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WEAK RIGIDITY THEORY AND ITS APPLICATION TO FORMATION STABILIZATION*

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Abstract. This paper introduces the notion of weak rigidity to characterize a framework by pairwise inner products of interagent displacements. Compared to distance-based rigidity, weak rigidity requires fewer constrained edges in the graph to determine a geometric shape in an arbitrarily dimensional space. A necessary and sufficient graphical condition for infinitesimal weak rigidity of planar frameworks is derived. As an application of the proposed weak rigidity theory, a gradient-based control law and a nongradient-based control law are designed for a group of single-integrator modeled agents to stabilize a desired formation shape, respectively. Using the gradient control law, we prove that an infinitesimally weakly rigid formation is locally exponentially stable. In particular, if the number of agents is one greater than the dimension of the space, a minimally infinitesimally weakly rigid formation is almost globally asymptotically stable. In the literature of rigid formation, the sensing graph is always required to be rigid. Using the nongradient control law based on weak rigidity theory, it is not necessary for the sensing graph to be rigid for local exponential stability of the formation. A numerical simulation is performed for illustrating the effectiveness of our main results.

Key words. graph rigidity, rigid formation, multiagent systems, matrix completion

AMS subject classifications. 05C10, 68M14, 93C10

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1. Introduction.

There is a rapidly growing interest in the study of distributed coordination of networked multiagent systems due to their wide applications and diverse mathematical challenges. As one of the most significant and challenging problems, the formation stabilization problem, which is concerned with the stabilization of a group of agents via local information to form a desired formation shape, has been studied in a vast body of references; see, e.g., the survey papers [26, 28, 1, 25].

In the formation stabilization problem, the formation shape is often characterized by specified constraints on agents' states. These constraints differ depending on the sensing graph which describes interaction relationships between agents and sensing capability of agents. In recent years, due to their advantages in alleviation of computational burden and enhancement of reliability, decentralized formation stabilization strategies based on relative displacement information have received a lot of attention [11, 26, 28, 36]. Control algorithms proposed in these references often guarantee global stability of the formation but, unfortunately, come at the cost of being implemented under a common coordinate system, which is often unavailable when the

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global positioning system (GPS) is disabled. References [24] and [16] proposed two distributed orientation alignment laws for agents to reach agreement on their local coordinate systems, which can efficiently solve the formation problem in the plane in the absence of a common orientation. However, their approaches require all agents to have the capability of communicating with each other and thus will be invalid when agents equip no communication sensors. Unlike these investigations, the author in [10] obtained global stability of formation with a discrete-time algorithm via initial orientation alignment. Another hot issue in distributed formation control is bearing-constrained formation. For example, in [5, 39], the authors studied how to encode the desired formation by bearing-only constraints. However, the proposed methods also require either the global coordinate system or an orientation synchronization law based on interagent communications. Reference [19] obtained a necessary and sufficient condition for global stability of the formation by a consensus-like control protocol, which allows agents to use relative displacements measured in their local coordinate frames, but the stabilizing gain matrix should be designed via a centralized approach.

In order to achieve distributed and communication-free formation in GPS-denied environments, intensive research efforts have been expended. Distance-based formation control, which determines the desired formation by interagent distance, and only requires each agent to sense local relative displacements with its local coordinate frame, was investigated in [1, 14, 37, 31, 32, 22, 23, 38, 33, 20, 35, 34, 13, 8]. By embedding the formation graph into a specified space, a framework consisting of a formation topology and certain coordinates of all vertices is employed to describe the desired formation shape. To solve the formation problem, one should answer the question of how many distance constraints are required to determine the formation framework, which turns out to be equivalent to a Euclidean distance matrix completion problem [15, 17, 29]. Moreover, this question is also relevant to network localization problems [4]. In the literature, graph rigidity theory [3, 12, 9] was often employed to answer this question and many interesting results have been obtained. In [14], the authors proposed a gradient-based control law for multiple autonomous agents to restore infinitesimally rigid formations under small perturbations from the desired formation shape. In [37] and [32], the authors solved minimally persistent formation problems under a directed sensing graph by introducing an appropriately designed gain matrix. In [23], the authors showed that rigidity of the formation framework is sufficient to ensure local stability of the desired formation. Besides the distance-based formation strategy, [2] proposed a displacement-based approach to achieving local and global stability of rigid planar formations under different graphs. Reference [18] introduced an affine formation strategy and obtained global stability of formation under universally rigid graphs. Reference [21] proposed a control strategy based on Henneberg vertex additions to achieve a minimally rigid acyclic formation. All these investigations require the target formation shape to be rigid. However, this restriction is not easy to satisfy in practice due to the demand for a large number of edges in the formation graph.

This paper aims to reduce the number of edges in a graph for determining an undirected formation framework in an arbitrary dimensional space. The fundamental method we propose is based on a modification of the rigidity function in graph rigidity theory. More specifically, we regard pairwise inner products of relative displacements as components of the rigidity function, which are actually constraints determining the desired formation shape. Accordingly, a generalized notion of rigidity, the *weak rigidity*, is introduced. Since a distance constraint is equal to the inner product of two identical displacements, weak rigidity can reduce to distance-based rigidity. In fact, one can intuitively observe that angles subtended at vertices are also helpful for determining a desired formation; unfortunately, this information is not efficiently utilized in distance-based formation control. As pointed out in [24], the angle information contained in the displacement measurements is difficult to directly utilize. In this paper, by employing inner products of relative displacements as constraints, angles subtended in the formation graph can be used to determine the desired formation shape. Moreover, the inner product of two vectors in any local coordinate frame is invariant and thus independent of the global coordinate system. As a result, weak rigidity requires fewer edges than distance-based rigidity to recognize a framework and provides a novel insight into decentralized formation controller synthesis in GPS-denied environments.

The main contributions of this paper are summarized as follows. (i) We define a generalized concept of rigidity called weak rigidity (Definition 3.3), by which a framework in an arbitrarily dimensional space can be determined with fewer constrained edges than distance-based rigidity. In Theorem 3.10, we prove that weak rigidity is necessary but not sufficient for distance-based rigidity and thus is a weaker condition for determining a framework. (ii) For frameworks embedded in the plane, a necessary and sufficient graphical condition is derived for infinitesimal weak rigidity (Theorem 3.9). It is shown that a framework is infinitesimally weakly rigid if and only if the graph is connected and for each vertex with more than two neighbors, the edges connected to this vertex are not all collinear. Based on the graphical condition, we present two algorithms for constructing a constraint set with a minimal number of elements for determining weak rigidity of the framework. (iii) From a matrix completion perspective, we show that by employing weak rigidity theory, the realization problem of a framework is equivalent to a positive semidefinite (PSD) matrix completion problem [15, 29, 17]. Once the PSD matrix is completed, the framework can be uniquely determined up to translations, rotations, and reflections; see Theorem 3.13 and Remark 3. (iv) We show in subsection 3.5 that both weak rigidity and infinitesimal weak rigidity are generic properties of graphs. More precisely, after fixing the graph, either all the frameworks with generic configurations are infinitesimally weakly rigid, or none of them are. (v) As an application, on the basis of weak rigidity theory proposed, we present a gradient-based control law for multiple autonomous agents to achieve a desired formation. It is shown that if the number of agents is one greater than the dimension of the space, then almost global asymptotic stability[1] of the minimally infinitesimally weakly rigid formation and collision avoidance can be ensured (Theorem 4.4). Otherwise the infinitesimally weakly rigid formation is locally exponentially stable (Theorem 4.3). (vi) A nongradient-based protocol is also proposed for achieving weakly rigid formation. It is shown that once a control gain matrix is properly designed, our control strategy can drive agents to form a locally exponentially stable weakly rigid formation, while the underlying sensing graph is only required to be infinitesimally weakly rigid rather than rigid. This is a relaxed condition for sensing graphs compared to [14, 23, 38, 33, 35, 34, 2, 18, 8].

This paper is structured as follows. Section 2 provides preliminaries of graph rigidity theory and center manifold theory. Section 3 presents the weak rigidity theory. As an application, section 4 discusses two control strategies for formation stabiliza-

[1] A shape is said to be almost globally asymptotically stable if it is asymptotically stable for almost all the initial conditions. That is, the initial conditions converging to incorrect shapes belong to a set of measure zero [31].

tion control and the corresponding stability analysis. Section 5 presents a numerical example. Finally, section 6 concludes the paper.

Notation. Throughout this paper, $\mathbb{R}$ denotes the set of real numbers; $\mathbb{R}^n$ is the n-dimensional Euclidean space; $||\cdot||$ stands for the Euclidean norm; X^T means the transpose of matrix X; $\otimes$ is the Kronecker product; range(X), null(X), and rank(X) denote the range space, null space, and the rank of matrix X; $A \setminus B$ is the set of those elements of A not belonging to B; I_n represents the $n \times n$ identity matrix; $\mathbf{1}_n \in \mathbb{R}^{n\times 1}$ is a vector with each component being 1; $\lambda(X)$ is the set of eigenvalues of matrix X.

An undirected graph with n vertices and m edges is denoted as $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \dots, n\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denote the vertex set and the edge set, respectively. As the graph considered in this paper is undirected, we will not distinguish (i, j) and (j, i). The incidence matrix is represented by $H = [h_{ij}]$, which is a matrix with rows and columns indexed by edges and vertices of $\mathcal{G}$ with an orientation. $h_{ij} = 1$ if the ith edge sinks at vertex j, $h_{ij} = -1$ if the ith edge leaves vertex j, and $h_{ij} = 0$ otherwise. It is well known that rank$(H) = n - 1$ if and only if graph $\mathcal{G}$ is connected.

2. Preliminaries.

2.1. Graph rigidity theory.

A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ can be embedded in $\mathbb{R}^d$ by an assignment of locations $p_i \in \mathbb{R}^d$, $i \in \mathcal{V}$, to the vertices. Graph rigidity theory is for answering whether partial length-constrained edges of graph $\mathcal{G}$ can determine the coordinates of the points $p_1, \dots, p_n$ uniquely up to rigid transformations (translations, rotations, reflections). Several basic definitions related to graph rigidity taken from [3] and [12] are stated as follows.

The vector $p = (p_1^T, \dots, p_n^T)^T \in \mathbb{R}^{nd}$ is called a realization or configuration of $\mathcal{G}$. The pair $(\mathcal{G}, p)$ is said to be a framework. The *rigidity function* $g_{\mathcal{G}}(\cdot) : \mathbb{R}^{nd} \to \mathbb{R}^m$ associated with the framework $(\mathcal{G}, p)$ is defined as

$$g_{\mathcal{G}}(p) = (\dots, ||e_{ij}||^2, \dots)^T, \quad (i, j) \in \mathcal{E}, \tag{1}$$

where $n = |\mathcal{V}|$, $m = |\mathcal{E}|$, $e_{ij} = p_i - p_j$, and $\|e_{ij}\|$ is the Euclidean distance between the vertices i and j.

We say two frameworks $(\mathcal{G}, p)$ and $(\mathcal{G}, q)$ are *equivalent* if $g_{\mathcal{G}}(p) = g_{\mathcal{G}}(q)$, i.e., $||p_i - p_j|| = ||q_i - q_j||$ for all $(i, j) \in \mathcal{E}$. They are *congruent* if $||p_i - p_j|| = ||q_i - q_j||$ for all $i, j \in \mathcal{V}$. A framework $(\mathcal{G}, p)$ is called *rigid* if there exists a neighborhood U_p of p such that for any $q \in U_p$, once $(\mathcal{G}, p)$ is equivalent to $(\mathcal{G}, q)$, then they are congruent. $(\mathcal{G}, p)$ is *globally rigid* in $\mathbb{R}^d$ if it is rigid with $U_p = \mathbb{R}^{nd}$. $(\mathcal{G}, p)$ is *minimally rigid* if no edges of $\mathcal{G}$ can be removed without losing rigidity of $(\mathcal{G}, p)$. For example, the framework in Figure 1(a) is both minimally and globally rigid, the framework in Figure 1(c) is minimally rigid, and the frameworks in Figures 1(b) and 1(d) are both nonrigid.

The rigidity function $g_{\mathcal{G}}(p)$ is the key to recognizing the framework $(\mathcal{G}, p)$. For a time-varying framework, an assignment of velocities that guarantees the invariance of $g_{\mathcal{G}}(p)$, i.e., $\dot{g}_{\mathcal{G}}(p) = 0$, is called an *infinitesimal motion.* That is,

$$(v_i - v_j)^T e_{ij} = 0, \quad (i, j) \in \mathcal{E}, \tag{2}$$

where $v_i = \dot{p}_i$ is the velocity of vertex i. Note that rotations, translations, and their combinations always satisfy (2). Such motions are said to be *trivial.* A framework is *infinitesimally rigid* if every infinitesimal motion is trivial. In a d-dimensional space, there are d independent translations and $d(d-1)/2$ independent rotations. Therefore, for a framework $(\mathcal{G}, p)$ with $n \geq d$, the dimension of the space formed by trivial motions

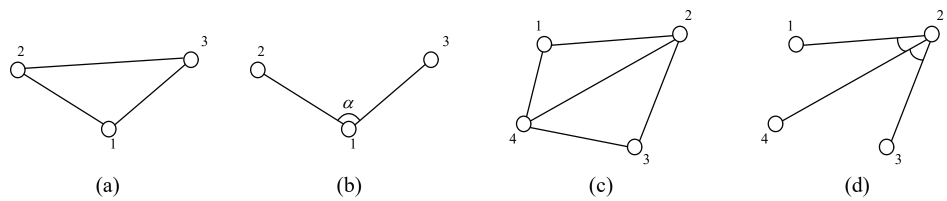

FIG. 1. *Four frameworks in* $\mathbb{R}^2$. (a) *A minimally and globally rigid framework.* (b) *A nonrigid framework.* (c) *A minimally rigid framework.* (d) *A nonrigid framework.*

is $T(n,d) = d + d(d-1)/2 = d(d+1)/2$. In fact, (2) is equivalent to $\dot{g}_{\mathcal{G}}(p) = R(p)\dot{p} = 0$, where $R(p) \triangleq \frac{\partial g_{\mathcal{G}}(p)}{\partial p} \in \mathbb{R}^{m\times nd}$ is called the *rigidity matrix*. Thus one can obtain that a framework $(\mathcal{G}, p)$ is infinitesimally rigid if $\text{rank}(R(p)) = nd - T(n,d)$.

