux-linkages and parameters of induction motors is presented. The formulas for computing the currents and flux-linkages of the rotor in quadrature-phase winding system are also presented. The computed results of the field-circuit-torque coupled multi-slice TS-FEM at each time step are used as the input data for dynamic parameter computation. The method of the least squares is used to computer the inductances. The solution of an 11 kW induction motor is presented as an example.

## II. Multi-Slice TS-FEM Model of Induction Motors

In the TS-FEM, the field equations and circuit equations are expressed in the natural reference frames. To consider the skewing of the rotor bars, the motor in the axial direction is divided into $M$ slices, with the rotor bars in each slice being offset from each other by $(1/M)$th of the total skewing angle [4]. In each slice the magnetic vector potential has an axial component only. The stator end-winding effect and the rotor end-ring effect are considered by coupling the electrical circuits into the FEM equations.

### A. Multi-Slice Modeling of Induction Motors

*1) Magnetic Field Equations:* The Maxwells equations applied to the domain of the $m$th slice will give rise to the following diffusion equation:

$$\left.\begin{array}{ll} \nabla\cdot(\nu\nabla A)=0 & \text{in iron and air-gap} \\ \nabla\cdot(\nu\nabla A)+i_s/S=0 & \text{in stator conductor} \\ \nabla\cdot(\nu\nabla A)-\sigma\partial A/\partial t & \\ \quad +(\sigma/l_M)u_{mn}=0 & \text{in rotor conductor} \end{array}\right\} \tag{1}$$

where

| | |
|---|---|
| $A$ | is the axial component of the magnetic vector potential, |
| $\nu$ | is the reluctivity of the material, |
| $i_s$ | is the stator phase current, and |
| $S$ | is the total cross-sectional area of one turn on one coil side of the stator. |

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The motor is divided into an even number of $M$ slices in the axial direction while $m$ stands for the $m$th slice. $l_M = l/M$ and $l$ is the axial length of the iron core. In the rotor conductor domain, $u_{mn}$is the voltage between the two terminals of the $n$th bar on the $m$th slice.

*2) Circuit Equations in Stator Windings:* The stator circuit equation of one phase is:

$$\frac{l_M N_{\phi 1} \sum_{m=1}^{M} \left( \iint_{\Omega_m^+} \frac{\partial A}{\partial t} d\Omega - \iint_{\Omega_m^-} \frac{\partial A}{\partial t} d\Omega \right)}{\sum_{m=1}^{M} \iint_{\Omega_m^+ + \Omega_m^-} d\Omega} + R_1 i_s + L_\sigma \frac{di_s}{dt} = u_s \tag{2}$$

where

- $R_1$ is the total stator resistance of one phase winding,
- $L_\sigma$ is the inductance of the end windings,
- $N_{\phi 1}$ is the total number of series connected conductors per phase,
- $u_s$ is the applied voltage on the stator phase, and
- $\Omega^+$ and $\Omega^-$ are the cross-sectional areas of the "go" and "return" side of the phase conductors of the coils, respectively.

The first item on the left of (2) is the induced electromotive force.

*3) Circuit Equations in Rotor Squirrel Cage:* The total current in the $n$th bar of the $m$th slice is:

$$i_{bmn} = \sigma \iint_{\Omega_{mn}} (-\partial A/\partial t + u_{mn}/l_M)\, d\Omega. \tag{3}$$

Another relationship between the $i_{bmn}$ and $u_{mn}$ can be obtained from the circuit equations of the rotor network. Two adjacent bars in the rotor are connected by the end-ring resistances and inductances at the two end rings as well as by the inter-bar resistances along the axial length of the rotor. The details of establishing the rotor circuit equations have been reported by the authors in [5].

*4) The Total System Equations:* By discretizing the field equation (1) and coupling the circuit equations (3), (4), the rotor cage circuit equations and the torque balance equation together, one obtains the following global system equations [4], [5]:

$$[C][A \quad i_s \quad u_r \quad i_r \quad \omega_r \quad \theta_r]^T + [D]\left[\frac{\partial A}{\partial t} \quad \frac{\partial i_s}{\partial t} \quad \frac{\partial u_r}{\partial t} \quad \frac{\partial i_r}{\partial t} \quad \frac{\partial \omega_r}{\partial t} \quad \frac{\partial \theta_r}{\partial t}\right]^T = [P] \tag{4}$$

where

- $A$, $i_s$, $u_r$ are the unknowns (the voltage of the rotor bars on each slice),
- $i_r$ is the mesh-current in the rotor network,
- $\omega_r$ is the rotor speed,
- $\theta_r$ is the rotor position,
- $[P]$ is associated with the exciting source $u_s$, and
- $[C]$ and $[D]$ are coefficient matrices [4], [5].

