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Control for Multiplicative Noise Systems With Intermittent Noise and Input Delay

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ABSTRACT In this paper, we consider the control problem for multiplicative noise system with intermittent noise and input delay. For the finite-horizon case, in virtue of the dynamic programming approach, the optimal output feedback controller is proposed for the first time. For the infinite-horizon case, it is shown that the multiplicative noise system can be stabilized if and only if the given modified Riccati equation has the unique solution.

INDEX TERMS Intermittent noise, input delay, output feedback control, stabilization.

I. INTRODUCTION

For the stochastic control problem, when the system is disturbed by the measurement noise such that the precise system state cannot be accessed directly, the output feedback controller should be designed. The distinguished Kalman filter was introduced in 1960s; see [1], [2]. From then on, the output feedback control problem has received much attentions and large progresses have been made in applications, such as signal processing, aerospace, networked control system (NCS) and so on; see [3], [4], [9], [10].

In this paper, we will focus on the output feedback control problem for multiplicative noise system with input delay and intermittent noise. In the considered system, the intermittent noise $\{\alpha_k\}$ satisfies the Bernoulli distribution, i.e., $\alpha_k = 1$ indicating that the state is transmitted successfully, otherwise the state being lost. Besides, the system suffers from the input delay and multiplicative noise.

It is stressed that the considered output feedback control problem was not thoroughly studied in previous literatures: On one hand, the optimal output feedback controller was not obtained. On the other hand, the output feedback stabilizing controller was not proposed. The relevant studies can be found in [5], [11]–[16], [18], [19]. The output feedback control problem for stochastic system with intermittent noise can be traced back to [11] where a suboptimal output feedback control was derived. [12] investigated the intermittent Kalman filter and the critical value of arrival probability rate was proposed. For deterministic systems with delay, the optimal controller was designed by smith predictor; see [20]–[24]; For the stochastic case with input delay, many works have been done; see [6], [7].

It is noted that the existence of the intermittent noise and input delay will cause fundamental difficulties to calculate the optimal output feedback controller. As pointed in [17], ''separation principle fails'' indicates that the control gain and the estimation gain are coupled, and cannot be calculated separately. The basic reasons are that the optimal estimation cannot be acquired, and error covariance matrix is related with the controller.

In this paper, we will investigate the output feedback control problem for both finite horizon case and infinite horizon case. Firstly, the optimal estimation will be proposed in the recursive method. By using the dynamic programming approach, the optimal output feedback controller will be derived. For the infinite horizon case, we will show the stabilization conditions (necessary and sufficient) for multiplicative system with intermittent noise and input delay. The innovations of this paper are two-fold: For the first time, the optimal output feedback controller is obtained for the

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multiplicative noise system with input delay and intermittent noise. The necessary and sufficient stabilization conditions are firstly explored in this paper.

The reminder of this paper is as below. The finite horizon output feedback control problem is formulated in Section II, and Section III considers the infinite horizon output feedback control and stabilization problems. The numerical examples are provided in Section IV to illustrate the main results in this paper. Finally, the conclusion is provided in Section V.

Notations: Superscript $'$ means the matrix transpose. Symmetric matrix $H > 0$ (≥ 0) represents the positive definiteness (positive semi-definiteness); $\mathbb{R}^n$ indicates $n-$dimensional Euclidean space; I denotes the unit matrix; $\mathcal{I}_{\{B\}}$ denotes the indicator function, with $\mathcal{I}_{\{B\}} = 1$ when $\omega \in B$,; otherwise, $\mathcal{I}_{\{B\}} = 0$; $\mathbb{E}[\cdot]$ means the mathematical expectation and $\mathbb{E}[X|Y]$ signifies conditional expectation.

II. FINITE HORIZON CASE

A. PROBLEM FORMULATION

In this paper, the following linear stochastic system is considered:

$$\chi_{k+1} = \mathcal{C}\chi_k + \mathcal{D}\upsilon_{k-d} + \vartheta_k(\bar{\mathcal{C}}\chi_k + \bar{\mathcal{D}}\upsilon_{k-d}), \tag{1}$$

$$\varrho_k = \alpha_k \chi_k, \tag{2}$$

where $\chi_k \in \mathbb{R}^n$ is the state process, $\upsilon_{k-d} \in \mathbb{R}^m$ denotes the control input with $d > 0$ being the input delay, ϑ_k is the 1-dimensional Gaussian white noise with zero mean and covariance σ^2. $\varrho_k \in \mathbb{R}^n$ signifies the measurement process and α_k obeys the Bernoulli distribution with probability $\mathcal{P}(\alpha_k = 1) = p \in [0, 1]$. $\mathcal{C}, \bar{\mathcal{C}}, \mathcal{D}, \bar{\mathcal{D}}$ are deterministic coefficient matrices with appropriate dimension. The initial state χ_0 is Gaussian random vector with mean μ and covariance $\mathcal{O}$.

The associated cost function is as

$$J_N = \mathbb{E}\left[\sum_{k=0}^{N} \chi_k' H \chi_k + \sum_{k=d}^{N} \upsilon_{k-d}' S \upsilon_{k-d}\right] \tag{3}$$

where weighting matrices $H \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{m\times m}$.

The problem to be dealt with for the finite horizon case is stated below.

Problem 1: For system (1)-(2) and cost function (3), find $\mathcal{F}_k$-measurable controller υ_k to minimize (3) in terms of measurements $\{\varrho_0, \cdots, \varrho_k\}$.

To ensure the solvability of Problem 1, we make the following standard assumption.

Assumption 1: $H \geq 0$ and $S > 0$.

B. OPTIMAL ESTIMATION

Firstly we shall show the optimal estimation $\hat{\chi}_{k+d/k} = \mathbb{E}[\chi_{k+d}|\varrho_0, \cdots, \varrho_k]$ in the following lemma.

Lemma 1: For system (1) and (2), the optimal estimation $\hat{\chi}_{k+d/k}$ obeys the following iteration:

$$\hat{\chi}_{k+d/k} = (1-\varpi_k)[\mathcal{C}^d \varrho_k + \sum_{i=1}^{d} \mathcal{C}^{i-1}\mathcal{D}\upsilon_{k-i}] + \varpi_k \hat{\chi}_{k+d/k-1}, \tag{4}$$

where $\varpi_k = \mathcal{I}_{\{\varrho_k=0\}}$ is a binary random variable with $\mathcal{P}(\varpi_k = 1) = q = 1 - p$.

Furthermore, we have

$$\hat{\chi}_{k+1/k-d+l} = \mathcal{C}\hat{\chi}_{k/k-d+l} + \mathcal{D}\upsilon_{k-d}, \quad l = 1, \cdots, d, \tag{5}$$

$$\hat{\chi}_{k+1/k+1} = \varpi_{k+1}\hat{\chi}_{k+1/k} + (1-\varpi_{k+1})\varrho_{k+1}, \tag{6}$$

and $\hat{\chi}_{0/0}$ satisfies

$$\hat{\chi}_{0/0} = (1-\varpi_0)\varrho_0 + \varpi_0 \mathbb{E}\chi_0. \tag{7}$$

Proof: The detailed proof of Lemma 1 is similar to that of Theorem 1 in our previous works [4]. Due to space limitation, we omit it here. ■

C. OPTIMAL OUTPUT FEEDBACK CONTROL

Now we are in the position to present the optimal controller for Problem 1.

