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A Discussion of Barabási-Albert's 1999 Paper[1]

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Abstract

Although the scale invariance is a main feature of growing networks, evidence from a few modelled networks show that the finite-size effect of network at a fixed time t cannot be ignored. We propose a concept of time-dependent scale-free networks, and prove the criteria for the stability and scale-free of growing networks by the degree-growing Markov chain. Our results show the importance of rigorous theoretical analysis for the network science.

Keywords: power-law, scale-free network, BA model, time-independent scale-free network, saturated model, time-dependent scale-free network, Markov chain, stability

1. Introduction

Ten years ago, the paper of Barabási and Albert published in *Science* [1] provoked a new direction of network science. Exploring several large databases of real network systems, they observed, along with similar observations by others around the same time, that "a common property of many large networks is that the vertex connectivities follow a scale-free power-law distribution." In other words, the decay probability $P(k)$ of the number of links of a vertex to other vertices in these networks follows a power law in k, i.e., $P(k) \sim k^{-\gamma}$, with, for example, $\gamma_{www} = 2.1 \pm 0.1$ for the WWW and $\gamma_{cite} = 3$ for the citation network of the scientific publications. To explain the power-law phenomenon, the authors proposed the well-known BA model: "Starting with a small number (m_0) of vertices, at every time step we add a new vertex with $m(\leq m_0)$ edges that link the new vertex to m different vertices already present in the system. To incorporate preferential attachment, we assume that the probability Π that a new vertex will be

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connected to a vertex i depends on the connectivity k_i of that vertex, so that $\Pi(k_i) = k_i/\sum_j k_j$." For the BA model, $\gamma = 2.9 \pm 0.1$ by computer simulation or $\gamma = 3$ by the mean-field argument. They commented that "This result indicates that large networks self-organize into a scale-free state, a feature unpredicted by all existing random network models." Thus, a network with power-law degree distribution is called the scale-free network. Moreover, "because the power law observed for real networks describes systems of rather different sizes at different stages of their development, it is expected that a correct model should provide a distribution whose main features are independent of time."

The idea of the BA paper is very novel, and it also leaves huge space for further development.

First, some details of the BA model are not very reasonable. Bollobás and Riordan [2] made a general comment on the BA model: "The first problem is getting started. The second problem is with the preferential attachment rule itself, and arises only for $m \geq 2$. In order to prove results about the BA model, one must first decide on the details of the model itself."

Three operable models have been proposed in the existing literature to refine the BA model. Bollobás *et al.* [3] recommended a Linearized Chord Diagram model with loops and multiple edges, briefly denoted LCD-m model. Dorogovtsev *et al.* [4] introduced an initial attraction model with multiple edges. Its a special case, where the initial attractiveness equal to m, called D-$w.m.e$ model. Holme and Kim [5] proposed a tunable clustering BA model. Consider the case of $p = 1$ in the model and called HK-$p = 1$ model. Each node of the initial m_0 nodes has at least $m - 1$ degrees, the first edge of a new vertex with $m(\leq m_0)$ edges connects to an existing vertex in the same way as the BA model; the rest $m - 1$ edges are randomly chosen from inside the neighborhood of the chosen vertex without allowing multiple edges.

The next problem is that the concept of the scale-free network and the scale-free power-law distribution implies the network performance, e.g., "robust yet fragile". Li *et al.* [6] point out that: (1) an important feature of power-law distributions is that they exhibit high variability; (2) preferential attachment is only one of several mechanisms that can produce graphs with power law degree distributions. Thus, they provided "one possible measure of the extent to which a graph is scale-free." By this measure, "it is now clear that either the Internet and biology networks are very far from scale-free," although their degree distributions all are of power-law.

In this paper, we discuss the problem that a network degree distribution is whether independent of time or not. Barabási and Albert expected that "a correct model should provide a distribution whose main features are independent of time." In section 2, we show that this is unrealistic for real networks and modelled networks. From a mathematical point of view, we extend time-independent scale-free to time-dependent scale-free in section 3. For rigorous theoretical analysis, we propose an appropriate Markov-chain model in section 4. In section 5, we derived stability and scale-free criteria of growing networks from the Markov-chain model. In section 6, we apply our results to a few network models and conclude the paper.

2. Time-dependence in a few networks

Although the BA model with linear growth and degree-preferential attachment is independent of time, can we say that degree distributions of real scale-free networks found in nature and modelled networks are all time-independent?

