New Iterative Method for Three-Dimensional Eddy-Current Problems

Heyun Lin, Siu-Lau Ho, Eric Ka-Wai Cheng, and S. Y. Yang

***Abstract*—A new iterative method for computing three-dimensional steady-state magnetic fields with eddy currents is presented. By using the proposed method, the numerical computation of eddy current fields can be divided into two successive stages on flux density and eddy current calculations. The convergent field solution is then obtained iteratively. The coefficient matrices arising from the proposed method contain relatively few variables and are real. As these matrices need to be eliminated only once in the iteration procedure, the requirement upon the computer resource can be reduced substantially. The convergence of the presented iterative method is also discussed in detail. The instructions for choosing the penalty factor and relaxation factor in order to obtain the globally convergent potentials with sufficiently accurate field solutions are also given. Some sample calculations show that the new iterative method is highly computationally efficient for studying large-scale unbounded eddy-current problems in engineering.**

***Index Terms*—Eddy current, FEM, iterative method, open boundary problem.**

I. INTRODUCTION

THE COMPUTATION of three-dimensional (3-D) eddy-current fields has recently become the subject of extensive researches by a lot of numerical analysts studying electromagnetic field problems. Many efficient methods and techniques are now available for finding the solutions of various practical eddy-current problems. Most methods have their characteristic advantages and disadvantages. Experience has shown that the type of problem generally dictates which method is most appropriate for that specific investigation.

For typical 3-D eddy-current problems, the conventional $\boldsymbol{A}$, φ, and $\boldsymbol{T}$, Ω formulations are the two most frequently used algorithms. No matter whether the formulation on $\boldsymbol{A}$, φ or $\boldsymbol{T}$, Ω is used, both the vector and scalar potentials should be introduced, at least in the conducting region [1], [2]. As a result, there are four degrees of freedom per node of the finite-element mesh if nodal finite elements are used. Thus, the boundary value problem as prescribed for the vector and scalar potentials becomes very complex. Besides, a complex linear equation system, which contains large number of variables having a large bandwidth, has to be solved when finding the solutions of the potentials. Hence, a large amount of computer resource will be required to store and/or eliminate the coefficient matrix, especially for large scale unbounded engineering problems.

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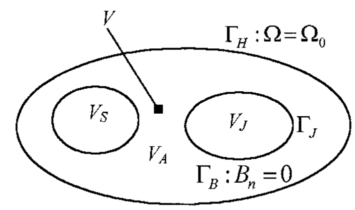


Fig. 1. Typical eddy-current problem.

The aim of this paper is to present a new iterative method that is suitable for studying large scale unbounded 3-D eddy-current fields economically. As the proposed method only requires a fraction of the computer resources that would otherwise be required, it would be a very useful algorithm for design engineers studying practical electromagnetic field problems [3]. The proposed method has been used to study an engineering-oriented loss model, and the computed results are compared and validated with experimental measurements as well as numerical results obtained by using the $\boldsymbol{A}$, $\varphi - \psi$ formulation.

II. MATHEMATICAL MODEL OF THE ITERATIVE METHOD

Consider a typical eddy-current problem as shown in Fig. 1, the entire field domain is denoted by V. It consists of a source current region V_s, an eddy-current region with nonzero conductivity V_J and a surrounding current free region V_A. The boundary of V_J as denoted by Γ_J is the interface between the conducting and nonconducting regions.

In the source region V_s, an electric vector potential $\boldsymbol{T}_S$ is defined to describe the known current density source and it can be analytically or numerically calculated from

$$\nabla \times \boldsymbol{T}_S = \boldsymbol{J}_S. \tag{1}$$

After setting an initial value for the eddy-current density in V_J, the other electric vector potential $\boldsymbol{T}_J$ can similarly be introduced to describe the eddy-current density

$$\nabla \times \boldsymbol{T}_J = \boldsymbol{J}_J. \tag{2}$$

Thus, a magnetic scalar potential Ω can be employed to compute the magnetic field in V by defining

$$\boldsymbol{H} = \boldsymbol{T} - \nabla\,\Omega \tag{3}$$

where $\boldsymbol{T}$ denotes $\boldsymbol{T}_S$ and $\boldsymbol{T}_J$ in V_s and V_J, respectively. $\boldsymbol{T}$ is equal to zero in V_A.