Theorem 1: Under Assumption 1, for system (1)-(2), the optimal output feedback controller is given by

$$\upsilon_k = -K_{k+d}\hat{\chi}_{k+d/k} = -\Pi_{k+d}^{-1}\Omega_{k+d}\hat{\chi}_{k+d/k}, \tag{8}$$

where $\hat{\chi}_{k+d/k}$ can be calculated via (4), and Π_{k+d}, Ω_{k+d} obey

$$\Pi_k = S + \mathcal{D}'\sum_{j=1}^{d+1} G_{k+1}^j \mathcal{D} + \sigma^2\bar{\mathcal{D}}' G_{k+1}^1 \bar{\mathcal{D}} + \mathcal{D}' I_{k+1}\mathcal{D} + (1-q)\sigma^2 \bar{\mathcal{D}}' I_{k+1}\bar{\mathcal{D}}, \tag{9}$$

$$\Omega_k = \mathcal{D}'\sum_{j=1}^{d+1} G_{k+1}^j \mathcal{C} + \sigma^2\bar{\mathcal{D}}' G_{k+1}^1 \bar{\mathcal{C}} + \mathcal{D}' I_{k+1}\mathcal{C} + (1-q)\sigma^2 \bar{\mathcal{D}}' I_{k+1}\bar{\mathcal{C}}. \tag{10}$$

Furthermore, G_k^i, $1 \leq i \leq d$ and I_k satisfy the modified Riccati equations:

$$G_k^1 = H + \mathcal{C}' G_{k+1}^1 \mathcal{C} + \sigma^2 \bar{\mathcal{C}}' G_{k+1}^1 \bar{\mathcal{C}} + (1-q) \times \mathcal{C}' I_{k+1}\mathcal{C} + (1-q)\sigma^2 \bar{\mathcal{C}}' I_{k+1}\bar{\mathcal{C}}, \tag{11}$$

$$G_k^2 = -\Omega_k' \Pi_k^{-1}\Omega_k, \tag{12}$$

$$G_k^i = \mathcal{C}' G_{k+1}^{i-1}\mathcal{C}, \ i = 3, \cdots, d+1, \tag{13}$$

$$I_k = \mathcal{C}' G_{k+1}^{d+1}\mathcal{C} + q\mathcal{C}' I_{k+1}\mathcal{C}, \tag{14}$$

where final conditions $G_{N+1}^i = 0$ for $i = 1, \cdots, d$ and $I_{N+1} = 0$.

The optimal cost function is as:

$$J_N^* = \mathbb{E}\Big\{\sum_{k=0}^{d-1} \chi_k' H \chi_k + \chi_d' G_d^1 \chi_d + \chi_d' \sum_{i=2}^{d+1} G_d^i \hat{\chi}_{d/i-2} + \chi_d' I_d \hat{\chi}_{d/d}\Big\}. \tag{15}$$

Before showing the proof of Theorem 1, we shall give the following lemma.

Lemma 2: Modified Riccati equations (11)-(14) can be reformulated as:

$$Z_k = H + \mathcal{C}' Z_{k+1}\mathcal{C} + \sigma^2 \bar{\mathcal{C}}' X_{k+1}\bar{\mathcal{C}} - \Omega_k' \Pi_k^{-1}\Omega_k + (1-q)\sigma^2 \bar{\mathcal{C}}' I_{k+1}\bar{\mathcal{C}}, \tag{16}$$

$$X_k = Z_k + \sum_{i=0}^{d-1}(\mathcal{C}')^i \Omega_k' \Pi_k^{-1}\Omega_k \mathcal{C}^i, \tag{17}$$

$$I_k = -(\mathcal{C}')^d \Omega_{k+d}' \Pi_{k+d}^{-1}\Omega_{k+d}\mathcal{C}^d + q\mathcal{C}' I_{k+1}\mathcal{C}. \tag{18}$$

where

$$\begin{aligned}\Omega_k &= \mathcal{D}' Z_{k+1}\mathcal{C} + \sigma^2 \bar{\mathcal{D}}' X_{k+1}\bar{\mathcal{C}} + \mathcal{D}' I_{k+1}\mathcal{C}\\ &\quad + (1-q)\sigma^2\bar{\mathcal{D}}' I_{k+1}\bar{\mathcal{C}},\end{aligned} \tag{19}$$

$$\begin{aligned}\Pi_k &= S + \mathcal{D}' Z_{k+1}\mathcal{D} + \sigma^2\bar{\mathcal{D}}' X_{k+1}\bar{\mathcal{D}} + \mathcal{D}' I_{k+1}\mathcal{D}\\ &\quad + (1-q)\sigma^2\bar{\mathcal{D}}' I_{k+1}\bar{\mathcal{D}}.\end{aligned} \tag{20}$$

Moreover, $\Pi_k > 0$ for $k = d, \cdots, N$.

Proof: Setting $Z_k = \sum_{i=1}^{d+1} G_k^i$ and $X_k = G_k^1$, for (11)-(14) and taking summation from $i = 1$ to $d+1$, (16)-(18) can be readily obtained. Moreover, (19)-(20) can be easily induced from (9)-(10).

Next, we will prove that $\Pi_k > 0$ for $k = d, \cdots, N$. In fact, noting $G_{N+1}^i = 0, i = 1, \cdots, d+1, I_{N+1} = 0$, it follows that $Z_{N+1} = X_{N+1} = 0$ and from (20), $\Pi_N > 0$ holds.

With (16), we have

$$\begin{aligned}Z_N &= I + K_N'\Pi_N K_N + [\mathcal{C} + (1-\bar{q})\mathcal{D}K_N]'\\ &\quad \times Z_{N+1}[\mathcal{C} + (1-\bar{q})\mathcal{D}K_N] \geq 0.\end{aligned} \tag{21}$$

With (12) we have $G_N^2 \leq 0$ and $X_N \geq Z_N \geq 0$. Then $\Pi_{N-1} > 0$ can be obtained.

In virtue of the induction method, we assume that $Z_{l+1} \geq 0, X_{l+1} \geq 0$ for $0 \leq l \leq N$, and $\Pi_l > 0$, which means (16)-(18) are well-defined for $k = l$. Then, similar to (21), we have that $X_l \geq Z_l \geq 0$. Hence, $\Pi_{l-1} \geq 0$ can be derived. Therefore, we have shown that $\Pi_k > 0$ for $k = d, \cdots, N$. ■

Remark 1: Compared with previous works [11], [15], [17], the optimal output feedback controller is derived in Theorem 1 for the first time. The control gain can be calculated off-line by the modified Riccati equations (16)-(18).

Now we are ready to show the proof of Theorem 1.