The scale-free power laws reported in [1] for the real networks are for topological structures at given time points of the network development. When we consider a series of time points instead of a fixed time point for a real system, empirical evidences show that the number of edges often grows faster than the number of vertices. Broder *et al.* [7] reported an experiment on the Connectivity Server 2 (CS2) software built at Compaq Systems Research Center. In May, 1999, the CS2 database contained 203 million URLs and 1466 million links. A later crawl (October, 1999) found 271 million URLs and 2130 million links. In other words, the average degree, i.e., m in the BA model in many real networks is not constant. Dorogovtsev and Mendes [8] considered an m-varying BA model, where except the increasing number mi^{θ}, $0 \leq \theta < 1$ of new edges added at time step i, everything else remains the same as in the BA model. The degree distribution from this accelerated growth model is of power law at any given time; but it is obviously time-dependent, as shown by the simulation results in Fig. 1(a) for $t = 10^3$, $t = 10^4$ and $t = 10^5$. Furthermore, its limiting distribution $P(k)$ does not exist.

Although the number of edges added each time increases in some observed real networks, the increase cannot be continuing forever and will likely slow down at some point. We use $[m(1-e^{-ri})]+1$ to replace mi^{θ}, where $m \geq 2$ is a growing upper boundary, r gives the accelerating rate, and $[x]$ represents integral part of x. We call this model saturated model. When t is large enough the saturated model converges to the BA model, as shown by the simulation results in Fig. 1(b) for $t = 10^3$, $t = 10^4$ and $t = 10^5$. Obviously, the degree distribution for this model is time-dependent initially; but the limiting distribution exists and is also of power-law.

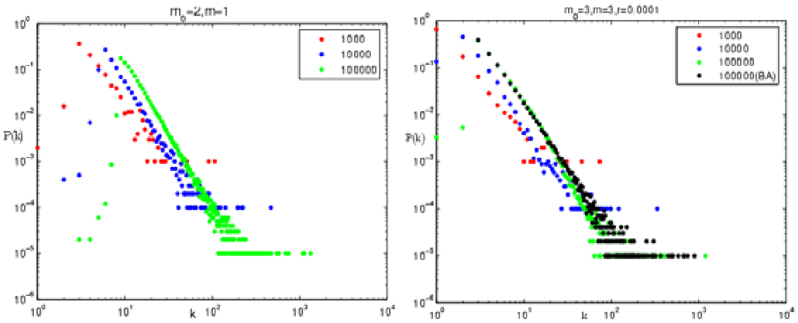

Figure 1: (a) The m-varying model; (b) The saturated model

Because new vertices do not have global information, Fortunato *et al.* [9] proposed a rank-preferential attachment mechanism to replace the degree-preferential attachment mechanism in the BA model. “If the nodes are sorted by age, from the oldest to the newest, the label of each node coincides with its rank, i.e., $R_t = t\ \forall t$.” For this special case, at each time step, the probability that the i-th vertex receives one of the m new edges is proportional to i^{-r}, $0 < r < \infty$. The authors found through simulation and heuristic arguments that the degree distribution of the modelled network with $r = 1$ is of power law with exponent $\gamma = 1 + (1/r) = 2$. However, the degree distribution for this modelled network is time-dependent, our simulation results for $t = 10^3$, $t = 10^4$ and $t = 10^5$ shown in Fig. 2(a). The degree distributions at three different time points move downwards to the left. In particular, all three distributions seem to be power-law, but the limiting distribution cannot because with $\gamma = 2$, the mean degree would diverge as the network size grows to infinity; but the mean degree for this network clearly is a constant $2m = 6$. In fact, if time-dependent degree distribution is of power law, a phase transition will occur at limit state. Hence, rigorous analysis still is needed for the modelled network.

To guarantee the degree distribution is independent of time, the BA model needs to ignore initial vertices and to assume m is a constant. For citation networks, Krapivsky and Redner [10] proposed copying instead of citing popular papers as the network growing mechanism. Obviously, the copying mechanism is equivalents to the degree preferential attachment. Their copying model successfully explained the logarithmic growth phenomenon which is observed in the average in-degree data from *Physical Review* citations. The in-degree distribution of the model is found time-independent and is of power-law with exponent $\gamma = 2$. The results obtained by simulation for $t = 50000$, $t = 100000$ and $t = 150000$ verify the conclusion, see Fig. 2(b).

