On Error Bound Moduli for Locally Lipschitz and Regular Functions

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Abstract: In this paper we study local error bound moduli for a locally Lipschitz and regular function via its outer limiting subdifferential set. We show that the distance from 0 to the outer limiting subdifferential of the support function of the subdifferential set, which is essentially the distance from 0 to the end set of the subdifferential set, is an upper estimate of the local error bound modulus. This upper estimate becomes tight for a convex function under some regularity conditions. We show that the distance from 0 to the outer limiting subdifferential set of a lower $\mathcal{C}^1$ function is equal to the local error bound modulus.

Keywords: error bound modulus, locally Lipschitz, outer limiting subdifferential, support function, end set, lower $\mathcal{C}^1$ function

1 Introduction

Error bounds play a key role in variational analysis. They are of great importance for subdifferential calculus, stability and sensitivity analysis, exact penalty functions, optimality conditions, and convergence of numerical methods, see the excellent survey papers [2, 19, 24] for more details. It should be noticed that the notion of error bounds is closely related to some other important concepts: weak sharp minima, calmness and metric subregularity, see [3, 4, 7, 14, 16, 23, 26].

In this paper, we study local error bound moduli in finite dimensional spaces. We say that a function $\phi : \mathbb{R}^n \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ has a local error bound at $\bar{x} \in [\phi \le 0]$ if there exist some $\tau > 0$ and some neighborhood U of $\bar{x}$ such that

$$\tau d(x, [\phi \le 0]) \le \phi(x)_+ \ \forall x \in U, \tag{1}$$

where $[\phi \le 0] := \{x \in \mathbb{R}^n | \phi(x) \le 0\}$ and $t_+ := \max\{t, 0\}$ for all $t \in \mathbb{R}$. The supremum of all possible constants τ in (1) (for some associated U) is called the local error bound modulus of ϕ at $\bar{x}$, denoted by $\mathrm{ebm}(\phi, \bar{x})$. We define $\mathrm{ebm}(\phi, \bar{x})$ as 0 if ϕ does not have a local error bound at $\bar{x}$. Clearly, the local error bound modulus of ϕ at $\bar{x}$ can be alternatively defined as

$$\mathrm{ebm}(\phi, \bar{x}) = \liminf_{x \to \bar{x}, \phi(x) > 0} \frac{\phi(x)}{d(x, [\phi \le 0])}.$$

We know from the literature [9, 15, 17] that the distance from 0 to the outer limiting subdifferential of a lower semicontinuous (lsc) function ϕ at $\bar{x}$, is a lower estimate of $\mathrm{ebm}(\phi, \bar{x})$, which becomes tight when ϕ is convex.

In this paper we consider the local error bound modulus of a locally Lipschitz and regular function ϕ and establish that the distance from 0 to the outer limiting subdifferential of the support function of the subdifferential $\partial\phi(\bar{x})$ at 0 is an upper estimate of $\mathrm{ebm}(\phi, \bar{x})$. We also investigate the geometric structure of this outer limiting subdifferential and show that it is equal to the closure of the end set of the subdifferential $\partial\phi(\bar{x})$, while the closure is surplus when the subdifferential set is a polyhedron. Thus the upper estimate is essentially the distance from 0 to the end set of $\partial\phi(\bar{x})$. We prove that, for convex function ϕ, under the Abadie's constraint qualification [20] and the assumption on exactness of tangent approximations [22] (see Remark 3.4), the upper estimate is tight. To the best of our knowledge, the first result of the kind is that Hu [13] proved for sublinear function ϕ that, the $\mathrm{ebm}(\phi, 0)$ is equal to the distance from 0 to the end set of $\partial\phi(0)$.

For lower $\mathcal{C}^1$ function ϕ, we show that the distance from 0 to the outer limiting subdifferential of ϕ at $\bar{x}$ is equal to $\mathrm{ebm}(\phi, \bar{x})$. This generalizes the corresponding results in [9, 15, 17] for convex function ϕ.

Throughout the paper we use the standard notations of variational analysis; see the seminal book [26] by Rockafellar and Wets. Let $A \subset \mathbb{R}^n$. We denote the interior, the closure, the boundary, the convex hull and the positive hull of A respectively by $\mathrm{int}\, A$, $\mathrm{cl}\, A$, $\mathrm{bdry}\, A$, $\mathrm{conv}\, A$ and $\mathrm{pos} A := \{0\} \cup \{\lambda x | x \in A \text{ and } \lambda > 0\}$.

The Euclidean norm of a vector x is denoted by $||x||$, and the inner product of vectors x and y is denoted by $\langle x, y \rangle$. Let $B(x, \varepsilon)$ be a closed ball centered at x with the radius

$\varepsilon > 0$. We say that A is locally closed at a point $x \in A$ if $A \cap U$ is closed for some closed neighborhood U of x. The polar cone of A is defined by

$$A^* := \{v \in \mathbb{R}^n | \langle v, x\rangle \leq 0 \ \forall x \in A\}.$$

The support function $\sigma_A : \mathbb{R}^n \to \overline{\mathbb{R}}$ of A is defined by

$$\sigma_A(w) := \sup_{x \in A} \langle x, w\rangle.$$

For a closed and convex set A with $0 \in A$, the gauge of A is the function $\gamma_A : \mathbb{R}^n \to \overline{\mathbb{R}}$ defined by

$$\gamma_A(x) := \inf\{\lambda \geq 0 | x \in \lambda A\}.$$

The distance from x to A is defined by

$$d(x, A) := \inf_{y \in A} ||y - x||.$$

For $A = \emptyset$, we define $d(x, A) = +\infty$. The projection mapping P_A is defined by

$$P_A(x) := \{y \in A | ||y - x|| = d(x, A)\}.$$

Let $x \in A$. We use $T_A(x)$ to denote the tangent cone to A at x, i.e. $w \in T_A(x)$ if there exist sequences $t_k \downarrow 0$ and $\{w_k\} \subset \mathbb{R}^n$ with $w_k \to w$ and $x + t_k w_k \in A \ \forall k$. We denote by $N_A^P(x)$ the proximal normal cone to A at x, i.e., $v \in N_A^P(x)$ if there exists some $t > 0$ such that $x \in P_A(x + tv)$. The regular normal cone $\hat{N}_A(x)$ to A at x is the polar cone of $T_A(x)$. A vector $v \in \mathbb{R}^n$ belongs to the normal cone $N_A(x)$ to A at x, if there exist sequences $x_k \to x$ and $v_k \to v$ with $x_k \in A$ and $v_k \in \hat{N}_A(x_k)$ for all k. The set A is said to be regular at x in the sense of Clarke if it is locally closed at x and $\hat{N}_A(x) = N_A(x)$.

Let $g : \mathbb{R}^n \to \overline{\mathbb{R}}$ be an extended real-valued function and x a point with $g(x)$ finite. We denote by $\ker g := \{x \in \mathbb{R}^n | g(x) = 0\}$ the kernel of g. The epigraph of g is the set

$$\mathrm{epi} g := \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} | g(x) \leq \alpha\}.$$

It is well known that g is lower semicontinuous (lsc) if and only if epi$g$ is closed. The vector $v \in \mathbb{R}^n$ is a regular subgradient of g at x, written $v \in \widehat{\partial} g(x)$, if

$$g(y) \geq g(x) + \langle v, y - x\rangle + o(||y - x||)$$

or equivalently,

$$\liminf_{y \to x,\, y \neq x} \frac{g(y) - g(x) - \langle v, y - x\rangle}{\|y - x\|} \geq 0.$$

The vector $v \in \mathbb{R}^n$ is a (general) subgradient of g at x, written $v \in \partial g(x)$, if there exist sequences $x_k \to x$ and $v_k \to v$ with $g(x_k) \to g(x)$ and $v_k \in \widehat{\partial} g(x_k)$. The outer limiting subdifferential of g at $\bar{x}$ is defined in [9, 15, 17] by

$$\partial^{>} g(\bar{x}) = \{\lim_{k\to+\infty} v_k \mid \exists x_k \underset{g}{\longrightarrow} \bar{x},\, g(x_k) > g(\bar{x}),\, v_k \in \partial g(x_k)\}.$$

The subderivative function $dg(x) : \mathbb{R}^n \to \overline{\mathbb{R}}$ is defined by

$$dg(x)(w) := \liminf_{t\downarrow 0, w'\to w} \frac{g(x+tw') - g(x)}{t} \;\forall w \in \mathbb{R}^n.$$

Note that the subderivative $dg(x)$ is a lsc and positively homogeneous function and that the regular subdifferential set can be derived from the subderivative as follows:

$$\widehat{\partial} g(x) = \{v \in \mathbb{R}^n | \langle v, w\rangle \leq dg(x)(w) \;\forall w \in \mathbb{R}^n\}.$$

The function g is said to be (subdifferentially) regular at $x \in \mathbb{R}^n$ if epi$g$ is regular in the sense of Clarke at $(x, g(x))$ as a subset of $\mathbb{R}^n \times \mathbb{R}$.

For a sequence $\{A_k\}$ of subsets of $\mathbb{R}^n$, the outer limit $\limsup_{k\to\infty} A_k$ is the set consisting of all possible cluster points of sequences x_k with $x_k \in A_k$ for all k, whereas the inner limit $\liminf_{k\to\infty} A_k$ is the set consisting of all possible limit points of such sequences. $\{A_k\}$ is said to converge to $A \subset \mathbb{R}^n$ in the sense of Painlevé-Kuratowski, written $A_k \to A$, if

$$\limsup_{k\to\infty} A_k = \liminf_{k\to\infty} A_k = A.$$

For a set-valued mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and a point $\bar{x} \in \mathbb{R}^n$, the outer limit of S at $\bar{x}$ is defined by

$$\limsup_{x\to\bar{x}} S(x) := \{u \in \mathbb{R}^m | \exists x_k \to \bar{x}, \exists u_k \to u \text{ with } u_k \in S(x_k)\}.$$

S is outer semicontinuous (osc, for short) at $\bar{x}$ if and only if

$$\limsup_{x\to\bar{x}} S(x) \subset S(\bar{x}).$$

Let gph$S$ denote the graph of S. We recall that S is said to be calm at $(\bar{x}, \bar{y}) \in$ gph$S$ if there exist a constant $\alpha > 0$ and neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$d(y, S(\bar{x})) \leq \alpha ||x - \bar{x}|| \;\forall x \in U \text{ and } y \in S(x) \cap V.$$

The infimum of all possible constants α (for some associated U and V) is called the calmness modulus of S at $(\bar{x}, \bar{y})$, denoted as clm$S(\bar{x}, \bar{y})$, defined as $+\infty$ if S is not calm at $(\bar{x}, \bar{y})$.