Proof: Firstly, we define the value function

$$V_N(k, x_k) = \mathbb{E}\Big[\chi_k' G_k^1 \chi_k + \chi_k' \sum_{i=2}^{d+1} G_k^i \hat{\chi}_{k/i+k-d-2} + \chi_k' I_k \hat{\chi}_{k/k}\Big]. \tag{22}$$

By applying (1), (22) and (11)-(13), it follows that

$$\begin{aligned}&V_N(k+1, \chi_{k+1})\\ &= \mathbb{E}\Big\{\chi_k'(\mathcal{C}' G_{k+1}^1\mathcal{C} + \sigma^2\bar{\mathcal{C}}' G_{k+1}^1\bar{\mathcal{C}})\chi_k\\ &\quad + 2\upsilon_{k-d}'(\mathcal{D}' G_{k+1}^1\mathcal{C} + \sigma^2\bar{\mathcal{D}}' G_{k+1}^1\bar{\mathcal{C}})\chi_k + \upsilon_{k-d}'(\mathcal{D}' G_{k+1}^1\\ &\quad \times \mathcal{D} + \sigma^2\bar{\mathcal{D}}' G_{k+1}^1\bar{\mathcal{D}})\upsilon_{k-d} + [\mathcal{C}\chi_k + \mathcal{D}\upsilon_{k-d} + \vartheta_k(\bar{\mathcal{C}}\chi_k\\ &\quad + \bar{\mathcal{D}}\upsilon_{k-d})]' \sum_{i=2}^{d+1} G_{k+1}^i(\mathcal{C}\hat{\chi}_{k/i+k-d-1} + \mathcal{D}\upsilon_{k-d}) + [\mathcal{C}\chi_k\\ &\quad + \mathcal{D}\upsilon_{k-d} + \vartheta_k(\bar{\mathcal{C}}\chi_k + \bar{\mathcal{D}}\upsilon_{k-d})]' I_{k+1}\varpi_{k+1}(\mathcal{C}\hat{\chi}_{k/k} + \mathcal{D}\\ &\quad \times \upsilon_{k-d}) + [\mathcal{C}\chi_k + \mathcal{D}\upsilon_{k-d} + \vartheta_k(\bar{\mathcal{C}}\chi_k + \bar{\mathcal{D}}\upsilon_{k-d})]' I_{k+1}\\ &\quad \times (1-\varpi_{k+1})[\mathcal{C}\chi_k + \mathcal{D}\upsilon_{k-d} + \vartheta_k(\bar{\mathcal{C}}\chi_k + \bar{\mathcal{D}}\upsilon_{k-d})]\Big\}\end{aligned}$$

$$\begin{aligned}&= \mathbb{E}\Big\{\chi_k'(\mathcal{C}' G_{k+1}^1\mathcal{C} + \sigma^2\bar{\mathcal{C}}' G_{k+1}^1\bar{\mathcal{C}})\chi_k + 2\upsilon_{k-d}'(\mathcal{D}' G_{k+1}^1\\ &\quad \times \mathcal{C} + \sigma^2\bar{\mathcal{D}}' G_{k+1}^1\bar{\mathcal{C}})\chi_k + \upsilon_{k-d}'(\mathcal{D}' G_{k+1}^1\mathcal{D} + \sigma^2\bar{\mathcal{D}}' G_{k+1}^1\bar{\mathcal{D}})\\ &\quad \times \upsilon_{k-d} + \chi_k'\mathcal{C}' \sum_{i=2}^{d+1} G_{k+1}^i \mathcal{C}\hat{\chi}_{k/i+k-d-1} + \chi_k'\mathcal{C}'\sum_{i=2}^{d+1} G_{k+1}^i\\ &\quad \times \mathcal{D}\upsilon_{k-d} + \upsilon_{k-d}'\mathcal{D}' \sum_{i=2}^{d+1} G_{k+1}^i \mathcal{C}\hat{\chi}_{k/i+k-d-1} + \upsilon_{k-d}'\mathcal{D}'\\ &\quad \times \sum_{i=2}^{d+1} G_{k+1}^i \mathcal{D}\upsilon_{k-d} + q\chi_k'\mathcal{C}' I_{k+1}\mathcal{C}\hat{\chi}_{k/k} + q\chi_k'\mathcal{C}' I_{k+1}\mathcal{D}\\ &\quad \times \upsilon_{k-d} + q\upsilon_{k-d}'\mathcal{D}' I_{k+1}\mathcal{C}\hat{\chi}_{k/k} + (1-q)\chi_k'(\mathcal{C}' I_{k+1}\mathcal{C}\\ &\quad + \sigma^2\bar{\mathcal{C}}' I_{k+1}\bar{\mathcal{C}})\chi_k + (1-q)\upsilon_{k-d}'(\mathcal{D}' I_{k+1}\mathcal{C} + \sigma^2\bar{\mathcal{D}}' I_{k+1}\\ &\quad \times \bar{\mathcal{C}})\chi_k + (1-q)\upsilon_{k-d}'(\mathcal{D}' I_{k+1}\mathcal{D} + \sigma^2\bar{\mathcal{D}}' I_{k+1}\bar{\mathcal{D}})\upsilon_{k-d}\Big\}\\ &= \mathbb{E}\Big\{\chi_k'[\mathcal{C}' G_{k+1}^1\mathcal{C} + \sigma^2\bar{\mathcal{C}}' G_{k+1}^1\bar{\mathcal{C}} + (1-q)(\mathcal{C}' I_{k+1}\mathcal{C}\\ &\quad + \sigma^2\bar{\mathcal{C}}' I_{k+1}\bar{\mathcal{C}})]\chi_k + 2\upsilon_{k-d}'[\mathcal{D}' \sum_{i=1}^{d+1} G_{k+1}^i\mathcal{C} + \sigma^2\bar{\mathcal{D}}' G_{k+1}^1\\ &\quad \times \bar{\mathcal{C}} + (1-q)(\mathcal{D}' I_{k+1}\mathcal{C} + \sigma^2\bar{\mathcal{D}}' I_{k+1}\bar{\mathcal{C}})]\chi_k + \upsilon_{k-d}'[\mathcal{D}'\\ &\quad \times \sum_{i=1}^{d+1} G_{k+1}^i\mathcal{D} + \sigma^2\bar{\mathcal{D}}' G_{k+1}^1\bar{\mathcal{D}} + (1-q)(\mathcal{D}' I_{k+1}\mathcal{D} + \sigma^2\\ &\quad \times \bar{\mathcal{D}}' I_{k+1}\bar{\mathcal{D}})]\chi_k + \chi_k'\mathcal{C}' G_{k+1}^{d+1}\mathcal{C}\hat{\chi}_{k/k} + \chi_k' \sum_{i=3}^{d+1} G_k^i\\ &\quad \times \hat{\chi}_{k/i+k-d-2} + q\chi_k'\mathcal{C}' I_{k+1}\mathcal{C}\hat{\chi}_{k/k}\Big\}\\ &= \mathbb{E}\Big\{\chi_k'(G_k^1 - H)\chi_k + \upsilon_{k-d}'(\Pi_k - S)\upsilon_{k-d} + 2\upsilon_{k-d}'\\ &\quad \times \Omega_k\hat{\chi}_{k/k-d} + \chi_k' \sum_{i=3}^{d+1} G_k^i\hat{\chi}_{k/i+k-d-2} + \chi_k'(\mathcal{C}' G_{k+1}^{d+1}\mathcal{C}\\ &\quad + q\mathcal{C}' I_{k+1}\mathcal{C})\hat{\chi}_{k/k}\Big\}\end{aligned} \tag{23}$$

Combining (22) and (23), we have

$$\begin{aligned}&V_N(k, \chi_k) - V_N(k+1, \chi_{k+1})\\ &\quad = \mathbb{E}\{\chi_k' H\chi_k + \upsilon_{k-d}' S\upsilon_{k-d} - (\upsilon_{k-d} + K_k\hat{\chi}_{k/k-d})'\\ &\quad\quad \times \Pi_k(\upsilon_{k-d} + K_k\hat{\chi}_{k/k-d})\}.\end{aligned} \tag{24}$$

Taking summation of (24) from $k = d$ to $k = N$, we get

$$\begin{aligned}&V_N(d, \chi_d) - V_N(N+1, \chi_{N+1})\\ &\quad = \mathbb{E}[\chi_d' G_d^1\chi_d + \chi_d' \sum_{i=2}^{d+1} G_d^i\hat{\chi}_{d/i-2} + \chi_d' I_d\hat{\chi}_{d/d}]\\ &\quad = \sum_{k=d}^{N}\mathbb{E}\{\chi_k' H\chi_k + \upsilon_{k-d}' S\upsilon_{k-d} - (\upsilon_{k-d} + K_k\hat{\chi}_{k/k-d})'\\ &\quad\quad \times \Pi_k(\upsilon_{k-d} + K_k\hat{\chi}_{k/k-d})\},\end{aligned} \tag{25}$$

where final conditions $G_{N+1}^i = 0$ for $i = 1, \cdots, d+1$ and $I_{N+1} = 0$ are used in the above equation.