### B. Time Stepping Formulations

By applying the Backward Eulers formula to approximate the time variable of (4), one obtains the following time stepping formulations for computing the transient performance of induction motors, i.e.,

$$\left[\frac{D_{n+1}}{h} + C_{n+1}\right]\{X\}_{n+1} = \{P\}_{n+1} + \left[\frac{D_n}{h}\right]\{X\}_n \tag{5}$$

where

- $\{X\} = [A \;\; i_s \;\; u_r \;\; i_r \;\; \omega_r \;\; \theta_r]^T$,
- $h$ is the step length of the time variables, and
- $n$ and $n+1$ denote the $n$th and $(n+1)$th time steps, respectively.

## III. Computation of Flux-Linkages

For the sake of convenience, the expressions of currents and flux-linkages in the quadrature-phase model in the stator reference frame ($DQ/dq$ model) will be deduced. In the $DQ/dq$ model, the stator is represented by two windings $D$ and $Q$ which are separated from each other by 90 electrical degrees. In a similar manner, the two rotor windings $d$ and $q$ are also separated from each other by 90 electrical degrees. Because of the symmetrical connections of the windings, it is assumed that there are no zero-sequence voltages and currents on the stator and rotor. When the solutions of the multi-slice TS-FEM are known, currents and flux-linkages in $DQ/dq$ frame can be obtained as described below.

### A. Stator Currents and Flux-Linkages in the $D$–$Q$ Frame

The stator currents in the $D$–$Q$ frame can be obtained according to the stator three phase currents $i_A$, $i_B$ and $i_C$:

$$\begin{bmatrix} i_D \\ i_Q \end{bmatrix} = \sqrt{\frac{2}{3}} \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & \sqrt{3}/2 & -\sqrt{3}/2 \end{bmatrix} \begin{bmatrix} i_A \\ i_B \\ i_C \end{bmatrix} \tag{6}$$

where the axis of stator winding $A$ coincides with the real axis of the $D$ axis in the stator. The flux-linkages of each stator phase are obtained from the distribution of magnetic vector $A$ in the cross-section of the motor:

$$\psi = l_M N_{\phi 1} \sum_{m=1}^{M} \left( \iint_{\Omega_m^+} A\, d\Omega - \iint_{\Omega_m^-} A\, d\Omega \right) \Bigg/ \sum_{m=1}^{M} \iint_{\Omega_m^+ + \Omega_m^-} d\Omega. \tag{7}$$

The stator flux-linkages in the $D$–$Q$ reference frame is:

$$\begin{bmatrix} \psi_D \\ \psi_Q \end{bmatrix} = \sqrt{2/3} \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & \sqrt{3}/2 & -\sqrt{3}/2 \end{bmatrix} \begin{bmatrix} \psi_A \\ \psi_B \\ \psi_C \end{bmatrix}. \tag{8}$$

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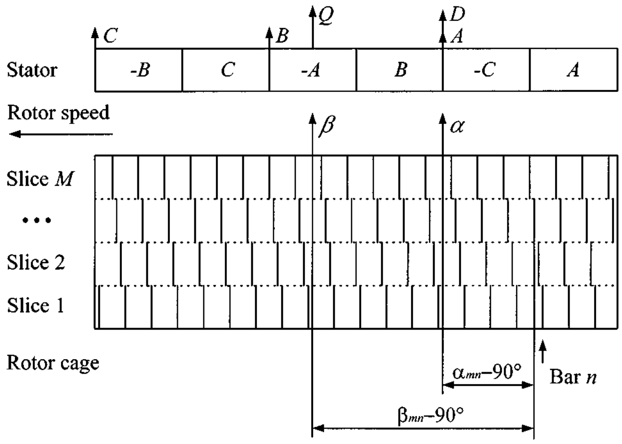


Fig. 1. Determination of rotor currents and flux-linkages in the $\alpha$–$\beta$ frame.