Two vertices of an infinitesimally rigid framework usually do not share identical positions. We present the following lemma to show a necessity condition for existence of overlaps in an infinitesimally rigid framework.

LEMMA 2.1. *Let* $(\mathcal{G}, p)$ *be infinitesimally rigid in* $\mathbb{R}^d$. *If there exists a vertex* i *colliding with another vertex, then vertex* i *has at least* d *neighbors not colliding with it.*

Proof. Consider a framework $(\tilde{\mathcal{G}}, \tilde{p})$, which is induced by deleting vertex i and all edges involving i from $(\mathcal{G}, p)$. Let $\tilde{R}(\tilde{p})$ be the rigidity matrix of $(\tilde{\mathcal{G}}, \tilde{p})$. It is easy to see that $\text{rank}(\tilde{R}(\tilde{p})) \leq (n-1)d - d(d+1)/2$. Note that when i is added into $(\tilde{\mathcal{G}}, \tilde{p})$, the corresponding rigidity function can be written as $g_{\mathcal{G}} = (g_{\tilde{\mathcal{G}}}^T, g^{iT})^T$, where $g_{\tilde{\mathcal{G}}}$ is the rigidity function of $(\tilde{\mathcal{G}}, \tilde{p})$, $g^i = (\ldots, ||e_{ij}||^2, \ldots)^T$, $j \in \mathcal{N}_i$. Hence, the rigidity matrix of $(\mathcal{G}, p)$ is

$$R(p) = \frac{\partial g_{\mathcal{G}}}{\partial p} = \left(\frac{\partial g_{\mathcal{G}}}{\partial \tilde{p}}, \frac{\partial g_{\mathcal{G}}}{\partial p_i} \right) = \begin{pmatrix} \tilde{R}(\tilde{p}) & \mathbf{0} \\ \frac{\partial g^i}{\partial \tilde{p}} & \frac{\partial g^i}{\partial p_i} \end{pmatrix}.$$

Infinitesimal rigidity of $(\mathcal{G}, p)$ implies that $\text{rank}(\frac{\partial g^i}{\partial p}) \geq \text{rank}(R(p)) - \text{rank}(\tilde{R}(\tilde{p})) \geq d$. Therefore, g^i should have at least d nonzero components. That is, there exist $N \geq d$ vertices $k_1, \ldots, k_N \in \mathcal{N}_i$ and $p_i \neq p_{k_j}$, $j \in \{1, \ldots, N\}$. ∎

Finally, it is worth noting that an infinitesimally rigid framework may or may not have overlapped vertices.

2.2. Center manifold theory. Center manifold theory is a tool of great utility in studying stability of nonhyperbolic equilibria of a nonlinear system. The details of center manifold theory can be found in [7]. Here we introduce a result for systems with an equilibrium manifold derived in [32], which will be employed to study stability of equilibria of the formation system.

LEMMA 2.2 (see [32]). *Consider the nonlinear autonomous system*

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \tag{3}$$

where f *is twice continuously differentiable almost everywhere in a neighborhood of the origin. Suppose there exists a smooth* m*-dimensional* $(m > 0)$ *manifold of the equilibrium set* M_1 *for* (3) *that contains the origin. If the Jacobian of* f *at the origin has* m *eigenvalues with zero real part and* $n - m$ *eigenvalues with negative real part, then* M_1 *is a center manifold for* (3). *Moreover, there exist compact neighborhoods*

Ω_1 and Ω_2 of the origin such that $M_2 = \Omega_2 \cap M_1$ is locally exponentially stable, and for each $x(0) \in \Omega_1$, it holds that $\lim_{t\to\infty} x(t) = q$ for some $q \in M_2$.

3. Weak rigidity. Generally speaking, a multiagent formation problem is about the stabilization of a desired geometric shape formed by a group of mobile agents in a d-dimensional space. In the literature, a distance-based formation strategy is often adopted since the global coordinate system is often absent for each agent. In addition to distances, the subtended angles are also available in determining the desired formation shape and are independent of the global coordinate system. The purpose of this paper is to show how to utilize such information in a multiagent formation problem. In this section, we present a novel approach to recognizing a framework. With the aid of subtended angle information, we show that the total number of edges for recognizing a framework can be reduced.

We look at a simple example first. Figure 1 presents several frameworks embedded in the plane. Observe that the framework in Figure 1(a) is minimally and globally rigid. One can see that if the information of edge $(2,3)$ is absent, as shown in Figure 1(b), the resulting framework is nonrigid. Nevertheless, the geometric shape can still be uniquely determined up to rigid transformations by $||e_{12}||$, $||e_{13}||$, and $e_{12}^T e_{13}$. This is due to the fact that $||e_{23}||^2 = ||-e_{12} + e_{13}||^2 = ||e_{12}||^2 + ||e_{13}||^2 - 2e_{12}^T e_{13}$. In fact, once $||e_{12}||$ and $||e_{13}||$ are both available, $e_{12}^T e_{13}$ is actually a constraint for the angle α subtended at vertex i. Similarly, the nonrigid framework with constraint set $\{||e_{21}||^2, e_{21}^T e_{23}, ||e_{23}||^2, e_{23}^T e_{24}, ||e_{24}||^2\}$ in Figure 1(d) is sufficient to determine the minimally rigid framework in Figure 1(c).

3.1. Definitions associated with weak rigidity. Now we introduce a generalized version of rigidity by utilizing a different rigidity function. Here $e_{ij}^T e_{ik}$ is employed as a component of the modified rigidity function. Let $\mathcal{T}_\mathcal{G} = \{(i,j,k) \in \mathcal{V}^3 : (i,j),(i,k) \in \mathcal{E}, j \le k\}$. In many cases, it is sufficient to recognize a framework when the information of $e_{ij}^T e_{ik}$ for partial $(i,j,k) \in \mathcal{T}_\mathcal{G}$ is available. We use $\mathcal{T}_\mathcal{G}^*$ with $|\mathcal{T}_\mathcal{G}^*| = s$ to denote a subset of $\mathcal{T}_\mathcal{G}$ such that for each triple $(i,j,k) \in \mathcal{T}_\mathcal{G}^*$, $e_{ij}^T e_{ik}$ is a component of the modified rigidity function. The modified rigidity function $r_\mathcal{G}(\cdot) : \mathbb{R}^{nd} \to \mathbb{R}^s$ is given by

$$r_\mathcal{G}(p) = (\ldots, e_{ij}^T e_{ik}, \ldots)^T, \quad (i,j,k) \in \mathcal{T}_\mathcal{G}^*. \tag{4}$$

Note that for a framework $(\mathcal{G}, p)$, the choice of $\mathcal{T}_\mathcal{G}^*$ is not unique. Moreover, whether $(\mathcal{G}, p)$ can be determined by (4) is directly dependent on $\mathcal{T}_\mathcal{G}^*$. Next, we give several definitions associated with weak rigidity.

Definition 3.1. *$(\mathcal{G}, p)$ and $(\mathcal{G}, q)$ are weakly equivalent for a given $\mathcal{T}_\mathcal{G}^*$ if $(p_i - p_j)^T (p_i - p_k) = (q_i - q_j)^T (q_i - q_k)$ for all $(i,j,k) \in \mathcal{T}_\mathcal{G}^*$.*

Definition 3.2. *$(\mathcal{G}, p)$ and $(\mathcal{G}, q)$ are weakly congruent if $(p_i - p_j)^T (p_i - p_k) = (q_i - q_j)^T (q_i - q_k)$ for all $i,j,k \in \mathcal{V}$.*

Definition 3.3. *A framework $(\mathcal{G}, p)$ is weakly rigid if there exists a neighborhood U_p of p such that for any $q \in U_p$ and some $\mathcal{T}_\mathcal{G}^*$, if $(\mathcal{G}, p)$ is weakly equivalent to $(\mathcal{G}, q)$, then they are weakly congruent.*

Definition 3.4. *A framework $(\mathcal{G}, p)$ is globally weakly rigid if for any $q \in \mathbb{R}^{nd}$ and some $\mathcal{T}_\mathcal{G}^*$, once $(\mathcal{G}, p)$ is weakly equivalent to $(\mathcal{G}, q)$, they are weakly congruent.*

Definition 3.5. *A framework $(\mathcal{G}, p)$ is minimally weakly rigid if $(\mathcal{G}, p)$ is weakly rigid, and deletion of any edge will make $(\mathcal{G}, p)$ not weakly rigid.*

By these definitions, the framework in Figure 1(b) with $\mathcal{T}_{\mathcal{G}}^* = \{(1,2,2),(1,3,3),(1,2,3)\}$ is globally and minimally weakly rigid, while the framework in Figure 1(d) with $\mathcal{T}_{\mathcal{G}}^* = \{(2,1,1),(2,3,3),(2,4,4),(2,1,4),(2,3,4)\}$ is minimally weakly rigid. Note that these two frameworks are both nonrigid.

In [30], the authors defined a concept of "generalized rigidity" for a multiagent formation by generalizing both the state space of each agent and the relative state constraints characterizing the formation. Since the constraint $(p_i - p_j)^T(p_i - p_k)$ can be viewed as a specific form of a function of all vertices' coordinates, weak rigidity is a special case of "generalized rigidity." The concept of "weak rigidity" is also defined in [27] by adding angle constraints into the distance-based rigidity function. Given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, an augmented graph $\bar{\mathcal{G}} = (\mathcal{V}, \bar{\mathcal{E}})$ is constructed in which $\bar{\mathcal{E}}$ is obtained by adding the edge (j,k) into the edge set $\mathcal{E}$ if the angle between $p_i - p_j$ and $p_i - p_k$ is used as an entry of the weak rigidity function. Then the relationship between weak rigidity of $(\mathcal{G}, p)$ and rigidity of $(\bar{\mathcal{G}}, p)$ is studied. Note that in [27], $\|p_i - p_j\|$ is always available for any edge (i,j). When $\|p_i - p_j\|$ and $\|p_i - p_k\|$ are both known, $(p_i - p_j)^T(p_i - p_k)$ is actually a constraint on the angle $\angle p_{jk}^i$ between $p_i - p_j$ and $p_i - p_k$. Therefore, their definition can be viewed as a special case of our definition. In our paper, the relationship between weak rigidity and rigidity is also discussed, but is given for the same framework—not by introducing an augmented graph. See Theorem 3.10 and its proof in subsection 3.3. Actually, our work focuses on exploring properties of weak rigidity in depth. We show that compared to rigidity, weak rigidity as defined in our paper has nice properties and is easier to check; see subsections 3.2–3.5. Moreover, the applications of the proposed weak rigidity theory on formation stabilization are studied; see section 4.

To preserve the invariance of $r_{\mathcal{G}}(p)$, an infinitesimal motion $v = (v_1^T, \ldots, v_n^T)^T \in \mathbb{R}^{nd}$ should satisfy

$$(v_i - v_j)^T e_{ik} + e_{ij}^T(v_i - v_k) = 0, \quad (i,j,k) \in \mathcal{T}_{\mathcal{G}}^*. \tag{5}$$

Equation (5) can be equivalently written as $\dot{r}_{\mathcal{G}} = \frac{\partial r_{\mathcal{G}}}{\partial p}\dot{p} = R_w(p)\dot{p} = 0$, where $R_w(p) \triangleq \frac{\partial r_{\mathcal{G}}}{\partial p} \in \mathbb{R}^{s \times nd}$ is called the weak rigidity matrix. Let $g_{\mathcal{K}}$ be the distance rigidity function corresponding to the complete graph $\mathcal{K}$; since it always holds that $e_{ij}^T e_{ik} = (||e_{ij}||^2 + ||e_{ik}||^2 - ||e_{jk}||^2)/2$, there exists a constant matrix $M \in \mathbb{R}^{|\mathcal{T}_{\mathcal{G}}^*| \times (n(n-1)/2)}$ such that $r_{\mathcal{G}} = M g_{\mathcal{K}}$. Note that a distance rigidity function in $\mathbb{R}^d$ is SE(d) invariant, i.e., invariant under translations and rotations. As a result, $r_{\mathcal{G}}$ is also SE(d) invariant. It is natural to obtain that the trivial motion space for weak rigidity, which consists of infinitesimal motions such that (5) always holds, is identical to the one for rigidity. We then have the following lemma directly.

LEMMA 3.6. *The trivial motion space for weak rigidity is $\mathcal{S} = \mathcal{S}_r \cup \mathcal{S}_t$, where $\mathcal{S}_r = \{(I_n \otimes A)p : A + A^T = 0, A \in \mathbb{R}^{d \times d}\}$ is the space including all infinitesimal motions that correspond to rotational motions, and $\mathcal{S}_t = \{\mathbf{1}_n \otimes q_i : q_i = (0, \ldots, 0, 1(ith), 0, \ldots, 0)^T \in \mathbb{R}^d, i = 1, \ldots, d\}$ is the space including all infinitesimal motions that correspond to translational motions.*

The specific forms of rotation motion space and translation motion space have also been given in [38] and [35] for the case when $d = 2, 3$. It is easy to see that the trivial motion space $\mathcal{S}$ always belongs to null(R_w); thus rank$(R_w) \leq nd - d(d+1)/2$. We present the following definition for infinitesimal weak rigidity.