A face of a convex set A is a convex subset A' of A such that every closed line segment in A with a relative interior point in A' has both endpoints in A'. An exposed face of A is the intersection of A and a non-trivial supporting hyperplane to A. See [25]. For a nonempty and convex set $A \subset \mathbb{R}^n$, the end set of A is defined in [11, 12] by

$$\mathrm{end}(A) := \{x \in \mathrm{cl}A | tx \notin \mathrm{cl}A \ \forall t > 1\}.$$

2 Support Function and Its Outer Limiting Subdifferential

In this section, we study the outer limiting subdifferential of the support function of a general compact and convex set and show that it is the closure of its end set. In the next section, we shall apply these results to study the error bound modulus for a locally Lipschitz and regular function as its subdifferential set is compact and convex. It is worth noting that the results presented in this section have their own interest in the field of convex analysis and optimization.

Let C be a compact and convex subset of $\mathbb{R}^n$. To begin with, we note from [26, Theorem 8.24, Proposition 8.29, Corollary 8.25] that

$$C = \{v \in \mathbb{R}^n \mid \langle v, w\rangle \leq \sigma_C(w) \ \forall w\} \tag{2}$$

and

$$\partial\sigma_C(w) = \arg\max_{v\in C}\langle v, w\rangle = C \cap \{v \in \mathbb{R}^n \mid \langle v, w\rangle = \sigma_C(w)\}. \tag{3}$$

Since C is compact and convex, we have $\sigma_C(w) < +\infty$ and $\partial\sigma_C(w) \neq \emptyset$ for all $w \in \mathbb{R}^n$.

We first consider the case that $0 \in C$ and then the general case that C may not contain 0. Some basic properties of the end set of C are listed in the following lemma.

Lemma 2.1 *If $C \subset \mathbb{R}^n$ is compact and convex with $0 \in C$, then the following properties hold:*

(i) $\mathrm{end}(C) \cap \mathrm{ri}\, C = \emptyset$*;*

(ii) $C = \bigcup_{v\in\mathrm{end}(C)}[0, v]$*;*

(iii) *For a subset $E \subset C$, $C = \bigcup_{v\in E}[0, v]$ if and only if $\mathrm{end}(C) \subset E$;*

(iv) *F is a nonempty exposed face of C if and only if $F = \partial\sigma_C(w)$ for some $w \neq 0$.*

Proof. (i) Suppose by contradiction that $v \in \mathrm{end}(C) \cap \mathrm{ri}\, C$. By the relative interior criterion [26, Exercise 2.41], there must exist some $v' \in C$ such that $v \in \mathrm{ri}[0, v']$, which contradicts to the fact that $v \in \mathrm{end}(C)$.

(ii) This equality holds because C is convex and compact with $0 \in C$.

(iii) The 'if' part is trivial due to (ii). As for the 'only if' part, we only need to show $\mathrm{end}(C) \subset E$. Let $v \in \mathrm{end}(C)$. Since C is compact, it is clear that $v \in C$. By $C = \bigcup_{v \in E}[0, v]$, there exists some $v' \in E$ such that $v \in [0, v']$. By the definition of the end set, we have $v = v'$. This entails that $\mathrm{end}(C) \subset E$.

(iv) Clearly, any $\partial\sigma_C(w)$ with $w \neq 0$ is an exposed face of C. Conversely, if $F \neq \emptyset$ is exposed in C, then by definition there exist some $w \neq 0$ and $\alpha \in \mathbb{R}$ such that $F = C \cap \{v \in \mathbb{R}^n \mid \langle w, v\rangle = \alpha\}$ and $C \subset \{v \in \mathbb{R}^n \mid \langle w, v\rangle \leq \alpha\}$. The latter inclusion holds if and only if $\sigma_C(w) \leq \alpha$. In view of (2) and the fact that $F \subset C$, we have $\alpha = \langle w, v\rangle \leq \sigma_C(w)$ for each $v \in F$. This entails that $\alpha = \sigma_C(w)$. In view of (3), we have $F = \partial\sigma_C(w)$. The proof is completed. □

Throughout this section, we use the following notation:

$$S := \bigcup_{\sigma_C(w) > 0} \partial\sigma_C(w).$$

According to Lemma 2.1 (iv), S is the union of all the exposed faces $\partial\sigma_C(w)$ of C with $\sigma_C(w) > 0$.

In next lemma, we prove that $\mathrm{end}(C)$ is sandwiched between S and its closure, the latter being exactly the same with the outer limiting subdifferential $\partial^{>}\sigma_C(0)$.

Lemma 2.2 *If $C \subset \mathbb{R}^n$ is compact and convex with $0 \in C$, then*

$$S \subset \mathrm{end}(C) = \gamma_C^{-1}(1) \subset \partial^{>}\sigma_C(0) = \mathrm{cl}\, S,$$

entailing that

$$\mathrm{cl}(\mathrm{end}(C)) = \mathrm{cl}(\gamma_C^{-1}(1)) = \partial^{>}\sigma_C(0).$$

Proof. First, we show $\mathrm{end}(C) = \gamma_C^{-1}(1)$ and $\mathrm{cl}\, S = \partial^{>}\sigma_C(0)$. The first equality follows from the definitions of the end set and the gauge function, while the second equality follows readily from the positive homogeneity of σ_C and the definition of outer limiting subdifferential.

Next, we show $S \subset \mathrm{end}(C)$. Let $v \in S$, i.e., $v \in \partial\sigma_C(w)$ for some $w \in \mathbb{R}^n$ with $\sigma_C(w) > 0$. In view of (3), we have $v \in C$ and $\langle v, w\rangle = \sigma_C(w)$, implying that $\langle tv, w\rangle > \sigma_C(w)$ for all $t > 1$. By (2), we have $tv \notin C$ for all $t > 1$. That is, $v \in \mathrm{end}(C)$.

Finally we show $\mathrm{end}(C) \subset \mathrm{cl}\, S$. To begin with, we show that $\mathrm{end}(C) \subset S \cup S^0$, where

$$S^0 = \bigcup_{\sigma_C(w)=0,\ \partial\sigma_C(w)\neq C} \partial\sigma_C(w). \tag{4}$$

Let $v \in \mathrm{end}(C)$. By Lemma 2.1 (i), $v \notin \mathrm{ri}\, C$. It then follows from [25, Theorem 11.6] that there exists a non-trivial supporting hyperplane H to C containing v. That is, we can find an exposed face $F := C \cap H$ of C such that $v \in F$ and $F \neq C$. By Lemma 2.1 (iv), we can find some $w \neq 0$ such that $F = \partial\sigma_C(w)$. This entails that $v \in S \cup S^0$. Therefore, we have $\mathrm{end}(C) \subset S \cup S^0$ as expected. By Lemma 2.1 (iii), we have $C = \bigcup_{v\in S\cup S^0}[0, v]$. Observing that $0 \in \partial\sigma_C(w)$ for all $w \in \mathbb{R}^n$ with $\sigma_C(w) = 0$, we have $\bigcup_{v\in S^0}[0, v] = S^0$. This entails that $C = A \cup S^0$, where $A := \bigcup_{v\in S}[0, v]$. Since each $\partial\sigma_C(w)$ with $\sigma_C(w) = 0$ and $\partial\sigma_C(w) \neq C$ is a non-trivial exposed face of C, we confirm that $\mathrm{ri}\, C \cap S^0 = \emptyset$ (implying that $\mathrm{ri}\, C \subset A$ and hence $C \subset \mathrm{cl}\, A$). Clearly, we have $A \subset C$ and hence $\mathrm{cl}\, A \subset C$. That is, we have $C = \mathrm{cl}\, A$. On the basis of the fact that $S \subset C$ is bounded, it's easy to verify that $\mathrm{cl}\, A = \bigcup_{v\in \mathrm{cl}\, S}[0, v]$. Thus, we have $C = \bigcup_{v\in\mathrm{cl}\, S}[0, v]$. By Lemma 2.1 (iii) again, we have $\mathrm{end}(C) \subset \mathrm{cl}\, S$.

To sum up, we have shown $S \subset \mathrm{end}(C) = \gamma_C^{-1}(1) \subset \partial^{>}\sigma_C(0) = \mathrm{cl}\, S$, which clearly implies that $\mathrm{cl}(\mathrm{end}(C)) = \mathrm{cl}(\gamma_C^{-1}(1)) = \partial^{>}\sigma_C(0)$. The proof is completed. □

Remark 2.1 *The closure operation in the equality* $\partial^{>}\sigma_C(0) = \mathrm{cl}(\mathrm{end}(C))$ *cannot in general be dropped, because* $\partial^{>}\sigma_C(0)$ *is always closed but* $\mathrm{end}(C)$ *may not be closed, taking for example the simple set* $C = \{x \in \mathbb{R}^2 \mid 0 \leq x_1 \leq 1, x_1^2 \leq x_2 \leq x_1\}$.

Under some further conditions on the faces of C, we show that S coincides with $\mathrm{end}(C)$.

Lemma 2.3 *Assume that* $C \subset \mathbb{R}^n$ *is compact and convex with* $0 \in C$. *If, for any* $w \in \mathbb{R}^n$ *with* $\sigma_C(w) = 0$ *and* $\partial\sigma_C(w) \neq C$, *all the faces of* $\partial\sigma_C(w)$ *that do not contain 0 are exposed in* C, *then*

$$\mathrm{end}(C) = S.$$

In particular, if C *is a polyhedral set, then*

$$S = \mathrm{end}(C) = \gamma_C^{-1}(1) = \partial^{>}\sigma_C(0) = \mathrm{cl}\, S,$$

implying that the sets S, $\mathrm{end}(C)$ *and* $\gamma_C^{-1}(1)$ *are all closed.*

Proof. We first show the equality $\mathrm{end}(C) = S$ under the assumed conditions on the faces of C. Let $v \in \mathrm{end}(C) \cap S^0$, where S^0 is given by (4). By the definition of S^0, there exists some $w \in \mathbb{R}^n$ with $\sigma_C(w) = 0$ and $\partial\sigma_C(w) \neq C$ such that $v \in \partial\sigma_C(w)$. By [25, Theorem 18.2], there exists a unique face F of C such that $v \in \mathrm{ri}\, F$. By [25, Theorem 18.1], we have $F \subset \partial\sigma_C(w) \subset C$. Clearly, F is also a face of $\partial\sigma_C(w)$. We claim that $0 \notin F$, for otherwise there must exist some $v' \in F$ such that $v \in \mathrm{ri}[0, v']$ (so that $t_0 v \in F \subset C$ for some $t_0 > 1$), contradicting to the assumption that $v \in \mathrm{end}(C)$. That is, F is a face of $\partial\sigma_C(w)$ containing no 0, which is assumed to be exposed in C. It then follows from Lemma 2.1 (iv) that $F = \partial\sigma_C(w')$ for some $w' \neq 0$. As $0 \notin F$, we have $\sigma_C(w') > 0$, implying that $F \subset S$. Then, we have $v \in S$. This entails that $\mathrm{end}(C) \cap S^0 \subset S$. As we have shown in the proof of Lemma 2.2 that $\mathrm{end}(C) \subset S \cup S^0$ and $S \subset \mathrm{end}(C)$, we get the equality $\mathrm{end}(C) = S$.