*B. Rotor Currents and Flux-Linkages in the $\alpha$–$\beta$ Rotating Reference Frame Fixed to the Rotor*

The squirrel-cage in the rotor can be considered as a $N_{bar}$ multi-phase winding system ($N_{bar}$ is the total number of rotor bars). This $N_{bar}$-phase winding system is firstly transformed to a fictitious quadrature-phase winding system which is fixed to the rotor ($\alpha$–$\beta$ frame). By resolving the instantaneous space-phase m.m.f. of the $N_{bar}$-phase winding onto the corresponding $\alpha$–$\beta$ axes, one could obtain the equivalent m.m.f.s of the $\alpha$–$\beta$ axes according to the following equations:

$$N_2 i_\alpha = [N_{bar}/(MN)] \sum_{m=1}^{M} \sum_{n=1}^{N} i_{bmn} \cos \alpha_{mn} \tag{9}$$

$$N_2 i_\beta = [N_{bar}/(MN)] \sum_{m=1}^{M} \sum_{n=1}^{N} i_{bmn} \cos \beta_{mn} \tag{10}$$

where $N$ is the number of rotor bars in the solved region; $\alpha_{\min}$ and $\beta_{\min}$ are the angles between the phase axis of the $n$th rotor bar at the $m$th slice and the $\alpha$ axis and $\beta$ axis, respectively (see Fig. 1). If $N_2$, the number of conductors in the fictitious rotor quadrature-winding systems, is assumed to be the same as the stator quadrature-winding system, one has

$$N_2 = \sqrt{3/2} N_{\phi 1} K_{dp1} \tag{11}$$

where $K_{dp1}$ is the winding distribution coefficient. Equation (9) and (10) can be further expressed in the matrix form:

$$\begin{bmatrix} i_\alpha \\ i_\beta \end{bmatrix} = C_i [i_{b11} \ \cdots \ i_{b1N} \ i_{b21} \ \cdots \ i_{b2N} \ \cdots \ i_{bM1} \ \cdots \ i_{bMN}]^T \tag{12}$$

where

$$C_i = \left[ N_{bar} \Big/ \left( MN \sqrt{3/2} N_{\phi 1} K_{dp1} \right) \right] C_\theta \tag{13}$$

and $C_\theta$ is defined in (14) as shown at the bottom of the page.

The flux-linkage at the $n$th rotor bar of the $m$th slice is:

$$\psi_{bmn} = l_M \iint_{\Omega_{mn}} A \, d\Omega \Big/ \iint_{\Omega_{mn}} d\Omega. \tag{15}$$

The rotor flux-linkages in $\alpha$–$\beta$ frame can be obtained by:

$$\begin{bmatrix} \psi_\alpha \\ \psi_\beta \end{bmatrix} = C_\psi [\psi_{b11} \cdots \psi_{b1N} \ \psi_{b21} \cdots \psi_{b2N} \cdots \psi_{bM1} \cdots \psi_{bMN}]^T \tag{16}$$

where the transformation matrix $C_\psi$ can be determined by the principle of the so-called classical, power-invariant form of the phase transformation from the $N_{bar}$-phase to the corresponding quadrature-phase components. In the power-invariant form of the phase transformation, $C_i$ and $C_\psi$ should satisfy:

$$C_i C_\psi^T = I \tag{17}$$

where $I$ is a $2 \times 2$ unit diagonal matrix. Notice that:

$$C_\theta C_\theta^T = (MN/2) I. \tag{18}$$

One obtains:

$$C_\psi = 2 \left( \sqrt{3/2} N_{\phi 1} K_{dp1} \Big/ N_{bar} \right) C_\theta. \tag{19}$$

*C. Rotor Currents and Flux-Linkages in the $d$–$q$ Stationary Reference Frame*

When the currents and the flux-linkages in the $\alpha$–$\beta$ frame fixed to the rotor axes are known, the currents and flux-linkages in the $d$–$q$ frame fixed to the stator can also be obtained easily by the following "commutator" transformation:

$$\begin{bmatrix} i_d \\ i_q \end{bmatrix} = \begin{bmatrix} \cos\theta_r & -\sin\theta_r \\ \sin\theta_r & \cos\theta_r \end{bmatrix} \begin{bmatrix} i_\alpha \\ i_\beta \end{bmatrix} \tag{20}$$

$$\begin{bmatrix} \psi_d \\ \psi_q \end{bmatrix} = \begin{bmatrix} \cos\theta_r & -\sin\theta_r \\ \sin\theta_r & \cos\theta_r \end{bmatrix} \begin{bmatrix} \psi_\alpha \\ \psi_\beta \end{bmatrix} \tag{21}$$

where $\theta_r = \omega_r t$ is the position of the rotor.