DEFINITION 3.7. *A framework $(\mathcal{G}, p)$ is infinitesimally weakly rigid if there exists a $\mathcal{T}_{\mathcal{G}}^*$ such that every infinitesimal motion satisfying* (5) *is trivial, or, equivalently,*

$rank(R_w) = nd - d(d+1)/2$.

Observe that if each component of $r_{\mathcal{G}}$ in (4) is a length constraint of an edge, or, equivalently, if $j = k$ for all $(i,j,k) \in \mathcal{T}^*_{\mathcal{G}}$, then the weak rigidity function becomes a rigidity function. Similar to infinitesimal rigidity, an implicit condition for infinitesimal weak rigidity of $(\mathcal{G}, p)$ is $s = |\mathcal{T}^*_{\mathcal{G}}| \geq nd - d(d+1)/2$. That is, compared to the distance-based rigidity, in characterizing a framework without nontrivial infinitesimal motions, the number of edges involved in the weak rigidity function can be reduced, but the number of entries in the weak rigidity function cannot be reduced.

Suppose $\mathcal{G}$ has a spanning tree $T_r = (\mathcal{V}, \mathcal{E}_{tr})$. Let $e_{tr} = (\ldots, e_{ij}^T, \ldots)^T$, $(i,j) \in \mathcal{E}_{tr}$, $r_{T_r} = (\ldots, e_{ij}^T e_{ik}, \ldots)^T$, $(i,j), (i,k) \in \mathcal{E}_{tr}$, $(i,j,k) \in \mathcal{T}^*_{\mathcal{G}}$. Define

$$R_e^{tr}(p) \triangleq \frac{\partial r_{T_r}}{\partial e_{tr}}, \quad R_w^{tr}(p) \triangleq \frac{\partial r_{T_r}}{\partial p}. \tag{6}$$

By the chain rule we have

$$R_w^{tr}(p) = \frac{\partial r_{T_r}}{\partial e_{tr}} \frac{\partial e_{tr}}{\partial p} = R_e^{tr}(p) \bar{H}_{tr}, \tag{7}$$

where $\bar{H}_{tr} \triangleq \frac{\partial e_{tr}}{\partial p} = H_{tr} \otimes I_d$ with H_{tr} being the incidence matrix of T_r. It is easy to see that R_w^{tr} is a submatrix of R_w. Hence $\text{rank}(R_w) \geq \text{rank}(R_w^{tr})$. That is, if $\text{rank}(R_w^{tr}) = 2n - d(d+1)/2$, infinitesimal weak rigidity can be guaranteed.

Now we try to fix $\text{rank}(R_w^{tr})$ by restricting $\text{rank}(R_e^{tr})$. Rewrite e_{tr} as $e_{tr} = (\alpha_1^T, \ldots, \alpha_{n-1}^T)^T$, where for each $i \in \{1, \ldots, n-1\}$, $\alpha_i = e_{jk}$ for some $(j,k) \in \mathcal{E}_{tr}$. In fact, by regarding $\alpha_i^T \alpha_j$ as the distance of edges α_i and α_j (see [29]), $R_e^{tr} \triangleq \frac{\partial r_{T_r}}{\partial e_{tr}}$ in (7) can be viewed as a distance rigidity matrix corresponding to the following rigidity function:

$$r_{T_r} = (\ldots, \alpha_i^T \alpha_j, \ldots)^T.$$

Different from Euclidean distance, $\alpha_i^T \alpha_j$ cannot be preserved during an identical translation of α_i and α_j, i.e., $\alpha_i^T \alpha_j \neq (\alpha_i + c)^T (\alpha_j + c)$ for some $c \in \mathbb{R}^d$. Therefore, a trivial motion of α_i and α_j for preserving $\alpha_i^T \alpha_j$ can only be rotation. Since the dimension of the space spanned by independent rotations is $d(d-1)/2$, one has $\text{rank}(R_e^{tr}) \leq |\mathcal{E}_{tr}| d - d(d-1)/2 = nd - d(d+1)/2$. In fact, if we restrict $\text{rank}(R_e^{tr}) = nd - d(d+1)/2$, then $\text{null}(R_e^{tr}) = \{\bar{H}_{tr} v \in \mathbb{R}^{(n-1)d} : v \in \mathcal{S}_r\}$, together with $\text{null}(\bar{H}_{tr}) = \mathcal{S}_t$, we have $\text{null}(R_w^{tr}) = \mathcal{S}$. Therefore, we have the following lemma.

LEMMA 3.8. *Given a framework $(\mathcal{G}, p)$, if there exist a spanning tree $T_r = (\mathcal{V}, \mathcal{E}_{tr})$ in graph $\mathcal{G}$ and a $\mathcal{T}^*_{\mathcal{G}}$ such that $rank(R_e^{tr}) = nd - d(d+1)/2$, then $(\mathcal{G}, p)$ is infinitesimally weakly rigid.*

Remark 1. The rank condition on R_e^{tr} cannot be used to check infinitesimal rigidity since a framework embedded by a tree can never be rigid. However, it is efficient to determine infinitesimal weak rigidity in many circumstances. This is a critical difference between distance-based rigidity and weak rigidity, also showing that fewer edges are required for guaranteeing weak rigidity of a framework.

3.2. Construction of a minimal $\mathcal{T}^*_{\mathcal{G}}$ for infinitesimal weak rigidity in the plane. In subsection 3.1, we show that for a framework $(\mathcal{G}, p)$, a subset $\mathcal{T}^*_{\mathcal{G}}$ of $\mathcal{T}_{\mathcal{G}}$ is often sufficient for the weak rigidity function to determine the weak rigidity of $(\mathcal{G}, p)$; thus there often exist redundant elements in $\mathcal{T}_{\mathcal{G}}$. In this subsection, we will show how to construct a minimal $\mathcal{T}_\mathcal{G}^*$ (i.e., $\mathcal{T}_\mathcal{G}^*$ contains minimal number of elements) such that a planar framework $(\mathcal{G}, p)$ with $\mathcal{T}_\mathcal{G}^*$ is infinitesimally weakly rigid. Before showing this, a necessary and sufficient graphical condition for infinitesimal weak rigidity in $\mathbb{R}^2$ is presented in the following theorem.

THEOREM 3.9. *A framework $(\mathcal{G}, p)$ with $n \geq 3$ vertices in $\mathbb{R}^2$ is infinitesimally weakly rigid if and only if $\mathcal{G}$ is connected, and for any $i \in \mathcal{V}$ with $|\mathcal{N}_i| \geq 2$, there exist at least two vertices $j, k \in \mathcal{N}_i$ such that e_{ij} and e_{ik} are not collinear.*

Proof. Necessity. Actually, the necessity condition holds for frameworks in $\mathbb{R}^d$ with any $d \geq 2$. Therefore, we give a proof in the general case. We first show that $\mathcal{G}$ is connected. Suppose this is not true; then for any selected $\mathcal{T}_\mathcal{G}^*$, each independent connected subgraph can rotate independently while preserving $R_w(p)\dot{p} = 0$. This conflicts with Definition 3.7. Therefore, graph $\mathcal{G}$ must be connected. Next we prove the second part. Suppose that vertex i has more than two neighbors, and all e_{ij}, $j \in \mathcal{N}_i$, are collinear. Notice that e_{ij} cannot all be zero; otherwise, $\text{rank}(R_w(p)) < nd - d(d+1)/2$, and so the framework $(\mathcal{G}, p)$ is not infinitesimally weakly rigid. Let e_{ik} be a nonzero vector; then $e_{ij} = c_j e_{ik}$ with a nonzero $c_j \in \mathbb{R}$ for all $j \in \mathcal{N}_i$. Let $A \in \mathbb{R}^{d\times d}$ be a nontrivial skew-symmetric matrix; we can obtain that $q = (\mathbf{0}, \ldots, \mathbf{0}, (Ae_{ik})^T, \mathbf{0}, \ldots, \mathbf{0})^T \in \text{null}(R_w)$, where the components of $(Ae_{ik})^T$ are $(i-1)d + 1$ to id components of q. Due to Lemma 3.6, q is not a trivial motion, and a contradiction with infinitesimal weak rigidity of $(\mathcal{G}, p)$ arises.

Sufficiency. We first claim that there exists a spanning tree $T_r = (\mathcal{V}, \mathcal{E}_{tr})$ of $\mathcal{G}$ satisfying that for any $i \in \mathcal{V}$, there are at least two vertices $j, k \in \mathcal{V}$ such that $(i, j), (i, k) \in \mathcal{E}_{tr}$, and e_{ij} and e_{ik} are not collinear. Suppose this is not true. Then there exists a vertex i with $|\mathcal{N}_i| \geq 3$ in a cycle of $\mathcal{G}$; the deletion of any edge (i, j) in this cycle will make e_{il}, $l \in \mathcal{N}_i \setminus \{j\}$, be all collinear. This implies that e_{ij} and e_{il} are not collinear for all $l \in \mathcal{N}_i \setminus \{j\}$. Note that there must be two edges involving i in the cycle. Without loss of generality, let $(i, j), (i, k)$ be these two edges. Then one can see that deleting (i, k) rather than (i, j) can also eliminate the cycle and make the vectors e_{il}, $l \in \mathcal{N}_i \setminus \{k\}$, be not all collinear, which is a contradiction. Hence the existence of T_r is proved. By Lemma 3.8, it suffices to show $\text{rank}(R_e^{tr}) = 2n - d(d+1)/2$.

By virtue of the above conclusion, the sufficiency can be proved in the case when $\mathcal{G}$ is a tree and generality is not lost. Now we regard $\mathcal{G}$ as a tree. It is only required to construct a set $\mathcal{T}_\mathcal{G}^*$, such that $\text{rank}(R_e) = \frac{\partial r_\mathcal{G}}{\partial e} = |\mathcal{E}|d - d(d-1)/2 = 2m - 1$, where $e = (\ldots, e_{ij}^T, \ldots)^T$, $(i, j) \in \mathcal{E}$. Let $\mathcal{T}_\mathcal{G}^* = \{(i, j, j) \in \mathcal{V}^3 : (i, j) \in \mathcal{E}\} \cup \mathcal{F}$; then $r_\mathcal{G} = (g_\mathcal{G}^T, \bar{r}^T)^T$, where $\bar{r} = (\ldots, e_{ij}^T e_{ik}, \ldots)^T$, $(i, j, k) \in \mathcal{F}$. Finding elements of $\mathcal{F}$ is equivalent to finding components of $\bar{r}$. Next we present an approach to constructing $\bar{r}$.

Let $\mathcal{H} \subseteq \mathcal{V}$ be the set of internal vertices, i.e., the vertices with more than two neighbors in $\mathcal{G}$. That is, $|\mathcal{N}_i| \geq 2$ for all $i \in \mathcal{H}$. Since $n \geq 3$ and $\mathcal{G}$ is connected, $|\mathcal{H}| \neq \varnothing$. Note that it always holds that

$$2|\mathcal{E}| = \sum_{i\in\mathcal{V}} |\mathcal{N}_i| = |\mathcal{V}| - |\mathcal{H}| + \sum_{i\in\mathcal{H}} |\mathcal{N}_i|.$$

Since $\mathcal{G}$ is a tree, it holds that $n = |\mathcal{V}| = |\mathcal{E}| + 1 = m + 1$. It follows that

$$\sum_{i\in\mathcal{H}} |\mathcal{N}_i| - |\mathcal{H}| = m - 1.$$

In fact, we can give $|\mathcal{N}_i| - 1$ components of $\bar{r}$ for each $i \in \mathcal{H}$, which are pairwise inner products of relative position vectors corresponding to the $|\mathcal{N}_i|$ edges. Note that for

$i \in \mathcal{H}$, we can always divide $\mathcal{N}_i$ into two disjoint sets $\hat{\mathcal{N}}_i$ and $\check{\mathcal{N}}_i$ such that e_{ij} and e_{ik} are not collinear for any $j \in \hat{\mathcal{N}}_i$, $k \in \check{\mathcal{N}}_i$.

We first select a vertex $j_i \in \hat{\mathcal{N}}_i$ randomly and let $e_{ij_i}^T e_{ik}$, $k \in \check{\mathcal{N}}_i$, be partial components of $\bar{r}$. Next we select a vertex $k_i \in \check{\mathcal{N}}_i$ randomly and let $e_{ij}^T e_{ik_i}$, $j \in \hat{\mathcal{N}}_i \setminus \{j_i\}$, be the components of $\bar{r}$. Then we have presented an approach to giving $|\hat{\mathcal{N}}_i| + |\check{\mathcal{N}}_i| - 1 = |\mathcal{N}_i| - 1$ components of $\bar{r}$ for a vertex $i \in \mathcal{H}$. By this approach, we can totally give $\sum_{i\in\mathcal{H}} \mathcal{N}_i - |\mathcal{H}| = m - 1$ components of $\bar{r}$. Now we prove that the rows of R_e corresponding to these $m-1$ constraints, which are actually the rows of $\frac{\partial \bar{r}}{\partial e}$, together with the rows of $\frac{\partial g_{\mathcal{G}}}{\partial e}$, are linearly independent. Suppose the following holds for some scalars l_{ij}, $\bar{l}_{ij}$:

$$\sum_{i\in\mathcal{H}} \left(\sum_{k\in\check{\mathcal{N}}_i} l_{ik} \frac{\partial e_{ij_i}^T e_{ik}}{\partial e} + \sum_{j\in\hat{\mathcal{N}}_i\setminus\{j_i\}} l_{ij} \frac{\partial e_{ij}^T e_{ik_i}}{\partial e} + \sum_{h\in\mathcal{N}_i} \bar{l}_{ih} \frac{\partial ||e_{ih}||^2}{\partial e} \right) = 0. \tag{8}$$

Let $\bar{\mathcal{H}} \subseteq \mathcal{H}$ be the set such that for any vertex $i \in \bar{\mathcal{H}}$, $\mathcal{N}_i$ includes at least one leaf vertex, where a leaf vertex is a vertex having only one neighbor. In fact, there must exist a leaf vertex $j \in \mathcal{N}_i$ for some $i \in \bar{\mathcal{H}}$, such that only one component of $\bar{r}$ involves e_{ij}. To show this, we claim that one of the following statements must be true.