To complete the proof, it suffices to note that any polyhedral set has only finitely many faces and all non-trivial faces are exposed ones. The proof is completed. □

Remark 2.2 *Without the conditions imposed on faces of C as in Lemma 2.3, the union set S may not be closed as can be seen from Example 2.1 below, demonstrating that the closure operation in the equality $\partial^{>}\sigma_C(0) = \mathrm{cl}\, S$ cannot in general be dropped, and that the equality* $\mathrm{end}(C) = S$ *does not hold in general.*

Example 2.1 *Let* $C = \mathrm{conv}(0 \cup \{x \in \mathbb{R}^2 \mid (x_1 - 1)^2 + (x_2 - 1)^2 \leq 1\})$. *Clearly, C is a compact and convex set with $0 \in C$. By some direct calculations, we have*

$$\mathrm{end}(C) = \{x \in \mathbb{R}^2 \mid (x_1 - 1)^2 + (x_2 - 1)^2 = 1, x_1 + x_2 \geq 1\}$$

and

$$S = \{x \in \mathbb{R}^2 \mid (x_1 - 1)^2 + (x_2 - 1)^2 = 1, x_1 + x_2 > 1\}.$$

Let $v = (0, 1)^T$ and $w = (-1, 0)^T$. Clearly, we have $v \in \mathrm{end}(C) \backslash S$, $\sigma_C(w) = 0$ *and* $\partial\sigma_C(w) = \{v \in \mathbb{R}^2 \mid v_1 = 0, 0 \leq v_2 \leq 1\}$. *Moreover, it is easy to verify that the singleton set $\{v\}$ is a face of $\partial\sigma_C(w)$, but is not exposed in C.*

In the following lemma, we present some equivalent conditions for the equality $\mathrm{end}(C) = \partial^{>}\sigma_C(0)$ to hold.

Lemma 2.4 *If $C \subset \mathbb{R}^n$ is compact and convex with $0 \in C$, then the following properties are equivalent:*

(i) $\mathrm{end}(C) = \partial^{>}\sigma_C(0)$*;*

(ii) $\mathrm{end}(C)$ *is closed;*

(iii) γ_C *is continuous at every $x \in \mathrm{pos}\, C$ relative to $\mathrm{pos}\, C$, i.e., for each $x_k \to x$ with $x_k \in \mathrm{pos}\, C$ for all k, it holds that $\gamma_C(x_k) \to \gamma_C(x)$.*

(iv) *C is a radiative subset of $\mathrm{pos}\, C$ in the sense of [27, Definition 4.1], i.e., $0 \in \mathrm{int}_{\mathrm{pos}\, C}\, C$ and for each $x \in \mathrm{pos}\, C$, the open ray $\{\lambda x \mid \lambda > 0\}$ does not intersect the boundary $\mathrm{bdry}_{\mathrm{pos}\, C}\, C$ more than once, where we denote by $\mathrm{int}_{\mathrm{pos}\, C}\, C$ and $\mathrm{bdry}_{\mathrm{pos}\, C}\, C$, respectively, the interior of C and its boundary in the induced topology on the subset $\mathrm{pos}\, C$ of $\mathbb{R}^n$.*

Proof. The equivalence of (i) and (ii) follows directly from Lemma 2.2, while the equivalence of (iii) and (iv) can be found in [27, Proposition 4.2]. It remains to show the equivalence of (ii) and (iii).

[(ii)$\Longrightarrow$(iii)]: Since $0 \notin \mathrm{end}(C)$ and $\mathrm{end}(C)$ is closed, we have $d(0, \mathrm{end}(C)) > 0$. From [22, Theorem 4.1], it then follows that $\mathrm{pos}\, C$ is closed, γ_C is continuous at 0 relative to $\mathrm{pos}\, C$, and there is no convergent sequence $\{x_k\} \subset \mathrm{pos}\, C$ such that $\gamma_C(x_k) \to +\infty$. Let $x \in \mathrm{pos}\, C$ with $x \neq 0$ and let $x_k \to x$ with $x_k \in \mathrm{pos}\, C$ for all k. Without loss of generality, we may assume that $x_k \neq 0$ for all k and that $\gamma_C(x_k) \to \beta$ (Note that the sequence $\gamma_C(x_k)$ is bounded). As γ_C is lower semi-continuous, we have $\beta \geq \gamma_C(x) > 0$. Since $\gamma_C(x_k/\gamma_C(x_k)) = 1$, we have $x_k/\gamma_C(x_k) \in \mathrm{end}(C)$. Since $\mathrm{end}(C)$ is assumed to be closed, we have $x_k/\gamma_C(x_k) \to x/\beta \in \mathrm{end}(C)$. Thus, we have $\gamma_C(x/\beta) = 1$ or $\beta = \gamma_C(x)$. This entails that γ_C is continuous at x relative to $\mathrm{pos}\, C$. Therefore, we have (ii)$\Longrightarrow$(iii).

[(iii)$\Longrightarrow$(ii)]: From [22, Theorem 4.1], it follows that $\mathrm{pos}\, C$ is closed and $d(0, \mathrm{end}(C)) > 0$. Let $v_k \to v$ with $v_k \in \mathrm{end}(C)$ (that is, $\gamma_C(v_k) = 1$) for all k. It's easy to verify that $v_k \in \mathrm{pos}\, C$ for all k and $v \in \mathrm{pos}\, C$. Moreover, we have $v \neq 0$, for otherwise we have $d(0, \mathrm{end}(C)) = 0$, a contradiction. By (iii), we have $\gamma_C(v_k) \to \gamma_C(v)$, implying that $\gamma_C(v) = 1$ or equivalently $v \in \mathrm{end}(C)$. This entails the closedness of $\mathrm{end}(C)$. The proof is completed. □

Now we present the results similar to the ones in Lemmas 2.2-2.4, for the case when C may not contain $0 \in \mathbb{R}^n$.

Theorem 2.1 *Let $C \subset \mathbb{R}^n$ be a compact and convex set not necessarily containing 0, and let $C' := \mathrm{conv}(C \cup \{0\})$. Then the following properties hold:*

(a) $S \subset \mathrm{end}(C) = \gamma_{C'}^{-1}(1) \subset \partial^{>}\sigma_C(0) = \mathrm{cl}\, S$, *entailing that* $\mathrm{cl}(\mathrm{end}(C)) = \mathrm{cl}(\gamma_{C'}^{-1}(1)) = \partial^{>}\sigma_C(0)$.

(b) *If, for any $w \in \mathbb{R}^n$ with $\sigma_C(w) = 0$ and $\partial\sigma_{C'}(w) \neq C'$, all the faces of $\partial\sigma_{C'}(w)$ that do not contain 0 are exposed in C', then* $\mathrm{end}(C) = S$. *In particular, if C is a polyhedral set, then* $S = \mathrm{end}(C) = \gamma_{C'}^{-1}(1) = \partial^{>}\sigma_C(0) = \mathrm{cl}\, S$, *implying that the sets S,* $\mathrm{end}(C)$ *and* $\gamma_{C'}^{-1}(1)$ *are all closed.*

(c) *The following properties are equivalent:*

- **(c1)** $\mathrm{end}(C) = \partial^{>}\sigma_C(0)$;
- **(c2)** $\mathrm{end}(C)$ *is closed;*
- **(c3)** $\gamma_{C'}$ *is continuous at every $x \in \mathrm{pos}\, C'$ relative to $\mathrm{pos}\, C'$, i.e., for each $x_k \to x$ with $x_k \in \mathrm{pos}\, C'$ for all k, it holds that $\gamma_{C'}(x_k) \to \gamma_{C'}(x)$;*
- **(c4)** *C' is a radiative subset of $\mathrm{pos}\, C'$ in the sense of [27, Definition 4.1]. See Lemma 2.4 (iv) for the description of a radiative subset.*

Proof. Clearly, $C' = \cup_{0 \le \lambda \le 1} \lambda C$ is a compact and convex set with $0 \in C'$, and C' is polyhedral if C is polyhedral. Moreover, it is easy to verify that $\sigma_{C'}(w) = \max\{\sigma_C(w), 0\}$ for all $w \in \mathbb{R}^n$, and that

$$\partial\sigma_{C'}(w) = \begin{cases} \partial\sigma_C(w) & \text{if } \sigma_C(w) > 0, \\ \cup_{0 \le \lambda \le 1} \lambda \partial\sigma_C(w) & \text{if } \sigma_C(w) = 0, \\ \{0\} & \text{if } \sigma_C(w) < 0. \end{cases}$$

This entails that $S = \cup_{\sigma_C(w) > 0} \partial\sigma_C(w) = \cup_{\sigma_{C'}(w) > 0} \partial\sigma_{C'}(w)$. By definition, we have $\mathrm{end}(C') = \mathrm{end}(C)$. All results then follow readily from Lemmas 2.2-2.4. □

By applying Theorem 2.1, we can give some formulas for calculating $\partial^{>}\sigma_C(0)$ when C is the convex hull of a compact subset of $\mathbb{R}^n$.

Corollary 2.1 *Let A be a nonempty compact subset of $\mathbb{R}^n$ such that $C = \mathrm{conv}\, A$. In terms of a collection of subsets of A defined by $\mathcal{A} := \{\arg\max_{a \in A} \langle a, w \rangle \mid \max_{a \in A} \langle a, w \rangle > 0\}$, we have*

$$\bigcup_{A' \in \mathcal{A}} \mathrm{conv}\, A' \subset \mathrm{end}(C) = \gamma_C^{-1}(1) \subset \mathrm{cl}\left(\bigcup_{A' \in \mathcal{A}} \mathrm{conv}\, A' \right) = \partial^{>}\sigma_C(0). \tag{5}$$

If A is a finite set, all the inclusions in (5) become equalities.