TABLE I
ITERATION NUMBERS OF DIFFERENT PENALTY FACTOR λ

Model	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
21.0	×	192	105	82	69	45	29	25	19	18	14
21.1	×	×	134	95	68	69	44	44	29	25	16
21.2	×	235	193	101	81	71	58	43	34	24	19
21.3	×	×	215	185	98	75	60	52	38	31	21

In terms of the Gauss law, the differential equations as well as the boundary conditions describing the magnetic fields are

$$\begin{cases} \nabla \cdot \mu \nabla \Omega = \nabla \cdot (\mu \boldsymbol{T}), & V \\ \Omega = \Omega_0, & \Gamma_H \, . \\ \partial \Omega / \partial n = 0, & \Gamma_B \end{cases} \tag{4}$$

The solution of the above boundary value problem is, in fact, very similar to that obtained in the analysis of magnetostatic fields with the exception that $\boldsymbol{T}$ and Ω are complex variables. Ω_o is a constant.

By solving the above boundary value problem, the magnetic field, which is induced by the source current and the assumed eddy current, can be obtained. Furthermore, the eddy-current distribution induced by this varying magnetic field can be calculated in terms of Faraday's law. In the eddy-current region, the electric vector potential $\boldsymbol{T}_J$ satisfies the following:

$$\nabla \times \frac{1}{\sigma} \nabla \times \boldsymbol{T}_J = -\boldsymbol{j}\omega \boldsymbol{B}. \tag{5}$$

Considering the conservation of charges, one has the boundary value problem for the electric vector potential $\boldsymbol{T}_J$

$$\begin{cases} \nabla \times \frac{1}{\sigma} \nabla \times \boldsymbol{T}_J - \nabla \left(\frac{\lambda}{\sigma} \nabla \cdot \boldsymbol{T}_J \right) = -\boldsymbol{j}\omega \boldsymbol{B}, & V_J \\ \left. \begin{array}{l} \boldsymbol{T}_J \times \boldsymbol{n} = 0 \\ \frac{\lambda}{\sigma} \nabla \cdot \boldsymbol{T}_J = 0 \end{array} \right\} \boldsymbol{T}_J \times \boldsymbol{n} = 0, & \Gamma_J \end{cases} \tag{6}$$

where λ is the penalty factor, which is generally taken within [0, 1]. Note that a penalty term is appended in the governing equation in order to ensure the uniqueness of the electric vector potential and the convergence of the iterative algorithm.

A new eddy-current distribution can be obtained by solving the above boundary value problem, which is used to repeat the computation of the magnetic field and eddy current until all the field quantities converge.

In the iterative method as described above, two relatively simple boundary value problems corresponding to the magnetic scalar and electric vector potentials, respectively, are employed. For 3-D boundary value eddy-current field problems, the conventional method is to solve the hybrid vector and scalar potentials simultaneously. Thus, the most distinct advantage of the proposed iterative method is that relatively fewer computer resources, which include both the computing time and memory occupation, are required when compared to that of the conventional algorithms. The reasons for such saving is because not only the number of the linear equations in the new method is less than that in the conventional methods, but also the coefficient matrices being drawn up in the new method are real instead of complex.

III. DISCUSSION ON THE CONVERGENCE

From the discussion in the previous section, it is easy to see that the iterative method is actually based on the conventional $\boldsymbol{T} - \Omega$ formulation. Substituting the solution of the first boundary problem (4) into the second one (6), one obtains the following governing equation of the electric vector potential in iterative form:

$$\nabla \times \frac{1}{\sigma} \nabla \times \tilde{\boldsymbol{T}}_J^i - \nabla \left(\frac{\lambda}{\sigma} \nabla \cdot \tilde{\boldsymbol{T}}_J^i \right) = -j\omega\, \mu \left(\boldsymbol{T}_J^i - \nabla \Omega^i \right) \tag{7}$$

where i stands for ith iteration, $\tilde{\boldsymbol{T}}$ is the eddy-current fields induced by the varying magnetic fields of (4), i.e., the solution of (6), $\boldsymbol{T}$ is the predetermined eddy-current fields obtained by solving (1) and (2).