(i) There exists a vertex $i \in \bar{\mathcal{H}}$ such that $|\mathcal{N}_i| = 2$.

(ii) There exists a vertex $i \in \bar{\mathcal{H}}$, such that $|\mathcal{N}_i| = 3$, and $\mathcal{N}_i$ includes at least two leaf vertices.

(iii) There exists a vertex $i \in \bar{\mathcal{H}}$, such that $|\mathcal{N}_i| \geq 4$, and $\mathcal{N}_i$ includes at least three leaf vertices.

Suppose that all the above statements are not true. Then $\bar{\mathcal{H}}$ can be divided into three sets, i.e., $\bar{\mathcal{H}} = \bar{\mathcal{H}}_1 \cup \bar{\mathcal{H}}_2 \cup \bar{\mathcal{H}}_3$, such that if $i \in \bar{\mathcal{H}}_1$, then $|\mathcal{N}_i| = 3$ and $\mathcal{N}_i$ includes one leaf vertex exactly; if $i \in \bar{\mathcal{H}}_2$, then $|\mathcal{N}_i| \geq 4$ and $\mathcal{N}_i$ includes one leaf vertex exactly; if $i \in \bar{\mathcal{H}}_3$, then $|\mathcal{N}_i| \geq 4$ and $\mathcal{N}_i$ includes two leaf vertices exactly. Let n_i be the number of leaf vertices having a neighbor in $\bar{\mathcal{H}}_i$, $i = 1, 2, 3$. Then we have $|\bar{\mathcal{H}}_1| = n_1$, $|\bar{\mathcal{H}}_2| = n_2$, $|\bar{\mathcal{H}}_3| = n_3/2$, and $n_1 + n_2 + n_3 = n - |\mathcal{H}|$. It follows that

$$\begin{aligned} 2m = \sum_{i\in\mathcal{V}} |\mathcal{N}_i| &\geq n - |\mathcal{H}| + 3|\bar{\mathcal{H}}_1| + 4|\bar{\mathcal{H}}_2| + 4|\bar{\mathcal{H}}_3| + 2(|\mathcal{H}| - |\bar{\mathcal{H}}|) \\ &= n + |\mathcal{H}| + n_1 + 2n_2 + n_3 \\ &\geq n + |\mathcal{H}| + n - |\mathcal{H}| = 2n. \end{aligned}$$

This conflicts with $m = n - 1$. Hence, there is at least one true statement among (i), (ii), and (iii). Now we discuss the following three cases.

Case 1. (i) holds. Let $\mathcal{N}_i = \{j, k\}$, where j is a leaf vertex. Then it is obvious that there is only one component $e_{ij}^T e_{ik}$ in $\bar{r}$ involving j.

Case 2. (ii) holds. There are $|\mathcal{N}_i| - 1 = 2$ components selected from $e_{ij}^T e_{ik}$. Since $\mathcal{N}_i$ includes two leaf vertices, there must exist one leaf vertex $l \in \mathcal{N}_i$ such that e_{il} is involved by only one of the two components.

Case 3. (iii) holds. Note that for any $i \in \mathcal{H}$, only two vertices $j_i, k_i \in \mathcal{N}_i$ are possibly involved by more than two components selected from $e_{ij}^T e_{ik}$, $j, k \in \mathcal{N}_i$. Hence, there must exist at least one leaf vertex $l \in \mathcal{N}_i$ such that e_{il} is involved by only one component of $\bar{r}$.

So far we have proved that there always exists a leaf vertex $j \in \mathcal{N}_i$ for some $i \in \bar{\mathcal{H}}$, such that only one component $e_{ij}^T e_{ik}$ of $\bar{r}$ involves e_{ij}. Observe that there are only two nonzero rows in $\frac{\partial r_{\mathcal{G}}}{\partial e_{ij}}$, i.e., e_{ij}^T and e_{ik}^T. Since e_{ij} and e_{ik} are not collinear, the

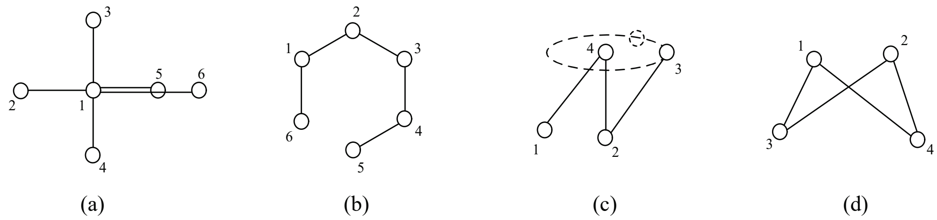

Fig. 2. *Four nonrigid frameworks.* (a) *A minimally infinitesimally weakly rigid framework in* $\mathbb{R}^2$ *with a minimal* $\mathcal{T}_{\mathcal{G}}^* = \{(1,2,3),(1,2,4),(1,3,5),(1,3,6),(1,j,j),\ j \in \mathcal{N}_1\}$. (b) *A minimally infinitesimally weakly rigid framework in* $\mathbb{R}^2$ *with* $\mathcal{T}_{\mathcal{G}}^* = \{(1,2,6),(2,1,3),(3,2,4),(4,3,5),(i,j,j),(i,j) \in \mathcal{E}, i > j\}$. (c) *A framework which is not weakly rigid in* $\mathbb{R}^3$. (d) *A minimally and globally weakly rigid framework in* $\mathbb{R}^3$ *with* $\mathcal{T}_{\mathcal{G}}^* = \{(1,3,3),(1,4,4),(2,3,3),(2,4,4),(1,3,4),(3,1,2)\}$.

validity of (8) implies $l_{ij} = \bar{l}_{ij} = 0$. Note that after deleting vertex j and edge (i,j), $\mathcal{G}' = (\mathcal{V} \setminus \{j\}, \mathcal{E} \setminus \{(i,j)\})$ is another tree. By the aforementioned approach, we can prove that once (8) holds, there is a leaf vertex $j' \in \mathcal{V} \setminus \{j\}$, such that $l_{i'j'} = \bar{l}_{i'j'} = 0$. By repeating this process, we can finally obtain that $l_{ij} = \bar{l}_{ij} = 0$ for all $(i,j) \in \mathcal{E}$. As a result, $\text{rank}(R_e) = m + m - 1 = 2m - 1$. That is, $(\mathcal{G}, p)$ is infinitesimally weakly rigid. ◻

Two infinitesimally weakly rigid frameworks are given in Figures 2(a) and 2(b) to demonstrate Theorem 3.9.

It is important to note that by virtue of Theorem 3.9, the sufficiency condition in Lemma 3.8 for infinitesimal weak rigidity is also necessary when $d = 2$. One may ask whether the sufficiency of Theorem 3.9 and necessity of Lemma 3.8 hold for frameworks in $\mathbb{R}^d$ with $d \geq 3$. The answer is no. We show two counterexamples as follows.

Two counterexamples. In Figure 2, two frameworks in $\mathbb{R}^3$ are shown in (c) and (d). In (c), for vertex 4, e_{41} and e_{42} are not collinear. Similarly, for vertex 2, e_{23} and e_{24} are not collinear. However, vertex 3 can move along the dotted circle continuously while preserving the invariance of $r_{\mathcal{G}} = (||e_{14}||^2, ||e_{24}||^2, ||e_{23}||^2, e_{41}^T e_{42}, e_{23}^T e_{24})^T$, which implies that $\text{null}(R_w)$ includes nontrivial infinitesimal motions. Hence, the sufficiency of Theorem 3.9 is invalid. The framework $(\mathcal{G}, p)$ in (d) is infinitesimally weakly rigid. However, each spanning tree T_r of $\mathcal{G}$ is isomorphic to the graph in (c); thus the framework (T_r, p) also has nontrivial infinitesimal motions in $\mathbb{R}^3$. This implies that the necessity of Lemma 3.8 does not hold in $\mathbb{R}^3$.

Remark 2. Observe that the necessary and sufficient condition in Theorem 3.9 can be easily verified instead of examining the rank of a matrix and is necessary but not sufficient for a framework to be infinitesimally rigid. This implies that infinitesimal weak rigidity is milder than infinitesimal rigidity for a framework. Moreover, the proof of Theorem 3.9 actually provides an idea for constructing a minimal set $\mathcal{T}_{\mathcal{G}}^*$ for infinitesimal weak rigidity of $(\mathcal{G}, p)$ and thus will be used later.

By Theorem 3.9, it is easy to design algorithms for examining infinitesimal weak rigidity of a planar framework. In the following, we will mainly focus on how to construct a minimal $\mathcal{T}_{\mathcal{G}}^*$ for a given infinitesimally weakly rigid framework $(\mathcal{G}, p)$. According to Theorem 3.9, there must exist a spanning tree T_r of $\mathcal{G}$ such that (T_r, p) is minimally infinitesimally weakly rigid. We present Algorithm 1 to find (T_r, p).

Notice that even for a minimally infinitesimally weakly rigid framework $(\mathcal{G}, p)$, it

Algorithm 1. Finding a minimally infinitesimally weakly rigid subframework.

Input: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, $p = (p_1^T, \dots, p_n^T)^T \in \mathbb{R}^{2n}$.
Output: (T_r, p)

Initialize $\mathcal{V}_{tr} \leftarrow \{a, b\}$, $\mathcal{E}_{tr} \leftarrow (a, b)$, where a, b are selected randomly such that $(a, b) \in \mathcal{E}$
while $|\mathcal{V}_{tr}| < n$ **do**
  Select an edge $(i, j) \in \mathcal{E}$ such that $i \in \mathcal{V}_{tr}$, $j \in \mathcal{V} \setminus \mathcal{V}_{tr}$, and there exists at least one edge $(i, k) \in \mathcal{E}_{tr}$ such that $p_i - p_j$ is not collinear with $p_i - p_k$
  $\mathcal{V}_{tr} \leftarrow \{j\}$, $\mathcal{E}_{tr} \leftarrow \{(i, j)\}$
end while
$T_r \leftarrow (\mathcal{V}_{tr}, \mathcal{E}_{tr})$
return (T_r, p)

may be possible that $|\mathcal{T}_{\mathcal{G}}| > nd - d(d+1)/2$. For example, in Figure 2(a), a minimal $\mathcal{T}_{\mathcal{G}}^*$ should have $2n - 3 = 9$ elements, but $|\mathcal{T}_{\mathcal{G}}| = 15$. In this case, we still have to choose suitable elements from $\mathcal{T}_{\mathcal{G}}$ to form a minimal $\mathcal{T}_{\mathcal{G}}^*$. We adopt $\mathcal{T}_{\mathcal{G}}^{\dagger}$ to denote the minimal $\mathcal{T}_{T_r}^*$ for infinitesimal weak rigidity of (T_r, p) generated by Algorithm 1, which is also the minimal $\mathcal{T}_{\mathcal{G}}^*$ for infinitesimal weak rigidity of $(\mathcal{G}, p)$. Algorithm 2 is designed to construct $\mathcal{T}_{\mathcal{G}}^{\dagger}$.

Algorithm 2. Construction of $\mathcal{T}_{\mathcal{G}}^{\dagger}$.

Input: $T_r = (\mathcal{V}_{tr}, \mathcal{E}_{tr})$, $p = (p_1^T, \dots, p_n^T)^T \in \mathbb{R}^{2n}$.
Output: $\mathcal{T}_{\mathcal{G}}^{\dagger}$

Initialize $\mathcal{T}_{\mathcal{G}}^{\dagger} \leftarrow \{(i, j, j) \in \mathcal{V}_{tr}^3 : (i, j) \in \mathcal{E}_{tr}\}$
for all $i \in \mathcal{V}_{tr}$ **do**
  Compute the neighbor set of i in T_r, i.e., $\mathcal{N}_i$. Proceed only if $|\mathcal{N}_i| \geq 2$
  $j_i \leftarrow \min \mathcal{N}_i$, $\hat{\mathcal{N}}_i \leftarrow \{j_i\} \cup \{k \in \mathcal{N}_i : p_i - p_{j_i}$ *is collinear with* $p_i - p_k\}$, $\check{\mathcal{N}}_i \leftarrow \mathcal{N}_i \setminus \hat{\mathcal{N}}_i$. Proceed only if $\check{\mathcal{N}}_i \neq \varnothing$
  $\mathcal{T}_{\mathcal{G}}^{\dagger} \leftarrow \mathcal{T}_{\mathcal{G}}^{\dagger} \cup (i, j_i, k)$ for all $k \in \check{\mathcal{N}}_i$. Proceed only if $|\hat{\mathcal{N}}_i| > 1$
  Select k_i from $\check{\mathcal{N}}_i$ randomly
  for all $j \in \hat{\mathcal{N}}_i \setminus \{j_i\}$ **do**
    $\mathcal{T}_{\mathcal{G}}^{\dagger} \leftarrow \mathcal{T}_{\mathcal{G}}^{\dagger} \cup (i, j, k_i)$ if $j < k_i$, $\mathcal{T}_{\mathcal{G}}^{\dagger} \leftarrow \mathcal{T}_{\mathcal{G}}^{\dagger} \cup (i, k_i, j)$ otherwise
  end for
end for
return $\mathcal{T}_{\mathcal{G}}^{\dagger}$

By Theorem 3.9, it is easy to see that $\mathcal{T}_{\mathcal{G}}^{\dagger}$ generated by Algorithm 2 is sufficient for determining infinitesimal weak rigidity of $(\mathcal{G}, p)$. Moreover, $\mathcal{T}_{\mathcal{G}}^{\dagger}$ contains $2n - 3$ elements exactly and thus is minimal for infinitesimal weak rigidity of $(\mathcal{G}, p)$. In Figure 2(a), the framework is minimally infinitesimally weakly rigid; thus $\mathcal{T}_{\mathcal{G}}^{\dagger}$ can be obtained by Algorithm 2 directly. A possible $\mathcal{T}_{\mathcal{G}}^{\dagger}$ generated by Algorithm 2 is shown in the caption. In Figure 2(b), $|\mathcal{T}_{\mathcal{G}}| = 9 = 2n - 3$; therefore, $\mathcal{T}_{\mathcal{G}}^{\dagger} = \mathcal{T}_{\mathcal{G}}$.

