Pricing American Options under Regime Switching

Kai Zhang[1] Xiaoqi Yang[2]

[1] Shenzhen Audencia Business School,
Shenzhen University, China

[2] Department of Applied Mathematics,
The Hong Kong Polytechnic University, Hong Kong

[1] kaizhang@szu.edu.cn

[2] mayangxq@polyu.edu.hk

Abstract

This work aims at studying a power penalty approach to the coupled system of differential complementarity problems arising from the valuation of American options under regime switching. We introduce a power penalty method to approximate the differential complementarity problems, which results in a set of coupled nonlinear partial differential equations. By virtue of variational inequality theory, we establish the unique solvability of the system of differential complementarity problems. Moreover, the convergence property of this power penalty method in an appropriate infinite dimensional space is explored, where an exponential convergence rate of the power penalty method is established and the monotonic convergence of the penalty method with respect to the penalty parameter is shown. Finally, some numerical experiments are presented to verify the convergence property of the power penalty method.

Keywords. American option pricing; Regime switching; Differential complementarity problem; Power penalty method; Convergence analysis

AMS subject classifications. 65N12, 65K10, 91B28.

1 Introduction

Since the famous Black–Scholes model [1] was established in 1970's, the study of option pricing under the framework of partial differential equation (PDE) has attracted more and more attention in mathematical finance. In the standard Black–Scholes model, under the risk neutral measure the dynamics of the underlying asset S is assumed to follow a standard geometric Brownian motion, where an important assumption is that the volatility of the underlying asset is constant. Nonetheless, this assumption is unrealistic in practical applications, see numerous empirical studies in [2]. In fact, the volatility of the underlying asset behaves randomly, especially when it is measured with high frequency data [3]. To remedy this defect, different stochastic volatility models [4] are proposed to the option pricing problem. Among these models, the regime switching model [5, 6] is getting more and more attractive due to its economic intuition and inexpensive computation. In fact, by assuming that the market switches from time to time among different regimes, the regime-switching framework allows one to account for certain periodic or cyclic patterns caused by, for example, short-term political or economic uncertainty. Therefore, the incorporation of a regime switching component with the log-normal dynamics of stock price can better fit the market dynamics.

A lot of works have been done to study the standard American option pricing models. In [7, 8, 9], variational inequality theory is employed to show the unique solvability using the equivalence between the complementarity problem and variational inequalities. Meanwhile, because of its well-behavior and fulfilled property [10], the penalty approach is commonly used to characterize the solution behavior by showing the convergence property of solutions of the penalized problem, cf. [7, 11, 12]. Compared with the standard Black-Scholes pricing model, the American pricing problem under regime switching is more complicated because of the existence of the coupled partial differential operators. So, more care must be taken in studying the solvability and convergence property of the American option pricing problem under regime switching.

In this paper, we focus on studying a power penalty penalty approach to American option pricing model (1) under the regime switching. Within the framework of variational inequality theory, we first transform the coupled system of differential complementarity problems (DCPs) into a system of differential variational inequalities. We will then study the properties of the resulting system of variational inequalities. Then, the unique solvability is established under an appropriate Sobolev space. After that, we propose a power penalty approach to the system of differential variational inequalities, which results in a coupled system of nonlinear PDEs. For the penalized nonlinear system, we will study its solvability and convergence behavior. To this end, we first use the nonlinear operator theory to investigate the unique solvability of the system of nonlinear PDEs. To study the convergence rate of solutions of penalized problems to that of the original problem (1), an error bound estimation is given in an appropriate Sobolev space. With this error bound, a convergence rate of the penalized problem will be established. Furthermore, the monotonicity of convergence of the power penalty method with respect to the penalty parameter λ is shown. Finally, we shall design some preliminary numerical experiments to illustrate the convergence rate of the power penalty method.

The rest of this paper is arranged as follows. In Section 2, some definitions and notations are first given. The differential variational inequalities reformulation of the DCPs (1), arising from the American option under regime switching is then described.

In Section 3, a power penalty approach to the complementarity problem is introduced. The unique solvability of the penalized problem is also established. In Section 4, an error bound estimation is established. A convergence rate of the power penalty approach is eventually developed. The monotonicity of convergence of the power penalty method is given as well in this section. Section 5 gives some numerical experiments for pricing American put options under two regimes switching to show the convergence property of the power penalty method.

2 Mathematical Formulation

2.1 Notations

Before proceeding, we give some standard notations which will be used for the theoretical analysis in the paper. For an open set $\boldsymbol{S} =]S_1, S_2[\subset \mathbb{R}$ and $1 \leq p \leq \infty$, let $L^p(\boldsymbol{S}) = \left\{v : (\int_{\boldsymbol{S}} |v(x)| \mathrm{d}x)^{1/p} < \infty\right\}$ denote the space of all p-power integrable functions on $\boldsymbol{S}$. We use the $\|\cdot\|_{L^p(\boldsymbol{S})}$ to denote the norm on $L^p(\boldsymbol{S})$. The weighted Sobolev space $H^1_{0,\varpi}(\boldsymbol{S})$ is defined as

$$H^1_{0,\varpi}(\boldsymbol{S}) = \left\{v : v \in L^2(\boldsymbol{S}), S\,\tfrac{\partial v}{\partial S} \in L^2(\boldsymbol{S}),\ \text{and } v(S_2) = 0\right\},$$

with $\|v\|_{H^1_{0,\varpi}(\boldsymbol{S})} = (\int_{\boldsymbol{S}}(v^2 + S^2(\frac{\partial v}{\partial S})^2)\mathrm{d}S)^{1/2}$. It is easy to prove that the pair $(H^1_{0,\varpi}(\boldsymbol{S}), (\cdot,\cdot)_{H^1_{0,\varpi}(\boldsymbol{S})})$ is a Hilbert space by defining a weighted inner product on $H^1_{0,\varpi}(\boldsymbol{S})$ with $(u,v)_{H^1_{0,\varpi}(\boldsymbol{S})} = (u,v)_{L^2(\boldsymbol{S})} + (S\frac{\partial u}{\partial S}, S\frac{\partial v}{\partial S})_{L^2(\boldsymbol{S})}$. Finally, for any Hilbert space $W(\boldsymbol{S})$, the norm of $L^p(0,T;W(\boldsymbol{S}))$ is denoted by $\|v\|_{L^p(0,T;W(\boldsymbol{S}))} = (\int_0^T \|v(\cdot,t)\|^p_{W(\boldsymbol{S})}\,\mathrm{d}t)^{1/p}$. From this definition, it is obvious that $L^p(0,T;L^p(\boldsymbol{S})) = L^p(\boldsymbol{S} \times]0,T[)$.

For clarity, we will often simply write $v(\cdot,t)$ as $v(t)$ when we regard $v(\cdot,t)$ as an element of $H^1_{0,\varpi}(\boldsymbol{S})$. From time to time, we will also suppress the independent time variable t when it causes no confusion in doing so.

2.2 Formulation

Assuming the underlying economy switches among a finite number of states $\mathcal{M} = \{1, \ldots, m\}$, which is modeled by a finite Markov chain α_t with generator Q. To simplify the presentation throughout the paper, we only consider the case that there are only two states. Hence, in this case, $m = 2$, $\alpha_t = 1, 2$ and $Q = \left(\begin{smallmatrix} -q_1 & q_1 \\ q_2 & -q_2 \end{smallmatrix}\right)$, where q_1 and q_2 are positive constants. Let r_i and σ_i, $(i = 1, 2)$ be a set of discrete risk-free interest rates and volatilities, respectively. Under the risk-neutral measure, the stochastic process for the underlying asset S is

$$\frac{dS}{S} = r_{\alpha_t}\mathrm{d}t + \sigma_{\alpha_t}\mathrm{d}W,$$

where r_{α_t} and σ_{α_t} can take different values depending on different regimes.