*D. Main Flux-Linkages in the $d$–$q$ Stationary Reference Frame*

The main flux-linkages can be obtained according to the distribution of the flux density in the air-gap. A closed integration along a circular ring that surrounds the rotor in free space along the air-gap should be chosen. The flux in the air-gap is:

$$\phi_m = \{ l_M / [(r_s - r_r) M] \} \iint_{\Omega_{ag}} (\pm 1) B_r \, d\Omega \tag{22}$$

where

$r_s$ and $r_r$ are the outer and inner radii of the circular ring,
$B_r$ is the $r$-component of the flux density, and
$\Omega_{ag}$ is the cross sectional area of the ring.

$$C_\theta = \begin{bmatrix} \cos\alpha_{11} & \cdots & \cos\alpha_{1N} & \cos\alpha_{21} & \cdots & \cos\alpha_{2N} & \cdots & \cos\alpha_{M1} & \cdots & \cos\alpha_{MN} \\ \cos\beta_{11} & \cdots & \cos\beta_{1N} & \cos\beta_{21} & \cdots & \cos\beta_{2N} & \cdots & \cos\beta_{M1} & \cdots & \cos\beta_{MN} \end{bmatrix} \tag{14}$$

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When computing the flux $\phi_{mX}(X = D, Q)$ of the $X$ axis for $(\theta_X - 90°) > \theta \geq (\theta_X + 90°)$, the sign "±" becomes a "+"; otherwise it becomes a "−." Here $\theta$ is the electrical angle along the circular ring; $\theta_D$ is the electrical angle of the $D$ axis; $\theta_Q$ is the electrical angle of the $Q$ axis. The main flux-linkages in $D$–$Q$ frame are:

$$\psi_{mD} = \sqrt{3/2} N_{\phi 1} K_{dp1} \phi_{mD} \tag{23}$$

$$\psi_{mQ} = \sqrt{3/2} N_{\phi 1} K_{dp1} \phi_{mQ}. \tag{24}$$

## IV. Estimation of Inductances

The flux linkage equations in the stationary $DQ/dq$ reference are:

$$\begin{bmatrix} \psi_D \\ \psi_Q \\ \psi_d \\ \psi_q \end{bmatrix} = \begin{bmatrix} L_s & 0 & L_m & 0 \\ 0 & L_s & 0 & L_m \\ L_m & 0 & L_r & 0 \\ 0 & L_m & 0 & L_r \end{bmatrix} \begin{bmatrix} i_D \\ i_Q \\ i_d \\ i_q \end{bmatrix}. \tag{25}$$

According to (25), because $i_D, i_Q, i_d, i_q, \psi_D, \psi_Q, \psi_d$ and $\psi_q$ are known, the inductances $L_m$, $L_s$ and $L_r$ can be computed by:

$$\begin{bmatrix} i_d & i_D & 0 \\ i_q & i_Q & 0 \\ i_D & 0 & i_d \\ i_Q & 0 & i_q \end{bmatrix} \begin{bmatrix} L_m \\ L_s \\ L_r \end{bmatrix} = \begin{bmatrix} \psi_D \\ \psi_Q \\ \psi_d \\ \psi_q \end{bmatrix}. \tag{26}$$

The least squares method is used to find the solution of (26) because there are three unknowns only in the equation.

## V. Estimation of Resistances

The rotor resistance is computed according to the copper loss in the rotor cage. The copper loss in the rotor bar should include the skin effect:

$$P_{Cu2\_bar} = \sum_{m=1}^{M} \sum_{n=1}^{N} \left( \sigma l_M \iint_{\Omega_{mn}} (-\partial A/\partial T + u_{mn}/l_M)^2 \, d\Omega \right). \tag{27}$$

The copper loss in the rotor end ring is:

$$P_{Cu2\_endring} = \sum_{i=1}^{N_{bar}} 2 i_i^2 R_{ring} \tag{28}$$

where $i_i$ is the end-ring current of the $i$th bar and $R_{ring}$ is the resistance of the rotor ring. Therefore, the rotor resistance can be obtained by

$$R_2 = (P_{Cu2\_bar} + P_{Cu2\_endring}) / (I_d^2 + I_q^2). \tag{29}$$

The total eddy-current loss of the motor is [6]:

$$P_{Fe\_eddy} = \sum_{i=1}^{N_{element}} \sigma_e \left[ \left(B_i^k - B_i^{k-1}\right) / \left(\sqrt{2}\pi \, \Delta t\right) \right]^2 S_i l \tag{30}$$

where

$N_{element}$ is the number of elements in the iron cores,

$B_i^k$ and $B_i^{k-1}$ are the magnetic flux densities of element $i$ at the $k$th and $(k-1)$th step, respectively,

$\Delta t$ is the time step size, and

$\sigma_e$ is determined by the specifications and characteristics of the material.