3.3. Comparisons between rigidity and weak rigidity. Compared to distance-based rigidity, the advantage of weak rigidity is that fewer edges are required to determine a shape. The following theorem shows that rigidity is sufficient for weak rigidity.

THEOREM 3.10. *If a framework $(\mathcal{G}, p)$ is (infinitesimally, globally, minimally) rigid, then it is (infinitesimally, globally, minimally) weakly rigid.*

Proof. We choose a $\mathcal{T}_{\mathcal{G}}^*$ such that $(i,j,j) \in \mathcal{T}_{\mathcal{G}}^*$ for all $(i,j) \in \mathcal{E}$; then it is obvious that $rank(R_w) \geq rank(R)$. Therefore, infinitesimal rigidity leads to infinitesimal weak rigidity. Note that the components of $r_{\mathcal{K}}$ can always be denoted by a linear combination of distance constraints, i.e., $e_{ij}^T e_{ik} = (||e_{ij}||^2 + ||e_{ik}||^2 - ||e_{jk}||^2)/2$; therefore, congruence implies weak congruence.

Suppose $(\mathcal{G}, p)$ is rigid, and $(\mathcal{G}, q)$ is an arbitrary framework which is weakly equivalent to $(\mathcal{G}, p)$ for the above-mentioned $\mathcal{T}_{\mathcal{G}}^*$. Then they are also equivalent. From rigidity of $(\mathcal{G}, p)$, there exists a neighborhood U_p of p such that for any $q \in U_p$, $(\mathcal{G}, p)$ and $(\mathcal{G}, q)$ are congruent, which in turn implies that $(\mathcal{G}, p)$ and $(\mathcal{G}, q)$ are weakly congruent. Therefore, $(\mathcal{G}, p)$ is weakly rigid. Similarly, one can obtain that global rigidity implies global weak rigidity, and minimal rigidity implies minimal weak rigidity. ◻

The converse of Theorem 3.10 is not true, which has been shown in Remarks 1 and 2. This also implies that weak rigidity requires fewer edges in the graph than rigidity does.

3.4. The connection between weak rigidity and rigidity: A matrix completion perspective. Using graph rigidity theory with the rigidity function (1), a graph realization problem is actually equivalent to a completion problem of a Euclidean distance matrix (EDM) completion problem; see [15, 29, 17]. A matrix completion problem asks whether the unspecified entries of partially defined matrix can be completed to obtain a fully defined matrix satisfying a desired property. An EDM is a matrix whose entries are the pairwise squared Euclidean distances among a set of n points in d-dimensional space [15]. For a framework $(\mathcal{G}, p)$, we denote the corresponding EDM by $D(p) \in \mathbb{R}^{n \times n}$. It is easy to see that two frameworks $(\mathcal{G}, p)$ and $(\mathcal{G}, q)$ are congruent if and only if $D(p) = D(q)$. Therefore, a framework can be determined up to rigid transformations if and only if the corresponding EDM can be uniquely completed. The following theorem shows a relationship between weak congruence and congruence.

THEOREM 3.11. *Two frameworks $(\mathcal{G}, p)$ and $(\mathcal{G}, q)$ are weakly congruent if and only if they are congruent (i.e., $D(p) = D(q)$).*

Proof. The necessity is obvious from Definition 3.2. The sufficiency has been proved in the proof of Theorem 3.10. ◻

It is straightforward to obtain the following corollary.

COROLLARY 3.12. *Given a framework $(\mathcal{K}, p)$ in $\mathbb{R}^d$, it holds that $g_{\mathcal{K}}^{-1}(g_{\mathcal{K}}(p)) = r_{\mathcal{K}}^{-1}(r_{\mathcal{K}}(p))$ for $\mathcal{T}_{\mathcal{K}}^* = \mathcal{T}_{\mathcal{K}}$.*

Proof. For any $q \in \mathbb{R}^{nd}$, it follows from Theorem 3.11 that

$$q \in g_{\mathcal{K}}^{-1}(g_{\mathcal{K}}(p)) \Leftrightarrow g_{\mathcal{K}}(p) = g_{\mathcal{K}}(q) \Leftrightarrow r_{\mathcal{K}}(p) = r_{\mathcal{K}}(q) \Leftrightarrow q \in r_{\mathcal{K}}^{-1}(r_{\mathcal{K}}(p)).$$

The proof is completed. ◻

Corollary 3.12 implies that given a globally weakly rigid framework $(\mathcal{G}, p^*)$ and a globally rigid framework $(\bar{\mathcal{G}}, p^*)$, although $\mathcal{G}$ may have fewer edges than $\bar{\mathcal{G}}$, it holds that $r_{\mathcal{G}}(p^*)$ and $g_{\bar{\mathcal{G}}}(p^*)$ determine an identical geometric shape up to translations, rotations, and reflections.

In fact, when we use the weak rigidity function (4) to recognize frameworks in $\mathbb{R}^d$, the graph realization problem can be transformed to a positive semidefinite (PSD) matrix completion problem [15]. More precisely, let $E(p) \in \mathbb{R}^{d\times m}$ be the corresponding matrix with each column being a relative location vector, i.e., $E = (\dots, e_{ij}, \dots) \in \mathbb{R}^{d\times m}$. We can observe that each component of $r_{\mathcal{G}}(\cdot)$ is actually an entry of the gram matrix $\mathbf{E} = E^T E$. If we regard $D(e_{ij}, e_{kl}) = e_{ij}^T e_{kl}$ as the distance between e_{ij} and e_{kl}, then $\mathbf{E}$ becomes the distance matrix to be completed. The following theorem shows that for a connected graph $\mathcal{G}$, the framework $(\mathcal{G}, p)$ can be determined up to rigid transformations if and only if the gram matrix $\mathbf{E}(p) \in \mathbb{R}^{m\times m}$ can be uniquely completed.

THEOREM 3.13. *Given two frameworks $(\mathcal{G}, p)$ and $(\mathcal{G}, q)$, if $\mathcal{G}$ is connected, then $D(p) = D(q)$ if and only if $\mathbf{E}(p) = \mathbf{E}(q)$.*

Proof. Necessity. Suppose that $||p_i - p_j|| = ||q_i - q_j||$ for any $i, j \in \mathcal{V}$. Due to the fact that $e_{ij}^T e_{kl} = \frac{1}{2}(||e_{jk}||^2 + ||e_{il}||^2 - ||e_{ik}||^2 - ||e_{jl}||^2)$, we have $(p_i - p_j)^T (p_k - p_l) = (q_i - q_j)^T (q_k - q_l)$ for any $(i, j), (k, l) \in \mathcal{E}$.

Sufficiency. Since $\mathcal{G}$ is connected, for any $i, k \in \mathcal{V}$, there exists a path $\mathcal{P} = \{(i, i_1), \dots, (i_r, k)\} \subseteq \mathcal{E}$. It follows that

$$\begin{aligned}
||p_i - p_k||^2 &= ||p_i - p_{i_1} + \cdots + p_{i_r} - p_k||^2 \\
&= \sum_{(j,l),(u,v)\in\mathcal{P}} (p_j - p_l)^T (p_u - p_v) \\
&= \sum_{(j,l),(u,v)\in\mathcal{P}} (q_j - q_l)^T (q_u - q_v) \\
&= ||q_i - q_k||^2. \qquad \square
\end{aligned}$$

Now we are ready to present two necessity conditions for infinitesimal weak rigidity as follows.

THEOREM 3.14. *If $(\mathcal{G}, p)$ with $n \geq d + 1$ is infinitesimally weakly rigid for $\mathcal{T}_{\mathcal{G}}^*$, then*

(i) *$p_1, \dots, p_n$ do not lie in a hyperplane of $\mathbb{R}^d$;*

(ii) *$(\mathcal{G}, p)$ is weakly rigid for $\mathcal{T}_{\mathcal{G}}^*$.*

Proof. (i) Suppose this is not true. Then there always exist a nonzero normal vector $\eta \in \mathbb{R}^d$ and some constant $c \in \mathbb{R}$ such that $\eta^T p_i = c$ for all $i \in \mathcal{V}$. Then $e_{ij}^T \eta = 0$ for any $i, j \in \mathcal{V}$, which implies that $q = (\eta^T, 0, \dots, 0)^T \in \text{null}(R_w)$. However, if $\dot{p} = q$, only vertex 1 has a nonzero velocity; hence q is obviously neither a rotational motion nor a translational motion. That is, q does not belong to the trivial motion space described in Lemma 3.6, and a contradiction arises.

(ii) For the differentiable map $r_{\mathcal{G}}(\cdot) : \mathbb{R}^{nd} \to \mathbb{R}^s$, infinitesimal weak rigidity of $(\mathcal{G}, p)$ implies that $\dim(\text{null}(R_w))$ reaches its minimum and $\frac{\partial r_{\mathcal{G}}}{\partial p}$ has a maximal rank at p. As a result, p is a regular point of $r_{\mathcal{G}}$. Reference [3, Proposition 2] shows that there exists a neighborhood U_p of p, such that $r_{\mathcal{G}}^{-1}(r_{\mathcal{G}}(p)) \cap U_p$ is a differentiable manifold of dimension $nd - \text{rank}(\frac{\partial r_{\mathcal{G}}}{\partial p}) = d(d+1)/2$. Let $M = g_{\mathcal{K}}^{-1}(g_{\mathcal{K}}(p))$, where $\mathcal{K}$ is the complete graph with vertex set $\mathcal{V}$. The proof in [3, Theorem] shows that M is a manifold of dimension $d(d+1)/2 - (d-a)(d-a-1)/2 = (a+1)(2d-a)/2$, where a is the dimension of the affine hull of $\{p_1, \dots, p_n\}$. According to Corollary 3.12, it holds that $M = r_{\mathcal{K}}^{-1}(r_{\mathcal{K}}(p))$ with $\mathcal{T}_{\mathcal{G}}^* = \mathcal{T}_{\mathcal{G}}$. Note that $M \cap U_p$ is a submanifold of $r_{\mathcal{G}}^{-1}(r_{\mathcal{G}}(p)) \cap U_p$ and they are equal if $a = d$ or $a = d - 1$. The validity of (i) implies that $a = d$. Hence, $(\mathcal{G}, p)$ is weakly rigid. ∎

Similar to traditional graph rigidity, weak rigidity cannot induce infinitesimal weak rigidity. A typical counterexample is a framework $(\mathcal{G}, p)$ with $|\mathcal{V}| \geq d+1$ in $\mathbb{R}^d$, where $\mathcal{G}$ is a complete graph, and e_{ij} for all $(i,j) \in \mathcal{E}$ lie on a hyperplane. In this case, $(\mathcal{G}, p)$ is globally rigid and globally weakly rigid. However, when we let the normal vector to the hyperplane be the velocity of one vertex and zero be the velocity of all the other vertices, a nontrivial motion is constructed. Hence infinitesimal weak rigidity is not guaranteed.

Remark 3. By virtue of Theorems 3.11, 3.13, and 3.14, once a framework $(\mathcal{G}, p)$ is infinitesimally weakly rigid, there exists a neighborhood U_p of $p \in \mathbb{R}^d$, such that if $q \in U_p$ and $r_{\mathcal{G}}(p) = r_{\mathcal{G}}(q)$, then $\mathbf{E}(p) = \mathbf{E}(q)$. Note that Theorem 3.14(i) implies $rank(E(p)) = d$. Hence, the Cholesky decomposition of $\mathbf{E}(p)$ determines $E(p)$ uniquely up to an orthogonal matrix $A \in \mathbb{R}^{d\times d}$. It follows that $p_i - p_j = A(q_i - q_j)$ for all $(i,j) \in \mathcal{E}$. We then have $p_i = Aq_i + c$ for some $c \in \mathbb{R}^d$. If the determinant of A is 1, then $A \in \mathrm{SO(d)}$ is a rotation matrix; otherwise A can be written as the product of a reflection matrix and a rotation matrix. The vector c can be regarded as a translation vector. As a result, $(\mathcal{G}, p)$ can be obtained by a rigid transformation from $(\mathcal{G}, q)$.

3.5. Generic property. In [3, 9], the authors show that rigidity is a generic property of the graph. In other words, for any graph $\mathcal{G}$, if $(\mathcal{G}, p)$ is rigid for some generic configuration $p \in \mathbb{R}^{nd}$, then $(\mathcal{G}, q)$ is rigid for any generic configuration $q \in \mathbb{R}^{nd}$. Here a configuration $p = (p_1^T, \ldots, p_n^T) \in \mathbb{R}^{nd}$ is said to be generic if its nd coordinates are algebraically independent over integers [9]. A vector $\alpha = (\alpha_1, \ldots, \alpha_{nd})$ is algebraically independent if there does not exist a nonzero polynomial $h(x_1, \ldots, x_{nd})$ with integer coefficients such that $h(\alpha_1, \ldots, \alpha_{nd}) = 0$. Since generic configurations form a dense subset of $\mathbb{R}^{nd}$, once $(\mathcal{G}, p)$ is rigid for some generic configuration $p \in \mathbb{R}^{nd}$, $(\mathcal{G}, q)$ is rigid for almost all configurations $q \in \mathbb{R}^{nd}$. In this subsection, we will show that for a framework $(\mathcal{G}, p)$, both infinitesimal weak rigidity and weak rigidity are generic properties and thus are primarily determined by the graph $\mathcal{G}$ rather than the configuration p. Note that we only consider the case when $n \geq d+1$.