Proof. It suffices to show that $A' \in \mathcal{A}$ if and only if there is some $w \in \mathbb{R}^n$ such that $\sigma_C(w) > 0$ and $\operatorname{conv} A' = \partial\sigma_C(w)$, and then apply Theorem 2.1 in a straightforward way. □

Remark 2.3 *It is easy to verify that $\mathcal{A}$ can be rewritten as*

$$\{A' \subset A \mid \exists w \in \mathbb{R}^n \text{ such that} \langle a, w\rangle = 1 \, \forall a \in A',\ \langle a, w\rangle < 1 \, \forall a \in A \backslash A'\},$$

which is in the spirit of the index collection defined in Cánovas et al. [6] for the case that A is a finite set. On the other hand, when A is a finite set, the equalities in (5) provide a complete characterization of the set $\operatorname{end}(C)$. From which, it is easy to see that

$$d(0, \operatorname{end}(C)) > 0. \tag{6}$$

It is worth noting that (6) has been proved in [11, 29].

3 Main Results

Throughout this section, for a given function $\phi : \mathbb{R}^n \to \overline{\mathbb{R}}$, which is regular and locally Lipschitz continuous at $\bar{x}$, a point on the boundary of the level set $[\phi \leq 0]$, we shall conduct some variational analysis on $\operatorname{ebm}(\phi, \bar{x})$, the error bound modulus of ϕ at $\bar{x}$. We first show that the distance from 0 to $\partial^{>}\phi(\bar{x})$, the outer limiting subdifferential of ϕ at $\bar{x}$, is a lower estimate of $\operatorname{ebm}(\phi, \bar{x})$, while the distance from 0 to $\partial^{>}\sigma_{\partial\phi(\bar{x})}(0)$, the outer limiting subdifferential of $\sigma_{\partial\phi(\bar{x})}$ (the support function of $\partial\phi(\bar{x})$) at 0, is an upper estimate of $\operatorname{ebm}(\phi, \bar{x})$. We then show that the lower estimate is tight for a lower $\mathcal{C}^1$ function and the upper estimate is tight for a convex function under some regularity conditions.

To begin with, we recall that the inequality

$$\operatorname{ebm}(f, x) \geq d\left(0, \partial^{>} f(x)\right) \tag{7}$$

holds for a lsc function f on $\mathbb{R}^n$ and a point x with $f(x)$ finite, and the equality

$$\operatorname{ebm}(f, x) = d\left(0, \partial^{>} f(x)\right) \tag{8}$$

holds if, in addition, f is convex. See [9, 15, 17].

Theorem 3.1 *Consider a function $\phi : \mathbb{R}^n \to \overline{\mathbb{R}}$ and a point $\bar{x}$ on the boundary of the level set $[\phi \le 0]$. If ϕ is regular and locally Lipschitz continuous at $\bar{x}$, then*

$$d\left(0, \partial^{>}\phi(\bar{x})\right) \le \mathrm{ebm}(\phi, \bar{x}) \le d(0, \partial^{>}\sigma_{\partial\phi(\bar{x})}(0)). \tag{9}$$

Proof. In view of (7) and the fact that both $d\left(0, \partial^{>}\phi(\bar{x})\right)$ and $\mathrm{ebm}(\phi, \bar{x})$ reflect only local properties of ϕ near $\bar{x}$, we get the inequality $d\left(0, \partial^{>}\phi(\bar{x})\right) \le \mathrm{ebm}(\phi, \bar{x})$ immediately. In view of (8) and the fact that $\sigma_{\partial\phi(\bar{x})} : \mathbb{R}^n \to \mathbb{R}$ is continuous and sublinear (hence convex) as $\partial\phi(\bar{x})$ is a nonempty compact and convex set, we have $\mathrm{ebm}(\sigma_{\partial\phi(\bar{x})}, 0) = d\left(0, \partial^{>}\sigma_{\partial\phi(\bar{x})}(0)\right)$. Therefore, to show the inequality $\mathrm{ebm}(\phi, \bar{x}) \le d\left(0, \partial^{>}\sigma_{\partial\phi(\bar{x})}(0)\right)$, it suffices to show $\mathrm{ebm}(\phi, \bar{x}) \le \mathrm{ebm}(\sigma_{\partial\phi(\bar{x})}, 0)$. This can be done by establishing that if there exist some $\tau > 0$ and some neighborhood O of $\bar{x}$ such that

$$\tau d(x, [\phi \le 0]) \le \phi(x)_+ \quad \forall x \in O, \tag{10}$$

then the following condition holds:

$$\tau d(w, [h \le 0]) \le h(w)_+ \quad \forall w \in U,$$

where $h := \sigma_{\partial\phi(\bar{x})}$, and U is a neighborhood of the origin of $\mathbb{R}^n$.

Without loss of generality, we may assume that O is open and that ϕ is locally Lipschitz continuous on O. Let $U = \mathbb{R}^n$. In what follows, let $w \in \mathbb{R}^n$ be arbitrarily given with $h(w) > 0$. Since ϕ is regular and locally Lipschitz continuous at $\bar{x}$, we get from [26, Corollary 8.19 and Exercise 9.15] that

$$\lim_{t \downarrow 0} \frac{\phi(\bar{x} + tw) - \phi(\bar{x})}{t} = d\phi(\bar{x})(w) = \sigma_{\partial\phi(\bar{x})} = h(w). \tag{11}$$

As $\bar{x}$ is on the boundary of the level set $[\phi \le 0]$ and ϕ is locally Lipschitz continuous at $\bar{x}$, we have $\phi(\bar{x}) = 0$ and thus

$$h(w) = \lim_{t \downarrow 0} \frac{\phi(\bar{x} + tw)}{t} > 0,$$

entailing that $\phi(\bar{x} + tw) > 0$ for all $t > 0$ sufficiently small. By (10), we have for all $t > 0$ sufficiently small,

$$\tau d(\bar{x} + tw, [\phi \le 0]) \le \phi(\bar{x} + tw). \tag{12}$$

Let $\kappa(x) := d(x, [\phi \le 0])$ for all $x \in \mathbb{R}^n$. Since the distance function κ is Lipschitz continuous on $\mathbb{R}^n$, we get from [26, Exercise 9.15] that

$$d\kappa(\bar{x})(w) = \liminf_{t \downarrow 0} \frac{d(\bar{x} + tw, [\phi \le 0]) - d(\bar{x}, [\phi \le 0])}{t} = \liminf_{t \downarrow 0} \frac{d(\bar{x} + tw, [\phi \le 0])}{t}.$$

In view of (11) and (12), we have

$$\tau d\kappa(\bar{x})(w) \leq \liminf_{t\downarrow 0} \frac{\phi(\bar{x}+tw)-\phi(\bar{x})}{t} = h(w). \tag{13}$$

By [26, Example 8.53], we have $d(w, T_{[\phi\leq 0]}(\bar{x})) = d\kappa(\bar{x})(w)$, which, together with (13), implies that $\tau d(w, T_{[\phi\leq 0]}(\bar{x})) \leq h(w)$. By definition, it is easy to verify that $T_{[\phi\leq 0]}(\bar{x}) \subset [h \leq 0]$ and hence $d(w, [h\leq 0]) \leq d(w, T_{[\phi\leq 0]}(\bar{x}))$. Therefore, we have $\tau d(w, [h \leq 0]) \leq h(w)$. This completes the proof. □

In view of Theorem 2.1, we have

$$\partial^{>}\sigma_{\partial\phi(\bar{x})}(0) = \mathrm{cl}(\bigcup_{\sigma_{\partial\phi(\bar{x})}(w)>0} \partial\sigma_{\partial\phi(\bar{x})}(w)) = \mathrm{cl}(\mathrm{end}(\partial\phi(\bar{x}))),$$

and

$$d(0, \partial^{>}\sigma_{\partial\phi(\bar{x})}(0)) = d(0, \bigcup_{\sigma_{\partial\phi(\bar{x})}(w)>0} \partial\sigma_{\partial\phi(\bar{x})}(w)) = d(0, \mathrm{end}(\partial\phi(\bar{x}))).$$

That is, the upper estimate $d\left(0, \partial^{>}\sigma_{\partial\phi(\bar{x})}(0)\right)$ in (9) is nothing else but the distance from 0 to the end set of $\partial\phi(\bar{x})$, or equivalently, the distance from 0 to the union of all the exposed faces of $\partial\phi(\bar{x})$ having normal vectors at which the support function $\sigma_{\partial\phi(\bar{x})}$ takes positive values.

The following examples show that both the lower estimate and upper estimate in (9) may not be tight, where the first example is taken from [28] (see also [21]) and the second one is taken from [6, Remark 3.6].

Example 3.1 *(underestimated lower estimate). Let $\bar{x} = 0$ and let $\phi : \mathbb{R} \to \mathbb{R}_+$ be defined by*

$$\phi(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 2^{-n} & \text{if } 2^{-n-1} \leq x \leq 2^{-n} \text{ with } n \text{ being an odd integer}, \\ 3x - 2^{-n} & \text{if } 2^{-n-1} \leq x \leq 2^{-n} \text{ with } n \text{ being an even integer}, \\ x & \text{otherwise}. \end{cases}$$

It is clear to see that ϕ is Lipschitz continuous and regular at $\bar{x} = 0$. By some direct calculations, we have $\partial\phi(\bar{x}) = \partial^{>}\phi(\bar{x}) = [0,1]$, $\partial^{>}\sigma_{\partial\phi(\bar{x})}(0) = \mathrm{end}(\partial\phi(\bar{x})) = \{1\}$, and $\mathrm{ebm}(\phi, \bar{x}) = 1$. It then follows that

$$0 = d\left(0, \partial^{>}\phi(\bar{x})\right) < \mathrm{ebm}(\phi, \bar{x}) = d\left(0, \partial^{>}\sigma_{\partial\phi(\bar{x})}(0)\right) = 1.$$

That is, the lower estimate in (9) is underestimated.

Example 3.2 *(overestimated upper estimate). Let* $\bar{x} = (0,0)^T$*, and let*

$$\phi(x) = \max\{f_1(x), f_2(x)\},$$

where $f_1(x) = x_1^2 + x_2^2 + \frac{1}{2}(x_1 + x_2)$ *and* $f_2(x) = x_1 + x_2$*. It is clear that* ϕ *is a convex function. Clearly,* $\partial\phi(\bar{x}) = \mathrm{conv}\{(\frac{1}{2}, \frac{1}{2})^T, (1,1)^T\}$*. From Corollary 2.1, it follows that* $\partial^{>}\sigma_{\partial\phi(\bar{x})}(0) = \mathrm{end}(\partial\phi(\bar{x})) = \{(1,1)^T\}$*. But from Remark 3.6 (i) of [6], we get* $\partial^{>}\phi(\bar{x}) = \mathrm{conv}\{(\frac{1}{2}, \frac{1}{2})^T, (1,1)^T\}$*. Therefore,*

$$\frac{\sqrt{2}}{2} = d(0, \partial^{>}\phi(\bar{x})) = \mathrm{ebm}(\phi, \bar{x}) < d(0, \partial^{>}\sigma_{\partial\phi(\bar{x})}(0)) = \sqrt{2}.$$

That is, the upper estimate in (9) is overestimated.