It is obvious that the convergence of the iteration depends on the property of the stiff matrix when applying the finite-element method to discretize (6).

To guarantee the global convergence of iterations is equivalent to ensure the uniqueness of the electric vector potential $\boldsymbol{T}$. Furthermore, it is worth noting that, in general, the curl–curl equation results in numerical instability in finite-element computations. An effective means to alleviate such difficulty is to enforce a gauge on the vector potential to ensure its convergence. In the proposed method, the Coulomb gauge, which satisfies the penalty function technique, is employed. It can be easily proven that the solution of the boundary value problem (6) yields unique potentials satisfying the differential equations and the boundary conditions pertaining to the electric vector potential $\boldsymbol{T}$ and the magnetic scalar potential Ω of the $\boldsymbol{T} - \Omega$ formulation [4].

The effect of the penalty term in (6) on the convergence of the iterative algorithm has been investigated by some sample calculations. The entries in Table I show that the number of iterations varies with the penalty factor λ in the calculations (the relaxation factor $\beta = 1.3$) of the example defined in Section V. It can be found from Table I that the number of iterations decreases monotonically with an increase of λ from the range of zero to one. In particular, none of the convergent solutions can be obtained in the calculations of the four models if the penalty factor λ is set to zero, i.e., when the Coulomb gauge of the electric vector potential $\boldsymbol{T}$ is removed from the boundary value problem (6), the algorithm fails to converge.

It is also worth noting that the compliance of the Coulomb gauge by the penalty function technique is relatively "weak." In fact, it is impossible to make $\nabla \cdot \boldsymbol{T}_J$ exactly equal to zero in numerical computations. Thus, when λ increases, the weighted residual of the penalty term in (6) increases accordingly. As a result, the accuracy of the field solution decreases with an increase in the penalty factor λ [5]. Therefore, when choosing the penalty factor, due considerations should be given to two aspects: convergence and solution accuracy. It has been shown by some

TABLE II
ITERATION NUMBERS WITH DIFFERENT RELAXATION FACTOR

Model	0.1	0.3	0.5	0.7	0.9	1.1	1.3	1.5	1.7	1.9
21.0	65	50	43	42	35	31	25	24	21	×
21.1	98	78	67	65	52	49	44	44	×	156
21.2	101	99	84	76	65	46	43	40	42	×
21.3	113	108	101	97	98	71	62	60	56	×

numerical investigations that the proposed iterative method is globally convergent and yields field solutions with acceptable accuracy if the penalty factor λ lies within [0.6, 0.8].

IV. IMPLEMENTATION AND ADVANTAGES OF THE ITERATIVE METHOD

Before the commencement of the iterative procedure, a set of eddy-current values is assumed first. Since the eddy-current distributions are generally very complex in most practical problems, it is difficult to assume reasonable initial eddy currents. Fortunately, sample calculations indicate that the iteration can commence by assuming zero eddy-current value everywhere. Then, $\boldsymbol{T}_S$ and $\boldsymbol{T_J}$ can be calculated from (1) and (2). They would then be used as the impressed field for the ensuing computation of the magnetic fields in the entire region in the boundary value problem (4). Subsequently, the magnetic fields can then be employed to calculate another approximation of the eddy currents in the conductors. The iterative process terminates once two successive solutions agree within some prescribed tolerance.

A weighted combination of the electric vector potential is used at the $(i+1)$th step

$$T_J^{i+1} = \tilde{T}_J^i + \beta\left(\tilde{T}_J^i - T_J^i\right) \tag{8}$$

where β, the relaxation factor, is generally taken in [0, 2].

The value of β, undoubtedly, affects the speed of convergence of the iteration significantly. In general, the iteration has good stability if β is small, although more iterations are needed to reach the convergent solution. On the other hand, large values of β might lead to oscillations during the iterative process to result in divergence, especially when β approaches 2.0. Table II gives the iteration numbers of the four models with different values of β. The penalty factor λ is taken as 0.7 for all cases. Experience shows, however, that the relaxation factor should be taken within [1.1, 1.5] so as to give rapidly convergent solutions.

