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LINEAR-QUADRATIC-GAUSSIAN MIXED MEAN-FIELD GAMES WITH HETEROGENEOUS INPUT CONSTRAINTS*

YING HU†, JIANHUI HUANG‡, AND TIANYANG NIE§

Abstract. We consider a class of linear-quadratic-Gaussian mean-field games having a major agent and numerous heterogeneous minor agents in the presence of mean-field interactions. The individual admissible controls are constrained in closed convex subsets Γ_k of $\mathbb{R}^m$. The decentralized strategies of individual agents and the consistency condition system are represented in a unified manner through a class of mean-field forward-backward stochastic differential equations involving projection operators on Γ_k. The well-posedness of the consistency system is established in both the local and global cases through the contraction mapping and discounting methods, respectively. A related ε-Nash-equilibrium property is also verified.

Key words. ε-Nash equilibrium, forward-backward stochastic differential equation, input constraint, projection operator, linear-quadratic mixed mean-field games

AMS subject classifications. 60H10, 60H30, 91A10, 91A23, 91A25, 93E20

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1. Introduction.

Mean-field games (MFGs) for stochastic large-population systems have been well-studied because of their breadth of applications in various fields, such as economics, engineering, social sciences, and operational research. Large-population systems are distinguished by having numerous agents (or players). The individual influence of any single agent on the overall population is negligible, but the effects of its statistical behaviors cannot be ignored at the population scale. Mathematically, all agents are weakly coupled in their dynamics or cost functionals through the state average (in a linear-state case) or the general empirical measure (in a nonlinear-state case), either of which characterizes the statistical effect generated by the population from a macroscopic perspective. Because of these features, when the number of agents is sufficiently high, complicated coupling features arise, and it is unrealistic for a given agent to obtain all other agents' information. Consequently, for an agent to design centralized strategies on the basis of information concerning all peers in a large-population system is an intractable problem. Alternatively, one reasonable and practical solution is to transform a high-dimensional and weakly coupled problem to a low-dimensional and decoupled one; thus, the complexity in both analysis and computation can be reduced. To accomplish this task, one method is

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†IRMAR, Université Rennes 1, Campus de Beaulieu, 35042 Rennes Cedex, France, and School of Mathematical Sciences, Fudan University, Shanghai 200433, People's Republic of China (Ying.Hu@univ-rennes1.fr).
‡Department of Applied Mathematics, Hong Kong Polytechnic University, Hong Kong, People's Republic of China (james.huang@polyu.edu.hk).
§Corresponding author. School of Mathematics, Shandong University, Jinan 250100, People's Republic of China (nietianyang@sdu.edu.cn).

to investigate relevant decentralized strategies based only on local information. The relevant strategies are based only on the individual state of a given agent and some mass-effect quantities that are computed offline.

In this context, and motivated by the theory of the propagation of chaos, Lasry and Lions [39, 40, 41] proposed distributed closed-loop strategies, which were formulated as a coupled nonlinear forward-backward system consisting of a Hamilton–Jacobi–Bellman (HJB) equation and a Fokker–Planck equation. Moreover, the limiting problem enabled the design of approximate Nash-equilibrium strategies. Independently, Huang, Malhamé, and Caines [37] developed a similar program called the Nash certainty equivalence (NCE) principle, which was motivated by the analysis of large communications networks. In principle, the MFGs procedure consists of the following four main steps (see [5, 17, 34, 36, 41]): (1) A limiting mass-effect term, which comes from the asymptotic mass-effect behavior when the agent number N tends to infinity, is introduced. This limiting term should be treated as an exogenous and undetermined "frozen" term at this moment. (2) Through the replacement of the mass-effect term with the frozen limiting term, a related limiting-optimization problem can be formulated. Thus, the initial, highly coupled problem can be decoupled and only parameterized by this generic frozen limit. Subsequently, an HJB equation can be obtained on the basis of the dynamic programming principle (DPP) using standard control techniques (see [57, 46]), or a Hamiltonian system can be obtained on the basis of the stochastic maximum principle (SMP); the obtained equations can characterize the decentralized optimal strategies. (3) A consistency condition is established to ensure that the set of decentralized optimal strategies collectively replicates the mass-effect. (4) The derived decentralized strategies are shown to be ε-Nash equilibrium, which justifies the aforementioned scheme for finding the approximate Nash equilibrium.

Recently, Cardaliaguet and Rainer [16] studied the efficiency of MFG Nash equilibria. For further analysis and technical details of MFGs, readers are referred to [1, 5, 15, 17, 26, 34, 54, 55], the comprehensive notes of Cardaliaguet [14], and the books [18, 19] authored by Carmona and Delarue. We mention that there exists a substantial body of literature on MFG in the linear-quadratic-Gaussian (LQG) setting. For example, [35] studied LQG MFGs and its closed-loop strategies using the standard Riccati equation approach; [36] studied LQG MFGs having nonuniform agents through state-aggregation using empirical distribution; [8] investigated the MFG strategies using the SMP method and how the well-posedness of closed-loop strategies is connected to that of a family of open-loop strategies by using a Hamiltonian system; and [2] and [42] studied ergodic and long-time LQG MFGs. For further research on MFG in various LQG settings, readers are referred to [3, 7, 51] and the references therein.

Almost all of the aforementioned works examine standard MFGs, which require that all the agents be statistically identical and that the individual influence on the overall population of a single agent be negligible as the number of agents tends to infinity. However, in the real world, some models exist in which a major agent exerts a significant influence on other agents (called minor agents), regardless of how numerous the minor agents may be. Such interactions appear in numerous socioeconomic problems (e.g., [18, 19, 38]). This type of game involving agents with different hierarchical levels is usually called a mixed-type game. Compared with MFGs having only minor agents, a distinctive feature of mixed-type MFGs is that the mean-field behavior of the minor agents is affected by the major agent; thus, it is a random process, and the influence of the major agent on the minor agents is not negligible in the limiting problem. To deal with such new features, conditional distribution with respect to the major agent's information flow is introduced (see [47, 20]), and additional analytic steps are required to approach the major and minor agents in a sequential manner. We also remark that there are essential analysis differences between major-minor games and leader-follower (Stackelberg) games, in particular in their response functionals and necessary fixed-point arguments, although both types of games involve some sequential optimization arguments.

For the literature review, we will now briefly describe some relevant works on MFGs involving a major (dominating) agent and minor agents. To the best of our knowledge, the first MFG work having a major agent and minor agents was [33], which studied the mixed game having a major agent and a total finite K class of minor agents in a LQG and infinite-time horizon framework. Using the state-augmentation technique, the related decentralized strategies were derived through the algebraic Riccati equation, and the approximate Nash-equilibrium property was also verified. In a subsequent study [48], the authors examined mean-field LQG mixed games having continuum-parameterized minor agents. Nourian and Caines [47] investigated nonlinear, stochastic dynamic systems having major and minor agents and introduced a coupled stochastic HJB system to MFG strategies because of the random state-average limit. Buckdhan, Li, and Peng [12] studied nonlinear stochastic differential games involving a major agent and numerous, collectively acting, minor agents, engaged in two-person zero-sum stochastic differential games of feedback type control against feedback control, and the limiting behaviors of the saddle-point controls were also studied. For further research on mixed MFG using more probabilistic methods, readers are referred to [4, 20] and the references therein.

In this study, we investigate a class of LQG MFGs with a major agent and minor agents acting in the presence of control constraints. In all of the aforementioned papers concerning linear-quadratic (LQ) control problems, the control was unconstrained (in this sense, it can be called "full control"), and the (feedback) control can be constructed through either DPP or SMP, both of which are automatically admissible. However, if we impose constraints on the admissible control, the entire LQ approach fails to apply (see, e.g., [21, 32]). We emphasize that the LQ control problems concerning control constraints have broad applications in finance and economics. For example, the mean-variance problem in relation to the prohibition of short-selling can be transferred to LQ control problems having positive control (see, e.g., [6, 32]). The optimal investment problems, where the agents have relative performance characteristics (i.e., their portfolio constraints have different half-space or polyhedron forms), can also be addressed through the approach of using LQ control problems having input constraint (see, e.g., [28, 24]). Remark 2.1 of the current paper provide several other constraint sets $\Gamma \subset \mathbb{R}^m$ as well as their applications. For an investigation of LQ problems having positive controls or a more general study where the control is constrained in a given convex cone, readers are referred to [9] for a deterministic case and [21, 32, 43] for a stochastic case.

As far as we know, the present study is the first to examine constrained LQG MFGs having a major agent and a large number of minor agents. In addition to the control constraint being a completely new feature, our study also has other novel features, and these distinguish it from other relevant studies. In [33, 48], the diffusion term directly assumes a constant; hence, the state is driven only by some additive noise. By contrast, the present study considers the mean-field LQG mixed games in which the diffusion term depends on the major agent's and the minor agent's states as well as the individual control strategy. This introduces additional difficulties, especially when applying the general SMP, because now the dynamics are driven by controlled multiplicative noise. In [47, 12], nonlinear stochastic differential games were studied in which the control domain may be an arbitrary (nonconvex) subset. We, however, adopt an LQ mean-field framework having individual controls constrained in a closed convex set; thus, we can explicitly present the optimal strategies through a projection operator. Moreover, we use SMP to obtain the optimal strategies through Hamiltonian systems that are fully coupled forward-backward stochastic differential equations (FBSDEs). This approach differs from [47], in which they used DPP and a verification theorem to characterize the optimal strategies. Here, we connect the consistency condition to a new type of conditional mean-field forward-backward stochastic differential equation (MF-FBSDE) involving projection operators. We establish its well-posedness under suitable conditions using a fixed-point theorem, in both the local case and the global case. Unlike in our previous paper [29], we now focus on the mixed game, which is more realistic and challenging. In this situation, the consistency condition is a conditional MF-FBSDE that does not satisfy the usual monotonicity condition of [31]. Moreover, we require an additional subtle analysis to analyze the major agent's influence and to establish the approximate Nash equilibrium. Finally, motivated by [24], we contend that our results can be applied to solving the optimal investment problems having a major agent and N minor agents.

The remainder of this paper is structured as follows. In section 2, we formulate the LQG MFGs with a control constraint involving a major agent and minor agents. Decentralized strategies are derived through an FBSDE having projection operators. A consistency condition is also established using some fully coupled FBSDEs that come from the SMP. In section 3, we prove the well-posedness of fully coupled conditional MF-FBSDEs, which characterize the consistency condition in the local time horizon case. In section 4, we ascertain the well-posedness of the global time case. In section 5, we verify the ε-Nash equilibrium of the decentralized strategies. Moreover, we examine the convergence rate of the empirical measure.

The main contributions of this paper can be summarized as follows:

- We introduce and analyze a new class of LQG mixed MFGs using SMP. In our setting, both the major agent and minor agents are constrained in their control inputs.
- The diffusion terms of the major and minor agents are dependent on their states and control variables.
- The consistency condition system or NCE principle is represented through a new type of conditional mean-field FBSDE having projection operators.
- We establish the existence and uniqueness of such an NCE system in the local case (i.e., small time horizon) using the contraction mapping method, and in the global case (i.e., arbitrary time horizon) using the discounting method.

2. LQG mixed games with control constraint. Consider a finite time horizon $[0, T]$ for fixed $T > 0$. We assume $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, \mathbb{P})$ is a complete, filtered probability space satisfying usual conditions. $\{W_i(t),\ 0 \le i \le N\}_{0\le t\le T}$ is an $(N+1)$-dimensional Brownian motion on this space. Let $\mathcal{F}_t$ be the natural filtration generated by $\{W_i(s), 0 \le i \le N, 0 \le s \le t\}$ and augmented by $\mathcal{N}_{\mathbb{P}}$ (the class of $\mathbb{P}$-null sets of $\mathcal{F}$). Let $\mathcal{F}_t^{W_0}$, $\mathcal{F}_t^{W_i}$, $\mathcal{F}_t^{i}$ be respectively the augmentation of $\sigma\{W_0(s), 0 \le s \le t\}$, $\sigma\{W_i(s), 0 \le s \le t\}$, $\sigma\{W_0(s), W_i(s), 0 \le s \le t\}$ by $\mathcal{N}_{\mathbb{P}}$. Here, $\{\mathcal{F}_t^{W_0}\}_{0\le t\le T}$ stands for the information of the major agent, while $\{\mathcal{F}_t^{W_i}\}_{0\le t\le T}$ represents the individual information of the ith minor agent.

Throughout the paper, x' denotes the transpose of a vector or a matrix x, and $\mathcal{S}^n$ denotes the set of symmetric $n \times n$ matrices with real elements. For a matrix

$M \in \mathbb{R}^{n\times d}$, we define the norm $|M| := \sqrt{Tr(M'M)}$. If $M \in \mathcal{S}^n$ is positive (semi) definite, we write $M > (\geq) 0$. Let $\mathcal{H}$ be a given Hilbert space, and the set of $\mathcal{H}$-valued continuous functions is denoted by $C(0,T;\mathcal{H})$. If $N(\cdot) \in C(0,T;\mathcal{S}^n)$ and $N(t) > (\geq)0$ for every $t \in [0,T]$, we say that $N(\cdot)$ is positive (semi) definite, which is denoted by $N(\cdot) > (\geq) 0$. Moreover, for a given Hilbert space $\mathcal{H}$ and a filtration $\{\mathcal{G}_t\}_{0\leq t\leq T}$, we introduce the following spaces: $L^2_{\mathcal{G}_T}(\Omega;\mathcal{H})$ denotes the space of $\mathcal{G}_T$-measurable random variables $\{\xi\}$ such that $\mathbb{E}|\xi|^2 < \infty$; $L^2_{\mathcal{G}}(0,T;\mathcal{H})$ denotes the space of $\mathcal{G}_t$-progressively measurable processes $\{x(s), s \in [0,T]\}$ such that $\mathbb{E}\int_0^T |x(t)|^2 dt < \infty$; $L^2_{\mathcal{G}}(\Omega; C(0,T;\mathcal{H}))$ denotes the space of $\mathcal{G}$-adapted continuous processes $\{x(s), s \in [0,T]\}$ such that $\mathbb{E}\sup_{0\leq t\leq T}|x(t)|^2 < \infty$.

Now, we consider an LQG mixed mean-field game involving a major agent $\mathcal{A}_0$ and a heterogeneous large population with N individual minor agents $\{\mathcal{A}_i : 1 \leq i \leq N\}$. Unlike other works of LQG mixed games, our control domain is constrained in a closed convex set (more details of constraints will be given later). The states x_0 for major agent $\mathcal{A}_0$ and x_i for each minor agent $\mathcal{A}_i$ are modeled by the following controlled linear stochastic differential equations (SDEs) with empirical state-average coupling:

$$\begin{aligned} dx_0(t) &= [A_0(t)x_0(t) + B_0(t)u_0(t) + F_0^1(t)x^{(N)}(t) + b_0(t)]dt \\ &\quad + [C_0(t)x_0(t) + D_0(t)u_0(t) + F_0^2(t)x^{(N)}(t) + \sigma_0(t)]dW_0(t), \quad x_0(0) = x_0 \in \mathbb{R}^n, \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} dx_i(t) &= [A_{\theta_i}(t)x_i(t) + B(t)u_i(t) + F_1(t)x^{(N)}(t) + b(t)]dt \\ &\quad + [C(t)x_i(t) + D_{\theta_i}(t)u_i(t) + F_2(t)x^{(N)}(t) + Hx_0(t) + \sigma(t)]dW_i(t), \; x_i(0)=x \in \mathbb{R}^n, \end{aligned} \tag{2.2}$$

where $x^{(N)}(\cdot) = \frac{1}{N}\sum_{i=1}^N x_i(\cdot)$ is the state-average of minor agents. Note that $\mathcal{F}_t^i$ is the individual decentralized information, while $\mathcal{F}_t$ is the centralized information driven by all Brownian motion components. We point out that the heterogeneous noise W_i is specific for individual agent $\mathcal{A}_i$, whereas $x_i(t)$ is adapted to $\mathcal{F}_t$ instead of $\mathcal{F}_t^i$ due to the coupling state-average $x^{(N)}$. The coefficients of (2.1) and (2.2) are deterministic matrix-valued functions with appropriate dimensions. The number θ_i is a parameter of agent $\mathcal{A}_i$ to model a heterogeneous population of minor agents, for more explanations, see [33]. For the sake of notational brevity, in (2.2), we only set the coefficients $A(\cdot)$ and $D(\cdot)$ (see also $R(\cdot)$ in (2.4)) to be dependent on θ_i. A similar analysis can proceed when all other coefficients depend also on θ_i. In this paper, we assume that θ_i takes values from a finite set $\Theta := \{1, 2, \ldots, K\}$, which means that totally K types of minor agents are considered. We call $\mathcal{A}_i$ a k-type minor agent if $\theta_i = k \in \Theta$.

In this paper, we are interested in the asymptotic behavior as N tends to infinity. This is essentially to consider a family of games with an increasing number of minor agents. To describe the related large-population system, let us first define

$$\mathcal{I}_k = \{i \mid \theta_i = k, 1 \leq i \leq N\}, \qquad N_k = |\mathcal{I}_k|,$$

where N_k is the cardinality of index set $\mathcal{I}_k, 1 \leq k \leq K$. Let $\pi_k^{(N)} = \frac{N_k}{N}$ for $k \in \{1, \ldots, K\}$; then $\pi^{(N)} = (\pi_1^{(N)}, \ldots, \pi_K^{(N)})$ is a probability vector to represent the empirical distribution of $\theta_1, \ldots, \theta_N$. The following assumption gives some statistical properties for θ_i. For more details, the reader is referred to [33].

A1. There exists a probability mass vector $\pi = (\pi_1, \pi_2, \ldots, \pi_K)$ such that $\lim_{N\to+\infty}\pi^{(N)} = \pi$ and $\min_{1\leq k\leq K}\pi_k > 0$.

From A1 we know that when $N \to +\infty$, the proportion of k-type agents becomes stable for each k and that the number of each type agents tends to infinity. Otherwise, the agents in a given type with bounded size should be excluded from consideration when analyzing asymptotic behavior as $N \to +\infty$. Throughout the paper we make the convention that N is sufficiently large such that $\min_{1\le k\le K} N_k \ge 1$.

Now let us specify the admissible control and cost functionals of our LQG mixed game with input control constraint. We call u_0 a centralized admissible control for the major agent if $u_0 \in \mathcal{U}_{ad}^{c,0}$, where $\mathcal{U}_{ad}^{c,0} := \{u(\cdot) \mid u(\cdot) \in L^2_{\mathcal{F}}(0,T;\Gamma_0)\}$. Here $\Gamma_0 \subset \mathbb{R}^m$ is a nonempty closed convex set. For each $1 \le i \le N$, we define centralized admissible control u_i for the minor agent $\mathcal{A}_i$ as $u_i \in \mathcal{U}_{ad}^{c,i}$, where for a nonempty closed convex set $\Gamma_{\theta_i} \subset \mathbb{R}^m$, $\mathcal{U}_{ad}^{c,i} := \{u_i(\cdot) \mid u_i(\cdot) \in L^2_{\mathcal{F}}(0,T;\Gamma_{\theta_i})\}$. Moreover, in contrast, we call u_0 a decentralized admissible control for the major agent if $u_0 \in \mathcal{U}_{ad}^0$, where $\mathcal{U}_{ad}^0 := \{u(\cdot) \mid u(\cdot) \in L^2_{\mathcal{F}^{W_0}}(0,T;\Gamma_0)\}$, and for each $1 \le i \le N$, we also define decentralized admissible control u_i for the minor agent $\mathcal{A}_i$ as $u_i \in \mathcal{U}_{ad}^i$, where $\mathcal{U}_{ad}^i := \{u_i(\cdot) \mid u_i(\cdot) \in L^2_{\mathcal{F}^i}(0,T;\Gamma_{\theta_i})\}$. Note that we have $\mathcal{U}_{ad}^i \subset \mathcal{U}_{ad}^{c,i}$, for $0 \le i \le N$.

Remark 2.1. We give the following typical examples for the closed convex constraint set Γ: $\Gamma^1 = \mathbb{R}^m_+$ represents that the control can only take positive values. It connects with the mean-variance portfolio selection problem with a no-shorting constraint; see [6, 32]. The linear subspace $\Gamma^2 = (\mathbb{R}e_i)^\perp$ (where $(e_1, e_2, \dots, e_m)$ is the canonical basis of $\mathbb{R}^m$) represents that the control can only take from a hyperplane. It is used to deal with the following situation: each manager for portfolio investment has access to the whole market except some fixed firm who has private information and thus linear constraint with segment arises. For more examples of linear constraints and their economic meaning, the reader may refer to [24]. Γ can also be some closed cone (i.e., Γ is closed and if $u \in \Gamma$, then $\alpha u \in \Gamma$, for all $\alpha \ge 0$), e.g., $\Gamma^3 = \{u \in \mathbb{R}^m : \Upsilon u = 0\}$ or $\Gamma^4 = \{u \in \mathbb{R}^m : \Upsilon u \le 0\}$, where $\Upsilon \in \mathbb{R}^{n\times m}$. For investigations on stochastic LQ problems with conic control constraint, the reader may refer to [21, 32].

Let $u = (u_0, u_1, \dots, u_N)$ be the set of strategies of all $N+1$ agents, $u_{-0} = (u_1, u_2, \dots, u_N)$ be the control strategies except $\mathcal{A}_0$, and $u_{-i} = (u_0, u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_N)$ be the set of strategies except the ith agent $\mathcal{A}_i$. We introduce the cost functional of the major agent as

(2.3)
$$\begin{aligned}\mathcal{J}_0(u_0,u_{-0}) = \frac{1}{2}\mathbb{E}\Big[\int_0^T &\Big\langle Q_0(t)\big(x_0(t)-\rho_0 x^{(N)}(t)\big), x_0(t)-\rho_0 x^{(N)}(t)\Big\rangle \\ &+ \langle R_0(t)u_0(t), u_0(t)\rangle dt + \Big\langle G_0\big(x_0(T)-\rho_0 x^{(N)}(T)\big), x_0(T)-\rho_0 x^{(N)}(T)\Big\rangle\Big]\end{aligned}$$

and the cost functional of the minor agent $\mathcal{A}_i$ as

$$\begin{aligned}\mathcal{J}_i(u_i,u_{-i}) = \frac{1}{2}\mathbb{E}\Bigg[\int_0^T &\Big(\Big\langle Q\big(x_i-\rho x^{(N)}-(1-\rho)x_0(t)\big), \\ &x_i-\rho x^{(N)}-(1-\rho)x_0(t)\Big\rangle + \big\langle R_{\theta_i}u_i, u_i\big\rangle\Big)\, dt \\ &+\Big\langle G\big(x_i(T)-\rho x^{(N)}(T)-(1-\rho)x_0(T)\big), x_i(T)-\rho x^{(N)}(T)-(1-\rho)x_0(T)\Big\rangle\Bigg]. \qquad (2.4)\end{aligned}$$

We mention that for notational brevity, the time argument is suppressed when necessary, and ρ does not depend on θ_i; a similar analysis can proceed when ρ depends on θ_i.

We impose the following assumptions:

A2. The coefficients of the states satisfy that, for $1 \le i \le N$,

$$\begin{aligned}&A_0(\cdot), A_{\theta_i}(\cdot), C_0(\cdot), C(\cdot), F_0^1(\cdot), F_1(\cdot), F_0^2(\cdot), F_2(\cdot), H(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times n}),\\ &B_0(\cdot), B(\cdot), D_0(\cdot), D_{\theta_i}(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times m}), b_0(\cdot), b(\cdot), \sigma_0(\cdot), \sigma(\cdot) \in L^\infty(0,T;\mathbb{R}^n).\end{aligned}$$

A3. The coefficients of cost functionals satisfy that, for $1 \le i \le N$,

$$\begin{aligned}&Q_0(\cdot), Q(\cdot) \in L^\infty(0,T;\mathcal{S}^n), R_0(\cdot), R_{\theta_i}(\cdot) \in L^\infty(0,T;\mathcal{S}^m), G_0, G \in \mathcal{S}^n,\\ &Q_0(\cdot) \ge 0, Q(\cdot) \ge 0, R_0(\cdot) > 0, R_{\theta_i}(\cdot) > 0, G_0 \ge 0, G \ge 0, \rho_0, \rho \in [0,1].\end{aligned}$$

Here $L^\infty(0,T;\mathcal{H})$ denotes the space of uniformly bounded functions mapping from $[0,T]$ to $\mathcal{H}$. It follows that, under assumptions A2 and A3, the system (2.1) and (2.2) admits a unique solution $x_0(\cdot), x_i(\cdot) \in L^2_{\mathcal{F}}(\Omega; C(0,T;\mathbb{R}^n))$ for given admissible control u_0 and u_i. Now, let us formulate the LQG mixed games with control constraint.