Analogously to the discussions of generic rigidity for graphs in [9, section 1.2], we present definitions of generic weak rigidity and generic infinitesimal weak rigidity for a graph as follows.

DEFINITION 3.15. *Graph $\mathcal{G}$ is said to be generically (infinitesimally) weakly rigid in $\mathbb{R}^d$ if for any generic configuration $p \in \mathbb{R}^{nd}$, $(\mathcal{G}, p)$ is (infinitesimally) weakly rigid.*

The following theorem shows that infinitesimal weak rigidity is a generic property of the graph.

THEOREM 3.16. *If $(\mathcal{G}, p)$ is infinitesimally weakly rigid for a generic configuration $p \in \mathbb{R}^{nd}$, then graph $\mathcal{G}$ is generically infinitesimally weakly rigid in $\mathbb{R}^d$.*

Proof. Let $p^* \in \mathbb{R}^{nd}$ be the generic configuration such that $(\mathcal{G}, p^*)$ is infinitesimally weakly rigid for $\mathcal{T}_{\mathcal{G}}^*$. Then we have $\mathrm{rank}(R_w(p^*)) = \mathrm{rank}(\frac{\partial r_{\mathcal{G}(p)}}{\partial p}|_{p=p^*}) = nd - d(d+1)/2$. Since $\mathrm{rank}(R_w(q)) \leq nd - d(d+1)/2$ for all $q \in \mathbb{R}^{nd}$, we have $\max_{q\in\mathbb{R}^{nd}} \mathrm{rank}(R_w(q)) = nd - d(d+1)/2$. Let $p \in \mathbb{R}^{nd}$ be a generic point distinct to p^*. The algebraic independence property of p implies that each $(nd - d(d+1)/2) \times (nd - d(d+1)/2)$ minor of $R_w(p)$ cannot be zero. As a result, p is a regular point, i.e., $\mathrm{rank}(R_w(p)) = \max_{q\in\mathbb{R}^{nd}} \mathrm{rank}(R_w(q)) = nd - d(d+1)/2$. Therefore, $(\mathcal{G}, p)$ is infinitesimally weakly rigid. The proof is completed. ∎

To show that weak rigidity is also a generic property, we give the following result.

THEOREM 3.17. *If $(\mathcal{G}, p)$ is weakly rigid for $\mathcal{T}_{\mathcal{G}}^*$ in $\mathbb{R}^d$, and $p \in \mathbb{R}^{nd}$ is generic, then $(\mathcal{G}, p)$ is infinitesimally weakly rigid for $\mathcal{T}_{\mathcal{G}}^*$ in $\mathbb{R}^d$.*

Proof. Let $\kappa = \max_{p \in \mathbb{R}^{nd}}\{\text{rank}(\frac{\partial r_{\mathcal{G}}(p)}{\partial p})\}$ with respect to $\mathcal{T}_{\mathcal{G}}^*$. From the proof of Theorem 3.16, a generic configuration is always a regular point (also shown in [9, Proposition 3.1]). Then we have $\text{rank}(R_w(p)) = \kappa$. It follows from [3, Proposition 2] that there exists a neighborhood U_p of p, such that $r_{\mathcal{G}}^{-1}(r_{\mathcal{G}}(p)) \cap U_p$ is a manifold of dimension $nd - \kappa$. By Corollary 3.12 and [3, Theorem], $r_{\mathcal{K}}^{-1}(r_{\mathcal{K}}(p)) = g_{\mathcal{K}}^{-1}(g_{\mathcal{K}}(p))$ is a manifold of dimension $(a+1)(2d-a)/2$, where $a = \text{rank}(p_1, \dots, p_n)$. Together with the fact that $(\mathcal{G}, p)$ is weakly rigid for $\mathcal{T}_{\mathcal{G}}^*$, there must hold $nd - \kappa = (a+1)(2d-a)/2$.

Note that there must hold $a = d$ for the generic configuration p; otherwise the determinant of each $d \times d$ minor of $P = (p_1, \dots, p_n)$ is zero, which conflicts with algebraic independence of p. It follows that $\text{rank}(R_w(p)) = \kappa = nd - d(d+1)/2$. That is, $(\mathcal{G}, p)$ is infinitesimally weakly rigid. ∎

From Theorem 3.14, infinitesimal weak rigidity implies weak rigidity. Together with Theorems 3.16 and 3.17, it is natural to obtain the following result.

THEOREM 3.18. *If $(\mathcal{G}, p)$ is weakly rigid for a generic configuration $p \in \mathbb{R}^{nd}$, then graph $\mathcal{G}$ is generically weakly rigid in $\mathbb{R}^d$.*

By Theorem 3.9, it is straightforward that any connected graph is generically weakly rigid in the plane. Moreover, Theorem 3.18 implies that for a generically weakly rigid graph $\mathcal{G}$ in $\mathbb{R}^d$, by randomizing a configuration $p \in \mathbb{R}^{nd}$, $(\mathcal{G}, p)$ is weakly rigid with probability 1.

4. Application to formation control. In this section, we aim to design distributed control laws for a multiagent system to solve the formation stabilization problem. The desired formation shape will be characterized by using weak rigidity theory. Since we have shown that weak rigidity requires fewer edges to recognize a framework, the restriction on the formation graph will be relaxed compared to [14, 23, 38, 33, 35, 34, 2, 18].

4.1. Control objective. Consider n autonomous agents moving in $\mathbb{R}^d$. In a given global coordinate frame, we denote the position of agent i as $p_i \in \mathbb{R}^d$. Each agent is considered to have single-integrator dynamics:

$$\dot{p}_i = u_i, \quad i \in \mathcal{V}, \tag{9}$$

where $u_i \in \mathbb{R}^d$ is a velocity input to be designed.

We denote the formation shape by $(\mathcal{G}_f, p^*)$ with $\mathcal{G}_f = (\mathcal{V}, \mathcal{E}_f)$, where $\mathcal{G}_f$ is called the formation graph, and $p^* = (p_1^{*T}, \dots, p_n^{*T})^T \in \mathbb{R}^{nd}$ is a configuration forming the desired formation shape. We represent the sensing graph by $\mathcal{G}_s = (\mathcal{V}, \mathcal{E}_s)$, which describes the interaction relationships between agents. It is natural to assume that $\mathcal{E}_f \subseteq \mathcal{E}_s$ since the desired edge information is useless in the design of the control input if the involved agents are unable to interact with each other. Let $\mathcal{N}_i^f$ and $\mathcal{N}_i^s$ denote the neighbor sets of agent i in $\mathcal{G}_f$ and $\mathcal{G}_s$, respectively. It is easy to see that $\mathcal{N}_i^f \subseteq \mathcal{N}_i^s$.

We always consider that all agents are in a GPS-denied environment. Each agent i can only achieve the relative position measurements from its neighbors in graph $\mathcal{G}_s$, i.e., $p_i^i - p_j^i$, $(i,j) \in \mathcal{E}_s$, where p_k^i denotes the position vector of agent k in the local coordinate system of agent i.

Differently from the distance-constrained formation control strategies, we encode the target formation $(\mathcal{G}_f, p^*)$ through a constraint set of pairwise inner products of relative position states, i.e., $\{(p_i^* - p_j^*)^T(p_i^* - p_k^*) \in \mathbb{R} : (i,j,k) \in \mathcal{T}^*_{\mathcal{G}_f}\}$. If $(\mathcal{G}_f, p^*)$ is infinitesimally weakly rigid, then the desired equilibrium can be described by the $d(d+1)/2$-dimensional manifold

$$\mathscr{E} = r_{\mathcal{K}}^{-1} r_{\mathcal{K}}(p^*) = \{p \in \mathbb{R}^{nd} : (p_i - p_j)^T(p_i - p_k) = (p_i^* - p_j^*)^T(p_i^* - p_k^*), i,j,k \in \mathcal{V}\}.$$

Remark 3 has shown that all agents with position states in $\mathscr{E}$ form the desired formation shape. As a result, our objective is to design distributed control strategies u_i for stabilizing agents' position states into $\mathscr{E}$ asymptotically.

A framework $(\mathcal{G}, p)$ is said to be realizable with $\mathcal{T}^*_{\mathcal{G}}$ if there exists some $q \in \mathbb{R}^{nd}$ such that $r_{\mathcal{G}}(q) = r_{\mathcal{G}}(p)$. Throughout this paper, we always assume that the framework characterizing the desired formation shape is realizable. The weak rigidity based formation stabilization problem is formally stated below.

Problem 1. *Given a realizable infinitesimally weakly rigid formation* $(\mathcal{G}_f, p^*)$, *design a distributed control protocol* (9) *for each agent* i *based on the relative position measurements* $\{p_i^i - p_j^i,\ j \in \mathcal{N}_i^s\}$, *such that the trajectories of agents asymptotically converge into manifold* $\mathscr{E}$.

4.2. A distributed control law. Given a framework $(\mathcal{G}_f, p^*)$ and $\mathcal{T}^*_{\mathcal{G}}$ describing the desired formation shape, let $\delta_{(i,j,k)} = e_{ij}^T e_{ik} - e_{ij}^{*T} e_{ik}^*$, $(i,j,k) \in \mathcal{T}^*_{\mathcal{G}}$. We aim to steer agents to cooperatively minimize the following cost function:

$$V(p) = \frac{1}{2} \sum_{(i,j,k) \in \mathcal{T}^*_{\mathcal{G}_f}} (e_{ij}^T e_{ik} - e_{ij}^{*T} e_{ik}^*)^2 = \frac{1}{2} \sum_{(i,j,k) \in \mathcal{T}^*_{\mathcal{G}_f}} \delta^2_{(i,j,k)}, \tag{10}$$

where $e_{ij}^* = p_i^* - p_j^*$. On the basis of function (10), a gradient-based control law can be induced as

$$u_i = -\sum_{(j,k) \in \mathcal{N}^f_{\mathcal{T}_i}} (e_{ij}^T e_{ik} - e_{ij}^{*T} e_{ik}^*)(e_{ij} + e_{ik}) - \sum_{(j,k) \in \mathcal{N}^f_{\mathcal{T}^i}} (e_{ji}^T e_{jk} - e_{ji}^{*T} e_{jk}^*)(e_{ij} - e_{ik}), \tag{11}$$

where $\mathcal{N}^f_{\mathcal{T}_i} = \{(j,k) \in \mathcal{V} \times \mathcal{V} : (i,j,k) \in \mathcal{T}^*_{\mathcal{G}_f}\}$, $\mathcal{N}^f_{\mathcal{T}^i} = \{(j,k) \in \mathcal{V} \times \mathcal{V} : (j,i,k) \in \mathcal{T}^*_{\mathcal{G}_f}$ or $(j,k,i) \in \mathcal{T}^*_{\mathcal{G}_f}\}$.

Note that for a triple $(i,j,k) \in \mathcal{T}^*_{\mathcal{G}_f}$, the gradient of $\delta_{(i,j,k)}$ with respect to p_j always involves the information of e_{ik}. That is, agent j should have access to e_{ik}, which can be computed by $e_{ij} - e_{kj}$. This implies that agent j should be able to sense information from i and k. Similarly, agent k should be able to sense information from i and j. Therefore, we require that $(i,j), (j,k), (i,k) \in \mathcal{E}_s$. To implement the control law (11) distributively, the set $\mathcal{T}^*_{\mathcal{G}_f}$, which includes the target inner products of displacements, is constructed by

$$\mathcal{T}^*_{\mathcal{G}_f} = \{(i,j,k) \in \mathcal{V}^3 : (i,j), (i,k) \in \mathcal{E}_f,\ j = k \text{ or } (j,k) \in \mathcal{E}_s, j \le k\}. \tag{12}$$

Remark 4. Observe that before implementing the control law (11), each agent i should be assigned with the elements of $\mathcal{T}^*_{\mathcal{G}_f}$ involving i and the target inner product constraints involving i. This can be viewed as a centralized distribution [32], which is similar to displacement- or distance-based strategies. When the decentralized controller (11) is implemented, each agent i only has to sense relative displacements from neighbors and compute the control input by simple inner products, additions, and subtractions of vectors. Therefore, the control law (11) is reasonable from a practical point of view.

We make the following assumption on the target formation $(\mathcal{G}_f, p)$.

Assumption 1. $(\mathcal{G}_f, p)$ with $\mathcal{T}^*_{\mathcal{G}_f}$ is infinitesimally weakly rigid.

In this paper, infinitesimal weak rigidity of $(\mathcal{G}_f, p)$ is the only condition for solving the formation stabilization problem. In fact, since $\mathcal{T}^*_{\mathcal{G}_f}$ is constructed as (12), to satisfy Assumption 1, $\mathcal{G}_s$ may be required to have more edges than $\mathcal{G}_f$. More specifically, for $(i,j,k) \in \mathcal{T}^*_{\mathcal{G}_f}$ and $j \neq k$, it should hold that $(j,k) \in \mathcal{E}_s$, but it is unnecessary that $(j,k) \in \mathcal{E}_f$. As an example, for the sensing graph shown in Figure 1(a), the edge $(2,3)$ can be reduced to generate the formation graph in Figure 1(b). It is important to note that using (12) as $\mathcal{T}^*_{\mathcal{G}_f}$ is mainly to guarantee effectiveness of the gradient controller (11). As a result, the restriction of $\mathcal{G}_s$ can be relaxed if some other distributed controller is applied. We will introduce the details in subsection 4.4.