The advantages of the iterative method over the conventional $\boldsymbol{A}, \varphi$ and $\boldsymbol{T}, \Omega$ formulations are summarized as follows.

1) Two relatively simple boundary value problems corresponding to the magnetic scalar and electric vector potentials, respectively, are employed in the proposed algorithm, instead of a hybrid one, in either the conventional formulation $\boldsymbol{A}$, φ or $\boldsymbol{T}$, Ω.
2) Relatively less computer resources, including computer time and memory occupation, are required with the proposed method. This is because the two boundary value problems produces two linear equation systems with fewer variables and real stiffness matrices. Hence, the proposed method is extremely suitable for unbounded problems.

TABLE III
COMPARISON OF THE ITERATIVE METHOD WITH THE $\boldsymbol{A}$, $\varphi - \psi$ FORMULATION IN TERMS OF THE COMPUTER RESOURCE REQUIRED

Model	Nodes	Elements	Iterations	Memory (M)		CPU Time (m)	
				Iterative	$A,\varphi-\psi$	Iterative	$A,\varphi-\psi$
21.0	18837	16720	25	80	115	65	108
21.1	19656	17480	44	83	130	88	117
21.2	18837	16720	43	80	115	84	108
21.3	19656	17480	62	83	130	95	117

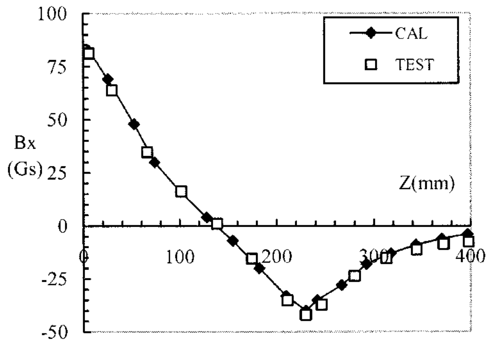
3) For linear problems, the two stiffness matrices arising from the two boundary problems never change in the iterative procedure. This is to say, it is unnecessary to assemble and eliminate them once more before getting the solution of the next step with the exception of the two backward substitutions. As for nonlinear problems, the proposed method is still feasible but then the corresponding iterative algorithm would be more complex. The authors are focusing on this researching topic and will report the findings in future papers.
4) Simplification of the two boundary value problems. For instance, each node has the same degrees of freedom, thereby simplifying the programming of the computer codes in numerical implementations.

V. NUMERICAL VERIFICATION

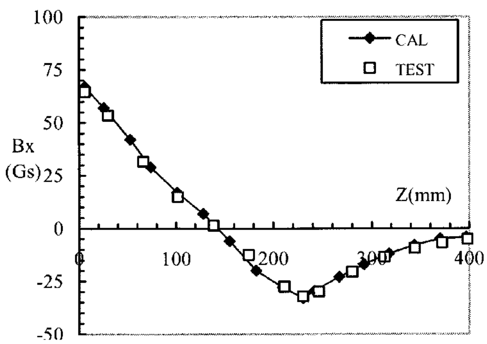
The example used to verify the proposed method is an industrial extended version of the TEAM Workshop Problem 21 and is referred as Problem 21+ [6]. It is well known that the tie plates, which are used to clamp the core laminations tightly of a large power transformer, are usually made of slotted and nonmagnetic steel plates in order to reduce the power loss and to eliminate the dangerous local overheating. The example being studied is, thus, a very practical problem since the determination of the eddy-current distributions in the tie plates and the investigation of the effects of the slot number on the eddy-current losses in the core plates of transformers are extremely important for transformer designers.

Problem 21+ consists of a set of product-based test models, each of which is a slotted nonmagnetic steel plate driven by Problem 21's source. Moreover, the different test models having different slot numbers in the steel plate, such as $0, 1, \ldots, 3$ slots, are referred as model $21.0, 21.1, \ldots, 21.3$, respectively [6].

The Galerkin form of the method of weighted residual equation is applied to set up the finite-element equation systems. The computations have been carried out using nodal finite-element method and isoparametric brick elements. Table III gives a comparison of computer resources needed in the iterative method with those in the $\boldsymbol{A}$, $\varphi - \psi$ formulation. In the calculations, the



(a)



(b)

Fig. 2. Flux densities along the two lines as specified in the study. (a) $X = 5.76$ mm, $Y = 0.0$ mm. (b) $X = 5.76$ mm. $Y = 0.0$ mm.