Let $V_i(S,t)$ be the value of an American put option with striking price K in regime i, where the holder can receive a given payoff $V^*(S)$ at the expiry date T. The standard no-arbitrage pricing method leads to the following coupled system of DCPs [13]: for $i = 1, 2$

$$\begin{cases} L_i V\left(S,t\right) \geq 0, \\ V_i\left(S,t\right) - V^* \geq 0, \\ L_i V\left(S,t\right) \cdot \left(V_i\left(S,t\right) - V^*\right) = 0, \end{cases} \tag{1}$$

a.e. in $]0,+\infty[\times]0,T[$, where $V\left(S,t\right) = \left(V_1\left(S,t\right), V_2\left(S,t\right)\right)^\top$, and

$$\begin{aligned} L_1 V &= -\frac{\partial V_1}{\partial t} - \left[\frac{1}{2}\sigma_1^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + r_1 S \frac{\partial V_1}{\partial S} - r_1 V_1 - q_1 V_1 + q_1 V_2\right], \\ L_2 V &= -\frac{\partial V_2}{\partial t} - \left[\frac{1}{2}\sigma_2^2 S^2 \frac{\partial^2 V_2}{\partial S^2} + r_2 S \frac{\partial V_2}{\partial S} - r_2 V_2 - q_2 V_2 + q_2 V_1\right], \end{aligned}$$

are two coupled degenerate parabolic partial differential operators with the final conditions

$$V_i(S,T) = V^*\left(S\right), \quad i = 1, 2,$$

and the following boundary conditions

$$V_i(0,t) = K, \quad \lim_{S\to+\infty} V_i(S,t) = 0, \quad i = 1, 2. \tag{2}$$

For computational purpose, we restrict S in a region $I = [0, X] \in \mathbb{R}$, where X is sufficiently large to ensure the accuracy of the solution ([4]). Thus, (2) becomes

$$V_i(0,t) = K, \quad V_i(X,t) = 0, \quad i = 1, 2. \tag{3}$$

By introducing two new variables

$$u_i(S,t) = e^{\beta t}\left(V_0(S) - V_i(S,t)\right), \quad \beta = \max_i \left\{\sigma_i^2\right\} + \max_i \left\{q_i\right\}, i = 1, 2 \tag{4}$$

with $V_0(S) = (1 - S/X)K$, we first transform (1) into the following equivalent standard form satisfying homogeneous Dirichlet boundary conditions.

Problem 2.1. For $i = 1, 2$,

$$\begin{cases} \mathcal{L}_i U(S,t) \leq f_i(S,t), \\ u_i\left(S,t\right) - u^*(S,t) \leq 0, \\ \left(\mathcal{L}_i U(S,t) - f_i(S,t)\right) \cdot \left(u_i(S,t) - u_i^*(S,t)\right) = 0, \end{cases} \tag{5}$$

a.e. in $[0,X]\times]0,T[$, where $U\left(S,t\right) = \left(u_1\left(S,t\right), u_2\left(S,t\right)\right)^\top$, and

$$\begin{aligned} \mathcal{L}_1 U &= -\frac{\partial u_1}{\partial t} - \frac{\partial}{\partial S}\left[a_1 S^2 \frac{\partial u_1}{\partial S} + b_1 u_1 S\right] + c_1 u_1 - q_1 u_2, \\ \mathcal{L}_2 U &= -\frac{\partial u_2}{\partial t} - \frac{\partial}{\partial S}\left[a_2 S^2 \frac{\partial u_2}{\partial S} + b_2 u_2 S\right] + c_2 u_2 - q_2 u_1, \end{aligned}$$

are the self-adjoint forms with

$$a_i = \frac{1}{2}\sigma_i^2,\ b_i = r_i - \sigma_i^2,\ c_i = r_i + b_i + q_i + \beta,\ \text{and}\ f_i(S,t) = e^{\beta t} L_i V_0(S). \tag{6}$$

The payoff function becomes

$$u^*(S,t) \doteq e^{\beta t}\left(V_0(S - V^*(S))\right), \tag{7}$$

and the new final and boundary conditions are, for $i = 1, 2$,

$$u_i(S,T) = u^*(S,T),$$
$$u_i(0,t) = u_i(X,t) = 0, \quad t \in]0,T[.$$

We will show the differential variational inequality problem corresponding to Problem 2.1 has a unique solution in $L^2(0,T; H^1_{0,\omega}(I)) \times L^2(0,T; H^1_{0,\omega}(I))$.

It is a standard result that the system of DCPs (5) can be reformulated as the following system of differential variational inequalities.

Problem 2.2. Find $U = (u_1(t), u_2(t))^\top \in \mathcal{K} \times \mathcal{K}$, such that, for all $v \in \mathcal{K}$,

$$\left(-\frac{\partial u_1}{\partial t}, v - u_1\right) + A_1(u_1, v - u_1; t) + B_1(u_2, v - u_1; t) \geq (f_1, v - u_1), \tag{8}$$

$$\left(-\frac{\partial u_2}{\partial t}, v - u_2\right) + A_2(u_2, v - u_2; t) + B_2(u_1, v - u_2; t) \geq (f_2, v - u_2), \tag{9}$$

a.e. in $]0,T[$, where $A_1(\cdot,\cdot;t)$, $A_2(\cdot,\cdot;t)$, $B_1(\cdot,\cdot;t)$ and $B_2(\cdot,\cdot;t)$ are bilinear forms defined on $H^1_{0,\varpi}(I) \times H^1_{0,\varpi}(I)$ by

$$A_1(u,v;t) = \left(a_1 S^2 \frac{\partial u}{\partial S} + b_1 S u, \frac{\partial v}{\partial S}\right) + (c_1 u, v), \quad B_1(u,v;t) = -(q_1 u, v),$$
$$A_2(u,v;t) = \left(a_2 S^2 \frac{\partial u}{\partial S} + b_2 S u, \frac{\partial v}{\partial S}\right) + (c_2 u, v), \quad B_2(u,v;t) = -(q_2 u, v),$$

and $\mathcal{K} = \{v \in H^1_{0,\varpi}(I) : v \leq u^*\}$ is a convex and closed subset of $H^1_{0,\varpi}(I)$.

For Problem 2.2, we establish the unique solvability result as follows:

Theorem 2.1. *The system of variational inequalities* (8) - (9) *has a unique solution.*

Proof For $v_1, v_2, \mu_1, \mu_2 \in H^1_{0,\varpi}(I)$, define the following global bilinear operator

$$\mathcal{A}(\mu_1, \mu_2; v_1, v_2) = A_1(\mu_1, v_1) + A_2(\mu_2, v_2) + B_1(\mu_2, v_1) + B_2(\mu_1, v_2). \tag{10}$$

It is easy to see that

$$\begin{aligned}A_1(v_1,v_1) &= \left(a_1 S^2 \frac{\partial v_1}{\partial S} + b_1 S v_1, \frac{\partial v_1}{\partial S}\right) + (c_1 v_1, v_1) \\ &\geq a_1 \left(S^2 \frac{\partial v_1}{\partial S}, \frac{\partial v_1}{\partial S}\right) + \left(\frac{3r_1 - \sigma_1^2}{2} + q_1 + \beta\right)(v_1, v_1) \\ &\geq \frac{\sigma_1^2}{2} \left\| S \frac{\partial v_1}{\partial S} \right\|^2 + \frac{1}{2}\left(3r_1 + \sigma_1^2 + 2q_1 + 2q_2\right)\|v_1\|^2 .\end{aligned}$$

In the same way, we also have

$$A_2(v_2,v_2) \geq \frac{\sigma_2^2}{2} \left\| S \frac{\partial v_2}{\partial S} \right\|^2 + \frac{1}{2}\left(3r_2 + \sigma_2^2 + 2q_2 + 2q_1\right)\|v_2\|^2 .$$

Moreover,

$$B_1(v_2,v_1) + B_2(v_1,v_2) = -(q_1 v_2, v_1) - (q_2 v_1, v_2) \geq -(q_1 + q_2)\|v_1\|\|v_2\| .$$