Problem (CC). Find a strategy profile $\bar{u} = (\bar{u}_0, \bar{u}_1, \ldots, \bar{u}_N)$ where $\bar{u}_i(\cdot) \in \mathcal{U}_{ad}^{c,i}$, $0 \le i \le N$, such that

$$\mathcal{J}_i(\bar{u}_i(\cdot), \bar{u}_{-i}(\cdot)) = \inf_{u_i(\cdot)\in\mathcal{U}_{ad}^{c,i}} \mathcal{J}_i(u_i(\cdot), \bar{u}_{-i}(\cdot)), \quad 0 \le i \le N.$$

We call $\bar{u}$ Nash equilibrium for Problem (CC).

For comparison, we also present the definition of ε-Nash equilibrium.

DEFINITION 2.1. *A strategy profile* $\bar{u} = (\bar{u}_0, \bar{u}_1, \ldots, \bar{u}_N)$ *where* $\bar{u}_i(\cdot) \in \mathcal{U}_{ad}^{c,i}$, $0 \le i \le N$, *is called an ε-Nash equilibrium with respect to costs* $\mathcal{J}^i$, $0 \le i \le N$, *if there exists an* $\varepsilon = \varepsilon(N) \ge 0, \lim_{N\to+\infty} \varepsilon(N) = 0$, *such that for any* $0 \le i \le N$, *we have*

$$\mathcal{J}_i(\bar{u}_i(\cdot), \bar{u}_{-i}(\cdot)) \le \mathcal{J}_i(u_i(\cdot), \bar{u}_{-i}(\cdot)) + \varepsilon,$$

when any alternative strategy $u_i \in \mathcal{U}_{ad}^{c,i}$ *is applied by* $\mathcal{A}_i$.

Remark 2.2. If $\varepsilon = 0$, Definition 2.1 reduces to the usual exact Nash equilibrium.

2.1. Stochastic optimal control problem for the major agent. As explained in the introduction, the centralized optimization strategies to Problem (CC) are rather complicated and infeasible to be applied when the number of the agents tends to infinity. Alternatively, we investigate the decentralized strategies via the limiting problem with the help of frozen limiting state-average. To this end, we first figure out the representation of limiting process using heuristic arguments. Based on it, we can find the decentralized strategies by the consistency condition and then rigorously verify the derived decentralized strategy profile is an ε-Nash equilibrium. We formalize the auxiliary limiting mixed game via the approximation of the average state $x^{(N)}$. Since $\pi_k^{(N)} \approx \pi_k$ for large N and

$$x^{(N)} = \frac{1}{N}\sum_{k=1}^{K}\sum_{i\in\mathcal{I}_k} x_i = \sum_{k=1}^{K} \pi_k^{(N)} \frac{1}{N_k}\sum_{i\in\mathcal{I}_k} x_i,$$

we may approximate $x^{(N)}$ by $\sum_{k=1}^K \pi_k m_k$, where $m_k \in \mathbb{R}^n$ is used to approximate $\frac{1}{N_k}\sum_{i\in\mathcal{I}_k} x_i$. Denote $m = (m_1', m_2', \ldots m_k')'$, which is called the set of aggregate quantities. Replacing $x^{(N)}$ of (2.1) and (2.3) by $\sum_{k=1}^K \pi_k m_k$, the major agent's dynamics is given by

$$
\begin{aligned}
dz_0(t) = &\left[A_0(t)z_0(t) + B_0(t)u_0(t) + F_0^1(t)\sum_{k=1}^K \pi_k m_k(t) + b_0(t)\right]dt \\
&+ \left[C_0(t)z_0(t) + D_0(t)u_0(t) + F_0^2(t)\sum_{k=1}^K \pi_k m_k(t) + \sigma_0(t)\right]dW_0(t), \\
z_0(0) = &\ x_0 \in \mathbb{R}^n, \qquad (2.5)
\end{aligned}
$$

and the limiting cost functional is

$$
\begin{aligned}
J_0(u_0) = \frac{1}{2}\mathbb{E}\Bigg[\int_0^T &\left\langle Q_0(t)\left(z_0(t) - \rho_0\sum_{k=1}^K \pi_k m_k(t)\right), z_0(t) - \rho_0\sum_{k=1}^K \pi_k m_k(t)\right\rangle \\
&+ \langle R_0(t)u_0(t), u_0(t)\rangle dt + \left\langle G_0\left(z_0(T) - \rho_0\sum_{k=1}^K \pi_k m_k(T)\right),\right. \\
&\left.z_0(T) - \rho_0\sum_{k=1}^K \pi_k m_k(T)\right\rangle\Bigg]. \qquad (2.6)
\end{aligned}
$$

For simplicity, let $\otimes$ be the Kronecker product of two matrix (see [25]) and we denote $F_0^{1,\pi} := \pi\otimes F_0^1$, $F_0^{2,\pi} := \pi\otimes F_0^2$, $\rho_0^\pi := \pi\otimes\rho_0 I_{n\times n}$. Then (2.5) and (2.6) respectively become

$$
\begin{aligned}
dz_0(t) = &[A_0(t)z_0(t) + B_0(t)u_0(t) + F_0^{1,\pi}(t)m(t) + b_0(t)]dt \\
&+ [C_0(t)z_0(t) + D_0(t)u_0(t) + F_0^{2,\pi}(t)m(t) + \sigma_0(t)]dW_0(t), \quad z_0(0) = x_0 \in \mathbb{R}^n,
\end{aligned} \tag{2.7}
$$

and

$$
\begin{aligned}
J_0(u_0) = \frac{1}{2}\mathbb{E}\bigg[\int_0^T &\Big\langle Q_0(t)\big(z_0(t) - \rho_0^\pi m(t)\big), z_0(t) - \rho_0^\pi m(t)\Big\rangle \\
&+ \langle R_0(t)u_0(t), u_0(t)\rangle dt + \Big\langle G_0\big(z_0(T) - \rho_0^\pi m(T)\big), z_0(T) - \rho_0^\pi m(T)\Big\rangle\bigg].
\end{aligned}
$$

We define the following auxiliary stochastic optimal control problem for the major agent with infinite population (note that the admissible control belongs to $\mathcal{U}_{ad}^0$ rather than $\mathcal{U}_{ad}^{c,0}$):

Problem (LCC-Major). For the major agent $\mathcal{A}_0$, find $u_0^*(\cdot) \in \mathcal{U}_{ad}^0$ satisfying

$$
J_0(u_0^*(\cdot)) = \inf_{u_0(\cdot)\in\mathcal{U}_{ad}^0} J_0(u_0(\cdot)).
$$

Then $u_0^*(\cdot)$ is called a decentralized optimal control for the auxiliary Problem (LCC-Major).

Now, similar to [29], in order to obtain the optimal control, we would like to apply the SMP to the above limiting LQG problem (LCC-Major) with input constraint. We introduce the first order adjoint equation

$$
\begin{cases}
dp_0(t) = -\big[A_0'(t)p_0(t) - Q_0(t)(z_0(t) - \rho_0^\pi m(t)) + C_0'(t)q_0(t)\big]dt + q_0(t)dW_0(t), \\
p_0(T) = -G_0\big(z_0(T) - \rho_0^\pi m(T)\big),
\end{cases} \tag{2.8}
$$

as well as the Hamiltonian function

$$H_0(t,p,q,x,u) = \langle p, A_0x + B_0u + F_0^{1,\pi}m + b_0\rangle + \langle q, C_0x + D_0u + F_0^{2,\pi}m + \sigma_0\rangle \\ - \frac{1}{2}\langle Q_0(x-\rho_0^\pi m), x - \rho_0^\pi m\rangle - \frac{1}{2}\langle R_0u, u\rangle.$$

Since Γ_0 is a closed convex set, for optimal control u_0^*, related optimal state z_0^*, and related solution (p_0^*, q_0^*) to (2.8), the SMP reads as the following local form:

$$\left\langle \frac{\partial H_0}{\partial u}(t, p_0^*, q_0^*, z_0^*, u_0^*), u - u_0^* \right\rangle \leq 0 \quad \text{for all } u \in \Gamma_0, \text{ a.e. } t \in [0,T], \ \mathbb{P}\text{-}a.s. \tag{2.9}$$

Similar to the argument in p. 5 of [29], using the well-known results of convex analysis (see Theorem 5.2 of [10] or Theorem 4.1 of [29]), (2.9) is equivalent to

$$u_0^*(t) = \mathbf{P}_{\Gamma_0}[R_0^{-1}(t)(B_0'(t)p_0^*(t) + D_0'(t)q_0^*)(t)], \qquad \text{a.e. } t \in [0,T], \ \mathbb{P}\text{-}a.s., \tag{2.10}$$

where $\mathbf{P}_{\Gamma_0}[\cdot]$ is the projection mapping from $\mathbb{R}^m$ to its closed convex subset Γ_0 under the norm $\|\cdot\|_{R_0}$ (where $\|x\|_{R_0}^2 = \langle\langle x, x\rangle\rangle := < R_0^{\frac{1}{2}}x, R_0^{\frac{1}{2}}x >$). Finally, by substituting (2.10) in (2.7) and (2.8), we get the following Hamiltonian system for the major agent:

$$\begin{cases} dz_0 = \Big(A_0z_0 + B_0\mathbf{P}_{\Gamma_0}\big[R_0^{-1}\big(B_0'p_0 + D_0'q_0\big)\big] + F_0^{1,\pi}m + b_0\Big)\,dt \\ \qquad + \Big(C_0z_0 + D_0\mathbf{P}_{\Gamma_0}\big[R_0^{-1}\big(B_0'p_0 + D_0'q_0\big)\big] + F_0^{2,\pi}m + \sigma_0\Big)\,dW_0(t), \\ dp_0 = -\big(A_0'p_0 - Q_0(z_0 - \rho_0^\pi m) + C_0'q_0\big)dt + q_0dW_0(t), \\ z_0(0) = x_0, \qquad p_0(T) = -G_0\big(z_0(T) - \rho_0^\pi m(T)\big). \end{cases} \tag{2.11}$$

Remark 2.3. We mention that since the cost functional for Problem (LCC-Major) is strictly convex and Γ_0 is compact, it admits a unique optimal control. Then u_0^* defined by (2.10) is the optimal control. Moreover, we have $J_0(u_0^*(\cdot)) = \inf_{u_0(\cdot)\in\mathcal{U}_{ad}^{c,0}} J_0(u_0(\cdot))$, due to the fact that $J_0(u_0^*(\cdot)) \leq J_0(u_0(\cdot))$ still holds even if the control u_0 is merely adapted to a larger filtration (e.g., $\{\mathcal{F}_t\}$) as long as the Wiener process W_0 remains a Brownian motion for this filtration; see Remark 2.3 of [17].

2.2. Stochastic optimal control problem for the minor agent. Denoting $F^{1,\pi} := \pi\otimes F_1$, $F^{2,\pi} := \pi\otimes F_2$, $\rho^\pi := \pi\otimes\rho I_{n\times n}$, the limiting state of minor agent $\mathcal{A}_i$ is

$$\begin{cases} dz_i = \big(A_{\theta_i}z_i + Bu_i + F^{1,\pi}m + b\big)\,dt + \big(Cz_i + D_{\theta_i}u_i + F^{2,\pi}m + Hz_0 + \sigma\big)\,dW_i(t), \\ z_i(0) = x. \end{cases}$$

The limiting cost functional is given by

$$J_i(u_i) = \frac{1}{2}\mathbb{E}\Bigg[\int_0^T \Bigg(\Bigg\langle Q\Bigg(z_i - \rho\sum_{k=1}^K \pi_k m_k - (1-\rho)z_0\Bigg), \\ z_i - \rho\sum_{k=1}^K \pi_k m_k - (1-\rho)z_0\Bigg\rangle + \big\langle R_{\theta_i}u_i, u_i\big\rangle\Bigg)dt \tag{2.12}$$

$$+ \Big\langle G\Big(z_i(T) - \rho\sum_{k=1}^{K}\pi_k m_k(T) - (1-\rho)z_0(T)\Big),\\ z_i(T) - \rho\sum_{k=1}^{K}\pi_k m_k(T) - (1-\rho)z_0(T)\Big\rangle\Big]$$

$$= \frac{1}{2}\mathbb{E}\Big[\int_0^T \Big(\Big\langle Q\big(z_i - \rho^\pi m - (1-\rho)z_0\big), z_i - \rho^\pi m - (1-\rho)z_0\Big\rangle + \big\langle R_{\theta_i}u_i, u_i\big\rangle\Big)dt\\ + \Big\langle G\big(z_i(T) - \rho^\pi m(T) - (1-\rho)z_0(T)\big), z_i(T) - \rho^\pi m(T) - (1-\rho)z_0(T)\Big\rangle\Big],$$

and the related limiting stochastic optimal control problem for the minor agents is the following.

Problem (LCC-Minor). For each minor agent $\mathcal{A}_i$, $1 \le i \le N$, find $u_i^*(\cdot) \in \mathcal{U}_{ad}^i$ satisfying

$$J_i(u_i^*(\cdot)) = \inf_{u_i(\cdot)\in\mathcal{U}_{ad}^i} J_i(u_i(\cdot)).$$

Then $u_i^*(\cdot)$ is called a decentralized optimal control for Problem (LCC-Minor).

Similar to the major agent, we obtain the following Hamiltonian system for minor agent $\mathcal{A}_i$:

(2.13)
$$\begin{cases} dz_i = \Big(A_{\theta_i}z_i + B\mathbf{P}_{\Gamma_{\theta_i}}\big[R_{\theta_i}^{-1}\big(B'p_i + D'_{\theta_i}q_i\big)\big] + F_1^\pi m + b\Big)\,dt \\ \qquad\quad + \Big(Cz_i + D_{\theta_i}\mathbf{P}_{\Gamma_{\theta_i}}\big[R_{\theta_i}^{-1}\big(B'p_i + D'_{\theta_i}q_i\big)\big] + F_2^\pi m + Hz_0 + \sigma\Big)\,dW_i(t), \\ dp_i = -\big(A'_{\theta_i}p_i - Q(z_i - \rho^\pi m - (1-\rho)z_0) + C'q_i\big)dt + q_i dW_i(t) + q_{i,0}dW_0(t), \\ z_i(0) = x, \qquad p_i(T) = -G\big(z_i(T) - \rho^\pi m(T) - (1-\rho)z_0(T)\big). \end{cases}$$

Here, $\mathbf{P}_{\Gamma_{\theta_i}}[\cdot]$ is the projection mapping from $\mathbb{R}^m$ to its closed convex subset Γ_{θ_i} under the norm $\|\cdot\|_{R_{\theta_i}}$. We mention that the limiting minor agent's state z_i also depends on the limiting major agent's state z_0; it makes that z_i is $\mathcal{F}^i$-adapted, and thus $q_{i,0}dW_0(t)$ appears in the adjoint equation.

2.3. Consistency condition system for the mixed game. Let us first focus on the k-type minor agent. When $i \in \mathcal{I}_k = \{i \mid \theta_i = k\}$, we denote $A_{\theta_i} = A_k$, $D_{\theta_i} = D_k$, $R_{\theta_i} = R_k$, and $\Gamma_{\theta_i} = \Gamma_k$. We would like to approximate x_i by z_i when $N \to +\infty$; thus m_k should satisfy the consistency condition (noticing that Assumption A1 implies that $N_k \to \infty$ if $N \to \infty$)

$$m_k(\cdot) = \lim_{N\to+\infty}\frac{1}{N_k}\sum_{i\in\mathcal{I}_k} z_i(\cdot).$$

Recall that for $i, j \in \mathcal{I}_k$, z_i and z_j are identically distributed and conditional independent (under $\mathbb{E}(\cdot\,|\mathcal{F}_\cdot^{W_0})$). Thus by the conditional strong law of large number, we have (the convergence is in the sense of almost surely; see, e.g., [45])

$$m_k(\cdot) = \lim_{N\to+\infty}\frac{1}{N_k}\sum_{i\in\mathcal{I}_k} z_i(\cdot) = \mathbb{E}(z_i(\cdot)|\mathcal{F}_\cdot^{W_0}), \tag{2.14}$$

where z_i is given by (2.13) with $A_{\theta_i} = A_k, D_{\theta_i} = D_k, R_{\theta_i} = R_k, \Gamma_{\theta_i} = \Gamma_k$. By combining (2.11), (2.13), and (2.14), we get the following consistency condition system or Nash certainty equivalence principle of k-type minor agent for $1 \leq k \leq K$ (as mentioned, for notational brevity, the time argument is suppressed in the following equations except $\mathbb{E}(\alpha_k(t)|\mathcal{F}_t^{W_0})$ to emphasize its dependence on conditional expectation under $\mathcal{F}_t^{W_0}$):

(2.15)
$$\begin{cases} d\alpha_k = \left(A_k\alpha_k + B\mathbf{P}_{\Gamma_k}\left[R_k^{-1}\left(B'\beta_k + D_k'\gamma_k\right)\right] + F_1\sum_{i=1}^K \pi_i\mathbb{E}(\alpha_i(t)|\mathcal{F}_t^{W_0}) + b\right)dt \\ +\left(C\alpha_k + D_k\mathbf{P}_{\Gamma_k}\left[R_k^{-1}\left(B'\beta_k + D_k'\gamma_k\right)\right] + F_2\sum_{i=1}^K \pi_i\mathbb{E}(\alpha_i(t)|\mathcal{F}_t^{W_0}) + H\alpha_0 + \sigma\right)dW_k(t), \\ d\beta_k = -\left(A_k'\beta_k - Q\left(\alpha_k - \rho\sum_{i=1}^K \pi_i\mathbb{E}(\alpha_i(t)|\mathcal{F}_t^{W_0}) - (1-\rho)\alpha_0\right) + C'\gamma_k\right)dt \\ +\gamma_k dW_k(t) + \gamma_{k,0}dW_0(t), \\ \alpha_k(0) = x, \qquad \beta_k(T) = -G\left(\alpha_k(T) - \rho\sum_{i=1}^K \pi_i\mathbb{E}(\alpha_i(T)|\mathcal{F}_T^{W_0}) - (1-\rho)\alpha_0(T)\right), \end{cases}$$

where α_0 satisfies the following FBSDE which is coupled with all k-type minor agents:

(2.16)
$$\begin{cases} d\alpha_0 = \left(A_0\alpha_0 + B_0\mathbf{P}_{\Gamma_0}\left[R_0^{-1}\left(B_0'\beta_0 + D_0'\gamma_0\right)\right] + F_0^1\sum_{i=1}^K \pi_i\mathbb{E}(\alpha_i(t)|\mathcal{F}_t^{W_0}) + b_0\right)dt \\ +\left(C_0\alpha_0 + D_0\mathbf{P}_{\Gamma_0}\left[R_0^{-1}\left(B_0'\beta_0 + D_0'\gamma_0\right)\right] + F_0^2\sum_{i=1}^K \pi_i\mathbb{E}(\alpha_i(t)|\mathcal{F}_t^{W_0}) + \sigma_0\right)dW_0(t), \\ d\beta_0 = -\left(A_0'\beta_0 - Q_0\left(\alpha_0 - \rho_0\sum_{i=1}^K \pi_i\mathbb{E}\left(\alpha_i(t)|\mathcal{F}_t^{W_0}\right)\right) + C_0'\gamma_0\right)dt + \gamma_0 dW_0(t), \\ \alpha_0(0) = x_0, \qquad \beta_0(T) = -G_0\left(\alpha_0(T) - \rho_0\sum_{i=1}^K \pi_i\mathbb{E}(\alpha_i(T)|\mathcal{F}_T^{W_0})\right). \end{cases}$$

We consider toge of minor agents, i.e., (2.16) and (2.15) for all $1 \leq k \leq K$; then there arise $2K+2$ fully coupled equations including $K+1$ forward equations and $K+1$ backward equations. Such fully coupled equations are called a consistency condition system. Once we can solve it, then $m_k = \mathbb{E}(\alpha_k(t)|\mathcal{F}_t^{W_0})$ which depends on the conditional distribution of α_k. This allows us to use arbitrary Brownian motion W_k in (2.15) which is independent of W_0. Finally, let us introduce the following notation, which will be used in the following sections:

$$\Phi(t) := \sum_{i=1}^K \pi_i m_i = \sum_{i=1}^K \pi_i\mathbb{E}(\alpha_i(t)|\mathcal{F}_t^{W_0}). \tag{2.17}$$

The consistency condition system (2.15)–(2.16) is a fully coupled conditional mean-field FBSDE with projection operator. Its solvability is not a direct consequence on current existing results, hence the next two sections investigate its well-posedness, in the local and global cases, respectively. Note that the maximal solvability horizon in the local case always takes some small time horizon, while the global solvability can admit an arbitrary time horizon. Before our discussions in sections 3 and 4, we end this section with the following remarks.

Remark 2.4. (1) For general classical fully coupled FBSDEs, the standard contraction mapping method (see [44]) can be utilized to derive local solvability on a small time horizon. In addition, one counterexample presented there (p. 11 in [44]) implies that such a small horizon feature cannot be relaxed in general. Pursuing this line of thought, section 3 is devoted to addressing the local solvability of system (2.15)–(2.16) in the presence of the new structures (conditional mean-field, projection operator) of the current work. To approach these new structures, some new analytic arguments are required.

In comparison with our work, [8] also establishes the well-posedness of a consistency condition system for a class of LQ MFGs on a small time horizon (Theorems 3.2 and 3.3 of the cited work). However, [8] only involves (symmetric) minor agents, no control constraint is imposed, and the diffusion term of minor agents in [8] is free of a control variable. Consequently, feedback representation using the Riccati equation is workable and takes the standard form, which is more tractable in analysis. Moreover, the solvability of the consistency condition system (i.e., the special FBSDE (1) of [8]) can be transformed to that of the Riccati equation, which is further equivalent to that of a family of FBSDEs in a local sense according to Radon's lemma.

In contrast to [8], our current work has the following features in setting and analysis: the diffusion term of our agents depends on both the major and minor agents' states as well as individual control strategies. In addition, the input constraint is imposed and thus full control is no longer available, and neither is feedback representation through the Riccati equation permitted. Therefore, as shown later, our arguments proceed differently from those of [8].

(2) From both theoretical and practical perspectives, it is more appealing to study the global solution to fully coupled FBSDEs (see [31, 49, 50]). One reason is from the following fact: the decision horizon is always prespecified without modeling flexibility, whereas the system parameters, to some extent of freedom degree, might be appropriately designed. In response, section 4 turns to a discussion of the global solvability of the system (2.15)–(2.16).

With respect to global solvability, one relevant work is [29], which also studies the global solvability of MFGs with input constraint. However, their FBSDE (see (9) of [29]) satisfies the monotonicity assumption addressed in [31]. More specifically, it is satisfied by one subtle argument leading to Theorem 2.1 in [29] that $\mathbb{E}p = 0$, where p is the solution to (9) of [29]. However, in our current work, $\mathbb{E}\beta_i = 0$, $0 \leq i \leq K$, cannot be ensured because of the presence of the coupling coefficient ρ in the terminal condition and the heterogeneities between major and minor agents' states. Hence, new analysis is required to handle our mixed mean-field game having a control constraint.

In addition, as mentioned before, the diffusion term of dynamics here depends on the states of both major and minor agents as well as individual control strategies. Such a structure is very different from [29], where the dependence is limited to the individual control strategy only. Consequently, the monotonicity analysis in [31, 29] cannot be implemented in the current setup. Instead, we need some particularized monotonicity conditions motivated by [49]. The related arguments are also different (see Theorem 4.2, assumption (H1), and Remarks 4.1 and 4.3 of the present paper).

(3) An intriguing remark was proposed in [15], whereby monotonicity conditions on the mean-field term can be interpreted in some "spatial" sense: the agents in a large population tend to dislike rather congested areas and to prefer configurations

in which they are more scattered (for details, see p. 4 and Remark 2.6 of [15]). For system (2.15)–(2.16) in our current work, a spatial monotonicity condition (different from [29]) on the mean-field term holds automatically (see Remark 4.1 $(H_1)-(i')$). Also, a crucial condition in our work is the relation on the norms of some matrices (see Theorem 4.2 and Remark 4.3). Such a relation, especially the condition on matrices $\mathbb{F}_1^{\Pi}, \mathbb{F}_2^{\Pi}$, has the same spatial interpretation as in [15]. Furthermore, such a relation, especially the condition on the eigenvalue value of matrix $\mathbb{A}$, can also be interpreted in a "temporal" sense because it is related to some stability property of the system (2.15)–(2.16) in an asymptotic time scale.