Noting (3), we have

$$J_N = \mathbb{E}\Big\{\chi_d' G_d^1 \chi_d + \chi_d' \sum_{i=2}^{d+1} G_d^i \hat{\chi}_{d/i-2} + \chi_d' I_d \hat{\chi}_{d/d} - \sum_{k=d}^{N} (\upsilon_{k-d} + K_k \hat{\chi}_{k/k-d})' \Pi_k (\upsilon_{k-d} + K_k \hat{\chi}_{k/k-d}) + \sum_{k=0}^{d-1} \chi_k' H \chi_k \Big\}. \quad (26)$$

From Lemma 2 we know $\Pi_k > 0$ for $k \geq d$. Therefore, the optimal cost function is as (15), and the optimal output feedback controller is shown by (8). ■

Remark 2: It should be pointed out that the methods used in this paper is dynamic programming, which differs from the previous works in [4]–[6].

III. INFINITE HORIZON CASE

In this section, the infinite horizon optimal output control problem and stabilization problems for system (1)-(2) shall be investigated.

The infinite horizon cost function is as

$$J = \sum_{k=d}^{\infty} \mathbb{E}\left(\chi_k' H \chi_k + \upsilon_{k-d}' S \upsilon_{k-d}\right) \quad (27)$$

Then the problem to be investigated in this section is given as follows.

Problem 2: Design the output feedback controller to minimize the cost function (27), and obtain the stabilization conditions for the system (1).

Firstly, we give the following definitions.

Definition 1: For output feedback controller $\upsilon_{k-d} = \mathbf{L}\hat{\chi}_{k/k-d}$, where $\mathbf{L}$ is constant matrix and $\hat{\chi}_{k/k-d}$ is the optimal estimation, if the closed-loop system (1) is mean square asymptotically stable, i.e., for any initial conditions $\chi_0, \upsilon_{-1}, \cdots, \upsilon_{-d}$, it holds $\lim_{k\to+\infty} \mathbb{E}(\chi_k'\chi_k) = 0$. We call system (1) is mean square stabilizable.

Definition 2: For system $(\mathcal{C}, \bar{\mathcal{C}}, C)$:

$$\begin{cases} \chi_{k+1} = (\mathcal{C} + \vartheta_k \bar{\mathcal{C}})\chi_k, \\ \varrho_k = C\chi_k, \end{cases} \quad (28)$$

if there holds

$$\varrho_k = 0, \text{ a.s., for } 0 \leq k \leq N, \Longrightarrow \chi_0 = 0.$$

Then system $(\mathcal{C}, \bar{\mathcal{C}}, C)$ is called exact observable.

In this section, for convenience, we remark the symbols Z_k, X_k, I_k, Π_k and Ω_k in (16)-(20) as $Z_k(N), X_k(N), I_k(N), \Pi_k(N)$ and $\Omega_k(N)$, respectively.

To solve Problem 2, we make the following standard assumption.

Assumption 2: System $(\mathcal{C}, \bar{\mathcal{C}}, H^{1/2})$ is exact observable.

Before showing the main results of this section, we shall introduce the following two lemmas.

Lemma 3: Under Assumption 2, there exists integer $N_0 > 0$ such that $X_d(N) \geq Z_d(N) > 0$ for arbitrary $N > N_0$.

Proof: Assuming that this is not true, then there exists nonzero $\chi_d \in \mathbb{R}^n$ satisfying $\chi_d' Z_d(N)\chi_d = 0$.

For the given χ_d, there holds $\hat{\chi}_{d/i} = \chi_d, i = 0, \cdots, d$. Then from (15) we have:

$$\sum_{k=d}^{N} \mathbb{E}(\chi_k' H \chi_k + \upsilon_{k-d}' S \upsilon_{k-d}) = \chi_d' [\sum_{i=1}^{d+1} G_d^i(N) + I_d(N)]\chi_d = \chi_d' Z_d(N)\chi_d = 0. \quad (29)$$

Noting Assumption 2, it can be induced from (29) that $\chi_k = 0, \upsilon_k = 0$, for $k = d, \cdots, N$ which contradicts with $\chi_d \neq 0$. In other words, there exists N_0 such that $Z_d(N) > 0$ for any $N > N_0$.

Finally, it can be deduced from (16)-(18) that there holds $X_d(N) \geq Z_d(N) > 0$ for $N > N_0$. ■

Lemma 4: The system (1) can be stabilized in the mean square sense if and only if $\sum_{k=0}^{\infty} \mathbb{E}(\chi_k'\chi_k) < +\infty$ holds.

Proof: 'Sufficiency': If $\sum_{k=0}^{\infty} \mathbb{E}(\chi_k'\chi_k) < +\infty$, it is obvious that there holds $\lim_{k\to+\infty} \mathbb{E}(\chi_k'\chi_k) = 0$. In other words, system (1) can be stabilized.

'Necessity': Suppose that the system (1) can be mean square stabilizable, we will prove that $\sum_{k=0}^{\infty} \mathbb{E}(\chi_k'\chi_k) < +\infty$.

Note Definition 1 that there exists $\upsilon_k = L\hat{\chi}_{k/k} + \sum_{i=1}^{d} L_i \upsilon_{k-i}$ such that $\lim_{k\to\infty} \mathbb{E}(\chi_k'\chi_k) = 0$, where $L, L_i, i = 1, \cdots, d$ are undetermined matrices. Then the system (1) can be rewritten as:

$$\bar{\chi}_{k+1} = (\mathcal{A}_k + \mathcal{B}_k \mathcal{L})\bar{\chi}_k. \quad (30)$$

where

$$\mathcal{A}_k = \begin{pmatrix} \mathcal{C}_k & 0 & 0 & \cdots & 0 & \mathcal{D}_k \\ (1-\varpi_{k+1})\mathcal{C}_k & \varpi_{k+1}\mathcal{C} & 0 & \cdots & 0 & \bar{\mathcal{D}}_k \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I & 0 \end{pmatrix},$$

$$\mathcal{L}' = \begin{pmatrix} 0 \\ L' \\ L_1' \\ \vdots \\ L_d' \end{pmatrix}', \quad \bar{\chi}_k = \begin{pmatrix} \chi_k \\ \hat{\chi}_{k/k} \\ \upsilon_{k-1} \\ \vdots \\ \upsilon_{k-d} \end{pmatrix}, \quad \mathcal{B}_k = \begin{pmatrix} 0 \\ 0 \\ I \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad (31)$$

where $\mathcal{C}_k = \mathcal{C} + \vartheta_k \bar{\mathcal{C}}, \mathcal{D}_k = \mathcal{D} + \vartheta_k \bar{\mathcal{D}}$ and $\bar{\mathcal{D}}_k = (1-\varpi_{k+1})\mathcal{D}_k + \varpi_{k+1}\mathcal{D}$.