Hence, using (4) and (6), we get

$$\begin{aligned}&\mathcal{A}(v_1,v_2;v_1,v_2) \\ =&A_1(v_1,v_1) + A_2(v_2,v_2) + B_1(v_2,v_1) + B_2(v_1,v_2) \\ \geq&\frac{\sigma_1^2}{2}\|v_1\|_{H^1_{0,\omega}}^2 + \frac{\sigma_2^2}{2}\|v_2\|_{H^1_{0,\omega}}^2 + (q_1+q_2)\left(\|v_1\|^2 + \|v_2\|^2 - \|v_1\|\|v_2\|\right) \\ \geq&\frac{\sigma_1^2}{2}\|v_1\|_{H^1_{0,\omega}}^2 + \frac{\sigma_2^2}{2}\|v_2\|_{H^1_{0,\omega}}^2 ,\end{aligned}$$

which shows that $\mathcal{A}$ is coercive. On the other hand, by virtue of the Cauchy-Schwartz and Poincaré inequalities, it is easy to show that there exists a positive constant C such that

$$\begin{aligned}&\mathcal{A}(\mu_1,\mu_2;v_1,v_2) \\ =&\left|A_1(\mu_1,v_1) + A_2(\mu_2,v_2) + B_1(\mu_2,v_1) + B_2(\mu_1,v_2)\right| \\ \leq&\left|A_1(\mu_1,v_1)\right| + \left|A_2(\mu_2,v_2)\right| + \left|B_1(\mu_2,v_1)\right| + \left|B_2(\mu_1,v_2)\right| \\ \leq&C\sqrt{\|\mu_1\|_{H^1_{0,\omega}}^2 + \|\mu_2\|_{H^1_{0,\omega}}^2}\sqrt{\|v_1\|_{H^1_{0,\omega}}^2 + \|v_2\|_{H^1_{0,\omega}}^2},\end{aligned}$$

which shows that $\mathcal{A}$ is continuous. Hence, it follows from the Lions-Stampaccia Theorem [14] that the existence and the uniqueness of (8) - (9) are guaranteed. $\square$

3 Power Penalty Approach

In this section, we propose a power penalty approach to the system of DCPs (5). To this effect, we first consider the following set of nonlinear variational inequalities.

Problem 3.1. Find $U^\lambda = \left(u_1^\lambda, u_2^\lambda\right)^\top \in H^1_{0,\omega}(I) \times H^1_{0,\omega}(I)$, such that, for all $v \in H^1_{0,\omega}(I)$,

$$\left(\mathcal{L}_i U^\lambda(S,t), v - u_i^\lambda\right) + \left(j(u_i^\lambda), v - u_i^\lambda\right) \geq \left(f_i, v - u_i^\lambda\right), \quad i = 1, 2,$$

a.e. in $]0,T[$, where

$$j\left(v\right) = \frac{\lambda k}{k+1}\left[v - u^*\right]_+^{\frac{k+1}{k}}, \quad k > 0, \lambda > 1, \tag{11}$$

and $[\cdot]_+ = \max\{0, \cdot\}$.

Since the bilinear operator $\mathcal{A}$ is coercive and continuous, and the operator j is lower semi-continuous, the unique solvability of Problem 3.1 is directly obtained (cf. [14]).

It follows from (11) that $j(v)$ is differentiable. Thus, we can get the following equivalent form of Problem 3.1.

Problem 3.2. Find $U^\lambda = \left(u_1^\lambda, u_2^\lambda\right)^\top \in H^1_{0,\omega}(I) \times H^1_{0,\omega}(I)$, such that, for all $v \in H^1_{0,\omega}(I)$,

$$\left(\mathcal{L}_i U^\lambda(S,t),\right) + \left(j'(u_i^\lambda), v\right) = f_i(S,t), \quad i = 1, 2$$

or equivalently

$$\left(-\frac{\partial u_1^\lambda}{\partial t}, v\right) + A_1(u_1^\lambda, v; t) + B_1(u_2^\lambda, v; t) + \left(j'(u_1^\lambda), v\right) = \left(f_1, v\right), \tag{12}$$

$$\left(-\frac{\partial u_2^\lambda}{\partial t}, v\right) + A_2(u_2^\lambda, v; t) + B_2(u_1^\lambda, v; t) + \left(j'(u_2^\lambda), v\right) = \left(f_2, v\right), \tag{13}$$

a.e. in $]0,T[$, where

$$j'(v) = \lambda[v - u^*]_+^{1/k},$$

is the power penalty term, also called $l_{1/k}$ penalty term with the power $1/k$.

It is worth noting that Problem 3.2 is a set of penalized variational equations corresponding to Problem 2.2, cf. [14]. We can easily write the strong form of (12)–(13), which defines the set of penalized system approximating Problem 2.1, as follows:

$$\mathcal{L}_i U^\lambda(S,t) + \lambda[u_i^\lambda - u^*]_+^{1/k} = f_i(S,t), \quad i = 1, 2, \tag{14}$$

with the given boundary and final conditions

$$\begin{aligned} u_i^\lambda(S,T) &= u^*(S,T), \\ u_i^\lambda(0,t) = u_i^\lambda(X,t) &= 0, t \in]0,T[, \end{aligned} \tag{15}$$

for $i = 1, 2$. By virtue of the variable changes in (4), (6) and (7), we can obtain the $l_{1/k}$ penalty approach to the original complementarity problem (1) from (14) - (15), which is given as follows:

$$L_i V^\lambda + \lambda[V^* - V^\lambda]_+^{1/k} = 0, \tag{16}$$

with terminal and boundary conditions

$$\begin{aligned} V_i^\lambda(S,T) &= V^*(S,T), \\ V_i^\lambda(0,t) = K, &\quad V_i^\lambda(X,t) = 0, \end{aligned} \tag{17}$$

for $i = 1, 2$, where V^λ is the power penalty approximation to V.

In the next section, we will investigate the convergence properties of u_λ to u as $\lambda \to \infty$ and establish a sharp convergence rate of the power penalty method with respect to the penalty parameter λ.

4 Convergence Analysis

4.1 An Error Bound of the Power Penalization

We now show that, under the assumption that

$$u_i^\lambda, \frac{\partial u_i^\lambda}{\partial t} \in L^2(0,T; H^1_{0,\omega}(I)) \cap L^\infty(0,T; L^2(I)),$$

for $i = 1, 2$, the solution to Problem 3.2 converges to that of Problem 2.1 in an appropriate norm as $\lambda \to \infty$. We start this discussion by the following Lemma. Let U^λ be the solution to Problem 3.2. If $U^\lambda \in L^p(0,T; L^p(I))$, then there exists a positive constant C, independent of U^λ and λ, such that for $i = 1, 2$,

$$\|[u_i^\lambda - u^*]_+\|_{L^p(0,T;L^p(I))} \leq \frac{C}{\lambda^k}, \tag{18}$$

$$\|[u_i^\lambda - u^*]_+\|_{L^\infty(0,T;L^2(I))} + \|[u_i^\lambda - u^*]_+\|_{L^2(0,T;H^1_{0,\omega}(I))} \leq \frac{C}{\lambda^{k/2}}, \tag{19}$$

where $1/k$ is the power of the $l_{1/k}$ penalty term and $p = 1 + 1/k$. *Proof* Assume that C is a generic positive constant, independent of U^λ and λ. To simply the notation, we denote $F = (f_1, f_2)^\top$, and

$$\Phi = (\phi_1, \phi_2)^\top = [U^\lambda - U^*]_+ = \left(\max\left[u_1^\lambda - u^*, 0\right], \max\left[u_2^\lambda - u^*, 0\right]\right)^\top.$$

Clearly, $\phi_i = \left[u_i^\lambda - u^*\right]_+ \in H^1_{0,\omega}(I)$, $i = 1, 2$ for almost all $t \in]0, T[$. Now, setting $v = \phi_1$ in (12), it follows that

$$\begin{aligned} &\left(-\frac{\partial\left(u_1^\lambda - u^*\right)}{\partial t}, \phi_1\right) + A_1(u_1^\lambda - u^*, \phi_1; t) + B_1(u_2^\lambda - u^*, \phi_1; t) + \lambda\left(\phi_1^{\frac{1}{k}}, \phi_1\right) \\ &= (f_1, \phi_1) - A_1(u^*, \phi_1; t) - B_1(u^*, \phi_1; t). \end{aligned} \tag{20}$$

In the same way, we have

$$\begin{aligned} &\left(-\frac{\partial\left(u_2^\lambda - u^*\right)}{\partial t}, \phi_2\right) + A_2(u_2^\lambda - u^*, \phi_2; t) + B_2(u_1^\lambda - u^*, \phi_2; t) + \lambda\left(\phi_2^{\frac{1}{k}}, \phi_2\right) \\ &= (f_2, \phi_2) - A_2(u^*, \phi_2; t) - B_2(u^*, \phi_2; t). \end{aligned} \tag{21}$$