The following lemma shows the implicit condition on $\mathcal{G}_s$ for validity of Assumption 1.

LEMMA 4.1. *Assumption* 1 *holds if and only if* $(\mathcal{G}_s, p)$ *is infinitesimally rigid.*

Proof. Let $r_{\mathcal{G}_f} = (\ldots, e_{ij}^T e_{ik}, \ldots)^T, (i,j,k) \in \mathcal{T}^*_{\mathcal{G}_f},\ R_w = \frac{\partial r_{\mathcal{G}_f}}{\partial p} = (\xi_1, \ldots, \xi_s)^T$, $g_{\mathcal{G}_s} = (\ldots, ||e_{ij}||^2, \ldots)^T, (i,j) \in \mathcal{E}$, be the distance-based rigidity function of $(\mathcal{G}_s, p)$, $R = \frac{\partial g_{\mathcal{G}_s}}{\partial p}$. It follows from (12) that $\mathcal{T}^*_{\mathcal{G}_f}$ always includes $\{(i,j,j) : (i,j) \in \mathcal{E}_s\}$, implying that $\text{rank}(R_w) \geq \text{rank}(R) = nd - \frac{d(d+1)}{2}$. Therefore, the sufficiency is obtained. Next we prove the necessity.

It suffices to show that each row of R_w can be denoted by a linear combination of rows of R. It is obvious that we only have to focus on $\xi_l^T = \frac{\partial(e_{ij}^T e_{ik})}{\partial p}$ for $j \neq k$. Note that it always holds that $e_{ij}^T e_{ik} = (||e_{ij}||^2 + ||e_{ik}||^2 - ||e_{jk}||^2)/2$; it follows that $\frac{\partial(e_{ij}^T e_{ik})}{\partial p} = (\frac{\partial(||e_{ij}||^2)}{\partial p} + \frac{\partial(||e_{ik}||^2)}{\partial p} - \frac{\partial(||e_{jk}||^2)}{\partial p})/2$, implying that each row in R_w can be denoted by several rows of R. Therefore, $\text{rank}(R) \geq \text{rank}(R_w) = nd - d(d+1)/2$. Recall that $\text{null}(R) \geq d(d+1)/2$; we have $\text{rank}(R) = nd - d(d+1)/2$. ☐

It is worthwhile to note that in a particular case when $\mathcal{G}_s$ is a triangulated Laman graph (see details in [8]), $(\mathcal{G}_s, p)$ can be a minimally infinitesimally rigid framework. Then Assumption 1 holds from Lemma 4.1.

Let $\delta(p) = (\ldots, \delta_{(i,j,k)}, \ldots)^T = r_{\mathcal{G}_f}(p) - r_{\mathcal{G}_f}(p^*)$; then $V(p) = \frac{1}{2}||\delta(p)||^2$. By the chain rule, the dynamic equation of multiagent system (9) with control law (11) can be written in the following compact form:

$$\dot{p} = -\nabla_p V(p) = -\left(\frac{\partial r_{\mathcal{G}_f}(p)}{\partial p}\right)^T \delta(p) = -R_w^T(p)\delta(p). \tag{13}$$

Under Assumption 1, each agent can achieve the required information for implementing controller (11) via local interactions with its neighbors. Therefore, our control strategy is a distributed one. In fact, we also have the following properties for the control law (11).

LEMMA 4.2. (i) *The controller* (11) *is independent of the global coordinate frame.*
(ii) *The centroid* $\bar{p} = \frac{1}{n}\sum_{i\in\mathcal{V}} p_i(t)$ *is invariant; i.e.,* $\dot{\bar{p}} = 0$.
(iii) *Let* $P = (p_1, \ldots, p_n)$, *rank*$(P(0)) =$ *rank*$(P(t))$ *for all* $t \geq 0$.

Proof. (i) and (ii) are straightforward by a proof similar to [22]; thus we omit the proofs here.

(iii) Let $R_e = \frac{\partial r_{\mathcal{G}_f}}{\partial e} \in \mathbb{R}^{s \times md}$ with $e = (\dots, e_{ij}^T, \dots)^T = (\alpha_1^T, \dots, \alpha_m^T)^T \in \mathbb{R}^{md}$, and $e^* = (\dots, e_{ij}^{*T}, \dots)^T = (\alpha_1^{*T}, \dots, \alpha_m^{*T})^T \in \mathbb{R}^{md}$. Note that we always have

$$R_e^T \delta = (\triangle \otimes I_d) e,$$

where $\delta = (\delta_1, \dots, \delta_s)^T = e^T e - e^{*T} e^*$, $\triangle = [\triangle_{ij}] \in \mathbb{R}^{m \times m}$, $\triangle_{ii} = ||\alpha_i||^2 - ||\alpha_i^*||^2$ if $||\alpha_i||^2 - ||\alpha_i^*||^2$ is a component of δ, and $\triangle_{ii} = 0$ otherwise; $\triangle_{ij} = \triangle_{ji} = \alpha_i^T \alpha_j - \alpha_i^{*T} \alpha_j^*$ if $\alpha_i^T \alpha_j - \alpha_i^{*T} \alpha_j^*$ is a component of δ, and $\triangle_{ij} = \triangle_{ji} = 0$ otherwise. As a result, (13) is transformed into

$$\dot{p} = -R_w^T \delta = -\bar{H}^T R_e^T \delta = -\bar{H}^T (\triangle \otimes I_d) \bar{H} p = -((H^T \triangle H) \otimes I_d) p,$$

which can be equivalently written as

$$\dot{P} = -P \bar{\triangle},$$

where $\bar{\triangle} = H^T \triangle H$. From the lemma on rank-preserving differential equations shown in [33], we can obtain that $\operatorname{rank}(P)$ is invariant during evolution of the formation system. ∎

4.3. Stability analysis. Notice that the cost function (10) is nonconvex and thus has multiple local minima. We will mainly concentrate on local stability of the formation system (13). We say a set M is locally exponentially stable if there exists an exponent $c > 0$, such that for any $x \in M$, there exists a neighborhood Ω of x, and any trajectory starting from Ω converges to M at least as fast as e^{-ct}.

THEOREM 4.3. *For a group of* $n > d + 1$ *agents with dynamics* (9) *and control law* (11) *moving in* $\mathbb{R}^d$*, under Assumption* 1*,* $\mathscr{E}$ *is locally exponentially stable.*

Proof. We observe that the gradient-based formation system (13) has a similar form to [14, equation (8)]. Let $z = (\bar{p}^T, \bar{z}^T)^T = Qp$, where $\bar{z} \in \mathbb{R}^{nd-d}$, and $Q \in \mathbb{R}^{nd \times nd}$ is an orthogonal matrix with its first d rows being $\frac{1}{n} \mathbf{1}_n^T \otimes I_d$. Using the centroid invariance property, i.e., $\dot{\bar{p}} = 0$, shown in Lemma 4.2, (13) can also be equivalently transformed into a reduced-order system $\dot{\bar{z}} = \bar{f}(\bar{z})$ with a compact manifold of equilibria $\bar{\mathscr{E}}$. Since the target formation $\mathscr{E} = r_{\mathcal{K}}^{-1}(r_{\mathcal{K}}(p^*))$ has been shown to be a $d(d+1)/2$-dimensional manifold in the proof of Theorem 3.14, $\bar{\mathscr{E}}$ is a $d(d-1)/2$-dimensional manifold characterized by rotations around the centroid $\bar{p}(0)$.

By simply following a procedure similar to that in [14], the traditional center manifold theory can be employed to show that each point in $\bar{\mathscr{E}}$ is locally exponentially stable. Due to compactness of $\bar{\mathscr{E}}$, there is a finite subcover forming a neighborhood of $\bar{\mathscr{E}}$. Therefore, there must exist an exponent $c > 0$ such that for each $\bar{z} \in \bar{\mathscr{E}}$, any trajectory converges to $\bar{\mathscr{E}}$ from a neighborhood of $\bar{z}$ at least as fast as e^{-ct}. Recall that $\bar{p}$ is invariant; it is straightforward that the same conclusion holds for $\mathscr{E}$. ∎

Note that although the desired equilibrium is $\mathscr{E}$, and $r_{\mathcal{G}_f}^{-1}(r_{\mathcal{G}_f}(p^*)) = \mathscr{E}$ only if $(\mathcal{G}_f, p)$ is globally weakly rigid, it is not necessary for the target formation to be globally weakly rigid since we only require local stability.

Local exponential stability actually characterizes the ability of control law (11) to restore the desired formation shape under a small perturbation from the desired equilibrium $\mathscr{E}$. In fact, when $n = d + 1$ and $(\mathcal{G}_f, p^*)$ with $\mathcal{T}_{\mathcal{G}_f}^*$ is minimally infinitesimally weakly rigid, almost global asymptotic stability of the formation system can be ensured, as given in the following theorem.

THEOREM 4.4. *For a group of $n = d+1$ agents with dynamics* (9) *and control law* (11) *moving in $\mathbb{R}^d$, if $p_1(0), \ldots, p_n(0)$ do not lie in a hyperplane and $(\mathcal{G}_f, p^*)$ with $\mathcal{T}^*_{\mathcal{G}_f}$ is minimally infinitesimally weakly rigid, then*

(i) $(\mathcal{G}_f, p(t))$ *is infinitesimally weakly rigid for all $t \geq 0$;*

(ii) *collisions between any agents are avoided; and*

(iii) *the stacked state vector p will converge into $\mathscr{E}$ exponentially.*

Proof. (i) According to Lemma 4.1, $\mathrm{rank}(R(p(0))) = \mathrm{rank}(R_w(p(0))) = nd - d(d+1)/2$, where $R(p(t)) = \frac{\partial g_{\mathcal{G}_s}}{\partial p}$. It follows that $|\mathcal{E}_s| \geq nd - d(d+1)/2 = n(n-1)/2 = C_n^2$. Thus $\mathcal{G}_s$ is a complete graph. By Lemma 4.1, we only have to prove that $R(p(t))$ is of full row rank for $t \geq 0$. Suppose this is not true for $t = t^*$. Let $R(p(t^*)) = (r_1, \ldots, r_{\bar{m}})^T = (c_1, \ldots, c_n)$, where $r_i \in \mathbb{R}^{nd}$, $i \in \{1, \ldots, \bar{m}\}$, $c_j \in \mathbb{R}^{\bar{m} \times d}$, $j \in \mathcal{V}$, $\bar{m} = n(n-1)/2 = d(d+1)/2$. Then there exist not all zero scalars $\tau_1, \ldots, \tau_{\bar{m}}$ such that $\tau_1 r_1 + \cdots + \tau_{\bar{m}} r_{\bar{m}} = 0$, which implies that $\tau^T R(p(t^*)) = \tau^T(c_1, \ldots, c_n) = 0$. Without loss of generality, suppose $\tau_k \neq 0$. Note that there must exist a submatrix c_l such that the kth row of c_l is associated with $e_{li}^T(t^*)$ for some $i \in \mathcal{N}_l^s$. Since $\mathcal{G}_s$ is complete, $c_l = (\frac{\partial g_{\mathcal{G}_s}}{\partial p_l})^T$ has exactly $n-1 = d$ nonzero rows associated with $e_{li}^T(t^*)$ for $i \in \mathcal{N}_l^s$. Together with the fact that $\tau^T c_l = 0$, where $\tau = (\tau_1, \ldots, \tau_{\bar{m}})^T$, these $e_{li}(t^*)$ for all $i \in \mathcal{N}_l^s$ are linearly dependent. Letting $E_l = (\ldots, e_{li}(t^*), \ldots)^T \in \mathbb{R}^{d \times d}, i \in \mathcal{N}_l^s$, we have $\mathrm{rank}(E_l) < d$.

Since $p_1(0), \ldots, p_n(0)$ do not lie on a hyperplane, there do not exist not all zero scalars $k_1, \ldots, k_d$ and $b \in \mathbb{R}$ such that $k_1 p_{i(1)}(0) + \cdots + k_d p_{i(d)}(0) = b$ for all $i \in \mathcal{V}$, where $p_{i(j)}(0) \in \mathbb{R}$ denotes the jth component of $p_i(0)$. As a result, the matrix $\bar{P}(0) = (\mathbf{1}_n, P^T(0)) \in \mathbb{R}^{(d+1)\times(d+1)}$ is of full rank. Due to Lemma 4.2, one has $\dot{P}^T = -\bar{\triangle}^T P^T = -\bar{\triangle} P^T$, and together with the fact that $\bar{\triangle}\mathbf{1}_n = H^T \triangle H \mathbf{1}_n = 0$, we have $\dot{\bar{P}} = -\bar{\triangle}\bar{P}$, which is a rank preserving differential equation. Hence, $\bar{P}(t^*)$ is of full rank, and $P^T(t^*)$ is of full column rank. Since $\mathcal{G}_s$ is complete, we have $E_l = H_l P^T(t^*)$, where H_l is an incidence matrix associated with a star topology with agent l as the root. Note that $\mathrm{null}(H_l) = \mathrm{span}\{\mathbf{1}_n\}$; if $E_l x = 0$ for some nontrivial $x \in \mathbb{R}^{nd}$, then either $P^T x = 0$ or $p_i^T x = p_j^T x$ for all $i, j \in \mathcal{V}$. Neither case can happen since $P^T(t^*)$ is of full column rank and $\bar{P}(t^*)$ is of full rank. As a result, $\mathrm{null}(E_l) = \varnothing$. This conflicts with $\mathrm{rank}(E_l) < d$. Consequently, we have $\mathrm{rank}(R_w(p(t))) = \mathrm{rank}(R(p(t))) = nd - d(d+1)/2$ for all $t \geq 0$.