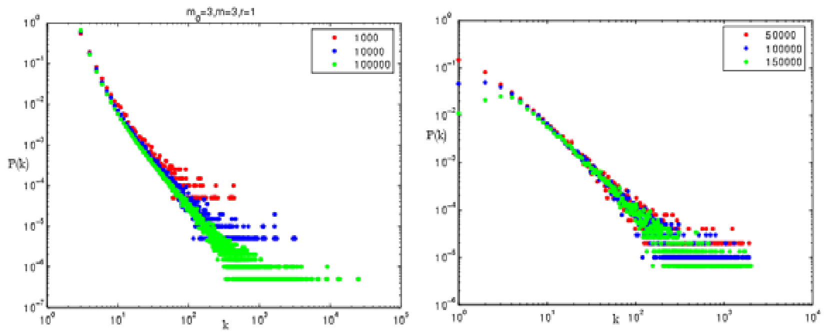

Figure 2: (a) The age-preferential model with $r = 1$; (b) The copying model

3. Time-dependent scale-free networks

The degree distribution of the scale-free network defined by Barabási and Albert is of the form $P(k) = Ck^{-\gamma}$, i.e, independent of time. However, the above examples show that many real and modelled networks are often not so. On the other hand, distributions $P(k,t) = C(t)k^{-\gamma}$, $P(k,t) = C(t)k^{-\gamma(t)}$ and $P(k,t) = C(t)[k + \kappa(t)]^{-\gamma(t)}$ are mathematically all power laws for any given time t. The degree distribution obtained by heuristic derivation is also $P(k,t)$ for any given time t. If $P(k,t)$ is time-dependent power-law, can we say it is not scale-free?

A natural question then arises in network science, what is exactly a scale-free network? To accommodate the effect of time factor in statistics and simulations, we should extend the concept of time-independent scale-free networks to time-dependent scale-free networks. Restricting ourselves to the simpler case, if the degree distribution of a network has the function form: $P(k,t) = C(t)k^{-\gamma}$, we call it a time-dependent scale-free network. In particular, when $P(k,t) \sim t^{\alpha}k^{-\gamma}$, e.g., for the m-varying BA model with $\alpha = \frac{2\theta}{1-\theta}$ and $\gamma = \frac{3-\theta}{1-\theta}$, the network is called non-stationary scale-free network except when $\alpha = 0$. The time-dependent definition greatly extends the applicable range of scale-free networks.

Because the finite size (time factor) effect can be very significant and the infinite network limit cannot be reached in real networks, various methods (statistical, simulating and heuristic) to predict the power laws of modelled networks depend on the thermodynamic limit. Does such a limit exist for a particular modelled network? Obviously, for the degree distribution $P(k,t)$ of a time-dependent scale-free network, the stability, i.e., $\lim_{t\to\infty} P(k,t) = P(k)$ depends on whether the limit of $C(t)$ exists or not. The m-varying BA model (a non-stationary scale-free network) is an example where the limit of $C(t)$ does not exist. If $\lim_{t\to\infty} P(k,t) = P(k)$ exists and $P(k)$ follows a power law with the same exponent, the time-dependent scale-free network is called tending-stable scale-free network.

Now we classify time-dependent scale-free networks according to two criteria: the time dependence and the existence of limiting degree distributions. This gives four types of scale-free networks: *time-independent SF network*, e.g., the BA model and the copying model; *tending-stable SF network*, e.g., the saturated model; *non-stationary SF network without limiting distribution*, e.g., the m-varying accelerated growth model; and *non-stationary SF network with phase transition*, whose limiting degree distribution exists but is not power law, e.g., the age-preferential model with $r = 1$.

Whether a modelled network is stable or not requires rigorous proof. Simulation results may be easily misread. In deriving the exact formula of degree distribution of the BA model, both Krapivsky *et al.* [11] with the rate equation approach and Dorogovtsev *et al.* [4] with the master equation approach simply assumed the existence of the limit of $P(k,t)$. Only recently, the stability for three similar BA modelled networks: LCD-m model, D-$w.m.e$ model and HK-$p = 1$ model has been proved [3, 12, 13] respectively. The key for the stability of these models is that the network-generating mechanism does not depend on time step of the evolving process.