Remark 3.1 *In the papers [9, 15, 17], the authors have obtained that the lower estimate is tight for convex function* ϕ*. In Subsection 3.1, we show that the lower estimate is also tight for lower* $\mathcal{C}^1$ *function* ϕ *which including convex functions. Since* ϕ *is regular and Lipschitz, the set* $\partial\phi(\bar{x})$ *is compact and convex. Thus, it follows from Theorem 2.1 that the upper estimate is essentially the distance from 0 to the end set of* $\partial\phi(\bar{x})$*. In Subsection 3.2, we show that the upper estimate is tight for a convex function under some regularity conditions. This generalizes the corresponding results in [5, 13]. In [5], the authors obtained that the upper estimate is tight for a finite linear system where* ϕ *is the maximum of a finite amount of linear functions. In [13], the authors got the similar result for sublinear function* ϕ *at* $\bar{x} = 0$.

3.1 Sharp Lower Estimation for Lower $\mathcal{C}^1$ Functions

Many functions expressed by pointwise max of infinite collections of smooth functions have the 'subsmoothness' property, which is between local Lipschitz continuity and strict differentiability. Our aim in this subsection is to show that the lower estimate in (9) is a tight one for lower $\mathcal{C}^1$ functions.

Throughout this subsection, let ϕ be lower-$\mathcal{C}^1$ on an open subset O of $\mathbb{R}^n$ (cf. [26, Definition 10.29]) and let $\bar{x} \in O$ be a fixed point on the boundary of the level set $[\phi \leq 0]$. Moreover, we assume that on some open neighborhood V of $\bar{x}$ there is a representation

$$\phi(x) = \max_{y \in Y} f(x, y) \tag{14}$$

in which the functions $f(\cdot, y)$ are of class $\mathcal{C}^1$ on V and the index set $Y \subset \mathbb{R}^m$ is a compact space such that $f(x, y)$ and $\nabla_x f(x, y)$ depend continuously not just on $x \in V$ but jointly on $(x, y) \in V \times Y$. In what follows, we shall show that the lower estimate $d(0, \partial^{>}\phi(\bar{x}))$ in (9) is equal to the error bound modulus $\text{ebm}(\phi, \bar{x})$.

To begin with, we list some nice properties of ϕ as follows (cf. [26, Theorem 10.31]).

(a) ϕ is locally Lipschitz continuous and regular on O.

(b) $\partial\phi(x) = \text{conv}\{\nabla_x f(x, y) | y \in Y(x)\}$ for all $x \in V$, where $Y : V \rightrightarrows \mathbb{R}^m$ is the active index set mapping defined by

$$Y(x) := \{y \in Y | f(x, y) = \phi(x)\}. \tag{15}$$

(c) $\sigma_{\partial\phi(x)}(w) = d\phi(x)(w) = \max_{y \in Y(x)} \langle \nabla_x f(x, y), w \rangle$ for all $x \in V$ and $w \in \mathbb{R}^n$.

(d) The set-valued mapping Y defined by (15) is outer semicontinuous at $\bar{x}$, i.e.,

$$\limsup_{x \to \bar{x}} Y(x) \subset Y(\bar{x}).$$

Next we obtain some equivalent properties for ϕ defined by (14) having a local error bound.

Proposition 3.1 *Let $\tau > 0$ and let*

$$\mathcal{Y}(\bar{x}) := \{Y' \subset Y(\bar{x}) \mid \exists \{x_k\} \subset [\phi > 0] \text{ with } x_k \to \bar{x} \text{ and } Y(x_k) \to Y'\}.$$

The following properties are equivalent:

(i) *There exists some $\varepsilon > 0$ such that for all $x \in \mathbb{R}^n$ with $\|x - \bar{x}\| \le \varepsilon$,*

$$\tau d(x, [\phi \le 0]) \le \phi(x)_+. \tag{16}$$

(ii) *For every $Y' \in \mathcal{Y}(\bar{x})$, there exists some $u \in \mathbb{R}^n$ with $\|u\| = 1$ such that*

$$\langle \nabla_x f(\bar{x}, y), u \rangle \ge \tau \quad \forall y \in Y'.$$

(iii) *For every $Y' \in \mathcal{Y}(\bar{x})$, $d(0, \text{conv}\{\nabla_x f(\bar{x}, y) | y \in Y'\}) \ge \tau$.*

(iv) *There exists some $\delta > 0$ such that the inequality $d(0, \partial\phi(x)) \ge \tau$ holds for all $x \in \mathbb{R}^n$ with $\phi(x) > 0$ and $||x - \bar{x}|| \le \delta$.*

Proof. For the sake of notation simplicity, we use C to denote the level set $[\phi \leq 0]$ in what follows. We shall prove step by step that (i)$\Longrightarrow$(ii)$\Longrightarrow$(iii)$\Longrightarrow$(iv)$\Longrightarrow$(i).

[(i)$\Longrightarrow$(ii)]: Assume that there exists some $\varepsilon > 0$ such that (16) holds for all $x \in \mathbb{R}^n$ with $\|x - \bar{x}\| \leq \varepsilon$. First, we show that for any $x \in \operatorname{bdry} C \cap B(\bar{x}, \frac{\varepsilon}{2})$ and any proximal normal vector u to C at x with $||u|| = 1$, there exists some $y \in Y(x)$ such that

$$\langle \nabla_x f(x, y), u \rangle \geq \tau. \tag{17}$$

By the definition of proximal normal vectors, there exist some $x' \in \mathbb{R}^n$ and $\beta > 0$ such that

$$u = \beta(x' - x) \text{ and } x \in P_C(x').$$

Take $\rho := \min\{\frac{\varepsilon}{2}, \|x' - x\|\}$. Then it is easy to verify that

$$x + tu \in B(\bar{x}, \varepsilon)\ \forall t \in (0, \rho] \text{ and } x \in P_C(x + tu)\ \forall t \in (0, \rho].$$

In view of (16), we have

$$\tau t = \tau ||x + tu - x|| = \tau d(x + tu, C) \leq \phi(x + tu)_+\ \forall t \in (0, \rho].$$

Thus, we have $\tau \leq \liminf_{t \to 0_+} \frac{\phi(x+tu)_+ - \phi(x)_+}{t}$. From [26, Theorems 9.16 and 10.31], it follows that $\phi(x)_+$ is locally Lipschitz continuous with

$$\liminf_{t \to 0_+} \frac{\phi(x + tu)_+ - \phi(x)_+}{t} = d\phi(x)(u)_+ = \max\{\max_{y \in Y(x)} \langle \nabla_x f(x, y), u \rangle, 0\}.$$

Therefore, we have $\tau \leq \max\{\max_{y \in Y(x)} \langle \nabla_x f(x, y), u \rangle, 0\}$. In view of $\tau > 0$, we have $\tau \leq \max_{y \in Y(x)} \langle \nabla_x f(x, y), u \rangle$. Since $Y(x)$ is compact, there exists some $y \in Y(x)$ such that (17) holds.

Next, we show (ii) by virtue of the previous result. Let $Y' \in \mathcal{Y}(\bar{x})$. By definition, there exists some sequence $\{x'_k\} \in \mathbb{R}^n \setminus C$ with $x'_k \to \bar{x}$ and $Y(x'_k) \to Y'$, entailing that each $y \in Y'$ corresponds to a sequence $y'_k \to y$ such that $y'_k \in Y(x'_k)$ for all k. Since C is a closed set, there exists some $x_k \in \operatorname{bdry} C$ such that $x_k \in P_C(x'_k)$. Clearly, $x_k \to \bar{x}$ and $u_k := \frac{x'_k - x_k}{||x'_k - x_k||}$ is a proximal normal vector to C at x_k. By taking a subsequence if necessary, we can assume that $u_k \to u$, implying that $||u|| = 1$. In what follows, let $y \in Y'$ be given arbitrarily. To show (ii), it suffices to show

$$\langle \nabla_x f(\bar{x}, y), u \rangle \geq \tau. \tag{18}$$

According to the previous result, we can find some $y_k \in Y(x_k)$ such that for all sufficiently large k,

$$\langle \nabla_x f(x_k, y_k), u_k \rangle \geq \tau. \tag{19}$$

Since all $Y(x_k)$ are subsets of the compact set Y, by taking a subsequence if necessary, we can assume that $y_k \to \bar{y}$. By the mean value theorem, there is some $\theta_k \in [0,1]$ such that

$$f(x_k', y_k') - f(x_k, y_k') = \langle \nabla_x f\left(x_k + \theta_k(x_k' - x_k), y_k'\right),\ x_k' - x_k \rangle,$$

which, by the continuity of $\nabla_x f$, implies that

$$\begin{aligned}
&\frac{|f(x_k', y_k') - f(x_k, y_k') - \langle \nabla_x f(x_k, y_k'), x_k' - x_k\rangle|}{||x_k' - x_k||} \\
&= \frac{\langle \nabla_x f\left(x_k + \theta_k(x_k' - x_k), y_k'\right) - \nabla_x f(x_k, y_k'),\ x_k' - x_k\rangle}{||x_k' - x_k||} \\
&\leq \|\nabla_x f\left(x_k + \theta_k(x_k' - x_k), y_k'\right) - \nabla_x f(x_k, y_k')\| \to 0.
\end{aligned}$$

Thus, we have

$$\lim_{k\to+\infty} \frac{f(x_k', y_k') - f(x_k, y_k')}{||x_k' - x_k||} = \lim_{k\to+\infty} \langle \nabla_x f(x_k, y_k'), u_k\rangle = \langle \nabla_x f(\bar{x}, y), u\rangle. \tag{20}$$

Similarly, we obtain

$$\lim_{k\to+\infty} \frac{f(x_k', y_k) - f(x_k, y_k)}{||x_k' - x_k||} = \lim_{k\to+\infty} \langle \nabla_x f(x_k, y_k), u_k\rangle \geq \tau, \tag{21}$$

where the inequality follows from (19). Observing that

$$f(x_k', y_k') - f(x_k, y_k') \geq \phi(x_k') - \phi(x_k) \geq f(x_k', y_k) - f(x_k, y_k),$$

we get from (20) and (21) that (18) holds. This completes the proof for (i)$\Longrightarrow$(ii).