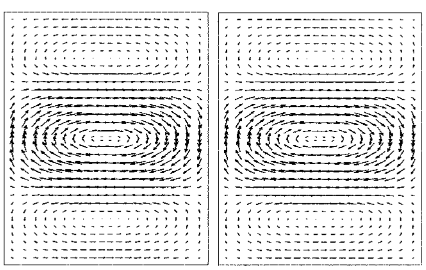


Fig. 3. Eddy currents on the surface of the plate in model 21.0.

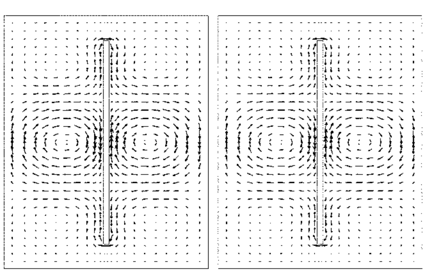


Fig. 4. Eddy currents on the surface of the plate in model 21.1.

penalty factor λ and the relaxation factor β are taken as 1.3 and 0.7, respectively. The magnetic flux densities along the two lines specified in model 21.2 together with the experimental results are shown in Fig. 2.

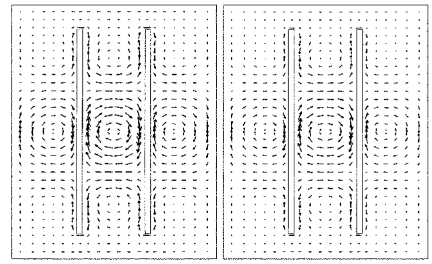


Fig. 5. Eddy currents on the surface of the plate in model 21.2.

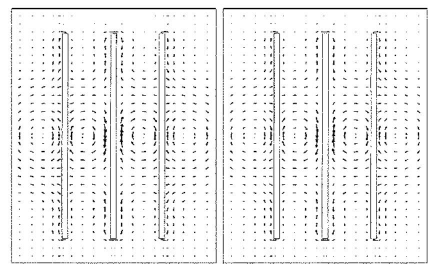


Fig. 6. Eddy currents on the surface of the plate in model 21.3.

The eddy-current distributions in the tie plates of models 21.0 to 21.3 are given in Figs. 3–6, respectively. It can be seen from the computations that the proposed iterative algorithm is effective and highly efficient.

VI. Conclusion

An efficient and globally convergent iterative method has been developed successfully for obtaining the solution of 3-D eddy-current problems in which the electric vector and the magnetic scalar potential are employed. Distinguished computational advantages over the conventional straightforward formulations can be accomplished with this algorithm. An engineering-oriented problem has been analyzed and reported to verify the validity and high efficiency of the new method. The method described has also been shown to produce reasonable results as compared with the $\boldsymbol{A}$, $\varphi - \psi$ formulation and measurement results.

References

[1] P. P. Silvester, *Finite Elements for Electrical Engineering*. Cambridge, U.K.: Cambridge Univ. Press, 1983.

[2] W. Renhart *et al.*, "Calculation of 3-D eddy current problem by finite element method using either an electric or magnetic vector potential," *IEEE Trans. Magn.*, vol. 24, pp. 122–125, Jan. 1988.

[3] I. D. Mayergoyz *et al.*, "Iterative solution of 3-D eddy current problems," *IEEE Trans. Magn.*, vol. 29, pp. 2335–2340, Nov. 1993.

[4] O. Biro *et al.*, "On the use of the magnetic vector potential in the finite element analysis of three-dimensional eddy current," *IEEE Trans. Magn.*, vol. 25, pp. 3145–3159, July 1989.

[5] L. Heyun *et al.*, "Numerical investigation of penalty factor for Coulomb gauge of magnetic vector potential for 3-D fields," *IEEE Trans. Magn.*, vol. 35, pp. 1354–1357, May 1999.

[6] S. Gao *et al.*, "Problem 21+: Slotted nonmagnetic steel plate driven by problem 21's source," in *Proc. TEAM Workshop*, Yichang, China, 1996, pp. 366–370.