Since the system (1) can be stabilized by $\upsilon_k = \mathbf{L}\hat{\chi}_{k+d/k}$, we have that $\lim_{k\to+\infty} \mathbb{E}(\chi_k'\chi_k) = 0$. In view of $0 \leq \mathbb{E}(\hat{\chi}_{k/k-l}'\hat{\chi}_{k/k-l}) \leq \mathbb{E}(\chi_k'\chi_k)$, $l = 0, \cdots, d$, we have that $\lim_{k\to+\infty} \mathbb{E}(\upsilon_k'\upsilon_k) = 0$. Besides, it can be concluded from (31) that $\lim_{k\to+\infty} \mathbb{E}(\bar{\chi}_k'\bar{\chi}_k) = 0$.

Similar to [8], Noting that $\lim_{k\to+\infty} \mathbb{E}(\bar{\chi}_k'\bar{\chi}_k) = 0$ in (30), it follows that $\sum_{k=0}^{+\infty} \mathbb{E}(\bar{\chi}_k'\bar{\chi}_k) < +\infty$. Finally, using (31), $\sum_{k=0}^{+\infty} \mathbb{E}(\chi_k'\chi_k) < +\infty$ can be derived. ■

Now the solution for Problem 2 shall be presented in the following theorem.

Theorem 2: The system (1) is mean square stabilized if and only if the following modified algebraic Riccati equation has the unique solution such that $X \geq Z > 0, I \leq 0$:

$$Z = H + \mathcal{C}'Z\mathcal{C} + \sigma^2\bar{\mathcal{C}}'X\bar{\mathcal{C}} - \Omega'\Pi^{-1}\Omega + (1-q)\sigma^2\bar{\mathcal{C}}'I\bar{\mathcal{C}}, \quad (32)$$

$$X = Z + \sum_{i=0}^{d-1}(\mathcal{C}')^i\Omega'\Pi^{-1}\Omega\mathcal{C}^i, \quad (33)$$

$$I = -(\mathcal{C}')^d\Omega'\Pi^{-1}\Omega\mathcal{C}^d + q\mathcal{C}'I\mathcal{C}. \quad (34)$$

where Ω, Π are

$$\Omega = \mathcal{D}'Z\mathcal{C} + \sigma^2\bar{\mathcal{D}}'X\bar{\mathcal{C}} + \mathcal{D}'I\mathcal{C} + (1-q)\sigma^2\bar{\mathcal{D}}'I\bar{\mathcal{C}}, \quad (35)$$

$$\Pi = S + \mathcal{D}'Z\mathcal{D} + \sigma^2\bar{\mathcal{D}}'X\bar{\mathcal{D}} + \mathcal{D}'I\mathcal{D} + (1-q)\sigma^2\bar{\mathcal{D}}'I\bar{\mathcal{D}}. \quad (36)$$

The system (1) can be stabilized by

$$\upsilon_k = K\hat{\chi}_{k+d/k} = -\Pi^{-1}\Omega\hat{\chi}_{k+d/k}. \quad (37)$$

Moreover, the infinite horizon cost function (27) can be minimized by (37).

Proof: 'Necessity': If the multiplicative noise system (1) is mean square stabilizable with controller (37), we will show the modified algebraic Riccati equations (32)-(34) have the unique positive definite solution such that $X \geq Z > 0$, $I \leq 0$.

The outline of the necessity proof is given as below:

- To show the monotonicity of $Z_d(N), X_d(N), I_d(N)$ with respect to N;
- To show the boundedness of $Z_d(N), X_d(N), I_d(N)$;
- To show the uniqueness of the solution to (32)-(34).

Firstly, in order to prove the monotonicity of $Z_d(N), X_d(N), I_d(N)$ with respect to N, we select the initial conditions $\upsilon_j = 0, j = -d, \cdots, -1$. From (15), the optimal J_N^* is as:

$$\begin{aligned}
J_N^* &= V_N(0,\chi_0) - \sum_{k=0}^{d-1}\mathbb{E}(\upsilon_{k-d}'S\upsilon_{k-d}) + \sum_{k=0}^{d-1}\mathbb{E}[(\upsilon_{k-d} \\
&\quad + \Pi_k^{-1}(N)\Omega_k(N)\hat{\chi}_{k/k-d})'\Pi_k(N)(\upsilon_{k-d} + \Pi_k^{-1}(N) \\
&\quad \times \Omega_k(N)\hat{\chi}_{k/k-d})] \\
&= \mathbb{E}[\chi_0'G_0^1(N)\chi_0 + \chi_0'\sum_{i=2}^{d+1}G_0^i(N)\hat{\chi}_{0/i-d-2} + \chi_0' \\
&\quad \times I_0(N)\hat{\chi}_{0/0} - \sum_{k=0}^{d-1}\mathbb{E}(\upsilon_{k-d}'S\upsilon_{k-d}) + \sum_{k=0}^{d-1}(\upsilon_{k-d} \\
&\quad + \Pi_k^{-1}(N)\Omega_k(N)\hat{\chi}_{k/k-d})'\Pi_k(N)(\upsilon_{k-d} \\
&\quad + \Pi_k^{-1}(N)\Omega_k(N)\hat{\chi}_{k/k-d})] \\
&= \mathbb{E}[\chi_0'G_0^1(N)\chi_0 + \chi_0'\sum_{i=2}^{d+1}G_0^i(N)\hat{\chi}_{0/i-d-2} \\
&\quad + \chi_0'I_0(N)\hat{\chi}_{0/0} - \sum_{k=0}^{d-1}\hat{\chi}_{k/k-d}'G_k^2(N)\hat{\chi}_{k/k-d}] \\
&= \mathbb{E}[\chi_0'G_0^1(N)\chi_0 + \chi_0'(\sum_{i=2}^{d+1}G_0^i(N) + I_0(N))\hat{\chi}_{0/0} \\
&\quad - \sum_{k=0}^{d-1}\hat{\chi}_{0/0}'(\mathcal{C}^k)'G_k^2(N)\mathcal{C}^k\hat{\chi}_{0/0}] \\
&= \mathbb{E}[\chi_0'G_0^1(N)\chi_0 + \chi_0'I_0(N)\hat{\chi}_{0/0}], \quad (38)
\end{aligned}$$

where $\hat{\chi}_{0/i} = \hat{\chi}_{0/0}$ for $i < 0$, $\hat{\chi}_{k/k-d} = \mathcal{C}\hat{\chi}_{k-1/k-d} + \mathcal{D}\upsilon_{k-d-1}$ for $k = 1, \cdots, d-1$ and (11)-(14) are used in the above equation.

Then we shall investigate the following three cases:

1) When $\chi_0 = \mathbb{E}\chi_0$, from Lemma 1 we have $\hat{\chi}_{0/0} = \chi_0$. Then (38) becomes

$$J_N^* = \mathbb{E}[\chi_0'G_0^1(N)\chi_0 + \chi_0'I_0(N)\chi_0]. \quad (39)$$

Since $J_N^* \leq J_{N+1}^*$, then

$$\chi_0'[G_0^1(N) + I_0(N)]\chi_0 \leq \chi_0'[G_0^1(N+1) + I_0(N+1)]\chi_0.$$

Therefore, $G_0^1(N) + I_0(N)$ increases with respect to N.