Because the waveforms of the magnetic flux density in each element are known, the hysteresis loss can be found by the integration of the area of the $B$–$H$ loop [6]. Then the iron loss resistor $R_m$ (connected in parallel with $L_m$) is:

$$R_m = (e_A^2 + e_B^2 + e_C^2)/P_{Fe} \tag{31}$$

where $e_A$, $e_B$ and $e_C$ are the induced electromotive force of phase $A$, $B$ and $C$ in stator windings, respectively.

## VI. Example

The proposed method has been used to estimate the flux-linkages and parameters of an induction motor (11 kW/380 V, 50 Hz, 4 poles, 48 stator slots, 44 rotor slots, Δ connection, rotor bars skew 1.3 stator slot pitch). The solution domain of the FEM is one pole pitch. The 2-D FEM mesh of each slice has 1626 nodes and 2655 elements. The motor is divided into 6 slice. The time step size is 0.038 ms. Typical computed and measured results of three-phase magnetizing inductance $L_m$, iron loss resistor $R_m$, lock-rotor inductance $L_k$ and resistor $R_k$ are given in Table I. When the impressed voltage is suddenly dipping from 380 V to 340 V but the rotor speed is maintained at 1480 rev/min due to the moment of inertia, the dynamic changing processes of the main flux-linkage, $L_m$ and $R_m$ are shown in Figs. 2 and 3, respectively.

TABLE I
COMPARISON OF COMPUTED AND MEASURED PARAMETERS

| | $L_m$ (H) | $R_m$ (Ω) | $L_k$ (H) | $R_k$ (Ω) |
|---|---|---|---|---|
| Computed | 0.2334 | 2272.8 | 0.01663 | 2.737 |
| Measured | 0.2368 | 2105.0 | 0.01719 | 2.665 |

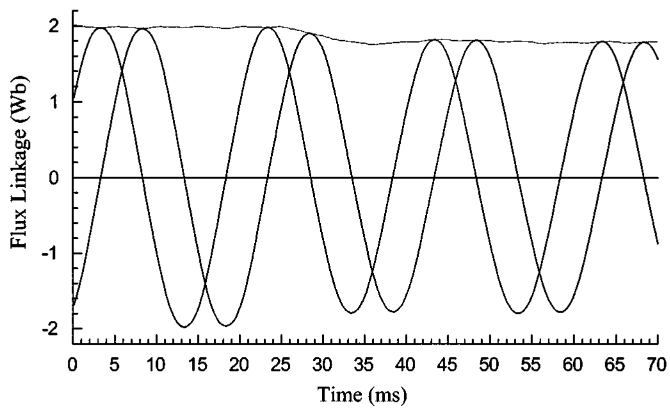


Fig. 2. Computed $\psi_{mD}$ and $\psi_{mQ}$ when the stator voltage suddenly dips.

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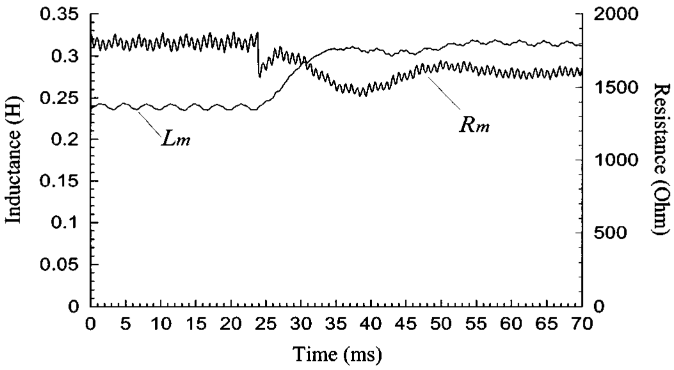


Fig. 3. Computed $L_m$ and $R_m$ when the stator voltage suddenly dips.

## VII. Conclusion

With the multi-slice TS-FEM model, the currents and the flux-linkages of induction motors can be computed directly. Therefore the parameters of the circuit model can be estimated successively. The proposed method of computing the flux-linkages can be further used in the evaluation of the performance of control methods of induction machines.

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