Integrating both sides of (20) and (21) from τ to T and adding them up deduces that

$$\begin{aligned}&\frac{1}{2}\sum_{i=1}^{2}(\phi_i,\phi_i)+\int_\tau^T \mathcal{A}\left(u_1^\lambda-u^*,u_2^\lambda-u^*;u_1^\lambda-u^*,u_2^\lambda-u^*;t\right)\mathrm{d}t\\&+\lambda\int_\tau^T\sum_{i=1}^{2}\left(\phi_i^{\frac{1}{k}},\phi_i\right)\mathrm{d}t\\=&\int_\tau^T\sum_{i=1}^{2}(f_i,\phi_i)\,\mathrm{d}\tau-\int_\tau^T \mathcal{A}\left(u^*,u^*;\phi_1,\phi_2;t\right)\mathrm{d}t.\end{aligned}$$

Employing the definition of $\mathcal{A}$ in (10), u^* in (7) and f_i in (6) and Hölder's inequality, it follows that

$$\begin{aligned}&\frac{1}{2}(\Phi,\Phi)+\frac{\gamma}{2}\int_\tau^T\|\Phi\|^2_{H^1_{0,\omega}(I)}\,\mathrm{d}t+\lambda\int_\tau^T||\Phi||^p_{L^p(I)}\mathrm{d}t\\ \le&\int_\tau^T(F,\Phi)\,\mathrm{d}t-\int_\tau^T\mathcal{A}\left(u^*,u^*;\phi_1,\phi_2;t\right)\mathrm{d}t\le C\left(\int_\tau^T||\Phi||^p_{L^p(I)}\mathrm{d}t\right)^{1/p},\end{aligned}\tag{22}$$

with $\gamma=\min\{\sigma_1^2,\sigma_2^2\}$. Here we used the coerciveness of the operator $\mathcal{A}$. Moreover, (22) implies that

$$\lambda\int_\tau^T||\Phi||^p_{L^p(I)}\mathrm{d}\tau\le C\left(\int_\tau^T||\Phi||^p_{L^p(I)}\mathrm{d}t\right)^{1/p}.\tag{23}$$

From this and $p=1+1/k$, we obtain

$$\left(\int_\tau^T||\Phi||^p_{L^p(I)}\mathrm{d}t\right)^{1/p}\le\frac{C}{\lambda^{1/(p-1)}},\ \text{ or }||\Phi||_{L^p(I)}\le\frac{C}{\lambda^{1/(p-1)}}=\frac{C}{\lambda^k}.\tag{24}$$

Thus,

$$\|[u_i^\lambda-u^*]_+\|_{L^p(0,T;L^p(I)}\le||\Phi||_{L^p(0,T;L^p(I)}\le\frac{C}{\lambda^k},\quad i=1,2,$$

because of $||\Phi||^p_{L^p}=\|\sum_{i=1}^2[u_i^\lambda-u^*]_+\|^p_{L^p}$ and $p>0$. This proves (18).

Now, from (22) and (23), we deduce

$$\frac{1}{2}(\Phi,\Phi)+\int_\tau^T\|\Phi\|^2_{H^1_{0,\omega}(I)}\,\mathrm{d}t\le\left(\int_\tau^T||\Phi||^p_{L^p(I)}\mathrm{d}t\right)^{1/p}\le\frac{C}{\lambda^k},$$

and hence

$$\frac{1}{2}(\Phi,\Phi)^{\frac{1}{2}}+\left(\int_\tau^T\|\Phi\|^2_{H^1_{0,\omega}(I)}\,\mathrm{d}t\right)^{\frac{1}{2}}\le\frac{C}{\lambda^{k/2}}.$$

Since for $i=1,2$

$$\begin{aligned}&\|[u_i^\lambda-u^*]_+\|_{L^\infty(0,T;L^2(I))}+\|[u_i^\lambda-u^*]_+\|_{L^2(0,T;H^1_{0,\omega}(I))}\\\le&\|[U^\lambda-U^*]_+\|_{L^\infty(0,T;L^2(I))}+\|[U^\lambda-U^*]_+\|_{L^2(0,T;H^1_{0,\omega}(I))},\end{aligned}$$

we finally obtain

$$\|[u_i^\lambda - u^*]_+\|_{L^\infty(0,T;L^2(I))} + \|[u_i^\lambda - u^*]_+\|_{L^2(0,T;H^1_{0,\omega}(I))} \leq \frac{C}{\lambda^{k/2}}.$$

$\square$

4.2 Rate of Convergence of the Power Penalization

On the basis of Lemma 4.1, we obtain the following convergence rate of the power penalty approach. Let assumptions of Lemma 4.1 be fulfilled. If $U_\lambda \in L^p(\boldsymbol{S}\times]0,T[)$ and $\frac{\partial U_\lambda}{\partial t} \in L^q(\boldsymbol{S}\times]0,T[)$, then there exists a positive constant C, independent of u_λ and λ, such that for $i = 1, 2$

$$\left\|u_i^\lambda - u_i\right\|_{L^\infty(0,T;L^2(I))} + \left\|u_i^\lambda - u_i\right\|_{L^2(0,T;H^1_0(I))} \leq \frac{C}{\lambda^{k/2}}, \tag{25}$$

where $1/k$ is the power of the $l_{1/k}$ penalty term, and $p = 1 + 1/k$, $1/p + 1/q = 1$. *Proof* We still use the notation of Lemma 4.1. First, define

$$\left[U^\lambda - U^*\right]_- = \left(-\min\left[u_1^\lambda - u^*, 0\right], -\min\left[u_2^\lambda - u^*, 0\right]\right)^\top,$$

then $U^\lambda - U^* = \left[U^\lambda - U^*\right]_+ - \left(U^\lambda - U^*\right)_- = \Phi - \left(U^\lambda - U^*\right)_-$. Hence, we can decompose $U - U^\lambda = \left(u_1 - u_1^\lambda, u_2 - u_2^\lambda\right)^\top$ as

$$U - U^\lambda = (U - U^*) - \left(U^\lambda - U^*\right) \triangleq R^\lambda - \Phi,$$

where

$$R^\lambda = \left(r_1^\lambda, r_2^\lambda\right)^\top = (U - U^*) + \left[U^\lambda - U^*\right]_- . \tag{26}$$

Then, it follows from (19) that, in order to prove (25), it is sufficient to show that

$$||r_1^\lambda||_{L^\infty(0,T;L^2(I))\cap L^2(0,T;H^1_{0,\omega}(I))} \leq \frac{C}{\lambda^{k/2}}.$$

Set $v = u_1 - r_1^\lambda$ in (8) and $v = r_1^\lambda$ in (12). Then, we have

$$\left(-\frac{\partial u_1}{\partial t}, -r_1^\lambda\right) + A_1(u_1, -r_1^\lambda; t) + B_1(u_2, -r_1^\lambda; t) \geq (f_1, -r_1^\lambda),$$

$$\left(-\frac{\partial u_1^\lambda}{\partial t}, r_1^\lambda\right) + A_1(u_1^\lambda, r_1^\lambda; t) + B_1(u_2^\lambda, -r_1^\lambda; t) + \lambda([u_1^\lambda - u^*]_+^{1/k}, r_1^\lambda) = (f_1, r_1^\lambda).$$

Combining the above two formulas gives

$$\left(-\frac{\partial(u_1^\lambda - u_1)}{\partial t}, r_1^\lambda\right) + A_1(u_1^\lambda - u_1, r_1^\lambda; t) + B_1(u_2^\lambda - u_2, r_1^\lambda; t) + \lambda(\phi_1^{1/k}, r_1^\lambda) \geq 0.$$