2) When $\mathbb{E}\chi_0 = 0$, from Lemma 1, we have $\hat{\chi}_{0/0} = (1-\varpi_0)\chi_0$. Then we can obtain

$$\begin{aligned}
&\mathbb{E}[\chi_0'(G_0^1(N) + (1-q)I_0(N))\chi_0] \\
&\quad \leq \mathbb{E}[\chi_0'(G_0^1(N+1) + (1-q)I_0(N+1))\chi_0], \quad (40)
\end{aligned}$$

i.e., $G_0^1(N) + qI_0(N)$ also increases with respect to N.

3) Noting $\sum_{k=d}^{N}\mathbb{E}(\chi_k'H\chi_k + \upsilon_{k-d}'S\upsilon_{k-d}) \leq \sum_{k=d}^{N+1}\mathbb{E}(\chi_k' \times H\chi_k + \upsilon_{k-d}'S\upsilon_{k-d})$, then from (29) we have $\sum_{i=1}^{d+1}G_d^i(N) + I_d(N) \leq \sum_{i=1}^{d+1}G_d^i(N+1) + I_d(N+1)$. In other words, $Z_d(N) = \sum_{i=1}^{d+1}G_d^i(N) + I_d(N)$ increases with N.

So far, we have derived the monotonically increasing of $G_0^1(N) + (1-q)I_0(N)$, $X_0(N) = G_0^1(N) + I_0(N)$ and $Z_d(N) = \sum_{i=1}^{d+1}G_d^i(N) + I_d(N)$.

In what follows, the boundedness of $G_0^1(N)$, $X_0(N)$ and $Z_d(N)$ will be proved.

Since system (1) is mean square stabilizable, i.e., $\lim_{k\to+\infty}\mathbb{E}(\chi_k'\chi_k) = 0$, and the stabilizing controller satisfies

$$\upsilon_k = \mathbf{L}\hat{\chi}_{k+d/k}, \quad \mathbf{L}'S\mathbf{L} \leq \lambda I, \quad H \leq \lambda I, \quad (41)$$

where λ is constant.

From Lemma 4, there exist constants c, c_1 satisfying

$$\begin{aligned}
\sum_{k=0}^{\infty}\mathbb{E}(\hat{\chi}_{k/k-d}'\hat{\chi}_{k/k-d}) &\leq \sum_{k=0}^{\infty}\mathbb{E}(\chi_k'\chi_k) \\
&< c\mathbb{E}(\chi_0'\chi_0) \leq c_1\mathbb{E}(\chi_d'\chi_d). \quad (42)
\end{aligned}$$

Also we have

$$\begin{aligned}
&\sum_{k=0}^{\infty}\mathbb{E}(\chi_k'H\chi_k + \upsilon_{k-d}'S\upsilon_{k-d}) \\
&= \sum_{k=0}^{\infty}\mathbb{E}(\chi_k'H\chi_k + \hat{\chi}_{k/k-d}'\mathbf{L}'S\mathbf{L}\hat{\chi}_{k/k-d})
\end{aligned}$$

$$\le \lambda \sum_{k=0}^{\infty} \mathbb{E}(\chi_k'\chi_k + \hat{\chi}_{k/k-d}'\hat{\chi}_{k/k-d})$$
$$\le (\lambda+1)\sum_{k=0}^{\infty}\mathbb{E}(\chi_k'\chi_k)$$
$$\le c(\lambda+1)\mathbb{E}(\chi_0'\chi_0) \le c_1(\lambda+1)\mathbb{E}(\chi_d'\chi_d). \quad (43)$$

Note that

$$\sum_{k=0}^{N}(\mathbb{E}\chi_k'H\chi_k+\mathbb{E}\upsilon_{k-d}'S\upsilon_{k-d})\le\sum_{k=0}^{\infty}\mathbb{E}(\chi_k'H\chi_k+\upsilon_{k-d}'S\upsilon_{k-d}).$$

Then, from (29) we get

$$\mathbb{E}\{\chi_d'[\sum_{i=1}^{d+1}G_d^i(N)+I_d(N)]\chi_d\} \le c_1(\lambda+1)\mathbb{E}(\chi_d'\chi_d), \quad (44)$$

which implies that $Z_d(N) = \sum_{i=1}^{d+1} G_d^i(N) + I_d(N)$ is bounded.

Similar to the above discussions, the following two cases are considered:

1) If $\chi_0 = \mathbb{E}\chi_0$, and $\upsilon_j = 0, j = -d, \cdots, -1$, from (38) we have

$$\chi_0'[G_0^1(N)+I_0(N)]\chi_0 \le c(\lambda+1)\mathbb{E}(\chi_0'\chi_0). \quad (45)$$

Thus, $X_0(N) = G_0^1(N) + I_0(N)$ is bounded.

2) If $\mathbb{E}\chi_0 = 0$ and $\upsilon_j = 0, j = -d, \cdots, -1$, (39) indicates

$$\mathbb{E}[\chi_0'(G_0^1(N)+(1-q)I_0(N))\chi_0] \le c(\lambda+1)\mathbb{E}(\chi_0'\chi_0),$$

i.e., the boundedness of $G_0^1(N)+(1-q)I_0(N)$ has been shown.

It is noted that

$$X_d(N)=X_0(N-d), Z_d(N)=Z_0(N-d), I_d(N)=I_0(N-d),$$

and $G_d^1(N) + (1-q)I_0(N), X_d(N), Z_d(N)$ are monotonically increasing with respect to N, and they are bounded. Thus, the convergences of $G_d^1(N), X_d(N), Z_d(N)$ have been shown, i.e., there exist G^1, X, Z satisfying

$$G^1=\lim_{N\to+\infty}G_d^1(N), X=\lim_{N\to+\infty}X_d(N), Z=\lim_{N\to+\infty}Z_d(N). \quad (46)$$

Noting that $X_d(N) = G_d^1(N)+I_d(N)$, then we have that $I_d(N)$ is convergent with N. It follows that $I = \lim_{N\to+\infty} I_d(N)$.

On the other hand, from (9), (10) and (12), the convergence of $G_d^2(N)$ can be derived. Furthermore, it can be easily verified from (13) that $G_d^i(N), i = 3, \cdots, d+1$ are convergent with N, i.e., there exists G^i satisfying $G^i = \lim_{N\to+\infty} G_d^i(N), i = 2, \cdots, d+1$.

Taking limitations with $N \to \infty$ of (11)-(14), we get

$$G^1 = H + \mathcal{C}'G^1\mathcal{C} + \sigma^2\bar{\mathcal{C}}'G^1\bar{\mathcal{C}} + (1-q)\mathcal{C}'I\mathcal{C} + (1-q)\sigma^2\bar{\mathcal{C}}'I\bar{\mathcal{C}}, \quad (47)$$
$$G^2 = -\Omega'\Pi^{-1}\Omega, \quad (48)$$
$$G^i = \mathcal{C}'G^{i-1}\mathcal{C},\ i = 3, \cdots, d+1, \quad (49)$$
$$I = \mathcal{C}'G^{d+1}\mathcal{C} + q\mathcal{C}'I\mathcal{C}, \quad (50)$$

where

$$\Pi = S + \mathcal{D}'\sum_{j=1}^{d+1}G^j\mathcal{D} + \sigma^2\bar{\mathcal{D}}'G^1\bar{\mathcal{D}} + \mathcal{D}'I\mathcal{D} + (1-q)\sigma^2\bar{\mathcal{D}}'I\bar{\mathcal{D}}, \quad (51)$$
$$\Omega = \mathcal{D}'\sum_{j=1}^{d+1}G^j\mathcal{C} + \sigma^2\bar{\mathcal{D}}'G^1\bar{\mathcal{C}}+\mathcal{D}'I\mathcal{C}+(1-q)\sigma^2\bar{\mathcal{D}}'I\bar{\mathcal{C}}. \quad (52)$$

Letting $X = G^1 + I, Z = \sum_{i=1}^{d} G^i + I$, then (32)-(34) and (35)-(36) can be obtained.