[(ii)$\Longrightarrow$(iii)]: Let $Y' \in \mathcal{Y}(\bar{x})$. By (ii), there exists some $u \in \mathbb{R}^n$ with $||u|| = 1$ that

$$\langle u, v\rangle \geq \tau \geq \langle u, w\rangle,\ \forall v \in \operatorname{conv}\{\nabla_x f(\bar{x}, y) | y \in Y'\}, \forall w \in B(0, \tau).$$

Then by a separation argument, we have

$$0 \notin \operatorname{int}(\operatorname{conv}\{\nabla_x f(\bar{x}, y) | y \in Y'\} - B(0, \tau)),$$

which clearly implies (iii).

[(iii)$\Longrightarrow$(iv)]: Let $\tau' \in (0, \tau)$ be given arbitrarily. First, we shall prove that, there exists some $\delta > 0$ such that for all $x \notin C$ with $||x - \bar{x}|| \leq \delta$,

$$d(0, \operatorname{conv}\{\nabla_x f(x, y) | y \in Y(x)\}) \geq \tau'. \tag{22}$$

Suppose by contradiction that (22) does not hold, i.e., there exists a sequence $\{x_k\} \subset \mathbb{R}^n \setminus C$ with $x_k \to \bar{x}$ and

$$d(0, \operatorname{conv}\{\nabla_x f(x_k, y) | y \in Y(x_k)\}) < \tau'.$$

It follows from the Carathéodory theorem that, there exist some $t_k^j \geq 0$ and $y_k^j \in Y(x_k)$ with $j = 1, 2, \cdots, n+1$ such that

$$\Sigma_{j=1}^{n+1} t_k^j = 1 \text{ and } ||\Sigma_{j=1}^{n+1} t_k^j \nabla_x f(x_k, y_k^j)|| \leq \tau'. \tag{23}$$

Since $Y(x_k) \subset Y$ for all k and Y is compact, it follows from [26, Theorem 4.18] that $Y(x_k)$ has a subsequence converging to Y^*, a subset of Y. By taking a subsequence if necessary, we assume that

$$Y(x_k) \to Y^*, \quad t_k^j \to t^j \geq 0, \quad \text{and} \quad y_k^j \to y^j \in Y^*.$$

Since $Y : V \rightrightarrows \mathbb{R}^m$ defined by (15) is osc at $\bar{x}$, it follows from [26, Exercise 5.3] that $Y^* \subset Y(\bar{x})$, entailing that $Y^* \in \mathcal{Y}(\bar{x})$. By (23) and the continuity of $\nabla_x f$, we have

$$\Sigma_{j=1}^{n+1} t^j = 1 \text{ and } ||\Sigma_{j=1}^{n+1} t^j \nabla_x f(\bar{x}, y^j)|| \leq \tau'.$$

Thus, we have $d(0, \text{conv}\{\nabla_x f(\bar{x}, y) | y \in Y^*\}) \leq \tau'$, contradicting to (ii). This contradiction implies that (22) holds. Since $\tau' \in (0, \tau)$ is given arbitrarily, we confirm that there exists some $\delta > 0$ such that the following inequality holds for all $x \notin C$ with $||x - \bar{x}|| \leq \delta$:

$$d(0, \text{conv}\{\nabla_x f(x, y) | y \in Y(x)\}) \geq \tau. \tag{24}$$

In view of (b), we can reformulate (24) as $d(0, \partial\phi(x)) \geq \tau$.

[(iv)$\Longrightarrow$(i)]: This implication follows readily from [21, Proposition 2.1]. □

Remark 3.2 *When Y is a finite set, the results in Proposition 3.1 can be found in [21, Theorem 2.1]. See also Kummer [18]. In the semi-infinite setting, Proposition 3.1 improves the corresponding results in Henrion and Outrata [10] and Zheng and Yang [31].*

Next theorem shows that the lower estimate in (9) is a tight one.

Theorem 3.2 *The following equalities hold:*

$$\partial^{>}\phi(\bar{x}) = \bigcup_{Y' \in \mathcal{Y}(\bar{x})} \text{conv}\{\nabla_x f(\bar{x}, y) | y \in Y'\}, \tag{25}$$

and

$$\text{ebm}(\phi, \bar{x}) = d(0, \partial^{>}\phi(\bar{x})).$$

Proof. The equality (25) follows readily from the definition of outer limiting subdifferential and the fact that all $Y(x)$ are compact and convex subsets of $Y(\bar{x})$ when x is close enough to $\bar{x}$. The equality $\mathrm{ebm}(\phi, \bar{x}) = d(0, \partial^{>}\phi(\bar{x}))$ follows from (25) and the equivalence of (i) and (iii) in Proposition 3.1. □

The upper estimate $d(0, \partial^{>}\sigma_{\partial\phi(\bar{x})}(0))$ in (9) has an alternative expression in terms of a collection of subsets of the index set $Y(\bar{x})$ defined by

$$\mathcal{Y}^{>}(\bar{x}) := \left\{ Y' \subset Y(\bar{x}) \;\middle|\; \exists w \in \mathbb{R}^n : Y' = \arg\max_{y \in Y(\bar{x})} \langle \nabla_x f(\bar{x}, y), w \rangle \,,\; \max_{y \in Y(\bar{x})} \langle \nabla_x f(\bar{x}, y), w \rangle > 0 \right\}.$$

By applying Corollary 2.1, we have

$$\begin{aligned} \bigcup_{Y' \in \mathcal{Y}^{>}(\bar{x})} \mathrm{conv}\{\nabla_x f(\bar{x}, y) | y \in Y'\} &\subset \mathrm{end}(\partial\phi(\bar{x})) = \gamma^{-1}_{\partial\phi(\bar{x})}(1) \\ &\subset \mathrm{cl}\left(\bigcup_{Y' \in \mathcal{Y}^{>}(\bar{x})} \mathrm{conv}\{\nabla_x f(\bar{x}, y) | y \in Y'\} \right) = \partial^{>}\sigma_{\partial\phi(\bar{x})}(0), \end{aligned} \tag{26}$$

where each $\mathrm{conv}\{\nabla_x f(\bar{x}, y) | y \in Y'\}$ is an exposed face of $\partial\phi(\bar{x})$. Thus

$$d(0, \partial^{>}\sigma_{\phi(\bar{x})}(0)) = d(0, \bigcup_{Y' \in \mathcal{Y}^{>}(\bar{x})} \mathrm{conv}\{\nabla_x f(\bar{x}, y) | y \in Y'\}).$$

If the index set $Y(\bar{x})$ is finite, all the inclusions in (26) become equalities.

When Y is finite, it follows from [18, 21] that $\mathcal{Y}(\bar{x})$ reduces to the following form:

$$\mathcal{Y}(\bar{x}) = \{Y' \subset Y(\bar{x}) \mid \exists \{x_k\} \subset [\phi > 0] \text{ with } x_k \to \bar{x} \text{ and } Y(x_k) \equiv Y'\}.$$

Combining with Corollary 2 in [8] one has the following result.

Corollary 3.1 *Let Y be finite. The following results hold:*

$$d(0, \bigcup_{Y' \in \mathcal{Y}(\bar{x})} \mathrm{conv}\{\nabla_x f(\bar{x}, y) | y \in Y'\}) = \mathrm{ebm}(\phi, \bar{x}) \le d(0, \bigcup_{Y' \in \mathcal{Y}^{>}(\bar{x})} \mathrm{conv}\{\nabla_x f(\bar{x}, y) | y \in Y'\})$$

and

$$\bigcup_{Y' \in \mathcal{Y}^{>}(\bar{x})} \mathrm{conv}\{\nabla_x f(\bar{x}, y) | y \in Y'\} \subset \bigcup_{Y' \in \mathcal{Y}(\bar{x})} \mathrm{conv}\{\nabla_x f(\bar{x}, y) | y \in Y'\}.$$

Remark 3.3 *The upper estimate and the inclusion in Corollary 3.1 may both be strict. See Example 3.2.*

As immediate consequence of Corollary 3.1, we obtain the following results which provide a tight bound and an lower bound for the calmness modulus of a finite $\mathcal{C}^1$ system defined by

$$f(x, y) \leq b_y \ \forall y \in Y,$$

where $f(\cdot, y) \in \mathcal{C}^1(\mathbb{R}^n)$, $b_y \in \mathbb{R}, y \in Y$ and Y is a finite set. The associated feasible set mapping $\mathcal{F} : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ given by

$$\mathcal{F}(b) := \{x \in \mathbb{R}^n | f(x, y) \leq b_y \ \forall y \in Y\}$$

with $b = (b_y)_{y \in Y}$ is the parameter to be perturbed. Associated with the finite $\mathcal{C}^1$ system, for a given nominal parameter $\bar{b}$, we consider the max-function

$$\phi(x) := \max_{y \in Y} f(x, y) - \bar{b}_y.$$

Let $\bar{x} \in \mathbb{R}^n$ with $\phi(\bar{x}) = 0$. Since $\mathrm{ebm}(\phi, \bar{x})^{-1} = \mathrm{clm}\mathcal{F}(\bar{b}, \bar{x})$, it follows from Corollary 3.1 that we easily get the following results for the finite $\mathcal{C}^1$ system which generalizes the corresponding results in [6].

Corollary 3.2 *Let Y be finite. One has*

$$[d(0, \bigcup_{Y' \in \mathcal{Y}^{>}(\bar{x})} \mathrm{conv}\{\nabla_x f(\bar{x}, y) | y \in Y'\})]^{-1} \leq \mathrm{clm}\mathcal{F}(\bar{b}, \bar{x}) = [d(0, \bigcup_{Y' \in \mathcal{Y}(\bar{x})} \mathrm{conv}\{\nabla_x f(\bar{x}, y) | y \in Y'\})]^{-1}.$$

3.2 Sharp Upper Estimation for Convex Functions

In the case of ϕ being finite and convex on some convex neighborhood of $\bar{x}$, entailing that ϕ is regular and locally Lipschitz continuous on some open neighborhood of $\bar{x}$ (cf. [26, Examples 7.27 and 9.14]), the lower estimate in (9) is tight, but the upper estimate in (9) could be overestimated, as seen in Example 3.2.

In general, we cannot expect that the upper estimate in (9) is a tight one, unless some regularity conditions are imposed as we have done in the following theorem.