3. Well-posedness of the consistency condition system: The local case.

This section aims to establish the well-posedness of consistency condition system (2.15)–(2.16) in small time duration. Similar to the classical results on FBSDEs (see, for example, Chapter 1, section 5, of Ma and Yong [44]), we need to introduce the following additional assumption:

A4. $R_0^{-1}(\cdot), R_{\theta_i}^{-1}(\cdot) \in L^\infty(0,T;\mathcal{S}^m)$ and $M_0|D|^2<1$, where $|D| := \max_{0\le k\le K}|D_k|$ and $M_0 := \max\{|G_0|^2(1+\rho_0^2),\ |G|^2(1+\rho^2+(1-\rho)^2)\}$.

For simplicity, we denote $\varphi_0(p,q) := \mathbf{P}_{\Gamma_0}[R_0^{-1}(B_0'p+D_0'q)]$ and $\varphi_{\theta_i}(p,q) := \mathbf{P}_{\Gamma_{\theta_i}}[R_{\theta_i}^{-1}(B'p+D_{\theta_i}'q)]$. We have the following theorem.

THEOREM 3.1. *Assume* A1–A4*; then there exists* $T_0>0$ *such that for any* $T\in(0,T_0]$*, the system* (2.15)–(2.16) *has a unique solution* $(\alpha_0,\beta_0,\gamma_0,\alpha_k,\beta_k,\gamma_k,\gamma_{k,0})$, $1\le k\le K$*, satisfying*

$$\begin{aligned}&\alpha_0,\beta_0\in L^2_{\mathcal{F}^{W_0}}(\Omega;C(0,T;\mathbb{R}^n)),\quad \alpha_k,\beta_k\in L^2_{\mathcal{F}^k}(\Omega;C(0,T;\mathbb{R}^n)),\\&\gamma_0\in L^2_{\mathcal{F}^{W_0}}(0,T;\mathbb{R}^n),\quad \gamma_k,\gamma_{k,0}\in L^2_{\mathcal{F}^k}(0,T;\mathbb{R}^n),\quad 1\le k\le K.\end{aligned}\tag{3.1}$$

Proof. Let $T_0\in(0,1]$ be undetermined and $0<T\le T_0$. We denote

$$\begin{aligned}\mathcal{N}[0,T] := {}& L^2_{\mathcal{F}^{W_0}}(\Omega;C(0,T;\mathbb{R}^n))\times\cdots\times L^2_{\mathcal{F}^K}(\Omega;C(0,T;\mathbb{R}^n))\\&\times L^2_{\mathcal{F}^{W_0}}(0,T;\mathbb{R}^n)\times\cdots\times L^2_{\mathcal{F}^K}(0,T;\mathbb{R}^n)\\&\times L^2_{\mathcal{F}^1}(0,T;\mathbb{R}^n)\times\cdots\times L^2_{\mathcal{F}^K}(0,T;\mathbb{R}^n).\end{aligned}$$

For $(Y_0,\dots,Y_K,Z_0,\dots,Z_K,\Upsilon_{1,0},\dots,\Upsilon_{K,0})\in\mathcal{N}[0,T]$, we introduce the following norm:

$$\begin{aligned}&\|(Y_0,\dots,Y_K,Z_0,\dots,Z_K,\Upsilon_{1,0},\dots,\Upsilon_{K,0})\|^2_{\mathcal{N}[0,T]}\\&:=\sup_{t\in[0,T]}\mathbb{E}\Big\{\sum_{k=0}^K|Y_k(t)|^2+\sum_{k=0}^K\int_0^T|Z_k(s)|^2ds+\sum_{k=1}^K\int_0^T|\Upsilon_{k,0}(s)|^2ds\Big\}.\end{aligned}\tag{3.2}$$

Let $\overline{\mathcal{N}}[0,T]$ be the completion of $\mathcal{N}[0,T]$ in $L^2_{\mathcal{F}^{W_0}}(0,T;\mathbb{R}^n)\times\cdots\times L^2_{\mathcal{F}^k}(0,T;\mathbb{R}^n)\times L^2_{\mathcal{F}^{W_0}}(0,T;\mathbb{R}^n)\times\cdots\times L^2_{\mathcal{F}^k}(0,T;\mathbb{R}^n)\times L^2_{\mathcal{F}^1}(0,T;\mathbb{R}^n)\times\cdots\times L^2_{\mathcal{F}^K}(0,T;\mathbb{R}^n)$ under norm (3.2). Now for any

$$(Y_0^j,\dots,Y_K^j,Z_0^j,\dots,Z_K^j,\Upsilon_{1,0}^j,\dots,\Upsilon_{K,0}^j)\in\mathcal{N}[0,T],\quad j=1,2,$$

we solve respectively the following system including $K+1$ SDEs for $1 \le k \le K$:

(3.3)
$$\begin{cases} d\alpha_0^j = \left(A_0\alpha_0^j + B_0\varphi_0(Y_0^j, Z_0^j) + F_0^1 \sum_{i=1}^K \pi_i \mathbb{E}[\alpha_i^j(t)|\mathcal{F}_t^{W_0}] + b_0 \right) dt \\ \qquad + \left(C_0\alpha_0^j + D_0\varphi_0(Y_0^j, Z_0^j) + F_0^2 \sum_{i=1}^K \pi_i \mathbb{E}[\alpha_i^j(t)|\mathcal{F}_t^{W_0}] + \sigma_0 \right) dW_0(t) \\ d\alpha_k^j = \left(A_k\alpha_k^j + B\varphi_k(Y_k^j, Z_k^j) + F_1 \sum_{i=1}^K \pi_i \mathbb{E}[\alpha_i^j(t)|\mathcal{F}_t^{W_0}] + b \right) dt \\ \qquad + \left(C\alpha_k^j + D_k\varphi_k(Y_k^j, Z_k^j) + F_2 \sum_{i=1}^K \pi_i \mathbb{E}[\alpha_i^j(t)|\mathcal{F}_t^{W_0}] + H\alpha_0^j + \sigma \right) dW_k(t) \\ \alpha_0^j(0) = x_0, \quad \alpha_k^j(0) = x. \end{cases}$$

Then (3.3) admits a unique solution for $j = 1, 2$,

$$(\alpha_0^j, \ldots, \alpha_K^j) \in L^2_{\mathcal{F}^{W_0}}(\Omega; C(0,T;\mathbb{R}^n)) \times \cdots \times L^2_{\mathcal{F}^K}(\Omega; C(0,T;\mathbb{R}^n)).$$

Indeed, (3.3) is an $n(K+1)$-dimensional SDE with the mean-field term $\sum_{i=1}^K \pi_i \mathbb{E}[\alpha_i(t)| \mathcal{F}_t^{W_0}]$, and we can prove the well-posedness of such SDEs system by constructing a fixed point using the classical contraction mapping method; we omit the proof here. Now let us denote for $0 \le k \le K$,

$$\hat{\alpha}_k := \alpha_k^1 - \alpha_k^2, \quad \hat{\varphi}_k := \varphi_k(Y_k^1, Z_k^1) - \varphi_k(Y_k^2, Z_k^2),$$
$$\hat{Y}_k := Y_k^1 - Y_k^2, \quad \hat{Z}_k := Z_k^1 - Z_k^2, \quad \hat{\Upsilon}_{k,0} := \Upsilon_{k,0}^1 - \Upsilon_{k,0}^2.$$

By applying Itô's formula and by using A2–A3, $\mathbb{E}|\mathbb{E}[\hat{\alpha}_i(s)|\mathcal{F}_s^{W_0}]|^2 \le \mathbb{E}|\hat{\alpha}_i(s)|^2$, as well as that φ_k is Lipschitz with Lipschitz constant 1 (see Proposition 4.2 of [29]), then for a constant C_ε independent of T which may vary line by line, we have

$$\begin{aligned} \mathbb{E}|\hat{\alpha}_0(t)|^2 \le{}& 2\mathbb{E}\int_0^t \left(|A_0||\hat{\alpha}_0|^2 + |B_0||\hat{\alpha}_0||\hat{\varphi}_0| + |F_0^1||\hat{\alpha}_0| \sum_{i=1}^K \left|\mathbb{E}[\hat{\alpha}_i(s)|\mathcal{F}_s^{W_0}]\right| \right) ds \\ & + \mathbb{E}\int_0^t \left| C_0\hat{\alpha}_0 + D_0\hat{\varphi}_0 + F_0^2 \sum_{i=1}^K \pi_i \mathbb{E}[\hat{\alpha}_i(s)|\mathcal{F}_s^{W_0}] \right|^2 ds \\ \le{}& C_\varepsilon \mathbb{E}\int_0^t \sum_{i=0}^K |\hat{\alpha}_i|^2 ds + \mathbb{E}\int_0^t (|D_0|^2 + \varepsilon)(|\hat{Y}_0|^2 + |\hat{Z}_0|^2) ds \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \mathbb{E}|\hat{\alpha}_k(t)|^2 \le{}& 2\mathbb{E}\int_0^t \left(|A_k||\hat{\alpha}_k|^2 + |B||\hat{\alpha}_k||\hat{\varphi}_k| + |F_1||\hat{\alpha}_k| \sum_{i=1}^K \left|\mathbb{E}[\hat{\alpha}_i(s)|\mathcal{F}_s^{W_0}]\right| \right) ds \\ & + \mathbb{E}\int_0^t \left| C\hat{\alpha}_k + D_k\hat{\varphi}_k + \sum_{i=1}^K F_2 \mathbb{E}[\hat{\alpha}_i(s)|\mathcal{F}_s^{W_0}] + H\hat{\alpha}_0 \right|^2 ds \\ \le{}& C_\varepsilon \mathbb{E}\int_0^t \sum_{i=0}^K |\hat{\alpha}_i|^2 ds + \mathbb{E}\int_0^t (|D_k|^2 + \varepsilon)(|\hat{Y}_k|^2 + |\hat{Z}_k|^2) ds. \end{aligned} \tag{3.5}$$

Adding up (3.4) and (3.5) for $1 \le k \le K$, we have

$$\mathbb{E}\sum_{i=0}^{K}|\hat{\alpha}_i(t)|^2 \le C_\varepsilon \mathbb{E}\int_0^t \sum_{i=0}^{K}|\hat{\alpha}_i(s)|^2 ds + \mathbb{E}\int_0^t \sum_{i=0}^{K}(|D_i|^2+\varepsilon)(|\hat{Y}_i|^2+|\hat{Z}_i|^2)ds,$$

and the Gronwall's inequality yields

$$\mathbb{E}\sum_{i=0}^{K}|\hat{\alpha}_i(t)|^2 \le e^{C_\varepsilon T}\mathbb{E}\int_0^T \sum_{i=0}^{K}(|D_i|^2+\varepsilon)(|\hat{Y}_i|^2+|\hat{Z}_i|^2)ds. \tag{3.6}$$

Next, we solve the following BSDEs, for $j=1,2$:

(3.7)

$$\begin{cases} d\beta_0^j = -\left[A_0'Y_0^j - Q_0\left(\alpha_0^j - \rho_0\sum_{i=1}^{K}\pi_i\mathbb{E}[\alpha_i^j(t)|\mathcal{F}_t^{W_0}]\right) + C_0'Z_0^j\right]dt + \gamma_0^j dW_0(t), \\ d\beta_k^j = -\left[A_k'Y_k^j - Q\left(\alpha_k^j - \rho\sum_{i=1}^{K}\pi_i\mathbb{E}[\alpha_i^j(t)|\mathcal{F}_t^{W_0}]\right) - (1-\rho)\alpha_0^j + C'Z_k^j\right]dt \\ \qquad\qquad + \gamma_k^j dW_k(t) + \gamma_{k,0}^j dW_0(t), \\ \beta_0^j(T) = -G_0\left(\alpha_0^j(T) - \rho_0\sum_{i=1}^{K}\pi_i\mathbb{E}[\alpha_i^j(T)|\mathcal{F}_T^{W_0}]\right), \\ \beta_k^j(T) = -G\left(\alpha_k^j(T) - \rho\sum_{i=1}^{K}\pi_i\mathbb{E}[\alpha_i^j(T)|\mathcal{F}_T^{W_0}] - (1-\rho)\alpha_0^j(T)\right). \end{cases}$$

Since A2–A3 hold and α_i, $0\le i\le K$, have been solved from (3.3), the classical result of BSDEs yields that (3.7) admits a unique solution

$$(\beta_0^j,\dots,\beta_K^j,\gamma_0^j,\dots,\gamma_K^j,\gamma_{1,0}^j,\dots,\gamma_{K,0}^j)\in\mathcal{N}[0,T]\subseteq\overline{\mathcal{N}}[0,T].$$

Thus we have defined a mapping through (3.3) and (3.7),

$$\mathcal{T}:\overline{\mathcal{N}}[0,T]\to\overline{\mathcal{N}}[0,T],$$

$$(Y_0^j,\dots,Y_K^j,Z_0^j,\dots,Z_K^j,\Upsilon_{1,0}^j,\dots,\Upsilon_{K,0}^j)\mapsto(\beta_0^j,\dots,\beta_K^j,\gamma_0^j,\dots,\gamma_K^j,\gamma_{1,0}^j,\dots,\gamma_{K,0}^j).$$

Similarly, we denote

$$\hat{\beta}_k := \beta_k^1-\beta_k^2,\quad \hat{\gamma}_k := \gamma_k^1-\gamma_k^2,\ \text{for } 0\le k\le K, \text{ and } \hat{\gamma}_{k,0} := \gamma_{k,0}^1-\gamma_{k,0}^2,\ \text{for } 1\le k\le K.$$

Applying Itô's formula to $|\hat{\beta}_0(t)|^2$, and noticing $\mathbb{E}|\mathbb{E}[\hat{\alpha}_i(s)|\mathcal{F}_s^{W_0}]|^2 \le \mathbb{E}|\hat{\alpha}_i(s)|^2$, we obtain

$$\begin{aligned} &\mathbb{E}\left(|\hat{\beta}_0(t)|^2 + \int_t^T|\hat{\gamma}_0|^2ds\right) \\ &\quad = \mathbb{E}|\hat{\beta}_0(T)|^2 + 2\mathbb{E}\int_t^T\left\langle\hat{\beta}_0, A_0'\hat{Y}_0 - Q_0\left(\hat{\alpha}_0 - \rho_0\sum_{i=1}^{K}\pi_i\mathbb{E}[\hat{\alpha}_i(s)|\mathcal{F}_s^{W_0}]\right) + C_0'\hat{Z}_0\right\rangle ds \\ &\quad \le |G_0|^2(1+\rho_0^2)\mathbb{E}\sum_{i=0}^{K}|\hat{\alpha}_i(T)|^2 + C_\varepsilon\mathbb{E}\int_t^T|\hat{\beta}_0|^2ds \\ &\qquad + \mathbb{E}\int_t^T\sum_{i=0}^{K}|\hat{\alpha}_i|^2ds + \varepsilon\int_t^T(|\hat{Y}_0|^2+|\hat{Z}_0|^2)ds. \end{aligned}$$

Substituting (3.6) into the above inequality, we have

$$
\begin{aligned}
&\mathbb{E}\left(|\hat\beta_0(t)|^2+\int_t^T|\hat\gamma_0|^2ds\right)\\
&\quad\le\left(|G_0|^2(1+\rho_0^2)+T\right)e^{C_\varepsilon T}\mathbb{E}\sum_{i=0}^K\int_0^T(|D_i|^2+\varepsilon)(|\hat Y_i|^2+|\hat Z_i|^2)ds\\
&\qquad+C_\varepsilon\mathbb{E}\int_t^T|\hat\beta_0|^2ds+\varepsilon\int_t^T(|\hat Y_0|^2+|\hat Z_0|^2)ds.
\end{aligned}
\tag{3.8}
$$

Similarly, by applying Itô's formula to $|\hat\beta_k(t)|^2$, $1\le k\le K$, we have

$$
\begin{aligned}
&\mathbb{E}\left|\hat\beta_k(t)\right|^2+\mathbb{E}\int_t^T|\hat\gamma_k|^2\,ds+\mathbb{E}\int_t^T|\hat\gamma_{k,0}|^2\,ds\\
&=\mathbb{E}\left|\hat\beta_k(T)\right|^2+2\mathbb{E}\int_t^T\left\langle\hat\beta_k,A_k'\hat Y_k-Q\left(\hat\alpha_k-\rho\sum_{i=1}^K\pi_i\mathbb{E}\left[\hat\alpha_i(s)|\mathcal{F}_s^{W_0}\right]-(1-\rho)\hat\alpha_0\right)-C'\hat Z_k\right\rangle ds\\
&\le|G|^2(1+\rho^2+(1-\rho)^2)\mathbb{E}\sum_{i=0}^K|\hat\alpha_i(T)|^2+C_\varepsilon\mathbb{E}\int_t^T|\beta_k|^2\,ds\\
&\qquad+\mathbb{E}\int_t^T\sum_{i=0}^K|\alpha_i|^2\,ds+\varepsilon\mathbb{E}\int_t^T(|\hat Y_k|^2+|\hat Z_k|^2)ds\\
&\le\left(|G|^2(1+\rho^2+(1-\rho)^2)+T\right)e^{C_\varepsilon T}\mathbb{E}\sum_{i=0}^K\int_0^T(|D_i|^2+\varepsilon)(|\hat Y_i|^2+|\hat Z_i|^2)ds\\
&\quad+C_\varepsilon\mathbb{E}\int_t^T|\hat\beta_k|^2ds+\varepsilon\mathbb{E}\int_t^T(|\hat Y_k|^2+|\hat Z_k|^2)ds,
\end{aligned}
\tag{3.9}
$$

where the second inequality comes from (3.6) and the first inequality is due to

$$
\begin{aligned}
\mathbb{E}\left|\hat\beta_k(T)\right|^2&\le|G|^2\left|\hat\alpha_k(T)-\rho\sum_{i=1}^K\pi_i\mathbb{E}[\hat\alpha_i(T)|\mathcal{F}_T^{W_0}]-(1-\rho)\hat\alpha_0(T)\right|^2\\
&\le|G|^2(1+\rho^2+(1-\rho)^2)\left(\mathbb{E}\left|\hat\alpha_k(T)-\rho\pi_k\mathbb{E}[\hat\alpha_k(T)|\mathcal{F}_T^{W_0}]\right|^2+\sum_{i=0,i\ne k}^K\mathbb{E}|\hat\alpha_i(T)|^2\right)\\
&\le|G|^2(1+\rho^2+(1-\rho)^2)\mathbb{E}\sum_{i=0}^K|\hat\alpha_i(T)|^2,
\end{aligned}
$$

and here we used the fact that

$$
\begin{aligned}
&\mathbb{E}\left|\hat\alpha_k(T)-\rho\pi_k\mathbb{E}[\hat\alpha_k(T)|\mathcal{F}_T^{W_0}]\right|^2\\
&\quad=\mathbb{E}|\hat\alpha_k(T)|^2+\rho^2\pi_k^2\mathbb{E}|\mathbb{E}[\hat\alpha_k(T)|\mathcal{F}_T^{W_0}]|^2-2\rho\pi_k\mathbb{E}\left[\hat\alpha_k(T)\mathbb{E}[\hat\alpha_k(T)|\mathcal{F}_T^{W_0}]\right]\\
&\quad=\mathbb{E}|\hat\alpha_k(T)|^2+(\rho^2\pi_k^2-2\rho\pi_k)\mathbb{E}|\mathbb{E}[\hat\alpha_k(T)|\mathcal{F}_T^{W_0}]|^2\le\mathbb{E}\left|\hat\alpha_k(T)\right|^2.
\end{aligned}
$$

Adding up (3.8) and (3.9) for $1 \le k \le K$, we obtain (recall $|D|^2 := \max_{0\le k\le K} |D_k|^2$ and $M_0 := \max\{|G_0|^2(1+\rho_0^2),\ |G|^2(1+\rho^2+(1-\rho)^2)\}$)

$$\begin{aligned}
&\mathbb{E}\sum_{i=0}^{K} \left|\hat\beta_i\right|^2 + \mathbb{E}\sum_{i=0}^{K}\int_t^T |\hat\gamma_i|^2\,ds + \mathbb{E}\sum_{i=1}^{K}\int_t^T |\hat\gamma_{i,0}|^2\,ds \\
&\le (M_0+T)\,e^{C_\varepsilon T}\mathbb{E}\sum_{i=0}^{K}\int_0^T (|D_i|^2+\varepsilon)(|\hat Y_i|^2+|\hat Z_i|^2)ds \\
&\qquad + C_\varepsilon \mathbb{E}\int_t^T \sum_{i=0}^{K}\left|\hat\beta_i\right|^2 ds + \varepsilon\mathbb{E}\int_0^T\sum_{i=0}^{K}(|\hat Y_i|^2+|\hat Z_i|^2)ds \\
&\le C_\varepsilon\mathbb{E}\int_t^T \sum_{i=0}^{K}\left|\hat\beta_i\right|^2 ds + \left[(M_0+T)\,e^{C_\varepsilon T}(|D|^2+\varepsilon)+\varepsilon\right]\mathbb{E}\sum_{i=0}^{K}\int_0^T(|\hat Y_i|^2+|\hat Z_i|^2)ds.
\end{aligned}$$

The Gronwall's inequality yields that

(3.10)

$$\begin{aligned}
&\mathbb{E}\sum_{i=0}^{K} \left|\hat\beta_i\right|^2 + \mathbb{E}\sum_{i=0}^{K}\int_t^T |\hat\gamma_i|^2\,ds + \mathbb{E}\sum_{i=1}^{K}\int_t^T |\hat\gamma_{i,0}|^2\,ds \\
&\le e^{C_\varepsilon T}\left[(M_0+T)\,e^{C_\varepsilon T}(|D|^2+\varepsilon)+\varepsilon\right]\mathbb{E}\sum_{i=0}^{K}\int_0^T(|\hat Y_i|^2+|\hat Z_i|^2)ds \\
&\le e^{C_\varepsilon T}(T+1)\left[(M_0+T)\,e^{C_\varepsilon T}(|D|^2+\varepsilon)+\varepsilon\right]\|(\hat Y_0,\dots,\hat Y_K,\hat Z_0,\dots,\hat Z_K,\hat\Upsilon_{1,0},\dots, \\
&\qquad\qquad \hat\Upsilon_{K,0})\|_{\overline{\mathcal{N}}[0,T]} \\
&= e^{C_\varepsilon T}(T+1)\left[M_0 e^{C_\varepsilon T}(|D|^2+\varepsilon)+\varepsilon+Te^{C_\varepsilon T}(|D|^2+\varepsilon)\right] \\
&\qquad \cdot\|(\hat Y_0,\dots,\hat Y_K,\hat Z_0,\dots,\hat Z_K,\hat\Upsilon_{1,0},\dots,\hat\Upsilon_{K,0})\|_{\overline{\mathcal{N}}[0,T]}.
\end{aligned}$$

Noticing assumption A4, by first choosing $\varepsilon > 0$ small enough such that $M_0(|D|^2+\varepsilon)+\varepsilon < 1$, then choosing $T > 0$ small enough, we obtain from (3.10) that for some $0 < \delta < 1$,

$$\begin{aligned}
&\|(\hat\beta_0,\dots,\hat\beta_K,\hat\gamma_0,\dots,\hat\gamma_K,\hat\gamma_{1,0},\dots,\hat\gamma_{K,0})\|_{\overline{\mathcal{N}}[0,T]} \\
&\quad \le \delta\|(\hat Y_0,\dots,\hat Y_K,\hat Z_0,\dots,\hat Z_K,\hat\Upsilon_{1,0},\dots,\hat\Upsilon_{K,0})\|_{\overline{\mathcal{N}}[0,T]}.
\end{aligned}$$

This means that the mapping $\mathcal{T} : \overline{\mathcal{N}}[0,T] \to \overline{\mathcal{N}}[0,T]$ is contractive. By the contraction mapping theorem, there exists a unique fixed point

$$(\beta_0,\beta_1,\dots,\beta_K,\gamma_0,\gamma_1,\dots,\gamma_K,\gamma_{1,0},\dots,\gamma_{K,0}) \in \overline{\mathcal{N}}[0,T].$$

Moreover, classical BSDE theory allows us to show that

$$(\beta_0,\beta_1,\dots,\beta_K,\gamma_0,\gamma_1,\dots,\gamma_K,\gamma_{1,0},\dots,\gamma_{K,0}) \in \mathcal{N}[0,T].$$

Let α_k, $0 \le k \le K$, be the corresponding solution of (3.3). Then, one can obtain that the system (2.15)–(2.16) has a unique solution $(\alpha_0,\beta_0,\gamma_0,\alpha_k,\beta_k,\gamma_k,\gamma_{k,0})$, $1 \le k \le K$, such that (3.1) holds. ∎

4. Well-posedness of the consistency condition system: The global case. This section aims to establish the well-posedness of consistency condition system (2.15)–(2.16) for arbitrary T. We first study one general kind of conditional MF-FBSDE by using the discounting method of Pardoux and Tang [49].