But, it follows from $u_i \leq u^*$ that

$$(\phi_1^{1/k}, r_1^\lambda) = (\phi_1^{1/k}, u_1 - u^*) + (\phi_1^{1/k}, [u_1^\lambda - u^*]_-) = ([u_1^\lambda - u^*]_+^{1/k}, u_1 - u_1^\lambda) \leq 0,$$

since $(\phi_1^{1/k}, u_1 - u^*) = ([u_1^\lambda - u^*]_+^{1/k}, u_1 - u^*) \equiv 0$. Thus,

$$\left(-\frac{\partial(u_1^\lambda - u_1)}{\partial t}, r_1^\lambda\right) + A_1(u_1^\lambda - u_1, r_1^\lambda; t) + B_1(u_2^\lambda, r_1^\lambda; t) \le 0,$$

and hence

$$\begin{aligned}&\left(-\frac{\partial r_1^\lambda}{\partial t}, r_1^\lambda\right) + A_1(r_1^\lambda, r_1^\lambda; t) + B_1(r_2^\lambda, r_1^\lambda; t)\\ \le&\left(-\frac{\partial \phi_1}{\partial t}, r_1^\lambda\right) + A_1(\phi_1, r_1^\lambda; t) + B_1(\phi_2, r_1^\lambda; t).\end{aligned} \tag{27}$$

In the same way, we also have

$$\begin{aligned}&\left(-\frac{\partial r_2^\lambda}{\partial t}, r_2^\lambda\right) + A_2(r_2^\lambda, r_2^\lambda; t) + B_1(r_1^\lambda, r_2^\lambda; t)\\ \le&\left(-\frac{\partial \phi_2}{\partial t}, r_2^\lambda\right) + A_2(\phi_2, r_2^\lambda; t) + B_2(\phi_1, r_2^\lambda; t).\end{aligned} \tag{28}$$

Then integrating both sides of (27) and (28) from τ to T and adding them up arrives at

$$\begin{aligned}&\frac{1}{2}\left(R^\lambda(t), R^\lambda(t)\right) + \int_\tau^T \mathcal{A}\left(r_1^\lambda, r_2^\lambda; r_1^\lambda, r_2^\lambda; t\right) \mathrm{d}t\\ \le&\left(-\frac{\partial \Phi}{\partial t}, R^\lambda(t)\right) + \int_\tau^T \mathcal{A}\left(\phi_1, \phi_2; r_1^\lambda, r_2^\lambda; t\right) \mathrm{d}t.\end{aligned}$$

It follows from Cauchy-Schwartz inequality that

$$\begin{aligned}&\frac{1}{2}\left(R^\lambda(t), R^\lambda(t)\right) + \int_\tau^T \mathcal{A}\left(r_1^\lambda, r_2^\lambda; r_1^\lambda, r_2^\lambda; t\right) \mathrm{d}t\\ \le&\left(\Phi(t), R^\lambda(t)\right) + \int_\tau^T \left(\Phi, \frac{\partial R^\lambda}{\partial t}\right) \mathrm{d}t + \int_\tau^T \mathcal{A}\left(\phi_1, \phi_2; r_1^\lambda, r_2^\lambda; t\right) \mathrm{d}t\\ \le&||R^\lambda||_{L^\infty(0,T;L^2(I))}||\Phi||_{L^\infty(0,T;L^2(I))} + C||R^\lambda||_{L^2(0,T;H^1_{0.\omega}(I))}||\Phi||_{L^2(0,T;H^1_{0.\omega}(I))}\\ &+\int_\tau^T \left(\Phi, \frac{\partial R^\lambda}{\partial t}\right) \mathrm{d}t.\end{aligned} \tag{29}$$

Noting that $(\Phi, [U - U^*]_-) = 0$ and employing (7), (26) and (24) reduces

$$\begin{aligned}\left|\int_\tau^T \left(\Phi, \frac{\partial R^\lambda}{\partial t}\right) \mathrm{d}t\right| \le& \left|\int_\tau^T \left(\Phi, \frac{\partial U}{\partial t}\right) \mathrm{d}t\right| + \left|\int_\tau^T \left(\Phi, \frac{\partial U^*}{\partial t}\right) \mathrm{d}t\right|\\ &\le C\|\Phi\|_{L^p(0,T;L^p(I))}\left\|\frac{\partial U}{\partial t}\right\|_{L^q(0,T;L^q(I))} \le \frac{C}{\lambda^k}.\end{aligned}$$

Hence, using the coerciveness of $\mathcal{A}$ and (29) gives

$$\begin{aligned}&||R^\lambda||^2_{L^\infty(0,T;L^2(I))} + ||R^\lambda||^2_{L^2(0,T;H^1_{0.\omega}(I))}\\ \le&(||R^\lambda||_{L^\infty(0,T;L^2(I))}||\Phi||_{L^\infty(0,T;L^2(I))}\\ &\quad+||R^\lambda||_{L^2(0,T;H^1_{0.\omega}(I))}||\Phi||_{L^2(0,T;H^1_{0.\omega}(I))} + \frac{C}{\lambda^k})\\ \le&\tfrac{C}{\lambda^{k/2}}(|R_\lambda(t)| + \int_t^T ||R_\lambda||^2_{H^1_0(I)}\mathrm{d}\tau) + \frac{C}{\lambda^k}\end{aligned}$$

with a generic constant $C > 0$, which infers to the following inequality

$$||R^\lambda||^2_{L^\infty(0,T;L^2(I))} + ||R^\lambda||^2_{L^2(0,T;H^1_{0.\omega}(I))} \leq \frac{C}{\lambda^{k/2}}.$$

Using the triangle inequality, the above inequality and (19), we obtain

$$||U - U^\lambda||^2_{L^\infty(0,T;L^2(I))} + ||U - U^\lambda||^2_{L^2(0,T;H^1_{0.\omega}(I))} \leq \frac{C}{\lambda^{k/2}}.$$

Since for $i = 1, 2$

$$\begin{aligned}&\|u_i - u_i^\lambda\|_{L^\infty(0,T;L^2(I))} + \|u_i - u_i^\lambda\|_{L^2(0,T;H^1_{0,\omega}(I))} \\ \leq &||U - U^\lambda||^2_{L^\infty(0,T;L^2(I))} + ||U - U^\lambda||^2_{L^2(0,T;H^1_{0.\omega}(I))},\end{aligned}$$

we immediately get the following inequality

$$\left\|u_i^\lambda - u_i\right\|_{L^\infty(0,T;L^2(I))} + \left\|u_i^\lambda - u_i\right\|_{L^2(0,T;H^1_0(I))} \leq \frac{C}{\lambda^{k/2}}, \quad i = 1, 2,$$

which proves (25). □

4.3 The Monotonic Convergence of the Penalization

In this subsection, we will show that, when the penalty parameter λ increases monotonically, the solution to the penalized problem decreases monotonically. Moreover, the solution of the penalized problem is bounded below by that of the original DCPs. The monotonicity of the solution sequence obtained by the power penalization is stated in the following theorem. Let $0 < \lambda_1 \leq \lambda_2$ be two different penalty parameters. Then,

$$u_i \leq u_i^{\lambda_2} \leq u_i^{\lambda_1}, \quad i = 1, 2,$$

where u_i is the solution to Problem 2.2, $u_i^{\lambda_1}$and $u_i^{\lambda_2}$ are the solutions to Problem 3.2 for $\lambda = \lambda_1$ and λ_2, respectively. *Proof* In (12), we set $v = [u_1^{\lambda_2} - u_1^{\lambda_1}]_+$ and $\lambda = \lambda_1$ and λ_2, respectively. Then, it follows that

$$\begin{aligned}&\left(-\frac{\partial u_1^{\lambda_1}}{\partial t}, [u_1^{\lambda_2} - u_1^{\lambda_1}]_+\right) + A_1(u_1^{\lambda_1}, [u_1^{\lambda_2} - u_1^{\lambda_1}]_+; t) + B_1(u_2^{\lambda_1}, [u_1^{\lambda_2} - u_1^{\lambda_1}]_+; t) \\ &+ \lambda_1([u_1^{\lambda_1} - u^*]_+^{\frac{1}{k}}, [u_1^{\lambda_2} - u_1^{\lambda_1}]_+) = (f_1, [u_1^{\lambda_2} - u_1^{\lambda_1}]_+),\end{aligned}$$

$$\begin{aligned}&\left(-\frac{\partial u_1^{\lambda_2}}{\partial t}, [u_1^{\lambda_2} - u_1^{\lambda_1}]_+\right) + A_1(u_1^{\lambda_2}, [u_1^{\lambda_2} - u_1^{\lambda_1}]_+; t) + B_1(u_2^{\lambda_2}, [u_1^{\lambda_2} - u_1^{\lambda_1}]_+; t) \\ &+ \lambda_2([u_1^{\lambda_2} - u^*]_+^{\frac{1}{k}}, [u_1^{\lambda_2} - u_1^{\lambda_1}]_+) = (f_1, [u_1^{\lambda_2} - u_1^{\lambda_1}]_+),\end{aligned}$$