Theorem 3.3 *Assume that ϕ is finite and convex on some convex neighborhood of $\bar{x}$. If there is a neighborhood V of $\bar{x}$ such that*

$$[\phi \leq 0] \cap V = (\bar{x} + [d\phi(\bar{x}) \leq 0]) \cap V, \tag{27}$$

then the following equalities hold:

$$d\left(0, \partial^{>}\phi(\bar{x})\right) = \mathrm{ebm}(\phi, \bar{x}) = d\left(0, \partial^{>}\sigma_{\partial\phi(\bar{x})}(0)\right). \tag{28}$$

Proof. By the definitions of tangent cone and subderivative, we can easily verify that $[\phi \le 0] \subset \bar{x} + T_{[\phi\le 0]}(\bar{x})$ and $T_{[\phi\le 0]}(\bar{x}) \subset [d\phi(\bar{x}) \le 0]$. Thus, the regularity condition (27) amounts to that

$$[d\phi(\bar{x}) \le 0] = T_{[\phi\le 0]}(\bar{x}), \tag{29}$$

and

$$[\phi \le 0] \cap V = (\bar{x} + T_{[\phi\le 0]}(\bar{x})) \cap V. \tag{30}$$

In view of (8) and the assumption that ϕ is finite and convex on some convex neighborhood of $\bar{x}$, we get the first equality in (28) immediately. It remains to show that the second equality holds under (29) and (30). Without loss of generality, we assume that there exists an open ball $O := \{x \in \mathbb{R}^n \mid \|x\| < \delta\}$ of radius $\delta > 0$ such that ϕ is finite and convex on $\bar{x} + O$ and that V in (30) can be replaced by $\bar{x} + O$. As ϕ is assumed to be finite and convex on some convex neighborhood of $\bar{x}$, it follows from [26, Examples 7.27 and 9.14, Theorem 9.16] that ϕ is regular and locally Lipschitz continuous on some open neighborhood of $\bar{x}$, and hence that $\sigma_{\partial\phi(\bar{x})} = d\phi(\bar{x})$ and

$$\mathrm{ebm}(d\phi(\bar{x}), 0) = \mathrm{ebm}(\sigma_{\partial\phi(\bar{x})}, 0) = d\left(0, \partial^{>}\sigma_{\partial\phi(\bar{x})}(0)\right). \tag{31}$$

Moreover, we get from Theorem 3.1 that $\mathrm{ebm}(\phi, \bar{x}) \le d\left(0, \partial^{>}\sigma_{\partial\phi(\bar{x})}(0)\right)$. In the case of $d\left(0, \partial^{>}\sigma_{\partial\phi(\bar{x})}(0)\right) = 0$, the second equality in (28) holds trivially. So in what follows we assume that $d\left(0, \partial^{>}\sigma_{\partial\phi(\bar{x})}(0)\right) > 0$.

Let $0 < \tau < d\left(0, \partial^{>}\sigma_{\partial\phi(\bar{x})}(0)\right)$. In view of (31) and the positive homogeneity of $d\phi(\bar{x})$, the following condition holds:

$$\tau d(w, [d\phi(\bar{x}) \le 0]) \le d\phi(\bar{x})(w)_+ \ \forall w \in \mathbb{R}^n. \tag{32}$$

Let $x \in \bar{x} + \frac{1}{2}O$ be arbitrarily chosen. It is straightforward to verify that

$$d(x - \bar{x}, T_{[\phi\le 0]}(\bar{x})) = d(x - \bar{x}, T_{[\phi\le 0]}(\bar{x}) \cap O) = d(x, (\bar{x} + T_{[\phi\le 0]}(\bar{x})) \cap (\bar{x} + O)),$$

and

$$d(x, [\phi \le 0]) = d(x, [\phi \le 0] \cap (\bar{x} + O)).$$

In view of (30), we have

$$d(x, [\phi \le 0]) = d(x - \bar{x}, T_{[\phi\le 0]}(\bar{x})),$$

which implies by (29) that

$$d(x, [\phi \le 0]) \le d(x - \bar{x}, [d\phi(\bar{x}) \le 0]).$$

By (32), we have

$$\tau d(x, [\phi \le 0]) \le d\phi(\bar{x})(x - \bar{x})_+. \tag{33}$$

Since ϕ is finite and convex on $\bar{x} + O$, we get from [26, Proposition 8.21] that

$$d\phi(\bar{x})(x - \bar{x}) \le \phi(x) - \phi(\bar{x}) = \phi(x). \tag{34}$$

In view of (33) and (34), we have $\tau d(x, [\phi \le 0]) \le \phi(x)_+$. Since $x \in \bar{x} + \frac{1}{2}O$ is chosen arbitrarily, we thus have $\tau \le \mathrm{ebm}(\phi, \bar{x})$, entailing that $d\left(0, \partial^{>}\sigma_{\partial\phi(\bar{x})}(0)\right) \le \mathrm{ebm}(\phi, \bar{x})$. This completes the proof. □

Remark 3.4 *Recall that the Abadie's constraint qualification [20] (ACQ, for short) holds at $\bar{x}$ if (29) holds, and that the level set $[\phi \le 0]$ admits exactness of tangent approximation (ETA, for short) at $\bar{x}$ if there exists some neighborhood V of $\bar{x}$ such that (30) holds. From the proof of Theorem 3.3, it is clear that the regularity condition (27) amounts to the ACQ plus the ETA. It turns out in the last section that, the outer limiting subdifferential set $\partial^{>}\sigma_{\partial\phi(\bar{x})}(0)$, unlike the outer limiting subdifferential set $\partial^{>}\phi(\bar{x})$, depends on the nominal point $\bar{x}$ only and does not get the nearby points involved. As can be seen from Theorem 3.3, it is the ETA property that makes it possible for $d(0, \partial^{>}\sigma_{\partial\phi(\bar{x})}(0))$ to serve as the error bound modulus $\mathrm{ebm}(\phi, \bar{x})$ which normally depends on not only $\bar{x}$ but its nearby points. Note that the idea of using the ETA property has already appeared in Zheng and Ng [30] and that various characterizations of the ETA property has been presented in [22]. If the ETA property (30) does not hold, the upper estimate $d(0, \partial^{>}\sigma_{\partial\phi(\bar{x})}(0))$ may be overestimated as can be seen from Example 3.2, in which $[\phi \le 0] = \{x \in \mathbb{R}^2 | x_1^2 + x_2^2 + \frac{1}{2}(x_1 + x_2) \le 0\}$ and the ETA property doest not hold at any $x \in [\phi \le 0]$.*

In the remainder of this subsection, we apply Theorem 3.3 to the linear system

$$\langle a_t, x\rangle \le b_t \quad \forall t \in T, \tag{35}$$

where $a_t \in \mathbb{R}^n$, $b_t \in \mathbb{R}$, and T is a compact space such that a_t and b_t depend continuously on $t \in T$. In what follows, let $\phi(x) := \max_{t\in T}\{\langle a_t, x\rangle - b_t\}$ and let $T(x) := \{t \in T \mid \langle a_t, x\rangle - b_t = \phi(x)\}$. Clearly, the level set $[\phi \le 0]$ is the solution set of the linear system (35), and the regularity condition (27) specified for $x \in [\phi \le 0]$ can be reformulated as

$$\{y \mid \langle a_t, y\rangle \le b_t \; \forall t \in T\} \cap V = (x + \{w \mid \langle a_t, w\rangle \le 0 \; \forall t \in T(x)\}) \cap V, \tag{36}$$

where V is a neighborhood of x.

Our first result for the linear system (35) assumes the regularity condition (36) on one nominal point in the solution set only.

Corollary 3.3 *Consider a solution x to the linear system (35). If the regularity condition (36) holds, then*

$$d\left(0, \partial^{>}\phi(x)\right) = \mathrm{ebm}(\phi, x) = d\left(0, \partial^{>}\sigma_{\partial\phi(x)}(0)\right) = d(0, \bigcup_{T' \in \mathcal{T}(x)} \mathrm{conv}\{a_t \mid t \in T'\}), \tag{37}$$

where

$$\mathcal{T}(x) := \{T' \subset T(x) \mid \exists w \in \mathbb{R}^n : \langle a_t, w\rangle = 1\, \forall t \in T',\ \langle a_t, w\rangle < 1\, \forall t \in T(x)\backslash T'\}.$$

Proof. Applying Theorem 3.3, we get the first two equalities in (37). Applying Corollary 2.1, we get the third equality in (37) by taking Remark 2.3 into account. This completes the proof. □

Our second result for the linear system (35) assumes the regularity condition (36) on the whole solution set, leading to a locally polyhedral linear system as defined in [1], which requires that

$$(\mathrm{pos}\,\mathrm{conv}\{a_t \mid t \in T(x)\})^{*} = \mathrm{pos}([\phi \le 0] - x) \quad \forall x \in [\phi \le 0]. \tag{38}$$

As a finite linear system is naturally locally polyhedral, our result below recovers [5, Theorem 4.1] for the case of a finite linear system.

Corollary 3.4 *Consider the linear system (35). The equalities in (37) hold for all $x \in [\phi \le 0]$ if one of the following equivalent properties is satisfied:*

(a) *The regularity condition (36) holds for all x in the solution set $[\phi \le 0]$;*

(b) *The linear system (35) is locally polyhedral, i.e., (38) holds.*

Proof. It suffices to show the equivalence of (a) and (b). To begin with, we point out that $d\phi(x)(w) = \max_{t\in T(x)}\langle a_t, w\rangle$ as can be seen from [26, Theorem 10.31], and that $[\phi \le 0]$ is convex (implying that $T_{[\phi\le 0]}(x) = \mathrm{cl}\,\mathrm{pos}([\phi \le 0] - x)$). Moreover, we have

$$[\phi \le 0] - x \subset \mathrm{pos}([\phi \le 0] - x) \subset T_{[\phi\le 0]}(x) \subset [d\phi(x) \le 0], \tag{39}$$

and

$$\begin{aligned}(\mathrm{pos}\,\mathrm{conv}\{a_t \mid t \in T(x)\})^{*} &= \{a_t \mid t \in T(x)\}^{*} \\ &= \{w \in \mathbb{R}^n \mid \langle a_t, w\rangle \le 0\ \forall t \in T(x)\} \\ &= [d\phi(x) \le 0].\end{aligned} \tag{40}$$

First, we show $(b) \Longrightarrow (a)$. Condition (38) implies that $\text{pos}([\phi \leq 0] - x)$ is closed for all $x \in [\phi \leq 0]$. In view of [22, Proposition 4.1], the level set $[\phi \leq 0]$ admits the ETA property (30) at every $x \in [\phi \leq 0]$. By (38) and (40), the ACQ (29) holds for all $x \in [\phi \leq 0]$. In view of Remark 3.4, the regularity condition (27) or its reformulation (36) holds for all $x \in [\phi \leq 0]$.