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete, filtered probability space satisfying usual conditions. $\{W_i(t),\ 0 \le i \le d\}_{0\le t\le T}$ is a $d+1$-dimensional Brownian motion on this space. Let $\mathcal{F}_t$ be the filtration generated by $\{W_i(s), 0 \le i \le d, 0 \le s \le t\}$ and augmented by $\mathcal{N}_\mathbb{P}$ (the class of $\mathbb{P}$-null sets of $\mathcal{F}$). Let $\mathcal{F}_t^{W_0}$ be the augmentation of $\sigma\{W_0(s), 0 \le s \le t\}$ by $\mathcal{N}_\mathbb{P}$. We consider the following general conditional MF-FBSDE:

(4.1)
$$\begin{cases} dX(s) = b(s, X(s), \mathbb{E}[X(s)|\mathcal{F}_s^{W_0}], Y(s), Z(s)])ds \\ +\sigma(s, X(s), \mathbb{E}[X(s)|\mathcal{F}_s^{W_0}], Y(s), Z(s))dW(s), \\ -dY(s) = f(s, X(s), \mathbb{E}[X(s)|\mathcal{F}_s^{W_0}], Y(s), Z(s))ds - Z(s)dW(s), \quad s \in [0, T], \\ X(0) = x, \quad Y(T) = g(X(T), \mathbb{E}[X(T)|\mathcal{F}_T^{W_0}]), \end{cases}$$

where the adapted processes X, Y, Z take their values in $\mathbb{R}^n, \mathbb{R}^l$, and $\mathbb{R}^{l\times(d+1)}$, respectively. The coefficients b, σ, and f are defined on $\Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^{l\times(d+1)}$, such that $b(\cdot, \cdot, x, m, y, z)$, $\sigma(\cdot, \cdot, x, m, y, z)$, and $f(\cdot, \cdot, x, m, y, z)$ are $\{\mathcal{F}_t\}$-progressively measurable processes for all fixed $(x, m, y, z) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^{l\times(d+1)}$. The coefficient g is defined on $\Omega \times \mathbb{R}^n \times \mathbb{R}^n$ and $g(\cdot, x, m)$ is $\mathcal{F}_T$-measurable for all fixed $(x, m) \in \mathbb{R}^n \times \mathbb{R}^n$. Moreover, the functions b, σ, f, and g are continuous w.r.t. $(x, m, y, z) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^{l\times(d+1)}$ and satisfy the following assumptions:

(H_1) There exist $\lambda_1, \lambda_2 \in \mathbb{R}$ and positive constants k_0, k_i, $i = 1, 2, \dots, 12$, such that for all t, x, x_1, x_2, m, m_1, m_2, y, y_1, y_2, z, z_1, z_2 a.s.

$$\begin{aligned}
&\text{(i)} && \langle b(t, x_1, m, y, z) - b(t, x_2, m, y, z), x_1 - x_2\rangle \le \lambda_1|x_1 - x_2|^2, \\
&\text{(ii)} && |b(t, x, m_1, y_1, z_1) - b(t, x, m_2, y_2, z_2)| \le k_1|m_1 - m_2| \\
& && +k_2|y_1 - y_2| + k_3|z_1 - z_2|, \\
&\text{(iii)} && |b(t, x, m, y, z)| \le |b(t, 0, m, y, z)| + k_0(1 + |x|), \\
&\text{(iv)} && \langle f(t, x, m, y_1, z) - f(t, x, m, y_2, z), y_1 - y_2\rangle \le \lambda_2|y_1 - y_2|^2, \\
&\text{(v)} && |f(t, x_1, m_1, y, z_1) - f(t, x_1, m_2, y, z_2)| \le k_4|x_1 - x_2| \\
& && +k_5|m_1 - m_2| + k_6|z_1 - z_2|, \\
&\text{(vi)} && |f(t, x, m, y, z)| \le |f(t, x, m, 0, z)| + k_0(1 + |y|), \\
&\text{(vii)} && |\sigma(t, x_1, m_1, y_1, z_1) - \sigma(t, x_2, m_2, y_2, z_2)|^2 \\
& && \le k_7^2|x_1 - x_2|^2 + k_8^2|m_1 - m_2|^2 + k_9^2|y_1 - y_2|^2 + k_{10}^2|z_1 - z_2|^2, \\
&\text{(viii)} && |g(x_1, m_1) - g(x_2, m_2)|^2 \le k_{11}^2|x_1 - x_2| + k_{12}^2|m_1 - m_2|.
\end{aligned}$$

(H_2) It holds that

$$\mathbb{E}\int_0^T \left(|b(s, 0, 0, 0, 0)|^2 + |\sigma(s, 0, 0, 0, 0)|^2 + |f(s, 0, 0, 0, 0)|^2\right) ds + \mathbb{E}|g(0, 0)|^2 < +\infty.$$

Remark 4.1. From the mean-field structure of (4.1), sometimes the following condition holds:

$(H_1)(i')$ There exist $\lambda_1, \widehat{k}_1 \in \mathbb{R}$, such that for all t, y, z, and process X_1, X_2, a.s.

$$\begin{aligned}
&\mathbb{E}\langle b(t, X_1, \mathbb{E}[X_1(t)|\mathcal{F}_t^{W_0}], y, z) - b(t, X_2, \mathbb{E}[X_2(t)|\mathcal{F}_t^{W_0}], y, z), X_1 - X_2\rangle \\
&\quad \le (\lambda_1 + \widehat{k}_1)\mathbb{E}|X_1 - X_2|^2.
\end{aligned}$$

For example, if $b(t,x,m,y,z) = \lambda_1 x + \widehat{k}_1 m + b_1(y,z)$, then it obviously satisfies the above assumption. Indeed, our mean-field FBSDE (2.15)–(2.16) satisfies this assumption.

Let $\mathcal{H}$ be a Hilbert space. Recall that $L^2_{\mathcal{F}}(0,T;\mathcal{H})$ denotes the space of $\mathcal{H}$-valued $\{\mathcal{F}_s\}$-progressively measurable processes $\{u(s),\ s\in[0,T]\}$ such that $\|u\|^2 := E\int_0^T |u(s)|^2 ds < \infty$. For $\lambda \in \mathbb{R}$, we define an equivalent norm on $L^2_{\mathcal{F}}(0,T;\mathcal{H})$:

$$\|u\|_\lambda := \left(E\int_0^T e^{-\lambda s}|u(s)|^2 ds \right)^{1/2}.$$

Now let us consider MF-FBSDE (4.1); its fully coupled structure brings difficulties for establishing its well-posedness. Similar to [49], when the coupling is weak enough, MF-FBSDE (4.1) should be solvable. The proof of the following theorem is given in the appendix (see also the appendix of [30]).

THEOREM 4.1. *Suppose that assumptions* (H_1) *and* (H_2) *hold. Then there exists a* $\delta_0 > 0$*, which depends on* $k_i, \lambda_1, \lambda_2, T$*, for* $i = 1,4,5,6,7,8,11,12$ *such that when* $k_2,k_3,k_9,k_{10} \in [0,\delta_0)$*, there exists a unique adapted solution* $(X,Y,Z) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^n) \times L^2_{\mathcal{F}}(0,T;\mathbb{R}^l) \times L^2_{\mathcal{F}}(0,T;\mathbb{R}^{l\times(d+1)})$ *to MF-FBSDE* (4.1)*. Further, if* $2(\lambda_1+\lambda_2) < -2k_1 - k_6^2 - k_7^2 - k_8^2$*, there exists a* $\delta_1 > 0$*, which depends on* $k_i, \lambda_1, \lambda_2$*, for* $i = 1,4,5,6,7,8,11,12$ *and is independent of* T*, such that when* $k_2,k_3,k_9,k_{10} \in [0,\delta_1)$*, there exists a unique adapted solution* (X,Y,Z) *to MF-FBSDE* (4.1)*.*

Remark 4.2. If in addition $(H_1)(i')$ holds (see Remark 4.1), by repeating the proof of the above theorem, one can show that if $2(\lambda_1+\lambda_2) < -2\widehat{k}_1 - k_6^2 - k_7^2 - k_8^2$, there exists a $\delta_1 > 0$, which depends on $\widehat{k}_1, k_i, \lambda_1, \lambda_2$, for $i = 4,5,6,7,8,11,12$ and is independent of T, such that when $k_2,k_3,k_9,k_{10} \in [0,\delta_1)$, there exists a unique adapted solution (X,Y,Z) to MF-FBSDE (4.1).

Now let us apply Theorem 4.1 to obtain the well-posedness of consistency condition system (2.15)–(2.16). Recall that

$$\varphi_0(p,q) = \mathbf{P}_{\Gamma_0}\left[R_0^{-1}\left(B_0'p + D_0'q\right)\right], \quad \varphi_k(p,q) = \mathbf{P}_{\Gamma_k}\left[R_k^{-1}\left(B'p + D_k'q\right)\right].$$

If we denote

$W = (W_0, W_1, \dots, W_K)'$, $\Pi = (0, \pi_1, \dots, \pi_K)$, $\alpha = (\alpha_0', \alpha_1', \dots, \alpha_K')'$, $\beta = (\beta_0', \beta_1', \dots, \beta_K')'$, $\mathbb{X} = (x_0', x', \dots, x')'$, $\mathbb{E}(\alpha(t)|\mathcal{F}_t^{W_0}) = (\mathbb{E}(\alpha_0(t)|\mathcal{F}_t^{W_0})', \mathbb{E}(\alpha_1(t)|\mathcal{F}_t^{W_0})', \dots, \mathbb{E}(\alpha_K(t)|\mathcal{F}_t^{W_0})')'$, $\Phi(\beta,\gamma) = (\varphi_0(\beta_0,\gamma_0), \varphi_1(\beta_1,\gamma_1), \dots, \varphi_K(\beta_K,\gamma_K))'$, $\rho_0^\Pi := \Pi \otimes \rho I_{n\times n}$, $\rho^\Pi := \Pi \otimes \rho I_{n\times n}$, $F_0^{1,\Pi} = \Pi \otimes F_0^1$, $F_0^{2,\Pi} := \Pi \otimes F_0^2$, $F^{1,\Pi} := \Pi \otimes F_1$, $F^{2,\Pi} := \Pi \otimes F_2$,

$$\gamma = \begin{pmatrix} \gamma_0 & 0 & \dots & 0 \\ \gamma_{1,0} & \gamma_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{K,0} & 0 & \dots & \gamma_K \end{pmatrix}, \quad \mathbb{B}_0 = \begin{pmatrix} b_0 \\ b \\ \vdots \\ b \end{pmatrix}, \quad \mathbb{D}_0 = \begin{pmatrix} \sigma_0 & 0 & \dots & 0 \\ 0 & \sigma & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma \end{pmatrix},$$

$$\mathbb{A} = \begin{pmatrix} A_0 & 0 & \dots & 0 \\ 0 & A_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_K \end{pmatrix}, \ \mathbb{B} = \begin{pmatrix} B_0 & 0 & \dots & 0 \\ 0 & B & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B \end{pmatrix}, \ \mathbb{R}^{-1} = \begin{pmatrix} R_0 & 0 & \dots & 0 \\ 0 & R_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & R_K \end{pmatrix},$$

$$\mathbb{Q}=\begin{pmatrix} Q_0 & 0 & \dots & 0\\ Q(1-\rho) & Q & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ Q(1-\rho) & 0 & \dots & Q\end{pmatrix}, \mathbb{G}=\begin{pmatrix} G_0 & 0 & \dots & 0\\ G(1-\rho) & G & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ G(1-\rho) & 0 & \dots & G\end{pmatrix},$$

$$\mathbb{F}_1^{\Pi}=\begin{pmatrix} F_0^{1,\Pi}\\ F^{1,\Pi}\\ \vdots\\ F^{1,\Pi}\end{pmatrix}, \mathbb{F}_2^{\Pi}=\begin{pmatrix} F_0^{2,\Pi}\\ F^{2,\Pi}\\ \vdots\\ F^{2,\Pi}\end{pmatrix},$$

$$\mathbb{Q}^{\Pi}=\begin{pmatrix} Q_0\rho_0^{\Pi}\\ Q\rho^{\Pi}\\ \vdots\\ Q\rho^{\Pi}\end{pmatrix},\quad \mathbb{G}^{\Pi}=\begin{pmatrix} G_0\rho_0^{\Pi}\\ G\rho^{\Pi}\\ \vdots\\ G\rho^{\Pi}\end{pmatrix},\quad \mathbb{H}=\begin{pmatrix} 0\\ H\\ \vdots\\ H\end{pmatrix},\quad \mathbb{H}(\alpha)=\begin{pmatrix} 0 & 0 & \dots & 0\\ 0 & H\alpha_0 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & H\alpha_0\end{pmatrix},$$

$$\mathbb{F}_2^{\Pi}(\mathbb{E}(\alpha(t)|\mathcal{F}_t^{W_0}))=\begin{pmatrix} F_0^{2,\Pi}\mathbb{E}(\alpha(t)|\mathcal{F}_t^{W_0}) & 0 & \dots & 0\\ 0 & F^{2,\Pi}\mathbb{E}(\alpha(t)|\mathcal{F}_t^{W_0}) & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & F^{2,\Pi}\mathbb{E}(\alpha(t)|\mathcal{F}_t^{W_0})\end{pmatrix},$$

$$\mathbb{C}=\begin{pmatrix} C_0 & 0 & \dots & 0\\ 0 & C & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & C\end{pmatrix},\quad \mathbb{C}(\alpha)=\begin{pmatrix} C_0\alpha_0 & 0 & \dots & 0\\ 0 & C\alpha & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & C\alpha\end{pmatrix},\quad \mathbb{C}(\gamma)=\begin{pmatrix} C_0'\gamma_0\\ C'\gamma_1\\ \vdots\\ C'\gamma_K\end{pmatrix},$$

$$\mathbb{D}=\begin{pmatrix} D_0 & 0 & \dots & 0\\ 0 & D_1 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & D_K\end{pmatrix},$$

$$\mathbb{D}(\beta,\gamma)=\begin{pmatrix} D_0\varphi_0(\beta_0,\gamma_0) & 0 & \dots & 0\\ 0 & D_1\varphi_1(\beta_1,\gamma_1) & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & D_K\varphi_K(\beta_K,\gamma_K)\end{pmatrix}.$$

Using the above notation, the system (2.15)–(2.16) can be written in compact form as

$$\text{(4.2)}\qquad \begin{cases} d\alpha =\Big(\mathbb{A}\alpha+\mathbb{B}\Phi(\beta,\gamma)+\mathbb{F}_1^{\Pi}\mathbb{E}(\alpha(t)|\mathcal{F}_t^{W_0})+\mathbb{B}_0\Big)dt\\ \qquad\qquad +\Big(\mathbb{C}(\alpha)+\mathbb{D}(\beta,\gamma)+\mathbb{F}_2^{\Pi}\left(\mathbb{E}(\alpha(t)|\mathcal{F}_t^{W_0})\right)+\mathbb{H}(\alpha)+\mathbb{D}_0\Big)dW(t),\\ d\beta =-\Big(\mathbb{A}'\beta-\mathbb{Q}\alpha+\mathbb{Q}^{\Pi}\mathbb{E}(\alpha(t)|\mathcal{F}_t^{W_0}))+\mathbb{C}(\gamma)\Big)dt+\gamma dW(t),\\ \alpha(0) =\mathbb{X},\qquad \beta(T)=-\mathbb{G}\alpha(T)+\mathbb{G}^{\Pi}\mathbb{E}(\alpha(T)|\mathcal{F}_T^{W_0}).\end{cases}$$

Now let λ^* be the largest eigenvalue of the symmetric matrix $\frac{1}{2}(\mathbb{A}+\mathbb{A}')$. Since the projection operator is Lipschitz continuous with Lipschitz constant 1, by comparing (4.2) with (4.1), one can check that the coefficients of assumption (H_1) can be chosen as

$$\begin{aligned}&\lambda_1=\lambda_2=\lambda^*,\quad k_0=\|\mathbb{A}\|,\quad k_1=\|\mathbb{F}_1^\Pi\|,\quad k_2=k_3=\|\mathbb{R}^{-1}\|\|\mathbb{B}\|(\|\mathbb{B}\|+\|\mathbb{D}\|),\\&k_4=\|\mathbb{Q}\|,\quad k_5=\|\mathbb{Q}^\Pi\|,\quad k_6=\|\mathbb{C}\|,\quad k_7^2=4(\|\mathbb{C}\|+\|\mathbb{H}\|)^2,\quad k_8^2=4\|\mathbb{F}_2^\Pi\|^2,\\&k_9=k_{10}=\|\mathbb{R}^{-1}\|\|\mathbb{D}\|(\|\mathbb{B}\|+\|\mathbb{D}\|),\quad k_{11}^2=2\|\mathbb{G}\|^2,\quad k_{12}^2=2\|\mathbb{G}^\Pi\|^2.\end{aligned}$$

Thus by applying Theorem 4.1, we obtain the following global well-posedness of (4.2).

THEOREM 4.2. *Suppose that*

$$4\lambda^*<-2\|\mathbb{F}_1^\Pi\|-\|\mathbb{C}\|^2-4(\|\mathbb{C}\|+\|\mathbb{H}\|)^2-4\|\mathbb{F}_2^\Pi\|^2;$$

then there exists a $\delta_1>0$, *which depends on* $\lambda^*,\|\mathbb{F}_1^\Pi\|,\|\mathbb{Q}\|,\|\mathbb{Q}^\Pi\|,\|\mathbb{C}\|,\|\mathbb{H}\|,\|\mathbb{F}_2^\Pi\|,$ $\|\mathbb{G}\|,\|\mathbb{G}^\Pi\|$, *and is independent of* T, *such that when* $\|\mathbb{R}^{-1}\|,\|\mathbb{B}\|,\|\mathbb{D}\|\in[0,\delta_1)$, *there exists a unique adapted solution* (α,β,γ) *to consistency condition system* (2.15)–(2.16).

Remark 4.3. Let $\lambda^*_{\mathbb{F}_1^\Pi}$ be the largest eigenvalue of $\frac{1}{2}(\mathbb{F}_1^\Pi+(\mathbb{F}_1^\Pi)')$. Noticing Remark 4.1, one can check that $(H_1)-(i')$ holds with $\widehat{k}_1=\lambda^*_{\mathbb{F}_1^\Pi}$. Thus, from Remark 4.2, we have that if

$$4\lambda^*<-2\lambda^*_{\mathbb{F}_1^\Pi}-\|\mathbb{C}\|^2-4(\|\mathbb{C}\|+\|\mathbb{H}\|)^2-4\|\mathbb{F}_2^\Pi\|^2,$$

then there exists a $\delta_1>0$, which depends on $\lambda^*,\lambda^*_{\mathbb{F}_1^\Pi},\|\mathbb{Q}\|,\|\mathbb{Q}^\Pi\|,\|\mathbb{C}\|,\|\mathbb{H}\|,\|\mathbb{F}_2^\Pi\|,$ $\|\mathbb{G}\|,\|\mathbb{G}^\Pi\|$, and is independent of T, such that when $\|\mathbb{R}^{-1}\|,\|\mathbb{B}\|,\|\mathbb{D}\|\in[0,\delta_1)$, there exists a unique adapted solution (α,β,γ) to consistency condition system (2.15)–(2.16).

5. ε-Nash equilibrium for Problem (CC). In section 2, we characterized the decentralized strategy profile $\bar{u}=(\bar{u}_0,\bar{u}_1,\ldots,\bar{u}_N)$ of Problem (CC) through the auxiliary Problem (LCC) and the consistency condition system. Now, we turn to verify the ε-Nash equilibrium of this decentralized strategy profile. Here, we proceed with our verification based on the assumptions of the local case (section 3). Note that it can also be verified based on the global case (section 4) without essential difficulties. For the major agent $\mathcal{A}_0$ and the minor agent $\mathcal{A}_i$, the decentralized states $\breve{x}_t^0$ and $\breve{x}_t^i$ are given, respectively, by

$$(5.1)\qquad \begin{cases}d\breve{x}_0=\Big[A_0\breve{x}_0+B_0\varphi_0(\bar{p}_0,\bar{q}_0)+F_0^1\breve{x}^{(N)}+b_0\Big]\,dt\\ \qquad\quad+\Big[C_0\breve{x}_0+D_0\varphi_0(\bar{p}_0,\bar{q}_0)+F_0^2\breve{x}^{(N)}+\sigma_0\Big]\,dW_0(t),\\ d\breve{x}_i=\Big[A_{\theta_i}\breve{x}_i+B\varphi_{\theta_i}(\bar{p}_i,\bar{q}_i)+F_1\breve{x}^{(N)}+b\Big]\,dt\\ \qquad\quad+\Big[C\breve{x}_i+D_{\theta_i}\varphi_{\theta_i}(\bar{p}_i,\bar{q}_i)+F_2\breve{x}^{(N)}+H\breve{x}_0+\sigma\Big]\,dW_i(t),\\ \breve{x}_0(0)=x_0,\quad \breve{x}_i(0)=x,\end{cases}$$

where $\breve{x}^{(N)} = \frac{1}{N}\sum_{i=1}^N \breve{x}^i$ and the processes $(\bar{p}_0, \bar{q}_0, \bar{p}_i, \bar{q}_i)$ are solved by

(5.2)

$$\begin{cases} d\bar{x}_0 = \left(A_0\bar{x}_0 + B_0\varphi_0(\bar{p}_0,\bar{q}_0) + F_0^1\sum_{k=1}^K \pi_k\mathbb{E}(\alpha_k(t)|\mathcal{F}_t^{W_0}) + b_0\right)dt \\ \qquad + \left(C_0\bar{x}_0 + D_0\varphi_0(\bar{p}_0,\bar{q}_0) + F_0^2\sum_{k=1}^K \pi_k\mathbb{E}(\alpha_k(t)|\mathcal{F}_t^{W_0}) + \sigma_0\right)dW_0(t), \\ d\bar{p}_0 = -\left(A'\bar{p}_0 - Q_0(\bar{x}_0 - \rho_0\sum_{i=1}^K \pi_k\mathbb{E}(\alpha_k(t)|\mathcal{F}_t^{W_0})) + C'\bar{q}_0\right)dt + \bar{q}_0 dW_0(t), \\ d\bar{x}_i = \left(A_{\theta_i}\bar{x}_i + B\varphi_{\theta_i}(\bar{p}_i,\bar{q}_i) + F_1\sum_{k=1}^K \pi_k\mathbb{E}(\alpha_k(t)|\mathcal{F}_t^{W_0}) + b\right)dt \\ \qquad + \left(C\bar{x}_i + D_{\theta_i}\varphi_{\theta_i}(\bar{p}_i,\bar{q}_i) + F_2\sum_{k=1}^K \pi_k\mathbb{E}(\alpha_k(t)|\mathcal{F}_t^{W_0}) + H\bar{x}_0 + \sigma\right)dW_i(t), \\ d\bar{p}_i = -\left(A'_{\theta_i}\bar{p}_i - Q\left(\bar{x}_i - \rho\sum_{k=1}^K \pi_k\mathbb{E}(\alpha_k(t)|\mathcal{F}_t^{W_0}) - (1-\rho)\bar{x}_0\right) + C'\bar{q}_i\right)dt \\ \qquad\qquad + \bar{q}_i dW_i(t) + \bar{q}1_{i,0}dW_0(t), \\ \bar{x}_0(0) = x_0, \qquad \bar{p}_0(T) = -G_0\left(\bar{x}_0(T) - \rho_0\sum_{k=1}^K \pi_k\mathbb{E}(\alpha_k(T)|\mathcal{F}_T^{W_0})\right), \\ \bar{x}_i(0) = x, \qquad \bar{p}_i(T) = -G\left(\bar{x}_i(T) - \rho\sum_{k=1}^K \pi_k\mathbb{E}(\alpha_k(T)|\mathcal{F}_T^{W_0}) - (1-\rho)\bar{x}_0(T)\right). \end{cases}$$

Here we recall that $\varphi_0(p,q) := \mathbf{P}_{\Gamma_0}[R_0^{-1}(B_0'p + D_0'q)]$, $\varphi_{\theta_i}(p,q) := \mathbf{P}_{\Gamma_{\theta_i}}[R_{\theta_i}^{-1}(B'p + D'_{\theta_i}q)]$, and α_k, $1 \le k \le K$, are given by (2.15) and (2.16). We mention that (5.2) gives also the dynamics of the limiting state $(\bar{x}_0, \bar{x}_1, \dots, \bar{x}_N)$ and one can check easily that $(\bar{x}_0, \bar{p}_0, \bar{q}_0) = (\alpha_0, \beta_0, \gamma_0)$. Now, we would like to show that for $\bar{u}_0 = \varphi_0(\bar{p}_0, \bar{q}_0)$ and $\bar{u}_i = \varphi_{\theta_i}(\bar{p}_i, \bar{q}_i)$, $1 \le i \le N$, $(\bar{u}_0, \bar{u}_1, \dots, \bar{u}_N)$ is an ε-Nash equilibrium of Problem (CC). Let us first present the following several lemmas.