and hence

$$\begin{aligned}
&\left(-\frac{\partial(u_1^{\lambda_2}-u_1^{\lambda_1})}{\partial t},[u_1^{\lambda_2}-u_1^{\lambda_1}]_+\right)+A_1(u_1^{\lambda_2}-u_1^{\lambda_1},[u_1^{\lambda_2}-u_1^{\lambda_1}]_+;t)\\
&\quad+B_1(u_2^{\lambda_2}-u_2^{\lambda_1},[u_1^{\lambda_2}-u_1^{\lambda_1}]_+;t)\\
=&\lambda_1([u_1^{\lambda_1}-u^*]_+^{1/k},[u_1^{\lambda_2}-u_1^{\lambda_1}]_+)-\lambda_2([u_1^{\lambda_2}-u^*]_+^{1/k},[u_1^{\lambda_2}-u_1^{\lambda_1}]_+)\\
=&(\lambda_1-\lambda_2)([u_1^{\lambda_1}-u^*]_+^{1/k},[u_1^{\lambda_2}-u_1^{\lambda_1}]_+)\\
&\quad+\lambda_2([u_1^{\lambda_1}-u^*]_+^{1/k}-[u_1^{\lambda_2}-u^*]_+^{1/k},[u_1^{\lambda_2}-u_1^{\lambda_1}]_+).
\end{aligned}\tag{30}$$

For the first term in (30), providing that $\lambda_1 \leq \lambda_2$, obviously,

$$\begin{aligned}
&(\lambda_1-\lambda_2)\left([u_1^{\lambda_1}-u^*]_+^{1/k},[u_1^{\lambda_2}-u_1^{\lambda_1}]_+\right)\\
=&(\lambda_1-\lambda_2)\int_0^X[u_1^{\lambda_1}-u^*]_+^{1/k}[u_1^{\lambda_2}-u_1^{\lambda_1}]_+\mathrm{d}x\leq 0.
\end{aligned}\tag{31}$$

For the second term in (30), we shall also show that

$$\begin{aligned}
&\lambda_2\left([u_1^{\lambda_1}-u^*]_+^{1/k}-[u_1^{\lambda_2}-u^*]_+^{1/k},[u_1^{\lambda_2}-u_1^{\lambda_1}]_+\right)\\
=&\lambda_2\int_0^X\left([u_1^{\lambda_1}-u^*]_+^{1/k}-[u_1^{\lambda_2}-u^*]_+^{1/k}\right)[u_1^{\lambda_2}-u_1^{\lambda_1}]_+\mathrm{d}x\leq 0.
\end{aligned}$$

In fact, $[u_1^{\lambda_2}-u_1^{\lambda_1}]_+=\max\{u_1^{\lambda_2}-u_1^{\lambda_1},0\}=0$ when $u_1^{\lambda_2}\leq u_1^{\lambda_1}$. To calculate

$$\int_0^X\left([u_1^{\lambda_1}-u^*]_+^{1/k}-[u_1^{\lambda_2}-u^*]_+^{1/k}\right)[u_1^{\lambda_2}-u_1^{\lambda_1}]_+\mathrm{d}x$$

we only need to integrate $u_1^{\lambda_2}-u_1^{\lambda_1}$ over the set for which $u_1^{\lambda_2}>u_1^{\lambda_1}$. On this set, by virtue of the monotonicity of the operator $[\cdot]_+^{1/k}=(\max\{\cdot,0\})^{1/k}$, we can infer that

$$\lambda_2\left([u_1^{\lambda_1}-u^*]_+^{1/k}-[u_1^{\lambda_2}-u^*]_+^{1/k},[u_1^{\lambda_2}-u_1^{\lambda_1}]_+\right)\leq 0.$$

Consequently, on the whole set $I=[0,X]$, we have

$$\lambda_2\left([u_1^{\lambda_1}-u^*]_+^{1/k}-[u_1^{\lambda_2}-u^*]_+^{1/k},[u_1^{\lambda_2}-u_1^{\lambda_1}]_+\right)\leq 0.\tag{32}$$

It then follows form (30) - (32) that

$$\begin{aligned}
&\left(-\frac{\partial(u_1^{\lambda_2}-u_1^{\lambda_1})}{\partial t},[u_1^{\lambda_2}-u_1^{\lambda_1}]_+\right)+A_1(u_1^{\lambda_2}-u_1^{\lambda_1},[u_1^{\lambda_2}-u_1^{\lambda_1}]_+;t)\\
&\quad+B_1(u_2^{\lambda_2}-u_2^{\lambda_1},[u_1^{\lambda_2}-u_1^{\lambda_1}]_+;t)\leq 0.
\end{aligned}\tag{33}$$

In the same way, we obtain

$$\begin{aligned}
&\left(-\frac{\partial(u_2^{\lambda_2}-u_2^{\lambda_1})}{\partial t},[u_2^{\lambda_2}-u_2^{\lambda_1}]_+\right)+A_2(u_2^{\lambda_2}-u_2^{\lambda_1},[u_2^{\lambda_2}-u_2^{\lambda_1}]_+;t)\\
&\quad+B_2(u_1^{\lambda_2}-u_1^{\lambda_1},[u_2^{\lambda_2}-u_2^{\lambda_1}]_+;t)\leq 0.
\end{aligned}\tag{34}$$

By adding (33) and (34) and using the coerciveness of $\mathcal{A}$, we have

$$\frac{1}{2}\left\|[U^{\lambda_2}(t) - U^{\lambda_1}(t)]_+\right\|^2_{L^2(I)} + \gamma \int_t^T ||[U^{\lambda_2} - U^{\lambda_1}]_+||^2_{H^1_{0,\omega}(I)} \mathrm{d}\tau \leq 0,$$

and hence

$$[U^{\lambda_2} - U^{\lambda_1}]_+ = 0 \text{ and } U^{\lambda_1} \geq U^{\lambda_2}.$$

Finally, passing to the limit as $\lambda_2 \to \infty$ (for a converging subsequence), we deduce that

$$U^{\lambda_1} \geq U^{\lambda_2} \geq U.$$

We complete the proof. □

Using Lemma 4.1 and Theorem 4.3 and noting that the transformation $u_i(S,t) = e^{\beta t}(V_0(S) - V_i(S,t))$ in (4), we can see that the power penalized approach (16) solves the following DCPs

$$\begin{cases} L_i V^\lambda \geq 0, \\ V_i^\lambda - V^* \leq \frac{C}{\lambda^k}, \\ L_i V^\lambda \cdot \left(V_i^\lambda - V^* - \frac{C}{\lambda^k}\right) = 0, \end{cases}$$

for $i = 1, 2$, which is an intuitive approximation of the original DCPs (1). We also have the sequence $\{V^\lambda\}$ is monotonically increasing when the penalty parameter $\lambda \to \infty$ and bounded above by the solution V of the DCPs (1).

5 Numerical Experiments

From the previous analysis, we establish a desirable theoretical result that the power penalty approach to the complementary system has a very sharp convergence rate with respect to the penalty parameter λ. In this section, we numerically verify this convergence rate via pricing an American option under two regimes switching. To this end, we first give a full discretization scheme of the penalized equation (16) - (17).

5.1 The Discretization Method

For brevity, we will omit the superscript λ in the discussions given below. But keep in mind that we refer V to as the solution to the penalized equation (16) - (17) rather than to the original complementarity problem (1). We apply the fitted finite volume scheme [15] to the space-discretization and a full implicit scheme in time-discretization. As the discretization of (16) - (17) is almost identical to that in [16], except the power penalty term, we omit the details and only give the final discrete form here.