4. Degree-growing Markov chains

We now consider growing network models in which multiple edges and loops are not permitted. Let $k_i(t)$ be the degree at time t of the vertex added at time step i. Obviously, $k_i(t)$ is nondecreasing and can increase at most by 1 at each time step. Hence, $k_i(t)$ for any i is a nonhomogeneous Markov chain [14] with transition probability

$$P\{k_i(t+1) = l \,|\, k_i(t) = k\} = \begin{cases} 1 - f_i(k,t), & l = k \\ f_i(k,t), & l = k+1 \\ 0, & \text{otherwise} \end{cases} . \tag{1}$$

Here, $f_i(k,t)$ is the probability that the degree of vertex i will increase by 1 at time step $t+1$. Let $\alpha_i(h)$ be the probability that the degree of vertex i will increase by h at time step i, $0 \le h \le i - 1 + m_0$, where m_0 is the number of vertices in the initial network. It is also the initial degree distribution of the Markov chain for vertex i. Clearly, $f_i(k,t)$, $\alpha_i(h)$, and equation (1) are determined by the network-generating mechanism of a network model and together, they completely define the Markov chain for vertex i.

Ignoring the initial vertices in the network, the family of Markov chains $\{k_i(t), i = 1, 2, \cdots; t = i, i+1, \cdots\}$ for all vertices characterizes a growing network model completely. We call this family *the degree-growing Markov chain* or DGMC for the fact that it captures the degree evolution of every individual vertex in a growing network model.

According to their network-generating mechanisms, we can easily define the DGMC for the existing growing-network models. For example,

Growth and degree-preferential model. This is the first growing network model and is commonly referred to as the BA model. Clearly, for the model, $\alpha_i(m) = 1$ and $f_i(k,t) = (k/2t) + o(1/t)$ for all i. For three similar BA models: (1) there is $\alpha_i(1) = (2i-2)/(2i-1)$, $\alpha_i(2) = 1/(2i-1)$ and $f_i(k,t) = [k/(2t+1)] + o(1/t)$ for all i in the LCD-1 model, but the case of $m > 1$ is more complex; (2) there is $\alpha_i(m) = 1$ and $f_i(k,t) = (k/2t) + o(1/t)$ for all i in D-$w.m.e$ model; and (3) there is $\alpha_i(m) = 1$ and $f_i(k,t) = k/2t$ for all i in HK-$p = 1$ model.

m-varying BA models. In this model, the total number of new edges added after time step t is $\int_0^t mi^\theta di = \frac{m}{\theta+1}t^{\theta+1}$, hence $\alpha_i(h) = \delta_{h[mi^\theta]}, i, h = 1, 2, \cdots$ and $f_i(k,t) = \frac{(\theta+1)k}{2t} + o(1/t)$ for all i.

Saturated growing models. In this model, the total number of new edges added after time step t is $\int_0^t \{[m(1-e^{-rx})] + 1\}dx$, hence $\alpha_i(h) = \delta_{h\{[m(1-e^{-ri})]+1\}}, i, h = 1, 2, \cdots$ and $f_i(k,t) = \frac{\{[m(1-e^{-rt})]+1\}k}{2\int_0^t \{[m(1-e^{-rx})]+1\}dx} + o(1/t)$ for all i. Since there is no analytic expression, neither the mean-field argument, nor the rate equation and master equation approaches are applicable here.

Nonlinear preferential BA models. Krapivsky *et al.* [11] discussed the impact of the nonlinear preferential attachment on the BA model. For this network model, $\alpha_i(m) = 1$ and $f_i(k,t) = [mk^r/M_r(t)] + o(1/t)$ for all i, where $M_r(t) = \mu_r t$ for $0 < r \le 1$ and $M_r(t) \propto t^r$ for $r > 1$.

Age-preferential attachment model. Here, $\alpha_i(m) = 1$ for all i and $f_i(k,t) = [mi^{-r}/\sum_{j=1}^t j^{-r}] + o(1/t)$ for all k, where $r > 0$.

Copying growing models. In this model, $k_i(t)$ represents the in-degree of vertex i after time step t. Hence, $\alpha_i(h) = \delta_{h0}$ and $f_i(k,t) = (k+1)/(t+1)$ for all i.

5. Stability and scale-free criteria

For a DGMC model with $f_i(k,t) \equiv f(k,t)$, e.g., the degree-preferential case, if

$$\lim_{i\to\infty} \alpha_i(h) = \alpha(h) \tag{2}$$

and there is a constant r such that for fixed k,

$$f_i(k,t) \equiv f(k,t) = g(k)O(t^{-r}),\ r \ge 1 \tag{3}$$

the steady-state degree distribution of the DGMC exists.