LEMMA 5.1. *Under* A1–A4*, there exists a constant M independent of N, which may vary line by line in the following, such that*

$$\sup_{0\le i\le N} \mathbb{E} \sup_{0\le t\le T} \left|\breve{x}^i(t)\right|^2 \le M.$$

Proof. From Theorem 3.1, we know that on a small time interval the system of fully coupled FBSDE (2.15)–(2.16) has a unique solution (for the global case, see Theorem 4.2 and Remark 4.3),

$$(\alpha_0, \beta_0, \gamma_0) \in L^2_{\mathcal{F}^{W_0}}(0,T;\mathbb{R}^{n\times 3}) \text{ and } (\alpha_k, \beta_k, \gamma_k, \gamma_{k,0}) \in L^2_{\mathcal{F}^k}(0,T;\mathbb{R}^{n\times 4}), \quad 1\le k\le K.$$

Then, the classical results on FBSDEs yield that (5.2) also has a unique solution,

$$(\bar{x}_0, \bar{p}_0, \bar{q}_0) \in L^2_{\mathcal{F}^{W_0}}(0,T;\mathbb{R}^{n\times 3}) \text{ and } (\bar{x}_i, \bar{p}_i, \bar{q}_i, \bar{q}_{i,0}) \in L^2_{\mathcal{F}^i}(0,T;\mathbb{R}^{n\times 4}), \quad 1 \le i \le N.$$

(Indeed, FBSDE (5.2) has a unique solution for arbitrary T by using similar arguments as in Theorem 2.1 of [29] and [31, 50]). Thus, SDEs system (5.1) has also a unique solution,

$$(\breve{x}_0, \breve{x}_1 \dots, \breve{x}_N) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^n) \times L^2_{\mathcal{F}}(0,T;\mathbb{R}^n) \times \cdots \times L^2_{\mathcal{F}}(0,T;\mathbb{R}^n).$$

Moreover, since $\{W_i\}_{i=1}^N$ is an N-dimensional Brownian motion whose components are independent and identically distributed, we have that for the k-type minor agents, under the conditional expectation $\mathbb{E}(\cdot|\mathcal{F}^{W_0}))$, for each $1 \le k \le K$, the processes $(\bar{x}_i, \bar{p}_i, \bar{q}_i)$, $i \in \mathcal{I}_k$, are independent and identically distributed. We also note that for each $1 \le k \le K$, $\breve{x}_i$, $i \in \mathcal{I}_k$, are identically distributed. Noticing that $(\bar{p}_0, \bar{q}_0) \in L^2_{\mathcal{F}^{W_0}}(0,T;\mathbb{R}^n) \times L^2_{\mathcal{F}^{W_0}}(0,T;\mathbb{R}^n)$, and $(\bar{p}_i, \bar{q}_i) \in L^2_{\mathcal{F}^i}(0,T;\mathbb{R}^n) \times L^2_{\mathcal{F}^i}(0,T;\mathbb{R}^n)$, $1 \le i \le N$, then the Lipschitz property of the projection onto the convex set yields that $\varphi_{\theta_0}(\bar{p}_0, \bar{q}_0) := \varphi_0(\bar{p}_0, \bar{q}_0) = \mathbf{P}_{\Gamma_0}(R_0^{-1}(B^T\bar{p}_0 + D^T\bar{q}_0)) \in L^2_{\mathcal{F}^{W_0}}(0,T;\Gamma_0)$ and $\varphi_{\theta_i}(\bar{p}_i, \bar{q}_i) := \mathbf{P}_{\Gamma_{\theta_i}}(R_{\theta_i}^{-1}(B^T\bar{p}_i + D^T\bar{q}_i)) \in L^2_{\mathcal{F}^i}(0,T;\Gamma_{\theta_i})$, $1 \le i \le N$. Moreover, there exists a constant M independent of N such that for all $0 \le i \le N$, $0 \le k \le K$,

$$\begin{aligned} &\mathbb{E}\sup_{0\le t\le T}(|\alpha_k(t)|^2 + |\beta_k(t)|^2 + |\bar{x}_i(t)|^2 + |\bar{p}_i(t)|^2) \\ &\qquad + \mathbb{E}\int_0^T (|\gamma_k(t)|^2 + |\bar{q}_i(t)|^2 + |\varphi_{\theta_i}(\bar{p}_i(t), \bar{q}_i(t))|^2) \le M. \end{aligned} \tag{5.3}$$

From (5.1), by using Burkholder–Davis–Gundy (BDG) inequality, it follows that for any $t \in [0,T]$,

$$\begin{aligned} \mathbb{E}\sup_{0\le s\le t}|\breve{x}_0(s)|^2 &\le M + M\mathbb{E}\int_0^t \left[|\breve{x}_0(s)|^2 + |\breve{x}^{(N)}(s)|^2\right]ds \\ &\le M + M\mathbb{E}\int_0^t \left[|\breve{x}_0(s)|^2 + \frac{1}{N}\sum_{i=1}^N |\breve{x}_i(s)|^2\right]ds, \end{aligned}$$

and by Gronwall's inequality, we obtain

$$\mathbb{E}\sup_{0\le s\le t}|\breve{x}_0(s)|^2 \le M + M\mathbb{E}\int_0^t \frac{1}{N}\sum_{i=1}^N |\breve{x}_i(s)|^2 ds. \tag{5.4}$$

Similarly, from (5.1) again and using (5.4), we have

$$\begin{aligned} \mathbb{E}\sup_{0\le s\le t}|\breve{x}_i(s)|^2 &\le M + M\mathbb{E}\int_0^t \left[|\breve{x}_0(s)|^2 + |\breve{x}_i(s)|^2 + \frac{1}{N}\sum_{i=1}^N |\breve{x}_i(s)|^2\right]ds \\ &\le M + M\mathbb{E}\int_0^t \left[|\breve{x}_i(s)|^2 + \frac{1}{N}\sum_{i=1}^N |\breve{x}_i(s)|^2\right]ds. \end{aligned} \tag{5.5}$$

Thus

$$\mathbb{E}\sup_{0\le s\le t}\sum_{i=1}^N |\breve{x}_i(s)|^2 \le \mathbb{E}\sum_{i=1}^N \sup_{0\le s\le t}|\breve{x}_i(s)|^2 \le MN + 2M\mathbb{E}\int_0^t \left[\sum_{i=1}^N |\breve{x}_i(s)|^2\right]ds.$$

By Gronwall's inequality, it follows that $\mathbb{E}\sup_{0\le t\le T}\sum_{i=1}^N |\breve{x}_i(t)|^2 = O(N)$, $1 \le i \le N$. Then, substituting this estimate to (5.5) and Gronwall's inequality yields $\mathbb{E}\sup_{0\le t\le T}|\breve{x}_i(t)|^2 \le M, 1 \le i \le N$. By applying this estimate to (5.4), we get $\mathbb{E}\sup_{0\le t\le T}|\breve{x}_0(t)|^2 \le M$. ∎

LEMMA 5.2. *Under* A1–A4*, there exists a constant M independent of N such that*

$$\mathbb{E}\sup_{0\le t\le T}\left|\breve{x}^{(N)}(t) - \Phi(t)\right|^2 \le M\left(\frac{1}{N} + \varepsilon_N^2\right), \text{ where } \varepsilon_N = \sup_{1\le k\le K}|\pi_k^{(N)} - \pi_k|.$$

Recall that $\breve{x}^{(N)} = \frac{1}{N}\sum_{k=1}^{K}\sum_{i\in\mathcal{I}_k}\breve{x}_i = \sum_{k=1}^{K}\pi_k^{(N)}\frac{1}{N_k}\sum_{i\in\mathcal{I}_k}\breve{x}_i$ *and* $\Phi(t) = \sum_{k=1}^{K}\pi_k \mathbb{E}(\alpha_k(t)|\mathcal{F}_t^{W_0})$.

Proof. For each fixed $1 \le k \le K$, we consider the k-type minor agents. We denote $\breve{x}^{(k)} := \frac{1}{N_k}\sum_{i\in\mathcal{I}_k}\breve{x}_i$. Let us add up N_k states of all k-type minor agents and then divide by N_k; we have

(5.6)
$$d\breve{x}^{(k)} = \left[A_k\breve{x}^{(k)} + \frac{B}{N_k}\sum_{i\in\mathcal{I}_k}\varphi_k(\bar{p}_i,\bar{q}_i) + F^1\breve{x}^{(N)} + b_0\right]dt + \frac{1}{N_k}\sum_{i\in\mathcal{I}_k}\left[C\breve{x}_i + D_k\varphi_k(\bar{p}_i,\bar{q}_i) + F_2\breve{x}^{(N)} + H\breve{x}_0 + \sigma_0\right]dW_i(t), \quad \breve{x}^{(k)}(0) = x.$$

Now we take conditional expectation $\mathbb{E}(\cdot|\mathcal{F}_t^{W_0})$ on the first equation of (2.15). Noticing that $\alpha_k(s)$ is $\mathcal{F}_s^k$-adapted and recalling $m_k = \mathbb{E}(\alpha_k(t)|\mathcal{F}_t^{W_0})$ and $\mathbb{E}(\varphi_k(\bar{p}_i,\bar{q}_i)|\mathcal{F}_t^{W_0}) = \mathbb{E}(\varphi_k(\beta_k,\gamma_k)|\mathcal{F}_t^{W_0})$ for any $i \in \mathcal{I}_k$, we have

$$dm_k = \Big(A_k m_k + B\mathbb{E}(\varphi_k(\bar{p}_i,\bar{q}_i)|\mathcal{F}_t^{W_0}) + F_1\Phi + b\Big)\,dt, \qquad m_k(0) = x. \tag{5.7}$$

From (5.6) and (5.7), by denoting $\Delta_k(t) := \breve{x}^{(k)}(t) - m_k(t)$, we have

$$d\Delta_k = \left[A_k\Delta_k + F_1(\breve{x}^{(N)} - \Phi) + \frac{B}{N_k}\sum_{i\in\mathcal{I}_k}\varphi_k(\bar{p}_i,\bar{q}_i) - B\mathbb{E}(\varphi_k(\bar{p}_i,\bar{q}_i)|\mathcal{F}_t^{W_0})\right]dt + \frac{1}{N_k}\sum_{i\in\mathcal{I}_k}\left[C\breve{x}_i + D_k\varphi_k(\bar{p}_i,\bar{q}_i) + F_2\breve{x}^{(N)} + H\breve{x}_0 + \sigma_0\right]dW_i(t), \qquad \Delta(0) = 0.$$

The Cauchy–Schwarz inequality and BDG inequality yield that

$$\begin{aligned}\mathbb{E}\sup_{0\le s\le t}|\Delta_k(s)|^2 &\le M\mathbb{E}\int_0^t\Big[|\Delta_k(s)|^2 + |\breve{x}^{(N)}(s) - \Phi(s)|^2\Big]ds \\ &\quad + M\mathbb{E}\int_0^t\Big|\frac{1}{N_k}\sum_{i\in\mathcal{I}_k}\varphi_k(\bar{p}_i(s),\bar{q}_i(s)) - \mathbb{E}(\varphi_k(\bar{p}_i(s),\bar{q}_i(s))|\mathcal{F}_s^{W_0})\Big|^2 ds \\ &\quad + \frac{M}{N_k^2}\mathbb{E}\sum_{i\in\mathcal{I}_k}\int_0^t\Big|C\breve{x}_i + D_k\varphi_k(\bar{p}_i,\bar{q}_i) + F_2(\breve{x}^{(N)} - \Phi) + F_2\Phi + H\breve{x}_0 + \sigma_0\Big|^2 ds.\end{aligned} \tag{5.8}$$

Let us first focus on the second term of the right-hand side of (5.8). Since for each fixed $s \in [0,T]$, under the conditional expectation $\mathbb{E}(\cdot|\mathcal{F}_s^{W_0}))$, for each $1 \le k \le K$, the processes $(\bar{x}_i(s),\bar{p}_i(s),\bar{q}_i(s))$, $i \in \mathcal{I}_k$, are independent and identically distributed, if we denote $\mu(s) = \mathbb{E}(\varphi_k(\bar{p}_i(s),\bar{q}_i(s))|\mathcal{F}_s^{W_0}))$, then μ does not depend on i and moreover we have

$$\begin{aligned}&\mathbb{E}\left|\frac{1}{N_k}\sum_{i\in\mathcal{I}_k}\varphi_k(\bar{p}_i(s),\bar{q}_i(s)) - \mu(s)\right|^2 \\ &= \frac{1}{N_k^2}\mathbb{E}\Bigg(\sum_{i\in\mathcal{I}_k}|\varphi_k(\bar{p}_i(s),\bar{q}_i(s)) - \mu(s)|^2 \\ &\qquad + \sum_{i,j\in\mathcal{I}_k, j\ne i}\langle\varphi_k(\bar{p}_i(s),\bar{q}_i(s)) - \mu(s), \varphi_k(\bar{p}_j(s),\bar{q}_j(s)) - \mu(s)\rangle\Bigg).\end{aligned}$$

Since $(\bar{p}_i(s), \bar{q}_i(s))$, $i \in \mathcal{I}_k$, are independent under $\mathbb{E}(\cdot|\mathcal{F}_s^{W_0}))$, we have

$$\begin{aligned}
&\mathbb{E} \sum_{i,j\in\mathcal{I}_k, j\neq i} \langle \varphi_k(\bar{p}_i(s), \bar{q}_i(s)) - \mu(s), \varphi_k(\bar{p}_j(s), \bar{q}_j(s)) - \mu(s)\rangle \\
&\quad = \mathbb{E}\left[\sum_{i,j\in\mathcal{I}_k, j\neq i} \mathbb{E}\left(\langle \varphi_k(\bar{p}_i(s), \bar{q}_i(s)) - \mu(s), \varphi_k(\bar{p}_j(s), \bar{q}_j(s)) - \mu(s)\rangle \Big| \mathcal{F}_s^{W_0}\right)\right] \\
&\quad = \mathbb{E}\left[\sum_{i,j\in\mathcal{I}_k, j\neq i} \Big\langle \mathbb{E}\left(\varphi_k(\bar{p}_i(s), \bar{q}_i(s)) - \mu(s) \Big| \mathcal{F}_s^{W_0}\right),\right. \\
&\qquad\qquad \left. \mathbb{E}\left(\varphi_k(\bar{p}_j(s), \bar{q}_j(s)) - \mu(s) \Big| \mathcal{F}_s^{W_0}\right)\Big\rangle\right] = 0.
\end{aligned}$$

Then, due to (5.3) and the fact that $(\bar{p}_i, \bar{q}_i)$, $1 \leq i \leq N$, are identically distributed, we obtain

$$\begin{aligned}
&\mathbb{E}\int_0^t \Big| \frac{1}{N_k}\sum_{i\in\mathcal{I}_k} \varphi_k\left(\bar{p}_i(s), \bar{q}_i(s)\right) - \mathbb{E}\left(\varphi_k(\bar{p}_i(s), \bar{q}_i(s))|\mathcal{F}_s^{W_0}\right)\Big|^2 ds \\
&\quad = \frac{1}{N_k^2}\int_0^t \mathbb{E}\sum_{i\in\mathcal{I}_k} |\varphi_k(\bar{p}_i(s), \bar{q}_i(s)) - \mu(s)|^2\, ds \\
&\quad = \frac{1}{N_k}\int_0^t \mathbb{E}\,|\varphi_k(\bar{p}_i(s), \bar{q}_i(s)) - \mu(s)|^2\, ds \leq \frac{M}{N_k}.
\end{aligned}$$

Now we focus on the third term of the right-hand side of (5.8). Recalling $\Phi(t) = \sum_{k=1}^K \pi_k \mathbb{E}(\alpha_k(t)|\mathcal{F}_t^{W_0})$, we have $\mathbb{E}\sup_{0\leq t\leq T}|\Phi(t)|^2 \leq M$; then using (5.3), Lemma 5.1, and that $(\breve{x}_i(s), \bar{p}_i(s), \bar{q}_i(s))$, $i \in \mathcal{I}_k$, are identically distributed, it follows that

$$\begin{aligned}
&\frac{M}{N_k^2}\sum_{i\in\mathcal{I}_k}\mathbb{E}\int_0^t \Big| C\breve{x}_i + D_k\varphi_k(\bar{p}_i, \bar{q}_i) + F_2(\breve{x}^{(N)} - \Phi) + F_2\Phi + H\breve{x}_0 + \sigma_0\Big|^2 ds \\
&\quad \leq \frac{M}{N_k^2}\sum_{i\in\mathcal{I}_k}\mathbb{E}\int_0^t \Big(|\breve{x}_i(s)|^2 + |\varphi_k(\bar{p}_i, \bar{q}_i)|^2 + |\breve{x}^{(N)}(s) \\
&\qquad\qquad - \Phi(s)|^2 + |\Phi(s)|^2 + |\breve{x}_0(s)|^2 + |\sigma(s)|^2\Big) ds \\
&\quad \leq \frac{M}{N_k}\mathbb{E}\int_0^t \mathbb{E}|\breve{x}^{(N)}(s) - \Phi(s)|^2 ds + \frac{M}{N_k}.
\end{aligned}$$

Therefore, from the above analysis, we get from (5.8) that

$$\mathbb{E}\sup_{0\leq s\leq t}|\Delta_k(s)|^2 \leq M\mathbb{E}\int_0^t \Big[|\Delta_k(s)|^2 + |\breve{x}^{(N)}(s) - \Phi(s)|^2\Big] ds + \frac{M}{N_k},$$

and Gronwall's inequality yields that

$$\mathbb{E}\sup_{0\leq s\leq t}|\Delta_k(s)|^2 \leq M\mathbb{E}\int_0^t |\breve{x}^{(N)}(s) - \Phi(s)|^2 ds + \frac{M}{N_k}. \tag{5.9}$$

Since

$$\begin{aligned}
&\breve{x}^{(N)}(s) - \Phi(s)\\
&\quad = \sum_{k=1}^{K} \left[\pi_k^{(N)} \frac{1}{N_k} \sum_{i\in\mathcal{I}_k} \breve{x}_i(s) - \pi_k \mathbb{E}(\alpha_k(s)|\mathcal{F}_s^{W_0}) \right]\\
&\quad = \sum_{k=1}^{K} \left[\pi_k^{(N)} \left(\frac{1}{N_k} \sum_{i\in\mathcal{I}_k} \breve{x}_i(s) - \mathbb{E}(\alpha_k(s)|\mathcal{F}_s^{W_0}) \right) + \left(\pi_k^{(N)} - \pi_k \right) \mathbb{E}(\alpha_k(s)|\mathcal{F}_s^{W_0}) \right]\\
&\quad = \sum_{k=1}^{K} \pi_k^{(N)} \Delta_k(s) + \sum_{k=1}^{K} \left(\pi_k^{(N)} - \pi_k \right) \mathbb{E}(\alpha_k(s)|\mathcal{F}_s^{W_0}),
\end{aligned}$$

by using (5.3), (5.9) and $\pi_k^{(N)} = \frac{N_k}{N} \le 1$ we obtain that for any $t \in [0,T]$,

$$\begin{aligned}
\mathbb{E} \sup_{0\le s\le t} |\breve{x}^{(N)}(s) - \Phi(s)|^2 &\le \mathbb{E} \sum_{k=1}^{K} \sup_{0\le s\le t} \pi_k^{(N)} |\Delta_k(s)|^2 + M\varepsilon_N^2\\
&\le M\mathbb{E} \int_0^t |\breve{x}^{(N)}(s) - \Phi(s)|^2 ds + \frac{M}{N} + M\varepsilon_N^2.
\end{aligned}$$

Finally, by using Gronwall's inequality, we complete the proof. ∎

LEMMA 5.3. *Under the assumptions of* A1–A4*, we have*

$$\sup_{0\le i\le N} \mathbb{E} \sup_{0\le t\le T} \left| \breve{x}_i(t) - \bar{x}_i(t) \right|^2 \le M \left(\frac{1}{N} + \varepsilon_N^2 \right).$$

Proof. On the one hand, from the first equations of both (5.1) and (5.2), we have

$$\begin{cases} d(\breve{x}_0 - \bar{x}_0) = \left[A_0(\breve{x}_0 - \bar{x}_0) + F_0^1(\breve{x}^{(N)} - \Phi) \right] dt + \left[C_0(\breve{x}_0 - \bar{x}_0) + F_0^2(\breve{x}^{(N)} - \Phi) \right] dW_i(t),\\ \breve{x}_0(0) - \bar{x}_0(0) = 0. \end{cases}$$

The classical estimate for the SDE yields that

$$\mathbb{E} \sup_{0\le t\le T} \left| \breve{x}_0(t) - \bar{x}_0(t) \right|^2 \le ME \int_0^T \left| \breve{x}^{(N)}(s) - \Phi(s) \right|^2 ds,$$

where M is a constant independent of N. Noticing Lemma 5.2, we obtain

$$\mathbb{E} \sup_{0\le t\le T} \left| \breve{x}_0(t) - \bar{x}_0(t) \right|^2 \le M \left(\frac{1}{N} + \varepsilon_N^2 \right). \tag{5.10}$$

On the other hand, from the second equation of (5.1) and the third equation of (5.2), with the help of the classical estimate for the SDE, we have that for $1 \le i \le N$,

$$\mathbb{E} \sup_{0\le t\le T} \left| \breve{x}_i(t) - \bar{x}_i(t) \right|^2 \le M\mathbb{E} \int_0^T \left(\left| \breve{x}^{(N)}(s) - \Phi(s) \right|^2 + |\breve{x}_0(s) - \bar{x}_0(s)|^2 \right) ds,$$

and noticing Lemma 5.2 and (5.10), we obtain $\mathbb{E} \sup_{0\le t\le T} |\breve{x}_i(t) - \bar{x}_i(t)|^2 \le M(\frac{1}{N} + \varepsilon_N^2)$, $0 \le i \le N$. ∎

LEMMA 5.4. *Under the assumptions of* A1–A4, *for all* $0 \le i \le N$, *we have*

$$\Big|\mathcal{J}_i(\bar u_i, \bar u_{-i}) - J_i(\bar u_i)\Big| = O\left(\frac{1}{\sqrt{N}} + \varepsilon_N\right).$$

Proof. Let us first consider the major agent. Recalling (2.3), (2.6), and (2.17), we have

$$\begin{aligned}
&\mathcal{J}_0(\bar u_0, \bar u_{-0}) - J_0(\bar u_0)\\
&\quad= \frac{1}{2}\mathbb{E}\Big[\int_0^T \Big(\Big\langle Q_0\big(\breve x_0 - \rho_0 \breve x^{(N)}\big), \breve x_0 - \rho_0\breve x^{(N)}\Big\rangle - \Big\langle Q_0\big(\bar x_0 - \rho_0\Phi\big), \bar x_0 - \rho_0\Phi\Big\rangle\Big)\,dt\\
&\qquad\qquad + \Big\langle G_0\big(\breve x_0(T) - \rho_0\breve x^{(N)}(T)\big), \breve x_0(T) - \rho_0\breve x^{(N)}(T)\Big\rangle\\
&\qquad\qquad\qquad - \Big\langle G_0\big(\bar x_0(T) - \rho_0\Phi(T)\big), \bar x_0(T) - \rho_0\Phi(T)\Big\rangle\Big].
\end{aligned}\tag{5.11}$$