It can be deduced from Lemma 3 that $X_d(N) \ge Z_d(N) > 0$ for any $N > N_0$ which implies that $X \ge Z > 0$. Besides, from (18) we have $I_d(N) \le 0$. Thus, $I = \lim_{N\to+\infty} I_d(N) \le 0$.

Finally, we shall show the uniqueness of the solution to (32)-(34). If this is not true, we assume (Ξ, Λ, Θ) with $\Xi \ge \Lambda > 0, \Theta \le 0$ satisfying (32)-(34).

In the case of $\chi_0 = \mathbb{E}\chi_0, \upsilon_j = 0, j = -d, \cdots, -1$, by taking limitations of (39), we have

$$J^* = \chi_0'X\chi_0 = \chi_0'\Xi\chi_0, \quad \Rightarrow X = \Xi. \quad (53)$$

Besides, if $\mathbb{E}\chi_0 = 0, \upsilon_j = 0, j = -d, \cdots, -1$, then there holds

$$J^*=\mathbb{E}[\chi_0'(X+(1-q)I)\chi_0]=\mathbb{E}[\chi_0'(\Xi+(1-q)\Theta)\chi_0]. \quad (54)$$

Therefore, $I = \Theta$ can be derived from (53) and (54).

Moreover, with the stabilizing controller (37), by taking limitations of (29), we have

$$\sum_{k=d}^{\infty}\mathbb{E}(\chi_k'H\chi_k + \upsilon_{k-d}'S\upsilon_{k-d}) = \mathbb{E}(\chi_d'Z\chi_d) = \mathbb{E}(\chi_d'\Lambda\chi_d),$$

Thus, it can be obtained that $Z = \Lambda$ which means that the solution to (32)-(34) is unique. This ends the necessity proof.

'Sufficiency': Suppose (32)-(34) admit the unique solution $X \ge Z > 0, I \le 0$. We will show that system (1) can be stabilized by controller (37).

We define the Lyapunov function candidate as

$$V(k,\chi_k)=\mathbb{E}\Big[\chi_k'G^1\chi_k+\chi_k'\sum_{i=2}^{d+1}G^i\hat{\chi}_{k/i+k-d-2}+\chi_k'I\hat{\chi}_{k/k}\Big], \quad (55)$$

where $G^i(i = 1, \cdots, d+1), I$ satisfy (47)-(50).

Using (1), $V(k+1, \chi_{k+1})$ can be reformulated as

$$V(k+1,\chi_{k+1})$$
$$= \mathbb{E}\Big\{\chi_k'[\mathcal{C}'G^1\mathcal{C} + \sigma^2\bar{\mathcal{C}}'G^1\bar{\mathcal{C}} + (1-q)$$
$$\times(\mathcal{C}'I\mathcal{C}+\sigma^2\bar{\mathcal{C}}'I\bar{\mathcal{C}})]\chi_k + 2\upsilon_{k-d}'[\mathcal{D}'\sum_{i=1}^{d+1}G^i\mathcal{C}+\sigma^2\bar{\mathcal{D}}'G^1\bar{\mathcal{C}}$$
$$+(1-q)(\mathcal{D}'I\mathcal{C}+\sigma^2\bar{\mathcal{D}}'I\bar{\mathcal{C}})]\chi_k + \upsilon_{k-d}'[\mathcal{D}'\sum_{i=1}^{d+1}G^i\mathcal{D}+\sigma^2$$
$$\times\bar{\mathcal{D}}'G^1\bar{\mathcal{D}}+(1-q)(\mathcal{D}'I\mathcal{D}+\sigma^2\bar{\mathcal{D}}'I\bar{\mathcal{D}})]\chi_k + \chi_k'\mathcal{C}'G^{d+1}\mathcal{C}$$

$$\times \hat{\chi}_{k/k} + \chi_k' \sum_{i=3}^{d+1} G^i \hat{\chi}_{k/i+k-d-2} + q\chi_k' \mathcal{C}' I \mathcal{C} \hat{\chi}_{k/k} \Big\}$$
$$= \mathbb{E}\Big\{ \chi_k'(G^1 - H)\chi_k + \upsilon_{k-d}'(\Pi - S)\upsilon_{k-d} + 2\upsilon_{k-d}'\Omega$$
$$\times \hat{\chi}_{k/k-d} + \chi_k' \sum_{i=3}^{d+1} G^i \hat{\chi}_{k/i+k-d-2} + \chi_k'(\mathcal{C}' G^{d+1}\mathcal{C}$$
$$+ q\mathcal{C}' I \mathcal{C})\hat{\chi}_{k/k} \Big\}. \quad (56)$$

Thus, we have

$$\begin{aligned} V(k, \chi_k) - V(k+1, \chi_{k+1}) &= \mathbb{E}[\chi_k' H \chi_k + \upsilon_{k-d}' S \upsilon_{k-d} - (\upsilon_{k-d} + K\hat{\chi}_{k/k-d})'\Pi \\ &\quad \times (\upsilon_{k-d} + K\hat{\chi}_{k/k-d})] \\ &= \mathbb{E}(\chi_k' H \chi_k + \upsilon_{k-d}' S \upsilon_{k-d}) \geq 0, \end{aligned} \quad (57)$$

Moreover, for $V(k, \chi_k)$, there holds

$$\begin{aligned} V(k, \chi_k) &= \mathbb{E}\Big[\chi_k' G^1 \chi_k + \chi_k' \sum_{i=2}^{d+1} G^i \hat{\chi}_{k/i+k-d-2} + \chi_k' I \hat{\chi}_{k/k}\Big] \\ &= \mathbb{E}\Big[\chi_k'(\sum_{i=1}^{d+1} G^1 + I)\chi_k - \sum_{i=2}^{d+1}(\chi_k - \hat{\chi}_{k/i+k-d-2})' G^i \\ &\quad \times (\chi_k - \hat{\chi}_{k/i+k-d-2}) - (\chi_k - \hat{\chi}_{k/k})' I (\chi_k - \hat{\chi}_{k/k})\Big] \\ &\geq \mathbb{E}\Big[\chi_k'(\sum_{i=1}^{d+1} G^1 + I)\chi_k - (\chi_k - \hat{\chi}_{k/k})' I (\chi_k - \hat{\chi}_{k/k})\Big] \\ &= \mathbb{E}\Big[\chi_k' Z \chi_k - (\chi_k - \hat{\chi}_{k/k})' I (\chi_k - \hat{\chi}_{k/k})\Big] \geq 0. \end{aligned} \quad (58)$$

where $G^i \leq 0$, $Z > 0$ and $I \leq 0$ are used in the above equation.

From (58), we have that $V(k, \chi_k) \geq 0$, and (57) means that $V(k, \chi_k)$ decreases with respect to k. Therefore, $V(k, \chi_k)$ is convergent.