We define a space partition of $I = [0, X]$ as $\Delta S_i =]S_i, S_{i+1}[$, $i = 0, \ldots N$ with $S_0 = 0$ and $S_{N+1} = X$, and a time partition of $]0, T[$ as a uniform mesh with mesh points

$\tau_n = n\Delta\tau$ for $n = 0, 1, ..., L$, where $\Delta\tau = T/L$. Also, we let $S_{i-1/2} = (S_{i-1} + S_i)/2$ and $S_{i+1/2} = (S_i + S_{i+1})/2$ for each $i = 2, \ldots, N$. These intervals $J_i =]S_{i-1/2}, S_{i+1/2}[$, $i = 0, ...N$, form a second partition of $I = [0, X]$ if we define $S_{-1/2} = S_0$ and $S_{N+1/2} = S_N$. Let $V_{1,i}^n$ and $V_{2,i}^n$ denote the approximation of $V_1(S_i, \tau_n)$ and $V_2(S_i, \tau_n)$, respectively. The fully implicit time-stepping scheme, coupled with the fitted finite volume discretization on space partitions, yields a fully discrete coupled system as follows:

$$[\boldsymbol{I} + \theta \boldsymbol{M}] \boldsymbol{V}^{n+1} - \lambda\Delta\tau \left[\boldsymbol{\Lambda} - \boldsymbol{V}^{n+1}\right]_+^{1/k} = [\boldsymbol{I} - (1-\theta)\boldsymbol{M}] \boldsymbol{V}^n + \boldsymbol{R}^n, \tag{35}$$

where $[\cdot]_+^{1/k}$ is a Hadamard power, $\boldsymbol{I}$ denotes the $(2N-2)\times(2N-2)$ unit matrix and

$$\boldsymbol{M} = \Delta\tau \begin{bmatrix} M_1 & q_1 I \\ q_2 I & M_2 \end{bmatrix}, \quad \boldsymbol{V}^n = \begin{bmatrix} V_1^n \\ V_2^n \end{bmatrix}, \quad \boldsymbol{\Lambda} = \begin{bmatrix} V^* \\ V^* \end{bmatrix}, \quad \boldsymbol{R}^n = \begin{bmatrix} R_1^n \\ R_2^n \end{bmatrix}.$$

Here, M_1 and M_2 denote two $(N-1)\times(N-1)$ matrices given by

$$M_1 = \begin{bmatrix} \gamma_{1,1} & \beta_{1,1} & & \\ \alpha_{1,2} & \gamma_{1,2} & \beta_{1,2} & \\ & \ddots & \ddots & \ddots \\ & & \alpha_{1,N-1} & \gamma_{1,N-1} \end{bmatrix}, \quad M_2 = \begin{bmatrix} \gamma_{2,1} & \beta_{2,1} & & \\ \alpha_{2,2} & \gamma_{2,2} & \beta_{2,2} & \\ & \ddots & \ddots & \ddots \\ & & \alpha_{2,N-1} & \gamma_{2,N-1} \end{bmatrix},$$

and V_i^n and R_i^n $(i = 1, 2)$ denote vectors given by

$$\begin{aligned} V_1^n &= [V_{1,1}^n, \cdots, V_{1,N-1}^n]^\top, \quad R_1^n = \left[\alpha_{1,1} V_{1,0}^n, 0, \cdots 0, \beta_{1,N-1} V_{1,N}^n\right]^\top, \\ V_2^n &= [V_{2,1}^n, \cdots, V_{2,N-1}^n]^\top, \quad R_2^n = \left[\alpha_{2,1} V_{2,0}^n, 0, \cdots 0, \beta_{2,N-1} V_{2,N}^n\right]^\top. \end{aligned}$$

In the above notations, we have for $j = 1, 2$

$$\alpha_{j,1} = \frac{S_1}{4l_1}(a_j - b_j), \quad \beta_{j,1} = \frac{b_j S_{3/2} S_2^{\eta_j}}{\left(S_2^{\eta_j} - S_1^{\eta_j}\right) l_1},$$

$$\gamma_{j,1} = -\frac{S_1}{4l_1}(a_j + b_j) - \frac{b_j S_{3/2} S_1^{\eta_j}}{(S_2^{\eta_1} - S_1^{\eta_1})\, l_1} - c_j,$$

and for $j = 1, 2$ and $i = 2, \ldots, N-1$

$$\alpha_{j,i} = \frac{b_j S_{i-1/2} S_{i-1}^{\eta_j}}{\left(S_i^{\eta_1} - S_{i-1}^{\eta_1}\right) l_i}, \quad \beta_{j,i} = \frac{b_j S_{i+1/2} S_{i+1}^{\eta_j}}{\left(S_{i+1}^{\eta_j} - S_i^{\eta_j}\right) l_i},$$

$$\gamma_{j,i} = -\frac{b_j S_{i-1/2} S_i^{\eta_j}}{\left(S_i^{\eta_j} - S_{i-1}^{\eta_j}\right) l_i} - \frac{b_j S_{i+1/2} S_i^{\eta_j}}{\left(S_{i+1}^{\eta_1} - S_i^{\eta_1}\right) l_i} - c_j,$$

where $a_j = \sigma_j^2/2$, $b_j = r_j - \sigma_j^2$, $c_j = r_i + b_j + q_j$ and $\eta_j = b_j/a_j$.

We comment that in (35) the Dirichlet boundary conditions (17) at $S = 0$ and X have been incorporated. Also, the initial condition is incorporated as the payoff function.

After giving the discrete system (35), we now present an iterative method to solve it numerically. Due to the power penalty term, (35) is nonlinear and nonsmooth, which make the classical Newton method inapplicable. To overcome this difficulty, we apply a nonlinear Jocobi method developed in [17] for the solution of the discrete system (35). Denoting by

$$\boldsymbol{A} = [\boldsymbol{I} + \theta \boldsymbol{M}] \quad \text{and} \quad \boldsymbol{f} = [\boldsymbol{I} - (1-\theta)\, \boldsymbol{M}]\, \boldsymbol{V}^n + \boldsymbol{R}^n,$$

we give the following decomposition

$$\boldsymbol{A} = \boldsymbol{D} + \boldsymbol{B},$$

where $\boldsymbol{D} = \operatorname{diag}(a_{11}, a_{22}, \ldots, a_{2N-2,2N-2})$ is the diagonal matrix composed of the diagonal elements of $\boldsymbol{A}$. With these notations, the nonlinear Jacobi iteration method is stated as the following algorithm.

Algorithm 1. *(Nonlinear Jacobi Algorithm)*

- *Step 1: Let* $n = 0$;
- *Step 2: Set* $l = 0, \widehat{\boldsymbol{V}}^0 = \boldsymbol{V}^n$, *where* $\widehat{\boldsymbol{V}}^0 = (\widehat{V}_1^0, \widehat{V}_2^0)^\top$ *with* $\widehat{V}_1^0 = V_1^n$, $\widehat{V}_2^0 = V_2^n$;
- *Step 3: Solve the following system of* n *one-dimensional nonlinear equations*

$$a_{ii}\widehat{\boldsymbol{V}}^{l+1} - \lambda\Delta t \left[\Lambda_i - \widehat{\boldsymbol{V}}^{l+1}\right]_+^{1/k} = \boldsymbol{f}_i - \left(\boldsymbol{B}\widehat{\boldsymbol{V}}^l\right)_i;$$

- *Step 4: If* $\max\limits_{1\le i\le 2N-2} \frac{\left|\widehat{\boldsymbol{V}}_i^{l+1} - \widehat{\boldsymbol{V}}_i^l\right|}{\max\left(1, \left|\widehat{\boldsymbol{V}}_i^{l+1}\right|\right)} < \mathrm{tol}$, *then stop. Otherwise, set* $l := l+1$ *and go to Step* 3.
- *Step 5: Set* $\boldsymbol{V}^{n+1} = \widehat{\boldsymbol{V}}^l$ *and* $n := n+1$ *and go to Step* 2.

5.2 Numerical Examples

In this subsection, we numerically solve an American put option under two regimes switching with parameters given in Table 1 by l_1 and $l_{1/2}$ penalty methods, respectively.

Table 1: Data used to value American options under regime switching

Regime 1	$q_1 = 6$	$r_1 = 10\%$	$\sigma_1 = 0.8$
Regime 2	$q_2 = 9$	$r_2 = 5\%$	$\sigma_2 = 0.3$
$T = 1$		$K = 9$	

For the put option with the parameters in Table 1, we choose $X = S_{\max} = 50$ to ensure the desirable accuracy. We keep the grid at a fixed uniform partitions of the solution domain $[0, X]\times]0, T[$ with $M = 450$, $N = 1000$. In Algorithm 1, the 'tol' is chosen to be 10^{-6}. All the numerical experiments were carried out under Matlab 2016a Environment on a Dual Core2 2.0GHz workstation.