From (5.3), we have $\mathbb{E}\sup_{0\le t\le T}|\bar x_0(t)|^2 \le M$ and $\mathbb{E}\sup_{0\le t\le T}|\alpha_i(t)|^2 \le M$ for any $0 \le i \le N$. Recalling $\mathbb{E}\sup_{0\le t\le T}|\Phi(t)|^2 \le M$ and Lemmas 5.2 and 5.3, as well as

$$\begin{aligned}
&\langle Q_0(a-b), a-b\rangle - \langle Q_0(c-d), c-d\rangle\\
&\quad = \langle Q_0(a-b-(c-d)), a-b-(c-d)\rangle + 2\langle Q_0(a-b-(c-d)), c-d\rangle,
\end{aligned}$$

we have

$$\begin{aligned}
&\Big|\mathbb{E}\int_0^T \Big(\big\langle Q_0\big(\breve x_0-\rho_0\breve x^{(N)}\big), \breve x_0-\rho_0\breve x^{(N)}\big\rangle - \Big\langle Q_0\big(\bar x_0-\rho_0\Phi\big), \bar x_0-\rho_0\Phi\Big\rangle\Big)\,dt\Big|\\
&\quad\le M\int_0^T \mathbb{E}\big|\breve x_0 - \rho_0\breve x^{(N)} - (\bar x_0 - \rho_0\Phi)\big|^2 dt\\
&\qquad + M\int_0^T \mathbb{E}\big|\breve x_0 - \rho_0\breve x^{(N)} - (\bar x_0 - \rho_0\Phi)\big|\big|\bar x_0 - \rho_0\Phi\big| dt\\
&\quad\le M\int_0^T \mathbb{E}\big|\breve x_0 - \bar x_0\big|^2 dt + M\int_0^T \mathbb{E}\big|\breve x^{(N)} - \Phi\big|^2 dt\\
&\qquad\qquad + M\int_0^T \Big(\mathbb{E}\big|\breve x_0 - \rho_0\breve x^{(N)} - (\bar x_0 - \rho_0\Phi)\big|^2\Big)^{\frac{1}{2}}\Big(\mathbb{E}\,|\bar x_0 - \rho_0\Phi|^2\Big)^{\frac{1}{2}}\,dt\\
&\quad\le M\int_0^T \mathbb{E}\big|\breve x_0 - \bar x_0\big|^2 dt + M\int_0^T \mathbb{E}\big|\breve x^{(N)} - \Phi\big|^2 dt\\
&\qquad + M\int_0^T \Big(\mathbb{E}\big|\breve x_0 - \bar x_0\big|^2 + \mathbb{E}\big|\breve x^{(N)} - \Phi\big|^2\Big)^{\frac{1}{2}}\,dt\\
&\quad = O\left(\frac{1}{\sqrt{N}} + \varepsilon_N\right).
\end{aligned}$$

A similar argument allows us to show that

$$\begin{aligned}
&\Big|\mathbb{E}\Big[\Big\langle G_0\big(\breve x_0(T)-\rho_0\breve x^{(N)}(T)\big), \breve x_0(T)-\rho_0\breve x^{(N)}(T)\Big\rangle\\
&\quad -\Big\langle G_0\big(\bar x_0(T)-\rho_0\Phi(T)\big), \bar x_0(T)-\rho_0\Phi(T)\Big\rangle\Big]\Big|\\
&\quad = O\left(\frac{1}{\sqrt{N}} + \varepsilon_N\right).
\end{aligned}$$

Thus, the proof for the major agent is completed by noticing (5.11). Let us now focus on the minor agents; for $1 \le i \le N$, recalling (2.4), (2.12), and (2.17), we have

$$\begin{aligned}&\mathcal{J}_i(\bar{u}_i,\bar{u}_{-i})\\ &\quad=\frac{1}{2}\mathbb{E}\Big[\int_0^T\Big(\big\langle Q\big(\breve{x}_i-\rho\breve{x}^{(N)}-(1-\rho)\breve{x}_0\big),\breve{x}_i-\rho\breve{x}^{(N)}-(1-\rho)\breve{x}_0\big\rangle+\big\langle R_{\theta_i}\bar{u}_i,\bar{u}_i\big\rangle\Big)dt\\ &\qquad+\big\langle G\big(\breve{x}_i(T)-\rho\breve{x}^{(N)}(T)-(1-\rho)\breve{x}_0(T)\big),\breve{x}_i(T)-\rho\breve{x}^{(N)}(T)-(1-\rho)\breve{x}_0(T)\big\rangle\Big]\end{aligned}$$

and

$$\begin{aligned}J_i(\bar{u}_i)=&\frac{1}{2}\mathbb{E}\Big[\int_0^T\big(\big\langle Q\big(\bar{x}_i-\rho\Phi-(1-\rho)\bar{x}_0\big),\bar{x}_i-\rho\Phi-(1-\rho)\bar{x}_0\big\rangle dt+\big\langle R_{\theta_i}\bar{u}_i,\bar{u}_i\big\rangle\big)\,dt\\ &+\big\langle G\big(\bar{x}_i(T)-\rho\Phi(T)-(1-\rho)\bar{x}_0(T)\big),\bar{x}_i(T)-\rho\Phi(T)-(1-\rho)\bar{x}_0(T)\big\rangle\Big].\end{aligned}$$

From (5.3), we have $\mathbb{E}\sup_{0\le t\le T}|\bar{x}_i(t)|^2 \le M$. Using such an estimate, $\mathbb{E}\sup_{0\le t\le T}|\Phi(t)|^2 \le M$, and Lemmas 5.2 and 5.3, similar to the major agent, it follows that $|\mathcal{J}_i(\bar{u}_i,\bar{u}_{-i})-J_i(\bar{u}_i)| = O(\frac{1}{\sqrt{N}}+\varepsilon_N)$. ◻

5.1. Major agent's perturbation. In this subsection, we will prove that the strategy profile $(\bar{u}_0,\bar{u}_1,\bar{u}_2,\dots,\bar{u}_N)$ is an ε-Nash equilibrium of Problem (CC) for the major agent, i.e., there exists an $\varepsilon=\varepsilon(N)\ge 0$, $\lim_{N\to\infty}\varepsilon(N)=0$ s.t.

$$\mathcal{J}_0(\bar{u}_0(\cdot),\bar{u}_{-0}(\cdot)) \le \mathcal{J}_0(u_0(\cdot),\bar{u}_{-0}(\cdot))+\varepsilon \quad \text{ for any } u_0\in\mathcal{U}_{ad}^{c,0}.$$

Let us consider that the major agent $\mathcal{A}_0$ uses an alternative strategy u_0 and each minor agent $\mathcal{A}_i$ uses the control $\bar{u}_i=\varphi_{\theta_i}(\bar{p}_i,\bar{q}_i)$, where $(\bar{p}_i,\bar{q}_i)$ are solved from (5.2). Then the realized state system with major agent's perturbation is, for $1\le i\le N$,

$$\text{(5.12)}\quad\begin{cases} dy_0=\big[A_0y_0+B_0u_0+F_0^1y^{(N)}+b_0\big]dt+\big[C_0y_0+D_0u_0+F_0^2y^{(N)}+\sigma_0\big]dW_0(t),\\ dy_i=\big[A_{\theta_i}y_i+B\varphi_{\theta_i}(\bar{p}_i,\bar{q}_i)+F_1y^{(N)}+b_0\big]dt\\ \qquad+\big[Cy_i+D_{\theta_i}\varphi_{\theta_i}(\bar{p}_i,\bar{q}_i)+F_2y^{(N)}+Hy_0+\sigma_0\big]dW_i(t),\\ y_0(0)=x_0,\quad y_i(0)=x,\end{cases}$$

where $y^{(N)}=\frac{1}{N}\sum_{i=1}^N y_i$. The well-posedness of the above SDE system is not hard to check. To prove $(\bar{u}_0,\bar{u}_1,\dots,\bar{u}_N)$ is an ε-Nash equilibrium for the major agent, we need to show that for possible alternative control u_0, $\inf_{u_0\in\mathcal{U}_{ad}^{c,0}}\mathcal{J}_0(u_0,\bar{u}_{-0})\ge\mathcal{J}_0(\bar{u}_0,\bar{u}_{-0})-\varepsilon$. Then we only need to consider the perturbation $u_0\in\mathcal{U}_{ad}^{c,0}$ such that $\mathcal{J}_0(u_0,\bar{u}_{-0})\le\mathcal{J}_0(\bar{u}_0,\bar{u}_{-0})$. Thus, noticing $Q_0\ge 0$ and $G_0\ge 0$, from Lemma 5.4, we have

$$\mathbb{E}\int_0^T\langle R_0u_0(t),u_0(t)\rangle dt\le\mathcal{J}_0(u_0,\bar{u}_{-0})\le\mathcal{J}_0(\bar{u}_0,\bar{u}_{-0})\le J_0(\bar{u}_0)+O\left(\frac{1}{\sqrt{N}}+\varepsilon_N\right),$$

which implies that (noting A4), $\mathbb{E}\int_0^T|u_0(t)|^2dt\le M$, where M is a constant independent of N. Then similar to Lemma 5.1, we can show that

$$\text{(5.13)}\qquad \sup_{0\le i\le N}\mathbb{E}\sup_{0\le t\le T}|y_i(t)|^2\le M.$$

LEMMA 5.5. *Under the assumptions of* A1–A4, *we have*

$$\mathbb{E}\sup_{0\le t\le T}\left|y^{(N)}(t)-\Phi(t)\right|^2 = O\left(\frac{1}{N}+\varepsilon_N^2\right).$$

Proof. For each fixed $1 \le k \le K$, we consider the k-type minor agents. We denote $y^{(k)} := \frac{1}{N_k}\sum_{i\in\mathcal{I}_k} y_i$. As there are N_k minor agents of the k-type, let us add up their states, and then divided by N_k, it follows that

$$\begin{aligned}dy^{(k)} &= \left[A_k y^{(k)} + \frac{B}{N_k}\sum_{i\in\mathcal{I}_k}\varphi_k(\bar p_i,\bar q_i) + F_1 y^{(N)} + b_0\right]dt\\ &\quad + \frac{1}{N_k}\sum_{i\in\mathcal{I}_k}\left[Cy_i + D_k\varphi_k(\bar p_i,\bar q_i) + F_2 y^{(N)} + Hy_0 + \sigma_0\right]dW_i(t), \qquad y^{(k)}(0)=x.\end{aligned}$$

Recall (5.7) and if we denote $\tilde\Delta_k(t) := y^{(k)}(t) - m_k(t)$, it follows that

$$\begin{aligned}d\tilde\Delta_k &= \left[A_k\tilde\Delta_k + \frac{B}{N_k}\sum_{i\in\mathcal{I}_k}\varphi_k(\bar p_i,\bar q_i) - B\mathbb{E}(\varphi_k(\bar p_i,\bar q_i)|\mathcal{F}_t^{W_0}) + F_1(y^{(N)}-\Phi)\right]dt\\ &\quad + \frac{1}{N_k}\sum_{i\in\mathcal{I}_k}\left[Cy_i + D_k\varphi_k(\bar p_i,\bar q_i) + F_2 y^{(N)} + Hy_0 + \sigma_0\right]dW_i(t), \qquad \tilde\Delta(0)=0.\end{aligned}$$

Similar to the argument in the proof of Lemma 5.2, using (5.3), (5.13) and $(y_i(s),\bar p_i(s), \bar q_i(s))$, $i\in\mathcal{I}_k$, are identically distributed, we can show that

$$\mathbb{E}\sup_{0\le s\le t}|\tilde\Delta_k(s)|^2 \le M\mathbb{E}\int_0^t\left[|\tilde\Delta_k(s)|^2 + |y^{(N)}(s)-\Phi(s)|^2\right]ds + \frac{M}{N_k},$$

and Gronwall's inequality yields that

$$\mathbb{E}\sup_{0\le s\le t}|\tilde\Delta_k(s)|^2 \le M\mathbb{E}\int_0^t\left[|y^{(N)}(s)-\Phi(s)|^2\right]ds.$$

Similar to the proof of Lemma 5.2 again, and using A1, we have for any $t\in[0,T]$,

$$\begin{aligned}\mathbb{E}\sup_{0\le s\le t}|y^{(N)}(s)-\Phi(s)|^2 &\le \mathbb{E}\sum_{k=1}^K\sup_{0\le s\le t}\pi_k^{(N)}|\Delta_k(s)|^2 + M\varepsilon_N^2\\ &\le M\mathbb{E}\int_0^t\left[|y^{(N)}(s)-\Phi(s)|^2\right]ds + \frac{M}{N} + M\varepsilon_N^2.\end{aligned}$$

Finally, Gronwall's inequality allows us to complete the proof. ∎

Now, we introduce the following system of the decentralized limiting state with the major agent's perturbation control for $1\le i\le N$:

(5.14)
$$\begin{cases} d\bar y_0 = \left[A_0\bar y_0 + B_0u_0 + F_0^1\Phi + b_0\right]dt + \left[C_0\bar y_0 + D_0u_0 + F_0^2\Phi + \sigma_0\right]dW_0(t)\\ d\bar y_i = [A_{\theta_i}\bar y_i + B\varphi_{\theta_i}(\bar p_i,\bar q_i) + F_1\Phi + b_0]\,dt\\ \qquad + [C\bar y_i + D_{\theta_i}\varphi_{\theta_i}(\bar p_i,\bar q_i) + F_2\Phi + H\bar y_0 + \sigma_0]\,dW_i(t)\\ \bar y_0(0) = x_0, \quad \bar y_i(0) = x.\end{cases}$$

LEMMA 5.6. *Under the assumptions of* A1–A4, *we have*

$$\sup_{0\le i\le N}\mathbb{E}\sup_{0\le t\le T}\left|y_i(t)-\bar{y}_i(t)\right|^2=O\left(\frac{1}{N}+\varepsilon_N^2\right).$$

Proof. From the first equations of both (5.12) and (5.14), we obtain

$$\begin{cases} d(y_0-\bar{y}_0)=\big[A(y_0-\bar{y}_0)+F_0^1(y^{(N)}-\Phi)\big]dt+\big[C(y_0-\bar{y}_0)+F_0^2(y^{(N)}-\Phi)\big]dW_0(t),\\ y_0(0)-\bar{y}_0(0)=0.\end{cases}$$

With the help of classical estimates of SDE and Lemma 5.5, it is easy to obtain

$$\mathbb{E}\sup_{0\le t\le T}\Big|y_0(t)-\bar{y}_0(t)\Big|^2=O\left(\frac{1}{N}+\varepsilon_N^2\right). \tag{5.15}$$

Now, for any $1\le i\le N$, from the second equation of both (5.12) and (5.14), we get

$$\begin{aligned} d(y_i-\bar{y}_i)&=\big[A_{\theta_i}(y_i-\bar{y}_i)+F_1(y^{(N)}-\Phi)\big]dt\\ &\quad+\big[C(y_i-\bar{y}_i)+F_2(y^{(N)}-\Phi)+H(y_0-\bar{y}_0)\big]dW_i(t),\quad y_i(0)-\bar{y}_i(0)=0.\end{aligned}$$

The classical estimates of SDE, Lemma 5.5, and (5.15) allow us to complete the proof. ∎

LEMMA 5.7. *Under* A1–A4, *for the major agent's perturbation control* u_0, *we have*

$$\left|\mathcal{J}_0(u_0,\bar{u}_{-0})-J_0(u_0)\right|=O\left(\frac{1}{\sqrt{N}}+\varepsilon_N\right).$$

Proof. Recalling (2.3), (2.6), and (2.17), we have

$$\begin{aligned} &\mathcal{J}_0(u_0,\bar{u}_{-0})-J_0(u_0)\\ &\quad=\frac{1}{2}\mathbb{E}\Bigg[\int_0^T\Big(\big\langle Q_0\big(y_0-\rho_0y^{(N)}\big),y_0-\rho_0y^{(N)}\big\rangle-\big\langle Q_0\big(\bar{y}_0-\rho_0\Phi\big),\bar{y}_0-\rho_0\Phi\big\rangle\Big)dt\\ &\qquad+\big\langle G_0\big(y_0(T)-\rho_0y^{(N)}(T)\big),y_0(T)-\rho_0y^{(N)}(T)\big\rangle\\ &\qquad\qquad-\big\langle G_0\big(\bar{y}_0(T)-\rho_0\Phi(T)\big),\bar{y}_0(T)-\rho_0\Phi(T)\big\rangle\Bigg]. \end{aligned} \tag{5.16}$$

Similar to Lemma 5.4, by using Lemmas 5.5, 5.6 and $\mathbb{E}(|\bar{y}_0(t)|^2+|\Phi(t)|^2)\le M$, we have

$$\begin{aligned} &\left|\mathbb{E}\int_0^T\Big(\big\langle Q_0\big(y_0-\rho_0y^{(N)}\big),y_0-\rho_0y^{(N)}\big\rangle-\big\langle Q_0\big(\bar{y}_0-\rho_0\Phi\big),\bar{y}_0-\rho_0\Phi\big\rangle\Big)dt\right|\\ &\quad\le M\int_0^T\mathbb{E}\left|y_0-\bar{y}_0\right|^2dt+M\int_0^T\mathbb{E}\big|y^{(N)}-\Phi\big|^2dt\\ &\qquad+M\int_0^T\Big(\mathbb{E}\left|y_0-\bar{y}_0\right|^2+\mathbb{E}\big|y^{(N)}-\Phi\big|^2\Big)^{\frac{1}{2}}dt\\ &\quad=O\left(\frac{1}{\sqrt{N}}+\varepsilon_N\right) \end{aligned}$$

and

$$\Big|\mathbb{E}\Big[\big\langle G_0\big(y_0(T)-\rho_0 y^{(N)}(T)\big), y_0(T)-\rho_0 y^{(N)}(T)\big\rangle - \big\langle G_0\big(\bar{y}_0(T)-\rho_0\Phi(T)\big), \bar{y}_0(T)-\rho_0\Phi(T)\big\rangle\Big]\Big| = O\left(\frac{1}{\sqrt{N}}+\varepsilon_N\right).$$

The proof is completed by noticing (5.16). ∎

THEOREM 5.8. *Under the assumptions of* A1–A4*, the strategy profile* $(\bar{u}_0, \bar{u}_1, \ldots, \bar{u}_N)$ *is an* ε*-Nash equilibrium of* Problem (CC) *for the major agent. More precisely, there exists a constant* $M > 0$ *and a sequence of positive numbers* $\{\varepsilon(N)\}_{N\ge 1}$ *such that for each* $N \ge 1$*,*

(i) $\varepsilon(N) \le M(\frac{1}{\sqrt{N}}+\varepsilon_N)$*, where* $\varepsilon_N := \sup_{1\le k\le K} |\pi_k^{(N)} - \pi_k|$*;*

(ii) *for any* $u_0 \in \mathcal{U}_{ad}^{c,0}$*, one has*

$$\mathcal{J}_0(\bar{u}_0(\cdot), \bar{u}_{-0}(\cdot)) \le \mathcal{J}_0(u_0(\cdot), \bar{u}_{-0}(\cdot)) + \varepsilon(N).$$

Proof. Combining Lemmas 5.4 and 5.7, we have

$$\begin{aligned}\mathcal{J}_0(\bar{u}_0, \bar{u}_{-0}) &\le J_0(\bar{u}_0) + O\left(\frac{1}{\sqrt{N}}+\varepsilon_N\right) \le J_0(u_0) + O\left(\frac{1}{\sqrt{N}}+\varepsilon_N\right)\\ &\le \mathcal{J}_0(u_0, \bar{u}_{-0}) + O\left(\frac{1}{\sqrt{N}}+\varepsilon_N\right),\end{aligned}$$

where the second inequality comes from the fact that $J_0(\bar{u}_0) = \inf_{u_0\in\mathcal{U}_{ad}^0} J_0(u_0)$ (we mention that even if u_0 is adapted to $\{\mathcal{F}_t\}$ other than $\{\mathcal{F}_t^{W_0}\}$, $J_0(\bar{u}_0) \le J_0(u_0)$ still works as pointed out in Remark 2.3). Consequently, Theorem 5.8 holds with $\varepsilon(N) = O(\frac{1}{\sqrt{N}}+\varepsilon_N)$. ∎

5.2. Minor agent's perturbation. Now, let us consider the following case: a given minor agent $\mathcal{A}_i$ uses an alternative strategy $u_i \in \mathcal{U}_{ad}^{c,i}$, the major agent uses $\bar{u}_0 = \varphi_0(\bar{p}_0, \bar{q}_0)$, while other minor agents $\mathcal{A}_j$ use the control $\bar{u}_j = \varphi_{\theta_j}(\bar{p}_j, \bar{q}_j)$, $j \ne i$, $1 \le j \le N$, where $(\bar{p}_j, \bar{q}_j)$, $0 \le j \le N$, $j \ne i$, are solved from (5.2). Then the realized state system with the minor agent's perturbation is, for $1 \le j \le N$, $j \ne i$,

(5.17)
$$\begin{cases} dl_0 = \Big[A_0 l_0 + B_0\varphi_0(\bar{p}_0, \bar{q}_0) + F_0^1 l^{(N)} + b_0\Big]dt \\ \qquad\quad + \Big[C_0 l_0 + D_0\varphi_0(\bar{p}_0, \bar{q}_0) + F_0^2 l^{(N)} + \sigma_0\Big]dW_0(t) \\ dl_i = \Big[A_{\theta_i} l_i + B u_i + F_1 l^{(N)} + b_0\Big]dt + \Big[C l_i + D_{\theta_i} u_i + F_2 l^{(N)} + H l_0 + \sigma_0\Big]dW_i(t), \\ dl_j = \Big[A_{\theta_j} l_j + B\varphi_{\theta_j}(\bar{p}_j, \bar{q}_j) + F_1 l^{(N)} + b_0\Big]dt \\ \qquad\quad + \Big[C l_j + D_{\theta_j}\varphi_{\theta_j}(\bar{p}_j, \bar{q}_j) + F_2 l^{(N)} + H l_0 + \sigma_0\Big]dW_i(t), \\ l_0(0) = x_0, \quad l_i(0) = l_j(0) = x, \end{cases}$$

where $l^{(N)} = \frac{1}{N}\sum_{i=1}^N l^i$. The well-posedness of the above SDE system is easy to obtain. Similar to the argument of the major agent, to prove the strategy profile $(\bar{u}_0, \bar{u}_1, \ldots, \bar{u}_N)$ is an ε-Nash equilibrium for the minor agent, noticing $Q \ge 0$, $G \ge 0$,

$R_{\theta_i} > 0$, and Lemma 5.4, we only need to consider the perturbation $u_i \in \mathcal{U}_{ad}^{c,i}$ satisfying

$$\mathbb{E}\int_0^T |u_i(t)|^2 dt \leq M, \tag{5.18}$$

where M is a constant independent of N. Similar to Lemma 5.1, we can show that

$$\sup_{0\leq i\leq N} \mathbb{E} \sup_{0\leq t\leq T} |l_i(t)|^2 \leq M. \tag{5.19}$$

We first present the following lemma.