To show that (2) and (3) provide a general existence condition, we first write the master equation for the degree distribution of vertex i by (1):

$$P(k,i,t+1) = f(k-1,t)P(k-1,i,t) + \left[1 - f(k,t)\right]P(k,i,t),\ P(k,i,i) = \alpha_i(k), \tag{4}$$

where $P(k,i,t) = P\{k_i(t) = k\}$.

Let $P(k,t) = \frac{1}{t}\sum_{i=1}^t P(k,i,t)$, we have, by summing over i on both sides of (4)

$$(t+1)P(k,t+1) - tP(k,t) + \frac{tf(k,t)}{t}tP(k,t) = tf(k-1,t)P(k-1,t) + \alpha_{t+1}(k). \tag{5}$$

For any fixed k, let $tP(k,t) = a_t$, $tf(k,t) = b_t$, $t = c_t$ and $tf(k-1,t)P(k-1,t) + \alpha_{t+1}(k) = d_t$. We obtain the following difference equation from (5)

$$a_{t+1} - a_t + b_t\frac{a_t}{c_t} = d_t. \tag{6}$$

This general difference equation satisfies the following limit theorem [12]: if $\lim_{t\to\infty} d_t = l$, $c_{t+1} - c_t = 1$ and $\lim_{t\to\infty} b_t = b \ge 0$,

$$\lim_{t\to\infty}\frac{a_t}{t} = \lim_{t\to\infty}(a_{t+1} - a_t) = \frac{l}{1+b}. \tag{7}$$

Let $m \geq 0$ be the minimum h such that $\alpha(h) > 0$. Since there are only finitely many initial vertices, where their degrees are possibly smaller than m, we have $\lim_{t\to\infty} P(k,t) = 0$ for $0 \leq k \leq m-1$. Furthermore,

$$\lim_{t\to\infty} tf(k,t) \equiv F(k) \geq 0 \tag{8}$$

from (3) and $\lim_{t\to\infty}\{tf(m-1,t)P(m-1,t) + \alpha_{t+1}(m)\} = \alpha(m)$ from (2). Thus, the conditions for the above limit theorem of the difference equation hold when $k = m$ for (5) and we have $\lim_{t\to\infty} P(m,t) = \frac{\alpha(m)}{1+F(m)} > 0$. By induction and applying the limit theorem repeatedly, we obtain the following recursive expression

$$P(m) \triangleq \lim_{t\to\infty} P(m,t) = \frac{\alpha(m)}{1+F(m)} > 0, \tag{9}$$

$$P(k) = \frac{F(k-1)P(k-1) + \alpha(k)}{1+F(k)}, \ k > m. \tag{10}$$

Again assume that there is a constant M such that $\alpha(h) = 0$ when $h > M$ and we have $F(k) = Ak + B$, where A and B are two constants. Then, when $0 < A \leq 1$, the network generated by the degree-preferential DGMC model is scale-free with degree distribution

$$P(k) \sim [k + (B/A)]^{-(1+\frac{1}{A})}. \tag{11}$$

Its degree exponent is $\gamma = 1 + \frac{1}{A}$ and dynamic exponent is $\beta = A$. We may also note that when $B = 0$, we have a *normal* scale-free network while when $B > 0$, we have a *shifted* scale-free network. When $A = 0$ and $B > 0$, the degree-preferential DGMC model predicts a random network and

$$P(k) \sim \frac{1}{B}e^{-\frac{k}{B}}. \tag{12}$$

The above existence and structural results can be easily shown. Without loss of generality, let $\alpha(n) = 1$. Substituting $F(k) = Ak + B$ into (10) and noting that $\Gamma(k) \sim \Gamma(k+\gamma)k^{-\gamma}$ for sufficiently large k, e.g., $k > M$ and $\lim_{j\to\infty}[1 + (x/j)]^j = e^x$, we have, for $A \neq 0$

$$P(k) = \frac{A(k-1)+B}{1+Ak+B}P(k-1) = \frac{\Gamma[k+(B/A)]}{\Gamma[k+(B/A)+1+(1/A)]}P(1) \sim [k+(B/A)]^{-(1+\frac{1}{A})},$$

and for $A = 0$

$$P(k) = \frac{B}{1+B}P(k-1) = \frac{1}{B}\left(\frac{B}{1+B}\right)^k = \frac{1}{B}\left(\left[1+\frac{k/B}{k}\right]^k\right)^{-1} \sim \frac{1}{B}e^{-\frac{k}{B}}.$$

Remark: By using *the first-passage probability* of Markov chain, Z. T. Hou *et al.* also obtained similar criteria (see //www.paper.edu.cn).