To examine the convergence rate of Algorithm 1 with respect to the penalty parameter λ, we choose an increasing sequence of penalty parameters in Algorithm 1 by successively doubling the penalty parameters. As analytical solution is unavailable, we use the solution computed by $l_{1/2}$ penalty method with the largest penalty parameter $\lambda = 256$ as the 'exact solution'. Then, we compute the following ratios of the numerical solutions of the consecutive penalty parameters:

$$\text{Rate} = \log_2 \frac{E^\lambda}{E^{2\lambda}},$$

where $E^\lambda = \sum_{j=1}^{2}(||V_j^\lambda - V_j||_{L^\infty(0,T;L^2(I))} + ||V_j^\lambda - V_j||_{L^2(0,T;H_0^1(I))})$ denotes the discrete error computed by the power penalty method with penalty parameter λ. All the computed results are listed in Table 2.

Table 2: Computed results by power penalty methods with fully implicit scheme on a uniform mesh with $M = 450$ and $N = 1000$.

	$l_{1/2}$ penalty method				l_1 penalty method			
λ	$V_1(K,T)$	$V_2(K,T)$	E^λ	Rate	$V_1(K,T)$	$V_2(K,T)$	E^λ	Rate
2^2	1.965111	1.875751	0.126334		1.935924	1.848558	0.149213	
2^3	1.969547	1.880041	0.061626	2.1	1.950162	1.861520	0.077461	1.9
2^4	1.970795	1.881259	0.029771	2.1	1.959700	1.870515	0.042007	1.8
2^5	1.971119	1.881577	0.013189	2.2	1.965186	1.875794	0.023980	1.7
2^6	1.971199	1.881656	0.006615	2.0	1.968135	1.878759	0.015128	1.6
2^7	1.971219	1.881675	0.003214	2.0	1.969667	1.880155	0.009752	1.5

In view of Table 2, we get several conclusions. First, the columns '$V_1(K,T)$' and '$V_2(K,T)$' in Table 2 indicates that both the l_1 and $l_{1/2}$ penalty methods converge monotonically under the given tolerance, which is consistent with the theoretical results in Theorem 4.3. Second, the columns 'Error' show that with the same penalty parameters the $l_{1/2}$ penalty method gives more accurate pricing results than l_1 penalty method does. Finally, the columns 'Rate' suggest that the errors of the l_1 and $l_{1/2}$ penalty methods are approximately of order $\mathcal{O}(1/\sqrt{\lambda})$ and $\mathcal{O}(1/\lambda)$ respectively, confirming the theoretical findings in Theorem 4.2.

We also plot the option values and Greeks (Delta and Gamma) of the numerical examples computed by $l_{1/2}$ penalty method in Figures 1-3, respectively. The figures show that numerical solutions computed by the the power penalty method is qualitatively very good. This shows that the power penalty method works very well.

Figure 1: Option values under regimes 1 and 2, computed by $l_{1/2}$ penalty method with penalty parameter $\lambda = 256$. The fully implicit scheme is used on a uniform mesh with $M = 450$ and $N = 1000$.

Figure 2: Deltas under regimes 1 and 2, computed by $l_{1/2}$ penalty method with penalty parameter $\lambda = 256$. The fully implicit scheme is used on a uniform mesh with $M = 450$ and $N = 1000$.

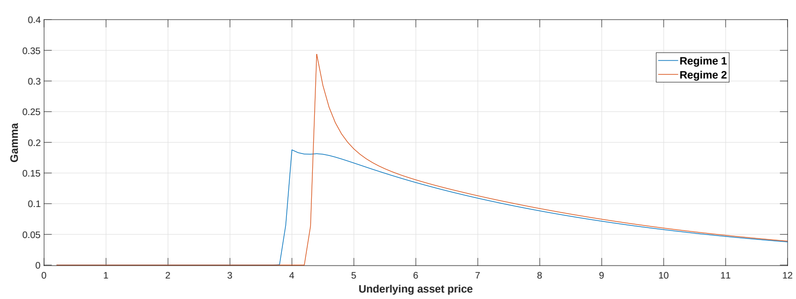

Figure 3: Gammas under regimes 1 and 2, computed by $l_{1/2}$ penalty method with penalty parameter $\lambda = 256$. The fully implicit scheme is used on a uniform mesh with $M = 450$ and $N = 1000$.

6 Conclusions

We have studied the convergence property of the power penalty method for pricing American options under regime switching. Using the equivalence of the system of DCPs and differential variational inequalities, we have developed a power penalty method to approximate the system of DCPs. The solvability of the power penalty problem and convergence properties of the power penalty method were established as well. Specifically, We have shown that the solution to the system of penalized nonlinear equations converges to that of the original DCPs at an exponential convergence rate with respect to the penalty parameter. We have also demonstrated the monotonicity of convergence of the penalization. Finally, numerical experiment showed that the numerical rate of convergence of the power penalty method confirms the theoretical result.

The authors would like to thank the anonymous referees and the editor for their helpful comments and suggestions toward the improvement of this paper. Kai Zhang wishes to thank the supports from the MOE Project of Key Research institute of Humanities and Social Sciences at Universities (Grant No. 14JJD790041). Project 11001178 partially supported by National Natural Science Foundation of China.

References

[1] Black, F., Scholes, M.: The pricing of options and corporate liabilities. Journal of Political Economy **81**(3), 637–654 (1973)

[2] Cont, R.: Volatility clustering in financial markets: empirical facts and agent-based models. Long memory in economics **2**, 289–309 (2007)

[3] Andersen, T.G., Bondarenko, O., Todorov, V., Tauchen, G.: The fine structure of equity-index option dynamics. Journal of Econometrics **187**(2), 532–546 (2014)

[4] Wilmott, P.: Paul Wilmott on Quantitative Finance. Wiley, New York (2000)

[5] Buffington, J., Elliott, R.J.: American options with regime switching. International Journal of Theoretical and Applied Finance **5**(05), 497–514 (2002)

[6] Dai, M., Zhang, Q., Zhu, Q.J.: Trend following trading under a regime switching model. SIAM Journal on Financial Mathematics **1**(1), 780–810 (2010)

[7] Achdou, Y., Pironneau, O.: Computational Methods for Option Pricing, Frontiers in Applied Mathematics. SIAM, Philadelphia (2005)

[8] Jaillet, P., Lamberton, D., Lapeyre, B.: Variational inequalities and the pricing of American options. Acta Applicandae Mathematicae **21**(3), 263–289 (1990)

[9] Yang, H.: A numerical analysis of American options with regime switching. Journal of Scientific Computing **44**(1), 69–91 (2010)

[10] Rubinov, A.M., Yang, X.: Lagrange-type Functions in Constrained Non-convex Optimization, vol. 85. Kluwer Academic Publishers, Boston (2003)

[11] Wang, S., Yang, X.Q., Teo, K.L.: Power penalty method for a linear complementarity problem arising from American option valuation. Journal of Optimization Theory and Applications **129**(2), 227–254 (2006)

[12] Zhang, K., Yang, X., Teo, K.L.: Convergence analysis of a monotonic penalty method for American option pricing. Journal of Mathematical Analysis and Applications **348**(2), 915–926 (2008)

[13] Kwok, Y.K.: Mathematical Models of Financial Derivatives. Springer, Berlin (1998)

[14] Glowinski, Roland: Numerical Methods for Nonlinear Variational Problems. Springer-Verlag, New York (1984)

[15] Wang, S.: A novel fitted finite volume method for the Black-Scholes equation governing option pricing. Ima Journal of Numerical Analysis **24**(4), 699–720 (2004)

[16] Zhang, K., Teo, K.L., Swartz, M.: A robust numerical scheme for pricing American options under regime switching based on penalty method. Computational Economics **3**(4), 463–483 (2014)

[17] Sun, Z., Liu, Z., Yang, X.: On power penalty methods for linear complementarity problems arising from American option pricing. Journal of Global Optimization **63**(1), 165–180 (2015)