LEMMA 5.9. *Under the assumptions of* A1–A4*, we have*

$$\mathbb{E}\sup_{0\leq t\leq T}\left|l^{(N)}(t)-\Phi(t)\right|^2 = O\left(\frac{1}{N}+\varepsilon_N^2\right).$$

Proof. We know that for each fixed i, there exists a unique $1\leq \bar{k}\leq K$ such that $i\in\mathcal{I}_{\bar{k}}$. Let us denote $l^{(k)} := \frac{1}{N_k}\sum_{i\in\mathcal{I}_k} l_i$, $1\leq k\leq K$. We first consider the k-type minor agents, where $k\neq\bar{k}$. Adding up their states and then dividing by N_k, similar to the proof of Lemma 5.2, for $k\neq\bar{k}$ and $m_k=\mathbb{E}(\alpha_k(t)|\mathcal{F}_t^{W_0})$, we have

$$\mathbb{E}\sup_{0\leq s\leq t}|l^{(k)}(s)-m_k(s)|^2 \leq M\mathbb{E}\int_0^t \left[|l^{(N)}(s)-\Phi(s)|^2\right]ds + \frac{M}{N_k}. \tag{5.20}$$

Now let us focus on the $\bar{k}$-type minor agents. Recall (5.7) and denote $\Xi := l^{(\bar{k})}-m_{\bar{k}}$; it follows from the Cauchy–Schwarz inequality and BDG inequality that

$$\begin{aligned}\mathbb{E}\sup_{0\leq s\leq t}|\Xi(s)|^2 &\leq M\mathbb{E}\int_0^t\left(|\Xi(s)|^2+\frac{1}{N_{\bar{k}}^2}|u_i(s)|^2+|l^{(N)}(s)-\Phi(s)|^2\right)ds\\ &\quad+M\mathbb{E}\int_0^t\Big|\frac{1}{N_{\bar{k}}}\sum_{j\in\mathcal{I}_{\bar{k}},j\neq i}\varphi_{\bar{k}}(\bar{p}_j,\bar{q}_j)-\mathbb{E}(\varphi_{\bar{k}}(\bar{p}_i,\bar{q}_i)|\mathcal{F}_t^{W_0})\Big|^2ds\\ &\quad+\frac{M}{N_{\bar{k}}^2}\mathbb{E}\sum_{j\in\mathcal{I}_{\bar{k}}}\int_0^t\big|F_2(l^{(N)}(s)-\Phi(s))+F_2\Phi(s)+Cl_j(s)+Hl_0(s)+\sigma(s)\big|^2ds\\ &\quad+\frac{M}{N_{\bar{k}}^2}\mathbb{E}\int_0^t|u_i(s)|^2ds+\frac{M}{N_{\bar{k}}^2}\mathbb{E}\sum_{j\in\mathcal{I}_{\bar{k}},j\neq i}\int_0^t|\varphi_{\bar{k}}(\bar{p}_j(s),\bar{q}_j(s))|^2ds.\end{aligned} \tag{5.21}$$

On the one hand, since for each fixed $s\in[0,T]$, under the conditional expectation $\mathbb{E}(\cdot|\mathcal{F}_s^{W_0})$, the processes $(\bar{p}_i(s),\bar{q}_i(s))$, $i\in\mathcal{I}_{\bar{k}}$, are independent and identically distributed. If we denote $\mu(s)=\mathbb{E}(\varphi_{\bar{k}}(\bar{p}_i(s),\bar{q}_i(s))|\mathcal{F}_s^{W_0}))$, then μ does not depend on i. Moreover, using (5.3), similar to the proof of Lemma 5.2, we can obtain

$$\begin{aligned}&\int_0^t\mathbb{E}\Big|\frac{1}{N_{\bar{k}}}\sum_{j\in\mathcal{I}_{\bar{k}},j\neq i}\varphi_{\bar{k}}(\bar{p}_j(s),\bar{q}_j(s))-\mu(s)\Big|^2ds\\ &\leq 2\frac{(N_{\bar{k}}-1)^2}{N_{\bar{k}}^2}\int_0^t\mathbb{E}\Big|\frac{1}{N_{\bar{k}}-1}\sum_{j\in\mathcal{I}_{\bar{k}},j\neq i}\varphi_{\bar{k}}(\bar{p}_j(s),\bar{q}_j(s))-\mu(s)\Big|^2ds+\frac{2}{N_{\bar{k}}^2}\int_0^t\mathbb{E}|\mu(s)|^2ds\\ &=2\frac{N_{\bar{k}}-1}{N_{\bar{k}}^2}\int_0^t\mathbb{E}\left|\varphi_{\bar{k}}(\bar{p}_j(s),\bar{q}_j(s))-\mu(s)\right|^2ds+\frac{2}{N_{\bar{k}}^2}\int_0^t\mathbb{E}|\mu(s)|^2ds\leq\frac{M}{N_{\bar{k}}}.\end{aligned}$$

On the other hand, due to (5.18) and (5.19), we get

$$\begin{aligned}&\frac{M}{N_{\bar k}^2}\mathbb{E}\int_0^t |u_i(s)|^2 ds\\ &\qquad+\frac{M}{N_{\bar k}^2}\mathbb{E}\sum_{j=1}^N\int_0^t \Big|F_2(l^{(N)}(s)-\Phi(s))+F_2\Phi(s)+Cl_j(s)+Hl_0(s)+\sigma(s)\Big|^2 ds\\ &\quad\le \frac{M}{N_{\bar k}}\mathbb{E}\int_0^t |l^{(N)}(s)-\Phi(s)|^2ds+\frac{M}{N_{\bar k}}.\end{aligned}$$

Moreover, since $(\bar p_i(s),\bar q_i(s))$, $i\in\mathcal{I}_{\bar k}$, are identically distributed under $\mathbb{E}(\cdot|\mathcal{F}_s^{W_0})$, we have $\frac{M}{N_{\bar k}^2}\mathbb{E}\sum_{j\in\mathcal{I}_{\bar k},j\neq i}\int_0^t|\varphi_{\bar k}(\bar p_j(s),\bar q_j(s))|^2ds\le\frac{M}{N_{\bar k}}$. Therefore, we get from (5.21) that, for any $t\in[0,T]$,

$$\mathbb{E}\sup_{0\le s\le T}|\Xi(s)|^2\le M\mathbb{E}\int_0^t |\Xi(s)|^2+|l^{(N)}(s)-\Phi(s)|^2\Big]ds+\frac{M}{N_{\bar k}},$$

which yields, by using Gronwall's inequality, that

$$\mathbb{E}\sup_{0\le s\le t}|l^{(\bar k)}(s)-m_{\bar k}(s)|^2\le M\mathbb{E}\int_0^t\Big[|l^{(N)}(s)-\Phi(s)|^2\Big]ds+\frac{M}{N_{\bar k}}.\tag{5.22}$$

Consequently, noticing (5.20) and (5.22), we have for each $1\le k\le K$,

$$\mathbb{E}\sup_{0\le s\le t}|l^{(k)}(s)-m_k(s)|^2\le M\mathbb{E}\int_0^t\Big[|l^{(N)}(s)-\Phi(s)|^2\Big]ds+\frac{M}{N_k}.$$

Then similar to the proof of Lemma 5.2, we can complete the proof. ∎

Now, we introduce the following system of decentralized limiting state with the perturbation strategy of the minor agent $\mathcal{A}_i$: for $1\le j\le N$, $j\neq i$,

$$\begin{cases} d\bar l_0=\big[A_0\bar l_0+B_0\varphi_0(\bar p_0,\bar q_0)+F_0^1\Phi+b_0\big]\,dt+\big[C_0\bar l_0+D_0\varphi_0(\bar p_0,\bar q_0)+F_0^2\Phi+\sigma_0\big]\,dW_0(t)\\ d\bar l_i=\big[A_{\theta_i}\bar l_i+Bu_i+F_1\Phi+b_0\big]\,dt+\big[C\bar l_i+D_{\theta_i}u_i+F_2\Phi+H\bar l_0+\sigma_0\big]\,dW_i(t),\\ d\bar l_j=\big[A_{\theta_j}\bar l_j+B\varphi_{\theta_j}(\bar p_j,\bar q_j)+F_1\Phi+b_0\big]\,dt\\ \qquad\qquad+\big[C\bar l_j+D_{\theta_j}\varphi_{\theta_j}(\bar p_j,\bar q_j)+F_2\Phi+H\bar l_0+\sigma_0\big]\,dW_i(t),\\ \bar l_0(0)=x_0,\quad \bar l_i(0)=\bar l_j(0)=x. \end{cases}$$

LEMMA 5.10. *Under the assumptions of* A1–A4*, we have*

$$\mathbb{E}\sup_{0\le t\le T}\Big(\big|l_0(t)-\bar l_0(t)\big|^2+\big|l_i(t)-\bar l_i(t)\big|^2\Big)=O\left(\frac{1}{N}+\varepsilon_N^2\right).$$

LEMMA 5.11. *Under the assumptions of* A1–A4*, for each* $1\le i\le N$*, for the minor agent* $\mathcal{A}_i$*'s perturbation control* u_i*, we have*

$$\Big|\mathcal{J}_i(u_i,\bar u_{-i})-J_0(u_i)\Big|=O\left(\frac{1}{\sqrt{N}}+\varepsilon_N\right).$$

THEOREM 5.12. *Under the assumptions of* A1–A4*, the strategy profile* $(\bar u_0,\bar u_1,\dots,\bar u_N)$ *is an* ε*-Nash equilibrium of* Problem (CC) *for the minor agents. More precisely,* *there exists a constant $M > 0$ and a sequence of positive numbers $\{\varepsilon(N)\}_{N\geq 1}$, such that for each $N \geq 1$,*

(i) $\varepsilon(N) \leq M(\frac{1}{\sqrt{N}}+\varepsilon_N)$*, where* $\varepsilon_N := \sup_{1\leq k\leq K} |\pi_k^{(N)} - \pi_k|$*;*

(ii) *for any minor agent $\mathcal{A}_i$ and any $u_i \in \mathcal{U}_{ad}^{c,i}$, $1 \leq i \leq N$, one has*

$$\mathcal{J}_i(\bar{u}_i(\cdot), \bar{u}_{-i}(\cdot)) \leq \mathcal{J}_i(u_i(\cdot), \bar{u}_{-i}(\cdot)) + \varepsilon(N).$$

We mention that, based on Lemma 5.9, similar to the proofs of Lemmas 5.6 and 5.7 and Theorem 5.8, respectively, we can easily prove Lemmas 5.10 and 5.11 and Theorem 5.12. The proofs are omitted here; one can find them in [30]. By combining Theorems 5.8 and 5.12, we obtain the following main result of this section.

THEOREM 5.13. *Under the assumptions of* A1–A4*, the strategy profile $(\bar{u}_0, \bar{u}_1, \ldots, \bar{u}_N)$ is an ε-Nash equilibrium of* Problem (CC)*, where $\bar{u}_0 = \varphi_0(\bar{p}_0, \bar{q}_0)$, $\bar{u}_i = \varphi_{\theta_i}(\bar{p}_i, \bar{q}_i)$, $1 \leq i \leq N$, for*

$$\varphi_0(p, q) := \mathbf{P}_{\Gamma_0}\big[R_0^{-1}\big(B_0'p + D_0'q\big)\big], \quad \varphi_{\theta_i}(p, q) := \mathbf{P}_{\Gamma_{\theta_i}}\big[R_{\theta_i}^{-1}\big(B'p+D_{\theta_i}'q\big)\big].$$

More precisely, there exists a constant $M > 0$ and a sequence of positive numbers $\{\varepsilon(N)\}_{N\geq 1}$ such that for each $N \geq 1$,

(i) $\varepsilon(N) \leq M(\frac{1}{\sqrt{N}}+\varepsilon_N)$*, where* $\varepsilon_N := \sup_{1\leq k\leq K} |\pi_k^{(N)} - \pi_k|$*;*

(ii) *for any agent $\mathcal{A}_i$ and any $u_i \in \mathcal{U}_{ad}^{c,i}$, $0 \leq i \leq N$, one has*

$$\mathcal{J}_i(\bar{u}_i(\cdot), \bar{u}_{-i}(\cdot)) \leq \mathcal{J}_i(u_i(\cdot), \bar{u}_{-i}(\cdot)) + \varepsilon(N).$$

5.3. Convergence of the empirical measure. In this subsection, we discuss the convergence rate of the empirical measure using the method of [15, 17] under the assumption that high order estimates of α_k hold, $1 \leq k \leq K$. More precisely, we suppose that

A5. $\mathbb{E}\int_0^T |\alpha_k(t)|^{n+5}dt < \infty$, $1 \leq k \leq K$, where α_k, $1 \leq k \leq K$ are given by (2.15)–(2.16).

Remark 5.1. The system (2.15)–(2.16) is a fully coupled FBSDE, and A5 holds true on a small time interval; see, e.g., [27]. For high order estimates of general FBSDEs on an arbitrary interval, readers are referred to [11, 23] et al. We mention that we cannot use the method of [23] to obtain the high order estimates for system (2.15)–(2.16). This is mainly because the diffusion term of the forward equation depends on γ_k. One may combine the methods of [11, 49] (applying the Itô formula to $e^{-\lambda t}|\alpha(t)|^p$ and $e^{-\lambda t}|\beta(t)|^p$) to prove the high order estimates for system (2.15)–(2.16). Nevertheless, here we just suppose that A5 holds to focus on our convergence rate of the empirical measure.

Let us first recall the following notation (see [15, 17]). Let E be a separable Banach space and $h \geq 1$ be an integer. $\mathcal{P}_h(E)$ stands for the space of probability measures of order h, i.e.,

$$\mathcal{P}_h(E) = \left\{\vartheta \text{ is a probability measure s.t. } M_{h,E}(\vartheta) := \left(\int_E \|x\|_E^h d\vartheta(x)\right)^{1/h} < +\infty\right\}.$$

For any integer $h \geq 1$, $\vartheta, \vartheta' \in \mathcal{P}_h(E)$, the Monge–Kantorovich distance $\mathcal{W}_h(\vartheta, \vartheta')$ is defined by

$$\mathcal{W}_h(\vartheta,\vartheta')$$
$$:= \inf\left\{\left[\int_{E\times E}\|x-y\|_E^h\chi(dx,dy)\right]^{1/h};\chi\in\mathcal{P}_h(E\times E)\text{ with marginals }\vartheta\text{ and }\vartheta'\right\}.$$

Let ϑ be a probability measure on $\mathbb{R}^n$ and $X_1, X_2, \dots$ be an independent and identically distributed random variable sequence with common probability law ϑ. Denote $\vartheta^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_i}$, where δ_X is the Dirac measure. Then (see Theorem 10.2.1 in [52]) we have the following.

LEMMA 5.14. *Given $\vartheta \in \mathcal{P}_{n+5}(\mathbb{R}^n)$, there exists a constant M depending only on n and $M_{d+5,\mathbb{R}^n}(\vartheta)$ such that $\mathbb{E}[\mathcal{W}_2^2(\vartheta^N,\vartheta)] \le MN^{-2/(n+4)}$.*

Moreover, if $X_1, X_2, \dots$ is an infinite exchangeable sequence with directing measure ϑ, we still denote $\vartheta^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_i}$ and let ν be the marginal distribution of X_n. Then (see Theorem 10.2.6 in [52]) we have the following.

LEMMA 5.15. *Suppose $M'_{d+5,\mathbb{R}^n}(\vartheta) := \int |x|^{n+5}\nu(dx) < +\infty$; there exists a constant M depending only on n and $M'_{d+5,\mathbb{R}^n}(\vartheta)$ such that $\mathbb{E}[\mathcal{W}_2^2(\vartheta^N,\vartheta)] \le MN^{-2/(n+4)}$.*

Now let us focus on our system (2.15)–(2.16). For each $1 \le k \le K$, we denote $\vartheta_k(t) := \mathcal{L}(\alpha_k(t)|\mathcal{F}_t^{W_0})$, where $\mathcal{L}(\alpha_k(t)|\mathcal{F}_t^{W_0})$ represents the conditional law of $\alpha_k(t)$ w.r.t. $\mathcal{F}_t^{W_0}$ (recall that we can always find a regular version of the conditional law $\mathcal{L}(\alpha_k(t)|\mathcal{F}_t^{W_0})$). By recalling (5.2) and (5.1), we introduce, for each $1 \le k \le K$ and $t \in [0,T]$,

$$\bar{\vartheta}_k^N(t) := \frac{1}{N_k}\sum_{i\in\mathcal{I}_k}\delta_{\bar{x}_i(t)}, \quad \breve{\vartheta}_k^N(t) := \frac{1}{N_k}\sum_{i\in\mathcal{I}_k}\delta_{\breve{x}_i(t)}.$$

Then we have (recall $\varepsilon_N := \sup_{1\le k\le K}|\pi_k^{(N)} - \pi_k|$) the following.

THEOREM 5.16. *Under the assumptions of* A1–A5*, we have, for each $1 \le k \le K$,*

$$\sup_{0\le t\le T}\mathbb{E}\left[\mathcal{W}_2^2(\bar{\vartheta}_k^N(t),\vartheta_k(t))\right] \le MN_k^{-2/(n+4)} \tag{5.23}$$

and

$$\sup_{0\le t\le T}\mathbb{E}\left[\mathcal{W}_2^2(\breve{\vartheta}_k^N(t),\vartheta_k(t))\right] \le M(N_k^{-2/(n+4)} + \varepsilon_N^2). \tag{5.24}$$

Proof. Since under the conditional expectation $\mathbb{E}(\cdot|\mathcal{F}^{W_0}))$, for each $1 \le k \le K$, the processes $\bar{x}_i$, $i \in \mathcal{I}_k$, are independent and identically distributed with the common conditional law $\vartheta_k(t) = \mathcal{L}(\alpha_k(t)|\mathcal{F}_t^{W_0})$. Then Lemma 5.15 yields that $\sup_{0\le t\le T}$ $\mathbb{E}[\mathcal{W}_2^2(\bar{\vartheta}_k^N(t),\vartheta_k(t))] \le MN_k^{-2/(n+4)}$, which is (5.23).

Moreover, for each $t \in [0,T]$, it follows that

$$\mathcal{W}_2^2(\breve{\vartheta}_k^N(t),\vartheta_k(t)) \le \frac{2}{N_k}\sum_{i\in\mathcal{I}_k}|\breve{x}_i(t)-\bar{x}_i(t)|^2 + 2\mathcal{W}_2^2(\bar{\vartheta}_k^N(t),\vartheta_k(t)).$$

Now we take expectation on both sides of the above inequality, and by using Lemma 5.3 and (5.23) we obtain that there exists a constant M independent of t which may vary line by line, such that

$$\begin{aligned}\mathbb{E}\left[\mathcal{W}_2^2(\breve{\vartheta}_k^N(t),\vartheta_k(t))\right] &\le M\mathbb{E}|\breve{x}_i(t)-\bar{x}_i(t)|^2 + MN_k^{-2/(n+4)}\\ &\le M(N^{-1}+\varepsilon_N^2) + MN_k^{-2/(n+4)} \le M(N_k^{-2/(n+4)}+\varepsilon_N^2),\end{aligned}$$

which yields (5.24) (noticing that M is independent of t). ∎

Finally, let us recall l_i defined by (5.17). Here we assume that $i \in \mathcal{I}_k$ for some $1 \leq k \leq K$. We only need to consider the perturbation $u_i \in \mathcal{U}_{ad}^{c,i}$ satisfying (5.18) (see the argument in section 5.2); thus (5.19) holds. Now, we focus on estimating $\sup_{0\leq t\leq T} \mathbb{E}[\mathcal{W}_2^2(\upsilon_k^N(t), \vartheta_k(t))]$, where $\upsilon_k^N(t) := \frac{1}{N_k}\sum_{j\in\mathcal{I}_k} \delta_{l_j(t)}$. We first give the following lemma.

Lemma 5.17. *Under the assumptions of* A1–A4*, we have*

$$\sup_{j\in\mathcal{I}_k, j\neq i} \mathbb{E} \sup_{0\leq t\leq T} \left(\left|l_0(t) - \bar{x}_0(t)\right|^2 + \left|l_j(t) - \bar{x}_j(t)\right|^2\right) = O\left(\frac{1}{N} + \varepsilon_N^2\right).$$

Proof. From the first equation of (5.2) and (5.17), we obtain

$$\begin{cases} d(l_0 - \bar{x}_0) = \Big[A_0(l_0 - \bar{x}_0) + F_0^1(l^{(N)} - \Phi)\Big]dt + \Big[C_0(l_0 - \bar{x}_0) + F_0^2(l^{(N)} - \Phi)\Big]dW_i(t), \\ l_0(0) - \bar{x}_0(0) = 0. \end{cases}$$

The classical estimate for the SDE and Lemma 5.9 yield that

$$\mathbb{E} \sup_{0\leq t\leq T} \left|l_0(t) - \bar{x}_0(t)\right|^2 \leq ME \int_0^T \left|l^{(N)}(s) - \Phi(s)\right|^2 ds \leq M\left(\frac{1}{N} + \varepsilon_N^2\right). \tag{5.25}$$

On the other hand, from the third equation of (5.17) and (5.2), with the help of classical estimate for the SDE, we have that for $j \in \mathcal{I}_k$ and $j \neq i$,

$$\mathbb{E} \sup_{0\leq t\leq T} \left|l_j(t) - \bar{x}_j(t)\right|^2 \leq M\mathbb{E} \int_0^T \left(\left|l^{(N)}(s) - \Phi(s)\right|^2 + |l_0(s) - \bar{x}_0(s)|^2\right) ds,$$

and by noticing Lemma 5.9 and (5.25), we complete the proof. ◻

Theorem 5.18. *Under the assumptions of* A1–A5*, we have*

$$\sup_{0\leq t\leq T} \mathbb{E}\left[\mathcal{W}_2^2(\upsilon_k^N(t), \vartheta_k(t))\right] \leq M(N_k^{-2/(n+4)} + \varepsilon_N^2).$$

Proof. By the triangle inequality, we have

$$\begin{aligned} \mathbb{E}\left[\mathcal{W}_2^2(\upsilon_k^N(t), \vartheta_k(t))\right] \leq M\Bigg\{\mathbb{E}\Bigg[&\mathcal{W}_2^2\Bigg(\frac{1}{N_k}\sum_{j\in\mathcal{I}_k} \delta_{l_j(t)}, \frac{1}{N_k - 1}\sum_{j\in\mathcal{I}_k, j\neq i} \delta_{l_j(t)}\Bigg) \\ &+ \mathcal{W}_2^2\Bigg(\frac{1}{N_k - 1}\sum_{j\in\mathcal{I}_k, j\neq i} \delta_{l_j(t)}, \frac{1}{N_k - 1}\sum_{j\in\mathcal{I}_k, j\neq i} \delta_{\bar{x}_j(t)}\Bigg) \\ &+ \mathcal{W}_2^2\Bigg(\frac{1}{N_k - 1}\sum_{j\in\mathcal{I}_k, j\neq i} \delta_{\bar{x}_j(t)}, \vartheta_k(t)\Bigg)\Bigg]\Bigg\}. \end{aligned} \tag{5.26}$$

We note that

$$\mathbb{E}\left[\mathcal{W}_2^2\left(\frac{1}{N_k}\sum_{j\in\mathcal{I}_k} \delta_{l_j(t)}, \frac{1}{N_k - 1}\sum_{j\in\mathcal{I}_k, j\neq i} \delta_{l_j(t)}\right)\right] \leq \frac{1}{N_k^2(N_k - 1)}\sum_{j\in\mathcal{I}_k, j\neq i} |l_i(t) - l_j(t)|^2,$$

and by noticing (5.19), we have

$$\sup_{0\leq t\leq T} \mathbb{E}\left[\mathcal{W}_2^2\left(\frac{1}{N_k}\sum_{j\in\mathcal{I}_k} \delta_{l_j(t)}, \frac{1}{N_k - 1}\sum_{j\in\mathcal{I}_k, j\neq i} \delta_{l_j(t)}\right)\right] \leq MN_k^{-2}.$$

For the second term in the right-hand side of (5.26), by using Lemma 5.17, we have

$$\begin{aligned}&\sup_{0\le t\le T}\mathbb{E}\left[\mathcal{W}_2^2\big(\tfrac{1}{N_k-1}\textstyle\sum_{j\in\mathcal{I}_k,j\neq i}\delta_{l_j(t)},\tfrac{1}{N_k-1}\sum_{j\in\mathcal{I}_k,j\neq i}\delta_{\bar{x}_j(t)}\big)\right]\\ &\quad\le \tfrac{1}{N_k-1}\sup_{0\le t\le T}\mathbb{E}\textstyle\sum_{j\in\mathcal{I}_k,j\neq i}|l_j(t)-\bar{x}_j(t)|^2\le M(\tfrac{1}{N}+\varepsilon_N^2)\le M(\tfrac{1}{N_k}+\varepsilon_N^2).\end{aligned}$$

Moreover, Lemma 5.15 yields that

$$\sup_{0\le t\le T}\mathbb{E}\left[\mathcal{W}_2^2\left(\frac{1}{N_k-1}\sum_{j\in\mathcal{I}_k,j\neq i}\delta_{\bar{x}_j(t)},\vartheta_k(t)\right)\right]\le M(N_k-1)^{-2/(n+4)}=O(N_k^{-2/(n+4)}).$$

Plugging the above estimates into (5.26), we complete the proof. ∎

Remark 5.2. If we suppose some mild assumptions (as in Theorem 10.2.7 of [52]) on α_k, we should obtain a uniform convergence rate but with order $N_k^{-2/(n+8)}$. The readers are referred to Lemma 6.8 in [15] for some similar results.

Appendix A. We give this appendix to prove Theorem 4.1. The fully coupled structure of MF-FBSDE (4.1) raises difficulties for establishing its well-posedness. Motivated by Pardoux and Tang [49, Theorem 3.1], we can establish the well-posedness of MF-FBSDE (4.1) for arbitrary time duration when it is weakly coupled.