6. Applications and conclusions

We now apply our results to a few network models.

For the BA model and three similar models, since $\alpha(m) = 1$ and $F(k) = k/2$, the steady-state degree distributions of exists and the model predicts a scale-free network with $\gamma = 3$. By (9) and (10), $P(m) = \frac{1}{1+m/2}$, $P(m+1) = \frac{m/2}{1+(m+1)/2}P(m), \cdots$. Thus, the degree distribution is explicitly given by $P(k) = \frac{2m(m+1)}{k(k+1)(k+2)} \sim 2m^2k^{-3}$.

By (2) and (3), the steady-state degree distribution of nonlinear preferential BA model for all r exists. However, when $r > 1$, $F(k) = 0$ by (8), $\alpha(m) = 1$, $P(m) = 1$ by (9), and $P(k) = 0$ for $k > m$ by (10); the generated network exhibits a "winner takes all" phenomenon and is not scale-free. Furthermore, when $0 < r < 1$, $F(k) = \frac{m}{\mu_r}k^r$, so that the condition to ensure a scale-free topology is not satisfied. To see if the modelled network is indeed not scale-free, we set $m = 1$, and obtain $P(1) = \frac{1}{1+1/\mu_r}$, $P(2) = \frac{1/\mu_r}{1+2^r/\mu_r}P(1)$, $\cdots$, $P(k) = \frac{\mu_r}{k^r}\prod_{j=1}^{k}\left(1+\frac{\mu_r}{j^r}\right)^{-1}$. Obviously, $P(k)$ does not follow a power law. Finally, $\mu_1 = 2mt$ when $r = 1$, and we have a normal scale-free network with $F(k) = k/2$. Thus, this model generates a scale-free network if and only if $r = 1$, i.e., when the degree-preferential mechanism is a linear preferential attachment.

A special case of corresponding to $r = 0$ is the *growth and random attachment model*. Thus, $\alpha_i(m) = 1, i = 1, 2, \cdots$ and $f_i(k, t) = m/t$ for all i and k. By (12), its degree distribution is $P(k) \sim \frac{1}{m}e^{-\frac{k}{m}}$.

For the copying model, we have $\alpha_i(h) = \delta_{h0}$ and $F(k) = \lim_{t\to\infty}\frac{t(k+1)}{t} = k+1$, hence its in-degree distribution is $P_{in}(0) = \frac{1}{1+1} = 1/2$ by (9), and (10), $P_{in}(k) = \frac{k}{k+2}P_{in}(k-1) = 1/(k+1)(k+2)$.

Finally, for the saturated model, we have $F(k) = \lim_{t\to\infty}\frac{t\{[m((1-e^{-rt})]+1\}k}{2\int_0^t\{[m(1-e^{-rx})]+1\}dx} = k/2$ and $\alpha(h) = \lim_{i\to\infty}\alpha_i(h) = \delta_{hm}$, i.e., $\alpha(m) = 1$ and $F(k) = k/2$. Hence the steady-state degree distribution of the saturated model is the same as the BA model.

Through simulation of modelled networks, we show that the scale-free connectivity for many growing networks can be time-dependent. This phenomenon contradicts to the time-independent definition of [1]. We propose the concept of time-dependent scale-free network to resolve this contradiction. We also demonstrate that rigorous analysis is needed to determine whether a time-dependent scale-free network has a thermodynamic limit. Based on a general Markov chain model framework, we provide some general criteria to judge whether a set of network generating mechanisms (without allowing multiple edges and loops) can ensure the existence of the steady-stage degree distribution and whether the steady-stage degree distribution is scale-free. The criteria are intuitive and their proofs are very simple.

Our results show that Markov chain provides a power framework for theoretical analysis of complex networks. We may similarly discuss the degree-evolving Markov chain (birth-and-death process) of evolving networks [15], the degree-growing Markov chain of the weighted-network, and other types of complex networks not discussed here.

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