Now we show $(a) \Longrightarrow (b)$. Let $x \in [\phi \leq 0]$. Assume that the regularity condition (36) or its reformulation (27) holds. It then follows from (39) that

$$\text{pos}([\phi \leq 0] - x) = [d\phi(x) \leq 0],$$

which together with (40) implies (38). This completes the proof. □

To end this subsection, we illustrate two examples selected from [5]. By Example 3.3, we demonstrate that (37) may not hold if the linear system (35) is not locally polyhedral, and by Example 3.4, we demonstrate that (37) may still hold even if the linear system (35) is not locally polyhedral.

Example 3.3 *Let* $\bar{x} = (1,0)^T$ *and* $\phi(x) = \max_{t\in T}\{\langle a_t, x\rangle - b_t\}$*, where* $T = [0, 2\pi]$*,* $a_t = (t\cos t, t\sin t)^T$ *and* $b_t = t$*. Clearly,* $[\phi \leq 0] = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$*. Thus,* $\text{pos}([\phi \leq 0] - \bar{x})$ *is not closed, implying that (38) does not hold at* $\bar{x}$ *and that the linear system (35) cannot be locally polyhedral. From Example 1 of [5], it follows that* $d\left(0, \partial^{>}\phi(\bar{x})\right) = \text{ebm}(\phi, \bar{x}) = 0$*. Observing that* $T(\bar{x}) = \{0, 2\pi\}$ *and* $\mathcal{T}(\bar{x}) = \{\{2\pi\}\}$*, we get*

$$d\left(0, \partial^{>}\sigma_{\partial\phi(\bar{x})}(0)\right) = d(0, \bigcup_{T'\in\mathcal{T}(\bar{x})} \text{conv}\{a_t \mid t \in T'\}) = 2\pi.$$

That is, the upper estimate $d\left(0, \partial^{>}\sigma_{\partial\phi(\bar{x})}(0)\right)$ *is overestimated.*

Example 3.4 *Let* $\bar{x} = (1,0)^T$ *and* $\phi(x) = \max_{t\in T}\{\langle a_t, x\rangle - b_t\}$*, where* $T = [0, 2\pi]$*,* $a_t = (\cos t, \sin t)^T$ *and* $b_t = 1$*. Clearly,* $[\phi \leq 0] = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$*. Thus,* $\text{pos}([\phi \leq 0] - \bar{x})$ *is not closed, implying that (38) does not hold at* $\bar{x}$ *and that the linear system (35) cannot be locally polyhedral. By some direct calculations, we have* $T(\bar{x}) = \{0, 2\pi\}$*,* $\mathcal{T}(\bar{x}) = \{\{0, 2\pi\}\}$*, and*

$$d\left(0, \partial^{>}\sigma_{\partial\phi(\bar{x})}(0)\right) = d(0, \bigcup_{T'\in\mathcal{T}(\bar{x})} \text{conv}\{a_t \mid t \in T'\}) = 1.$$

Moreover, we have $\partial\phi(\bar{x}) = \text{conv}\{a_t \mid t \in T(\bar{x})\} = (1,0)^T$ *and hence* $\partial^{>}\phi(\bar{x}) = (1,0)^T$*, entailing that*

$$d\left(0, \partial^{>}\phi(\bar{x})\right) = \text{ebm}(\phi, \bar{x}) = 1.$$

That is, (37) still holds even when the linear system (35) is not locally polyhedral.

4 Conclusions and Perspectives

When ϕ is regular and locally Lipschitz continuous on some neighborhood of $\bar{x} \in \text{bdry}([\phi \leq 0])$, we obtained in Theorem 3.1 a lower estimate and an upper estimate of the local error bound modulus $\text{ebm}(\phi, \bar{x})$ as follows:

$$d\left(0, \partial^{>}\phi(\bar{x})\right) \leq \text{ebm}(\phi, \bar{x}) \leq d\left(0, \partial^{>}\sigma_{\partial\phi(\bar{x})}(0)\right).$$

In particular, when ϕ is finite and convex on some convex neighborhood of $\bar{x} \in \text{bdry}([\phi \leq 0])$, we obtained in Theorem 3.3 under the ACQ and ETA properties the following:

$$d\left(0, \partial^{>}\phi(\bar{x})\right) = \text{ebm}(\phi, \bar{x}) = d\left(0, \partial^{>}\sigma_{\partial\phi(\bar{x})}(0)\right),$$

and when ϕ is a lower $\mathcal{C}^1$ functions, we obtained in Theorem 3.2 the following:

$$d\left(0, \partial^{>}\phi(\bar{x})\right) = \text{ebm}(\phi, \bar{x}) \leq d\left(0, \partial^{>}\sigma_{\partial\phi(\bar{x})}(0)\right).$$

One open question is whether the inclusion

$$\partial^{>}\sigma_{\partial\phi(\bar{x})}(0) \subset \partial^{>}\phi(\bar{x}) \tag{41}$$

holds or not in the general case or in some particular settings. By trying to find answers to this open question, one may need to look into the differential structure of the functions in question and need to apply some delicate modern variational tools. It is worth noting that [6, Theorem 3.1] shows that (41) holds as an equality when ϕ is the pointwise max of a finite collection of affine functions. When ϕ is the pointwise max of a finite collection of smooth functions, [6, Theorem 3.2] shows that a subset of the set $\partial^{>}\sigma_{\partial\phi(\bar{x})}(0)$ is included in $\partial^{>}\phi(\bar{x})$.

References

[1] E.J. Anderson, M.A. Goberna, M.A. López, Locally polyhedral linear inequality systems, Linear Algebra Appl. 270(1998)231-253.

[2] D. Azé, A survey on error bounds for lower semicontinuous functions, in: Proceedings of 2003 MODE-SMAI Conference of ESAIM Proc., EDP Sci., Les Ulis, 13(2003)1-17.

[3] D. Azé, J.-N. Corvellec, Characterizations of error bounds for lower semicontinuous functions on metric spaces, ESAIM, Control Optim. Calc. Var., 10(2004)409-425.

[4] J.V. Burke, M.C. Ferris, Weak sharp minima in mathematical programming, SIAM J. Control Optim., 31(1993)1340-1359.

[5] M.J. Cánovas, M.A. López, J. Parra, F.J. Toledo, Calmness of the feasible set mapping for linear inequality systems, Set-Valued Var. Anal., 22(2014)375-389.

[6] M.J. Cánovas, R. Henrion, M.A. López, J. Parra, Outer limit of subdifferentials and calmness moduli in linear and nonlinear programming, J. Optim. Theory Appl., 169(2016)925-952.

[7] A.L. Dontchev, R.T. Rockafellar, Regularity and conditioning of solution mappings in variational analysis, Set-Valued Anal., 12(2004)79-109.

[8] A. Eberhard, V. Roshchina, T. Sang, Outer limits of subdifferentials for min-max type functions, arXiv:1701.02852v1, 2017.

[9] M.J. Fabian, R. Henrion, A.Y. Kruger, J.V. Outrata, Error bounds: necessary and sufficient conditions, Set-Valued Var. Anal., 18(2010)121-149.

[10] R. Henrion, J. Outrata, Calmness of constraint systems with applications, Math. Program., 104(2005)437-464.

[11] H. Hu, Characterizations of the strong basic constraint qualifications, Math. Oper. Res., 30(2005)956-965.

[12] H. Hu, Characterizations of local and global error bounds for convex inequalities in Banach spaces, SIAM J. Optim., 18(2007)309-321.

[13] H. Hu, Q. Wang, Closedness of a convex cone and application by means of the end set of a convex set, J. Optim. Theory Appl., 150(2011)52-64.

[14] A.D. Ioffe, Necessary and sufficient conditions for a local minimum, I. A reduction theorem and first order conditions, SIAM J. Control Optim., 17(1979)245-250.

[15] A.D. Ioffe, Metric regularity-a survey, Part 1, theory, J. Aust. Math. Soc., 101(2016)188-243.

[16] D. Klatte, W. Li, Asymptotic constraint qualifications and error bounds for convex inequalities, Math. Program., 84(1999)137-160.

[17] A.Y. Kruger, H.V. Ngai, M. Théra, Stability of error bounds for convex constraint systems in Banach spaces, SIAM J. Optim. 20(2010)3280-3296.

[18] B. Kummer, Inclusions in general spaces: Hölder stability, solution schemes and Ekeland's principle, J. Math. Anal. Appl., 358(2009)327-344.

[19] A.S. Lewis, J.S. Pang, Error bounds for convex inequality systems, In: J.P. Crouzeix, J.E. Martinez-Legaz, M. Volle, (eds.) Generalized Convexity, Generalized Monotonicity: Recent Results, Nonconvex Optim. Appl., 27(1998)75-110.

[20] W. Li, Abadie's constraint qualification, metric regularity, and error bounds for differentiable convex inequalities, SIAM J. Optim., 7(1997)966-978.

[21] K.W. Meng, X.Q. Yang, Equivalent conditions for local error bounds, Set-Valued Var. Anal., 20(2012)617-636.

[22] K.W. Meng, V. Roshchina, X.Q. Yang, On local coincidence of a convex set and its tangent cone, J. Optim. Theory Appl., 164(2015)123-137.

[23] K.F. Ng, X.Y. Zheng, Error bounds for lower semicontinuous functions in normed spaces, SIAM J. Optim., 12(2001)1-17.

[24] J.S. Pang, Error bounds in mathematical programming, Math. Program., 79(1997)299-332.

[25] R.T. Rockafellar, Convex Analysis, Princeton University Press, Princeton, NJ, (1970).

[26] R.T. Rockafellar, R.J.-B. Wets, Variational Analysis, Springer, Berlin, (1998).

[27] A.M. Rubinov, Radiant sets and their gauges, in Quasidifferentiability and Related Topics, Nonconvex Optim. Appl., 43(2000)235-261.

[28] M. Studniarski, D.E. Ward, Weak sharp minima: characterizations and sufficient conditions. SIAM J. Control Optim., 38(1)(1999)219-236.

[29] X.Y. Zheng, K.F. Ng, Metric regularity and constraint qualifications for convex inequalities on Banach spaces, SIAM J. Optim., 14(2004)757-772.

[30] X.Y. Zheng, K.F. Ng, Metric subregularity and constraint qualifications for convex generalized equations in Banach spaces, SIAM J. Optim., 18(2007)437-460.

[31] X.Y. Zheng, X.Q. Yang, Weak sharp minima for semi-infinite optimization problems with applications, SIAM J. Optim., 18(2007)573-588.