Let us first note that for a given $(Y(\cdot),Z(\cdot))\in L^2_{\mathcal{F}}(0,T;\mathbb{R}^l)\times L^2_{\mathcal{F}}(0,T;\mathbb{R}^{l\times(d+1)})$, the forward equation in the MF-FBSDE (4.1) has a unique solution $X(\cdot)\in L^2_{\mathcal{F}}(0,T;\mathbb{R}^n)$; thus we introduce a map $\mathcal{M}_1: L^2_{\mathcal{F}}(0,T;\mathbb{R}^l)\times L^2_{\mathcal{F}}(0,T;\mathbb{R}^{l\times(d+1)})\to L^2_{\mathcal{F}}(0,T;\mathbb{R}^n)$, through

$$\begin{aligned}X(t)=x&+\int_0^t b(s,X(s),\mathbb{E}[X(s)|\mathcal{F}_s^{W_0}],Y(s),Z(s)])ds\\ &+\int_0^t\sigma(s,X(s),\mathbb{E}[X(s)|\mathcal{F}_s^{W_0}],Y(s),Z(s))dW(s).\end{aligned}\tag{A.1}$$

We mention that the well-posedness of (A.1) can be established by using the contraction mapping method under the assumptions (H_1) and (H_2), although it has the term $\mathbb{E}[X_s|\mathcal{F}_s^{W_0}]$. We omit the proof here. Moreover, with the help of the BDG inequality, it follows that $\mathbb{E}\sup_{t\in[0,T]}|X(t)|^2<\infty$.

Lemma A.1. *Let X_i be the solution of* (A.1) *corresponding to* $(Y_i(\cdot),Z_i(\cdot))\in L^2_{\mathcal{F}}(0,T;\mathbb{R}^l)\times L^2_{\mathcal{F}}(0,T;\mathbb{R}^{l\times(d+1)})$, $i=1,2$. *Then for all $\lambda\in\mathbb{R}$, K_1, $K_2>0$, we have*

$$\begin{aligned}&e^{-\lambda t}\mathbb{E}|X_1(t)-X_2(t)|^2+\overline{\lambda}_1\int_0^t e^{-\lambda s}\mathbb{E}|X_1(s)-X_2(s)|^2ds\\ &\quad\le(k_2K_1+k_9^2)\int_0^t e^{-\lambda s}\mathbb{E}|Y_1(s)-Y_2(s)|^2ds+(k_3K_2+k_{10}^2)\int_0^t e^{-\lambda s}\mathbb{E}|Z_1(s)-Z_2(s)|^2ds,\end{aligned}\tag{A.2}$$

where $\overline{\lambda}_1:=\lambda-2\lambda_1-k_2K_1^{-1}-k_3K_2^{-1}-2k_1-k_7^2-k_8^2$. *Moreover,*

$$\begin{aligned}e^{-\lambda t}\mathbb{E}|X_1(t)-X_2(t)|^2\le{}&(k_2K_1+k_9^2)\int_0^t e^{-\overline{\lambda}_1(t-s)}e^{-\lambda s}\mathbb{E}|Y_1(s)-Y_2(s)|^2ds\\ &+(k_3K_2+k_{10}^2)\int_0^t e^{-\overline{\lambda}_1(t-s)}e^{-\lambda s}\mathbb{E}|Z_1(s)-Z_2(s)|^2ds.\end{aligned}\tag{A.3}$$

Proof. We denote $\overline{X} := X_1 - X_2$, $\overline{Y} := Y_1 - Y_2$, $\overline{Z} := Z_1 - Z_2$, $\bar{b} := b(X_1, \mathbb{E}[X_1|\mathcal{F}^{W_0}], Y_1, Z_1) - b(X_2, \mathbb{E}[X_2|\mathcal{F}^{W_0}], Y_2, Z_2)$, $\overline{\sigma} := \sigma(X_1, \mathbb{E}[X_1|\mathcal{F}^{W_0}], Y_1, Z_1) - \sigma(X_2, \mathbb{E}[X_2|\mathcal{F}^{W_0}], Y_2, Z_2)$. Applying Itô's formula to $e^{-\lambda t}|X_1(t) - X_2(t)|^2$ and taking expectation, we obtain

$$e^{-\lambda t}\mathbb{E}|\overline{X}(t)|^2 = -\lambda \int_0^t e^{-\lambda s}\mathbb{E}|\overline{X}(s)|^2 ds + 2\mathbb{E}\int_0^t e^{-\lambda s}\langle \overline{X}(s), \bar{b}(s)\rangle ds + \mathbb{E}\int_0^t e^{-\lambda s}|\overline{\sigma}(s)|^2 ds. \tag{A.4}$$

Notice that

$$\begin{aligned}
2\langle \overline{X}(s), \bar{b}(s)\rangle &= 2\langle \overline{X}(s), b(s, X_1(s), \mathbb{E}[X_1(s)|\mathcal{F}_s^{W_0}], Y_1(s), Z_1(s)) \\
&\quad - b(X_2(s), \mathbb{E}[X_1(s)|\mathcal{F}_s^{W_0}], Y_1(s), Z_1(s))\rangle \\
&\quad + 2\langle \overline{X}(s), b(s, X_2(s), \mathbb{E}[X_1(s)|\mathcal{F}_s^{W_0}], Y_1(s), Z_1(s)) \\
&\quad - b(X_2(s), \mathbb{E}[X_2(s)|\mathcal{F}_s^{W_0}], Y_2(s), Z_2(s))\rangle \\
&\le 2\lambda_1|\overline{X}(s)|^2 + 2|\overline{X}(s)|(k_1|\mathbb{E}[\overline{X}(s)|\mathcal{F}_s^{W_0}]| + k_2|\overline{Y}(s)| + k_3|\overline{Z}(s)|) \\
&\le (2\lambda_1 + k_2K_1^{-1} + k_3K_2^{-1})|\overline{X}(s)|^2 + 2k_1|\overline{X}(s)|\mathbb{E}[|\overline{X}(s)||\mathcal{F}_s^{W_0}]| \\
&\quad + k_2K_1|\overline{Y}(s)|^2 + k_3K_2|\overline{Z}(s)|^2
\end{aligned}$$

and

$$\begin{aligned}
|\overline{\sigma}(s)|^2 &\le k_7^2|\overline{X}(s)|^2 + k_8^2|\mathbb{E}[\overline{X}(s)|\mathcal{F}_s^{W_0}]|^2 + k_9^2|\overline{Y}(s)|^2 + k_{10}^2|\overline{Z}(s)|^2 \\
&\le k_7^2|\overline{X}(s)|^2 + k_8^2\mathbb{E}[|\overline{X}(s)|^2|\mathcal{F}_s^{W_0}] + k_9^2|\overline{Y}(s)|^2 + k_{10}^2|\overline{Z}(s)|^2.
\end{aligned}$$

Then, from (A.4), $\mathbb{E}[\mathbb{E}[|\overline{X}(s)|^2|\mathcal{F}_s^{W_0}]] = \mathbb{E}|\overline{X}(s)|^2$ and

$$\begin{aligned}
\mathbb{E}\left[|\overline{X}(s)|\mathbb{E}[|\overline{X}(s)||\mathcal{F}_s^{W_0}]\right] &= \mathbb{E}\left[\mathbb{E}\left[|\overline{X}(s)|\mathbb{E}[|\overline{X}(s)||\mathcal{F}_s^{W_0}]\big|\mathcal{F}_s^{W_0}\right]\right] \\
&= \mathbb{E}\left[(\mathbb{E}[|\overline{X}(s)||\mathcal{F}_s^{W_0}])^2\right] \le \mathbb{E}\left[\mathbb{E}[|\overline{X}(s)|^2|\mathcal{F}_s^{W_0}]\right] = \mathbb{E}|\overline{X}(s)|^2,
\end{aligned}$$

we can obtain (A.2).

Now, we apply Itô's formula to $e^{-\overline{\lambda}_1(t-s)}e^{-\lambda s}|X_1(s) - X_2(s)|^2$ for $s \in [0, t]$ and taking expectation, it follows that

$$\begin{aligned}
&e^{-\lambda t}\mathbb{E}|\overline{X}(t)|^2 \\
&\quad = -(\lambda - \overline{\lambda}_1)\int_0^t e^{-\overline{\lambda}_1(t-s)}e^{-\lambda s}\mathbb{E}|\overline{X}(s)|^2 ds + 2\mathbb{E}\int_0^t e^{-\overline{\lambda}_1(t-s)}e^{-\lambda s}\langle \overline{X}(s), \bar{b}(s)\rangle ds \\
&\qquad + \mathbb{E}\int_0^t e^{-\overline{\lambda}_1(t-s)}e^{-\lambda s}|\overline{\sigma}(s)|^2 ds.
\end{aligned} \tag{A.5}$$

From the above estimates and (A.5), one can prove (A.3). ∎

Remark A.1. By integrating both sides of (A.3) on $[0, T]$ and using $\frac{1-e^{-\overline{\lambda}_1(T-s)}}{\overline{\lambda}_1} \le \frac{1-e^{-\overline{\lambda}_1 T}}{\overline{\lambda}_1}$ for all $s \in [0, T]$, we have

$$\|X_1 - X_2\|_\lambda^2 \le \frac{1-e^{-\overline{\lambda}_1 T}}{\overline{\lambda}_1}\left[(k_2K_1 + k_9^2)\|Y_1 - Y_2\|_\lambda^2 + (k_3K_2 + k_{10}^2)\|Z_1 - Z_2\|_\lambda^2\right]. \tag{A.6}$$

Let $t = T$ in (A.3) and notice that $e^{-\overline{\lambda}_1(T-s)} \leq 1 \vee e^{-\overline{\lambda}_1 T}$ for all $s \in [0,T]$; thus

$$e^{-\lambda T}\mathbb{E}|X_1(T) - X_2(T)|^2 \leq \left[1 \vee e^{-\overline{\lambda}_1 T}\right]\left[(k_2K_1 + k_9^2)\|Y_1 - Y_2\|_\lambda^2 + (k_3K_2 + k_{10}^2)\|Z_1 - Z_2\|_\lambda^2\right]. \tag{A.7}$$

In particular, if $\overline{\lambda}_1 > 0$, we have

$$e^{-\lambda T}\mathbb{E}|X_1(T) - X_2(T)|^2 \leq (k_2K_1 + k_9^2)\|Y_1 - Y_2\|_\lambda^2 + (k_3K_2 + k_{10}^2)\|Z_1 - Z_2\|_\lambda^2. \tag{A.8}$$

Similarly, for a given $X(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^n)$, the backward equation in the MF-FBSDE (4.1) has a unique solution $(Y(\cdot), Z(\cdot)) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^l) \times L^2_{\mathcal{F}}(0,T;\mathbb{R}^{l\times(d+1)})$, thus we can introduce another map $\mathcal{M}_2 : L^2_{\mathcal{F}}(0,T;\mathbb{R}^n) \to L^2_{\mathcal{F}}(0,T;\mathbb{R}^l) \times L^2_{\mathcal{F}}(0,T;\mathbb{R}^{l\times(d+1)})$, through

$$\begin{aligned} Y(t) &= g(X(T), \mathbb{E}[X(T)|\mathcal{F}_T^{W_0}]) \\ &\quad + \int_t^T f(s, X(s), \mathbb{E}[X(s)|\mathcal{F}_s^{W_0}], Y(s), Z(s))ds - \int_t^T Z(s)dW(s). \end{aligned} \tag{A.9}$$

The well-posedness of (A.9) under assumptions $(H_1), (H_2)$ is referred to in Darling and Pardoux [22, Theorem 3.4] and Buckdahn and Nie [13, Lemma 2.2]. Moreover, we have $\mathbb{E}\sup_{t\in[0,T]}|Y(t)|^2 < \infty$.

LEMMA A.2. *Let $(Y_i(\cdot), Z_i(\cdot))$ be the solution of* (A.9) *corresponding to $X_i \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^n)$, $i = 1, 2$. Then for all $\lambda \in \mathbb{R}$, K_3, $K_4 > 0$, we have*

(A.10)

$$\begin{aligned} &e^{-\lambda t}\mathbb{E}|Y_1(t) - Y_2(t)|^2 + \overline{\lambda}_2\int_t^T e^{-\lambda s}\mathbb{E}|Y_1(s) - Y_2(s)|^2 ds \\ &\qquad + (1 - k_6K_4)\int_t^T e^{-\lambda s}\mathbb{E}|Z_1(s) - Z_2(s)|^2 ds \\ &\quad \leq (k_{11}^2 + k_{12}^2)e^{-\lambda T}\mathbb{E}|X_1(T) - X_2(T)|^2 + (k_4 + k_5)K_3\int_t^T e^{-\lambda s}\mathbb{E}|X_1(s) - X_2(s)|^2 ds, \end{aligned}$$

where $\overline{\lambda}_2 := -\lambda - 2\lambda_2 - (k_4 + k_5)K_3^{-1} - k_6K_4^{-1}$. Moreover,

$$\begin{aligned} &e^{-\lambda t}\mathbb{E}|Y_1(t) - Y_2(t)|^2 + (1 - k_6K_4)\int_t^T e^{-\overline{\lambda}_2(s-t)}e^{-\lambda s}\mathbb{E}|Z_1(s) - Z_2(s)|^2 ds \\ &\quad \leq (k_{11}^2 + k_{12}^2)e^{-\overline{\lambda}_2(T-t)}e^{-\lambda T}\mathbb{E}|X_1(T) - X_2(T)|^2 \\ &\qquad + (k_4 + k_5)K_3\int_t^T e^{-\overline{\lambda}_2(s-t)}e^{-\lambda s}\mathbb{E}|X_1(s) - X_2(s)|^2 ds. \end{aligned} \tag{A.11}$$

Proof. We denote $\overline{X} := X_1 - X_2$, $\overline{Y} := Y_1 - Y_2$, $\overline{Z} := Z_1 - Z_2$, $\overline{f} := f(X_1, \mathbb{E}[X_1|\mathcal{F}^{W_0}], Y_1, Z_1) - f(X_2, \mathbb{E}[X_2|\mathcal{F}^{W_0}], Y_2, Z_2)$. Applying Itô's formula to $e^{-\lambda t}|Y_1(t) - Y_2(t)|^2$ and taking expectation, we obtain

$$\begin{aligned} &e^{-\lambda t}\mathbb{E}|\overline{Y}(t)|^2 - \lambda\int_t^T e^{-\lambda s}\mathbb{E}|\overline{Y}(s)|^2 ds + \mathbb{E}\int_t^T e^{-\lambda s}|\overline{Z}(s)|^2 ds \\ &\quad = e^{-\lambda T}\mathbb{E}|\overline{Y}(T)|^2 + 2\mathbb{E}\int_t^T e^{-\lambda s}\langle \overline{Y}(s), \overline{f}(s)\rangle ds. \end{aligned} \tag{A.12}$$

Noticing that

$$\begin{aligned}
2\langle \overline{Y}(s), \overline{f}(s)\rangle &= 2\langle \overline{Y}(s), f(s, X_1(s), \mathbb{E}[X_1(s)|\mathcal{F}_s^{W_0}], Y_1(s), Z_1(s)) \\
&\quad - f(X_1(s), \mathbb{E}[X_1(s)|\mathcal{F}_s^{W_0}], Y_2(s), Z_1(s))\rangle \\
&\quad + 2\langle \overline{Y}(s), f(s, X_1(s), \mathbb{E}[X_1(s)|\mathcal{F}_s^{W_0}], Y_2(s), Z_1(s)) \\
&\quad - f(X_2(s), \mathbb{E}[X_2(s)|\mathcal{F}_s^{W_0}], Y_2(s), Z_2(s))\rangle \\
&\le 2\lambda_2|\overline{Y}(s)|^2 + 2|\overline{Y}(s)|(k_4|\overline{X}(s)| + k_5|\mathbb{E}[\overline{X}(s)|\mathcal{F}_s^{W_0}]| + k_6|\overline{Z}(s)|) \\
&\le (2\lambda_2 + k_4K_3^{-1} + k_5K_3^{-1} + k_6K_4^{-1})|\overline{Y}(s)|^2 + k_4K_3|\overline{X}(s)|^2 \\
&\quad + k_5K_3\mathbb{E}[|\overline{X}(s)|^2|\mathcal{F}_s^{W_0}]| + k_6K_4|\overline{Z}(s)|^2
\end{aligned}$$

and

$$\begin{aligned}
|\overline{Y}(T)|^2 &= |g(X_1(T), \mathbb{E}[X_1(T)|\mathcal{F}_T^{W_0}]) - g(X_2(T), \mathbb{E}[X_2(T)|\mathcal{F}_T^{W_0}])|^2 \\
&\le k_{11}^2|\overline{X}(T)|^2 + k_{12}^2\mathbb{E}[|\overline{X}(s)|^2|\mathcal{F}_s^{W_0}].
\end{aligned}$$

Then, from (A.12) and $\mathbb{E}[\mathbb{E}[|\overline{X}(s)|^2|\mathcal{F}_s^{W_0}]] = \mathbb{E}|\overline{X}(s)|^2$, we can obtain (A.10).

Now, we apply Itô's formula to $e^{-\overline{\lambda}_2(s-t)}e^{-\lambda s}|Y_1(s) - Y_2(s)|^2$ for $s \in [t, T]$ and taking expectation, it follows that

(A.13)

$$\begin{aligned}
&e^{-\lambda t}\mathbb{E}|\overline{Y}(t)|^2 - (\lambda + \overline{\lambda}_2)\int_t^T e^{-\overline{\lambda}_2(s-t)}e^{-\lambda s}\mathbb{E}|\overline{Y}(s)|^2 ds + \mathbb{E}\int_t^T e^{-\overline{\lambda}_2(s-t)}e^{-\lambda s}|\overline{Z}(s)|^2 ds \\
&\quad = e^{-\overline{\lambda}_2(T-t)}e^{-\lambda T}\mathbb{E}|\overline{Y}(s)|^2 + 2\mathbb{E}\int_t^T e^{-\overline{\lambda}_2(s-t)}e^{-\lambda s}\langle \overline{Y}(s), \overline{f}(s)\rangle ds.
\end{aligned}$$

From the above estimates and (A.13), one can prove (A.11). ∎

Remark A.2. Now we choose K_4 satisfying $0 < K_4 \le k_6^{-1}$; then by integrating both sides of (A.11) on $[0, T]$ and using $\frac{1-e^{-\overline{\lambda}_2 s}}{\overline{\lambda}_2} \le \frac{1-e^{-\overline{\lambda}_2 T}}{\overline{\lambda}_2}$ for all $s \in [0, T]$, we have

(A.14)

$$\|Y_1 - Y_2\|_\lambda^2 \le \frac{1-e^{-\overline{\lambda}_2 T}}{\overline{\lambda}_2}\left[(k_{11}^2 + k_{12}^2)e^{-\lambda T}\mathbb{E}|X_1(T) - X_2(T)|^2 + (k_4 + k_5)K_3\|X_1 - X_2\|_\lambda^2\right].$$

Let $t = 0$ in (A.11) and notice that $1 \wedge e^{-\overline{\lambda}_2 T} \le e^{-\overline{\lambda}_2 s} \le 1 \vee e^{-\overline{\lambda}_2 T}$ for all $s \in [0, T]$; thus

(A.15)

$$\begin{aligned}
&\|Z_1 - Z_2\|_\lambda^2 \\
&\quad \le \frac{(k_{11}^2 + k_{12}^2)e^{-\overline{\lambda}_2 T}e^{-\lambda T}\mathbb{E}|X_1(T) - X_2(T)|^2 + (k_4 + k_5)K_3(1 \vee e^{-\overline{\lambda}_2 T})\|X_1 - X_2\|_\lambda^2}{(1 - k_6K_4)(1 \wedge e^{-\overline{\lambda}_2 T})}.
\end{aligned}$$

On the other hand, if $\overline{\lambda}_2 > 0$, letting $t = 0$ in (A.10), we have

(A.16)

$$\|Z_1 - Z_2\|_\lambda^2 \le \frac{(k_{11}^2 + k_{12}^2)e^{-\lambda T}\mathbb{E}|X_1(T) - X_2(T)|^2 + (k_4 + k_5)K_3\|X_1 - X_2\|_\lambda^2}{1 - k_6K_4}.$$

Now, we present the proof of Theorem 4.1.

Proof of Theorem 4.1. We define $\mathcal{M} := \mathcal{M}_2 \circ \mathcal{M}_1$, recalling that $\mathcal{M}_1$ is defined through (A.1) and $\mathcal{M}_2$ is defined through (A.9). Thus $\mathcal{M}$ maps $L^2_{\mathcal{F}}(0,T;\mathbb{R}^l) \times L^2_{\mathcal{F}}(0,T;\mathbb{R}^{l\times(d+1)})$ into itself. To prove the theorem, it is only needed to show that $\mathcal{M}$ is a contraction mapping for some equivalent norm $\|\cdot\|_\lambda$. In fact, for $(Y_i, Z_i) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^l) \times L^2_{\mathcal{F}}(0,T;\mathbb{R}^{l\times(d+1)})$, let $X_i := \mathcal{M}_1(Y_i, Z_i)$ and $(\overline{Y}_i, \overline{Z}_i) := \mathcal{M}((Y_i, Z_i))$; from (A.6), (A.7), (A.14), and (A.15), we have

$$
\begin{aligned}
&\|\overline{Y}_1 - \overline{Y}_2\|^2_\lambda + \|\overline{Z}_1 - \overline{Z}_2\|^2_\lambda \\
&\quad \le \left[\frac{1-e^{-\overline{\lambda}_2 T}}{\overline{\lambda}_2} + \frac{1 \vee e^{-\overline{\lambda}_2 T}}{(1-k_6K_4)(1\wedge e^{-\overline{\lambda}_2 T})}\right] \\
&\qquad \times \left[(k_{11}^2 + k_{12}^2)e^{-\lambda T}\mathbb{E}|X_1(T) - X_2(T)|^2 + (k_4+k_5)K_3\|X_1 - X_2\|^2_\lambda\right] \\
&\quad \le \left[\frac{1-e^{-\overline{\lambda}_2 T}}{\overline{\lambda}_2} + \frac{1 \vee e^{-\overline{\lambda}_2 T}}{(1-k_6K_4)(1\wedge e^{-\overline{\lambda}_2 T})}\right] \\
&\qquad \times \left[(k_{11}^2 + k_{12}^2)(1 \vee e^{-\overline{\lambda}_1 T}) + (k_4+k_5)K_3 \frac{1-e^{-\overline{\lambda}_1 T}}{\overline{\lambda}_1}\right] \\
&\qquad \times \left[(k_2K_1 + k_9^2)\|Y_1 - Y_2\|^2_\lambda + (k_3K_2 + k_{10}^2)\|Z_1 - Z_2\|^2_\lambda\right].
\end{aligned}
$$

Recall that $\overline{\lambda}_1 := \lambda - 2\lambda_1 - k_2K_1^{-1} - k_3K_2^{-1} - 2k_1 - k_7^2 - k_8^2$ and $\overline{\lambda}_2 := -\lambda - 2\lambda_2 - (k_4+k_5)K_3^{-1} - k_6K_4^{-1}$. Then by choosing suitable λ (e.g., we can easily choose λ big enough such that $\overline{\lambda}_1 \ge 1$ and $\overline{\lambda}_2 \le 0$), the first assertion of Theorem 4.1 is immediate.

Now let us prove the second assertion. Since $2(\lambda_1 + \lambda_2) < -2k_1 - k_6^2 - k_7^2 - k_8^2$, we can choose $\lambda \in \mathbb{R}$, $0 < K_4 \le k_6^{-1}$, and sufficiently large K_1, K_2, K_3 such that

$$
\overline{\lambda}_1 > 0, \qquad \overline{\lambda}_2 > 0, \qquad 1 - K_4k_6 > 0.
$$

Then from (A.6), (A.8), (A.14), and (A.16), we have

$$
\begin{aligned}
&\|\overline{Y}_1 - \overline{Y}_2\|^2_\lambda + \|\overline{Z}_1 - \overline{Z}_2\|^2_\lambda \\
&\le \left[\frac{1}{\overline{\lambda}_2} + \frac{1}{1-k_6K_4}\right] \times \left[(k_{11}^2 + k_{12}^2)e^{-\lambda T}\mathbb{E}|X_1(T) - X_2(T)|^2 + (k_4+k_5)K_3\|X_1 - X_2\|^2_\lambda\right] \\
&\le \left[\frac{1}{\overline{\lambda}_2} + \frac{1}{1-k_6K_4}\right] \times \left[k_{11}^2 + k_{12}^2 + (k_4+k_5)K_3\frac{1}{\overline{\lambda}_1}\right] \\
&\quad \times \left[(k_2K_1 + k_9^2)\|Y_1 - Y_2\|^2_\lambda + (k_3K_2 + k_{10}^2)\|Z_1 - Z_2\|^2_\lambda\right].
\end{aligned}
$$

This completes the second assertion of Theorem 4.1. ∎

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