For any $m > 0$, we have

$$\begin{aligned} &\lim_{m\to+\infty} \sum_{k=m+d}^{m+N} \mathbb{E}(\chi_k' H \chi_k + \upsilon_{k-d}' S \upsilon_{k-d}) \\ &= \lim_{m\to+\infty} V(m+d, \chi_{m+d}) - V(m+N+1, \chi_{m+N+1}) = 0. \end{aligned} \quad (59)$$

Note that

$$\begin{aligned} \mathbb{E}[\chi_d' I_d(N)\hat{\chi}_{d/d}] &= \mathbb{E}[\chi_d' I_d(N)\chi_d] - \mathbb{E}[\tilde{\chi}_{d/d}' I_d(N)\tilde{\chi}_{d/d}] \\ &\geq \mathbb{E}(\chi_d' I_d(N)\chi_d), \\ \mathbb{E}[\chi_d' G_d^i(N)\hat{\chi}_{d/i-2}] &\\ &= \mathbb{E}[\chi_d' G_d^i(N)\chi_d] - \mathbb{E}[\tilde{\chi}_{d/i-2}' G_d^i(N)\tilde{\chi}_{d/i-2}] \\ &\geq \mathbb{E}(\chi_d' G_d^i(N)(N)\chi_d),\ i = 2, \cdots, d+1, \end{aligned}$$

where $\tilde{\chi}_{d/j} = \chi_d - \hat{\chi}_{d/j}, j = 0, \cdots, d$, and $I_d(N) \leq 0$, $G_d^i(N) \leq 0$, $i = 2, \cdots, d+1$ have been used in the above equation.

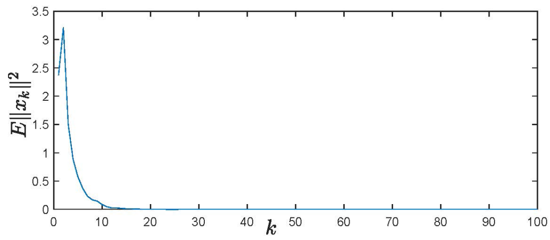

FIGURE 1. Dynamic behavior of $\mathbb{E}\|\chi_k\|^2$.

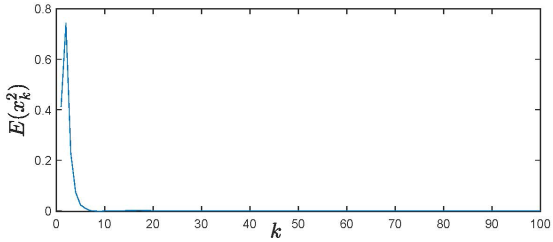

FIGURE 2. Dynamic behavior of the first coordinate of $\mathbb{E}(\chi_k)$.

Using Theorem 1, we get

$$\begin{aligned} &\sum_{k=d}^{N} \mathbb{E}(\chi_k' H \chi_k + \upsilon_{k-d}' S \upsilon_{k-d}) \\ &\geq \mathbb{E}\Big\{\chi_d' G_d^1(N)\chi_d + \chi_d' \sum_{i=2}^{d+1} G_d^i(N)\hat{\chi}_{d/i-2} + \chi_d' I_d(N)\hat{\chi}_{d/d}\Big\} \\ &\geq \mathbb{E}\{\chi_d'[\sum_{i=1}^{d+1} G_d^i(N) + I_d(N)]\chi_d\}. \end{aligned} \quad (60)$$

Since the coefficients are time-invariance, we have

$$\begin{aligned} &\sum_{k=m+d}^{m+N} \mathbb{E}(\chi_k' H \chi_k + \upsilon_{k-d}' S \upsilon_{k-d}) \\ &\qquad \geq \mathbb{E}\{\chi_{m+d}'[\sum_{i=1}^{d+1} G_d^i(N) + I_d(N)]\chi_{m+d}\} \geq 0. \end{aligned}$$

Finally, with (59), it follows

$$\lim_{m\to+\infty} \mathbb{E}\{\chi_{m+d}'[\sum_{i=1}^{d+1} G_d^i(N) + I_d(N)]\chi_{m+d}\} = 0. \quad (61)$$

Noting $\sum_{i=1}^{d+1} G_d^i(N) + I_d(N) > 0$ for $N > N_0$, $\lim_{m\to+\infty} \mathbb{E}(\chi_m' \chi_m) = 0$ can be derived. Therefore, we have proved that system (1) can be stabilized by (37). The proof is completed. ∎

Remark 3: The stabilization conditions (necessary and sufficient) for system (1)-(2) are proposed in Theorem 2. It is stressed that only some sufficient conditions were given in previous works [13], [25].

IV. NUMERICAL EXAMPLE

In this section, we will provide numerical examples to show the effectiveness of the main results obtained in this paper.

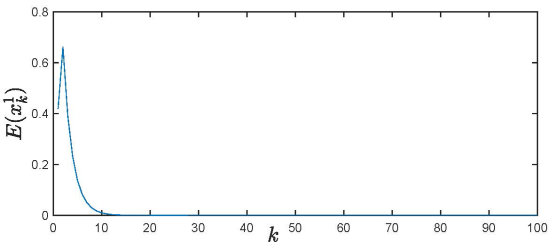

FIGURE 3. Dynamic behavior of the second coordinate of $\mathbb{E}(\chi_k)$.

Consider system (1)-(2) with delay $d = 2$, coefficients $\mathcal{C} = \begin{pmatrix} 0.7294 & 0 \\ 0 & 0.7175 \end{pmatrix}$, $\bar{\mathcal{C}} = \begin{pmatrix} 0.1334 & 0 \\ 0 & 0.4458 \end{pmatrix}$, $\mathcal{D} = \begin{pmatrix} 0.5088 & 0 \\ 0 & 0.5305 \end{pmatrix}$, $\bar{\mathcal{D}} = \begin{pmatrix} 0.8597 & 0 \\ 0 & 0.6777 \end{pmatrix}$, $\mu = 0$, $\sigma^2 = 1$, $p = 0.5$, $\upsilon_{-2} = \begin{pmatrix} 0.5415 \\ 0.2817 \end{pmatrix}$, $\upsilon_{-1} = \begin{pmatrix} 0.4809 \\ 0.6849 \end{pmatrix}$, and cost function (3) with $H = \begin{pmatrix} 0.8058 & 0 \\ 0 & 0.5312 \end{pmatrix}$, $S = \begin{pmatrix} 0.9559 & 0 \\ 0 & 0.0667 \end{pmatrix}$. Via iterative method, we can obtain the solutions of Z, X, I in (32)-(36) as

$$\begin{cases} Z \approx \begin{pmatrix} 1.3931 & 0 \\ 0 & 0.7177 \end{pmatrix}, \quad X \approx \begin{pmatrix} 1.7270 & 0 \\ 0 & 1.5677 \end{pmatrix}, \\ I \approx \begin{pmatrix} -0.0709 & 0 \\ 0 & -0.1704 \end{pmatrix}, \end{cases}$$

which implies that $X \geq Z > 0, I \leq 0$. Observe the dynamic behavior of χ_k in Fig.1, Fig.2 and Fig.3. Obviously, it can be obtained that the state χ_k is asymptotically mean-square stable.

V. CONCLUSION

The output feedback control problem has been investigated for the multiplicative noise system with intermittent noise and input delay. For the finite time horizon case, the optimal output feedback controller has been derived; For the infinite horizon case, it has been shown the multiplicative noise system with intermittent noise and input delay is mean square stabilizable if and only if the given algebraic modified Riccati equation has the unique solution. For future research, the obtained results are expected to solve the random time delay case.

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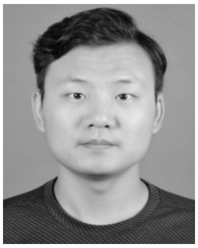

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