(ii) Suppose that there are two agents i, j colliding with each other at some $t \geq 0$. From (i) and Lemma 2.1, agent i should have d neighbors other than j. It follows that $|\mathcal{N}_i^s| \geq d+1 > d$, which conflicts with $|\mathcal{N}_i^s| = d$. Hence, collision avoidance is guaranteed during the formation process.

(iii) We first claim that $r_{\mathcal{G}_f}$ has exactly $\bar{m} = nd - d(d+1)/2 = d(d+1)/2$ components. It follows from Theorem 3.9 that $\mathcal{G}_f$ is a tree. From the form of $\mathcal{T}^*_{\mathcal{G}_f}$ given by (12), we have $|\mathcal{T}^*_{\mathcal{G}_f}| = n - 1 + C_{n-1}^2 = d(d+1)/2 = \bar{m}$; therefore, $r_{\mathcal{G}_f} \in \mathbb{R}^{\bar{m}}$. Next we show that $r_{\mathcal{G}_f}(p) = r_{\mathcal{G}_f}(p^*)$ implies $p \in \mathscr{E}$; i.e., $(\mathcal{G}_f, p)$ is globally weakly rigid. Note that it always holds that $e_{ij}^T e_{ik} = 1/2(||e_{ij}||^2 + ||e_{ik}||^2 - ||e_{jk}||^2)$; i.e., any component of $r_{\mathcal{G}_f}$ can be denoted by a linear combination of several components of $g_{\mathcal{G}_s}$. Therefore, there exists a constant matrix $M \in \mathbb{R}^{\bar{m} \times \bar{m}}$ such that $r_{\mathcal{G}_f}(p) = M g_{\mathcal{G}_s}(p)$ for any $p \in \mathbb{R}^{nd}$. It follows that $R_w = \frac{\partial r_{\mathcal{G}_f}}{\partial p} = M \frac{\partial g_{\mathcal{G}_s}}{\partial p} = MR$. From the well-known inequality $\mathrm{rank}(M) + \mathrm{rank}(R) - \bar{m} \leq \mathrm{rank}(R_w) \leq \min\{\mathrm{rank}(M), \mathrm{rank}(R)\}$, we have $\mathrm{rank}(M) = \bar{m}$; i.e., M is nonsingular. It follows that $g_{\mathcal{G}_s} = M^{-1} r_{\mathcal{G}_f}$. That is, once $r_{\mathcal{G}_f}(p) = r_{\mathcal{G}_f}(p^*)$, it holds that $g_{\mathcal{G}_s}(p) = g_{\mathcal{G}_s}(p^*)$. Recall that $\mathcal{G}_s$ is complete; we have

$D(p) = D(p^*)$. According to Theorem 3.13, it holds that $\mathbf{E}(p) = \mathbf{E}(p^*)$; i.e., $p \in \mathscr{E}$.

Next we prove exponential stability of $\{p \in \mathbb{R}^{nd} : r_{\mathcal{G}_f}(p) = r_{\mathcal{G}_f}(p^*)\}$. Let $\delta = r_{\mathcal{G}_f}(p) - r_{\mathcal{G}_f}(p^*)$. Due to the fact that $\dot{\delta} = \dot{r}_{\mathcal{G}_f} = \frac{\partial r_{\mathcal{G}_f}}{\partial p}\dot{p} = R_w\dot{p}$, together with (13), the formation system can be described by

$$\dot{\delta} = -R_w R_w^T \delta. \tag{14}$$

Now we show that (14) is a self-contained system. It suffices to show that $R_w R_w^T$ is a function of δ. Note that each entry of $R_w R_w^T$ is a linear combination of $e_{ij}^T e_{kl}$, $(i,j),(k,l) \in \mathcal{E}$. It is easy to see that $e_{ij}^T e_{kl}$ can always be denoted by a linear combination of components of $g_{\mathcal{G}_s}$, i.e., $e_{ij}^T e_{kl} = \frac{1}{2}(||e_{jk}||^2 + ||e_{il}||^2 - ||e_{ik}||^2 - ||e_{jl}||^2)$. Together with $g_{\mathcal{G}_s} = M^{-1} r_{\mathcal{G}_f}$, it becomes certain that $R_w R_w^T$ can be written as a smooth function of $r_{\mathcal{G}_f}$ and therefore is also a smooth function of δ.

Let $\phi = ||\delta||^2$; it follows from (14) that $\dot{\phi} = -2\delta^T R_w R_w^T \delta \leq 0$. This implies that δ always stays in the compact set $\Psi = \{\delta \in \mathbb{R}^{\bar{m}} : ||\delta||^2 \leq \phi(0)\}$. Since we have shown in (i) that $\operatorname{rank}(R_w R_w^T) = \operatorname{rank}(R_w) = \bar{m}$ for all $t \geq 0$, together with the fact that $R_w R_w^T$ is totally determined by δ, there must exist $\kappa > 0$ such that $\kappa = \min_{\delta \in \Psi} \lambda(R_w R_w^T)$. It follows that $\dot{\phi} \leq -2\kappa\phi$. Then $\phi \leq \exp(-2\kappa)\phi(0)$. That is, δ vanishes exponentially.

Recall that $(\mathcal{G}_f, p)$ is globally weakly rigid; hence, p must converge into $\mathscr{E}$ exponentially. ◻

Remark 5. In the case when $n \geq d+1$, let $\mathcal{C}$ be the set of configurations in a hyperplane of $\mathbb{R}^d$. Then $\mathcal{C} = \{p \in \mathbb{R}^{nd} : f(p) = 0\}$, where $f(p)$ is the sum of the squares of all the $(d+1) \times (d+1)$ minors of $\bar{P} = (\mathbf{1}_n, P^T) \in \mathbb{R}^{n \times (d+1)}$. Note that $f(p)$ is a nontrivial polynomial; thus $\mathcal{C}$ is either equal to $\mathbb{R}^{nd}$ or of measure zero [6]. Since p^* is a configuration such that $\operatorname{rank}(\bar{P}) = d+1$, we have $p^* \notin \mathcal{C}$. Therefore, the measure of $\mathcal{C}$ is zero. That is, if $n = d+1$, Theorem 4.4 implies that for almost any given initial configuration, the control law (11) will exponentially stabilize a minimally infinitesimally weakly rigid formation. However, since the exponent 2κ is dependent on $p(0) \in \mathbb{R}^{nd} \setminus \mathcal{C}$, and $\mathbb{R}^{nd} \setminus \mathcal{C}$ is not compact, it is uncertain whether a uniform κ exists. Thus, we can only conclude that if $n = d+1$, the minimally infinitesimally weakly rigid formation is almost globally asymptotically stable.

4.4. Formation control under nonrigid sensing graphs. Lemma 4.1 shows that the control strategy proposed in the previous subsections can only be implemented on the premise that $(\mathcal{G}_s, p^*)$ is infinitesimally rigid. In this subsection, we consider $\mathcal{G}_f = \mathcal{G}_s = \mathcal{G} = (\mathcal{V}, \mathcal{E})$ and try to solve the formation stabilization problem when $(\mathcal{G}, p^*)$ is only infinitesimally weakly rigid, which is a weaker graph condition compared to rigidity. Differently from (12), we use $\mathcal{T}_{\mathcal{G}}^*$ as follows:

$$\mathcal{T}_{\mathcal{G}}^* = \{(i,j,k) \in \mathcal{V}^3 : (i,j),(i,k) \in \mathcal{E}, j \leq k\}. \tag{15}$$

To achieve the goal of formation, we consider the following independent cost function for each agent:

$$V_i(p) = \frac{1}{2}\sum_{(j,k)\in\mathcal{N}_{\mathcal{T}_i}} \delta_{(i,j,k)}^2 + \frac{1}{2}\sum_{j\in\mathcal{N}_i} \delta_{(j,i,i)}^2, \qquad i \in \mathcal{V}, \tag{16}$$

where $\mathcal{N}_{\mathcal{T}_i} = \{(j,k) \in \mathcal{V} \times \mathcal{V} : (i,j,k) \in \mathcal{T}_{\mathcal{G}}^*\}$.

The distributed control law is

$$u_i = -K_i \nabla_{p_i} V_i, \qquad i \in \mathcal{V}, \tag{17}$$

where $K_i \in \mathbb{R}^{d\times d}$ is a control gain matrix for agent i to be designed. Note that the control law (17) for agent i induced by function (16) only requires information of $p_i - p_j$ for $(i,j) \in \mathcal{E}_f$. Hence a sensing graph $\mathcal{G}_s = \mathcal{G}_f$ is sufficient for each agent to implement the control law. Moreover, (17) is no longer a normal gradient-based control law. As a result, the centroid $\bar{p}$ is dynamic during the formation process. However, it can be easily verified that the control law (17) is still independent of the global coordinate frame.

With (17), the formation system can be written as

$$\dot{p} = -K\bar{R}_w^T\delta, \tag{18}$$

where $K = \mathrm{diag}(K_1, \ldots, K_n) \in \mathbb{R}^{nd\times nd}$.

Different from (13), the nongradient-based formation system (18) has a dynamic centroid and thus cannot be transformed into a reduced-order system with a compact manifold of equilibria. Due to the noncompactness of $\mathcal{E}$, there does not exist an open cover of $\mathcal{E}$ having a finite subcover. In the following, we will employ Lemma 2.2 to establish local exponential stability of the target formation shape.

THEOREM 4.5. *For a group of $n \geq d+1$ agents with dynamics* (9) *and control law* (17) *moving in $\mathbb{R}^d$, under Assumption* 1*, if for some $p^* \in \mathscr{E}$, the gain matrix K can be chosen such that $J^* = K\bar{R}_w^T R_w|_{p=p^*}$ has $d(d+1)/2$ zero eigenvalues and the rest have positive real parts, then for any $\tilde{p} \in \mathscr{E}$, there exists a compact neighborhood Ω of $\tilde{p}$, such that $M = \Omega \cap \mathscr{E}$ is locally exponentially stable.*

Proof. For any $\tilde{p} \in \mathscr{E}$, let $\rho = p - \tilde{p}$. Rewrite (18) as $\dot{p} = f(p)$; expanding in a Taylor series about $\tilde{p}$, we have $f(p) = f(\tilde{p}) + \frac{\partial f(\tilde{p})}{\partial p}\rho + g(\rho)$. Due to the fact that $\mathscr{E}$ is a manifold of equilibria, we have $\dot{\tilde{p}} = f(\tilde{p})$. Then (18) can be equivalently written as

$$\dot{\rho} = \frac{\partial f(\tilde{p})}{\partial p}\rho + g(\rho) = -J_f(\tilde{p})\rho + g(\rho), \tag{19}$$

where $J_f(\tilde{p}) = K\bar{R}_w^T R_w\big|_{p=\tilde{p}}$. Note that each entry of $\bar{R}_w^T R_w$ is either zero or an inner product of relative positions, implying that $\bar{R}_w^T R_w$ is invariant under translations and rotations of p. Due to the assumption, $J_f(\tilde{p}) = J^*$ has $d(d+1)/2$ zero eigenvalues and the rest have positive real parts. Let $M_1 = \{\rho \in \mathbb{R}^{nd} : \rho + \tilde{p} \in \mathscr{E}\}$; M_1 is obviously a manifold of dimension $d(d+1)/2$. Applying Lemma 2.2 to system (19), there exists a compact neighborhood Ω_1 of the origin, such that $M_2 = \Omega_1 \cap M_1$ is locally exponentially stable. Let $\Omega = \{\tilde{p} + \rho : \rho \in \Omega_1\}$; it is straightforward that $M = \Omega \cap \mathscr{E}$ is locally exponentially stable. ∎

Observe that if $(j,k) \in \mathcal{E}$ for all $(i,j,k) \in \mathcal{T}_{\mathcal{G}}^*$, then $\bar{R}_w = R_w$. Let $\bar{R}_w^* = \bar{R}_w(p^*)$, $R_w^* = R_w(p^*)$. It follows that $\mathrm{rank}(\bar{R}_w^{*T}R_w^*) = \mathrm{rank}(R_w^*) = nd - d(d+1)/2$, and all nonzero eigenvalues of $\bar{R}_w^{*T}R_w^*$ are positive. Hence J^* satisfies our condition by setting $K = I_{nd}$. Otherwise, we have $\mathrm{null}(R_w^*) \subseteq \mathrm{null}(\bar{R}_w^{*T}R_w^*)$, implying that $\mathrm{rank}(\bar{R}_w^{*T}R_w^*) \leq nd - d(d+1)/2$. Since it always holds that $\mathrm{rank}(J^*) \leq \min\{\mathrm{rank}(K), \mathrm{rank}(\bar{R}_w^{*T}R_w^*)\}$, to make $\mathrm{rank}(J^*) = nd - d(d+1)/2$, there should hold $\mathrm{rank}(\bar{R}_w^{*T}R_w^*) \geq nd - d(d+1)/2$. Therefore, a necessity condition for validity of the condition in Theorem 4.5 is $\mathrm{rank}(\bar{R}_w^{*T}R_w^*) = nd - d(d+1)/2$. In particular, consider a formation stabilization problem in the plane; if $\mathcal{T}_{\mathcal{G}}^*$ is selected by Algorithms 1 and 2, then $\bar{R}_w^*, R_w^* \in \mathbb{R}^{(2n-3)\times 2n}$. From the proof of Theorem 3.9, there is always one edge $(i,j) \in \mathcal{E}$ such that only two components of $r_{\mathcal{G}}$ involve e_{ij}^*, i.e., $||e_{ij}^*||^2$ and $e_{ij}^{*T}e_{ik}^*$ for some $k \in \mathcal{N}_i$. The corresponding two rows in $\bar{R}_w^*$ form the following submatrix:

$$\begin{array}{ccccccc} & i & & j & & k & \\ \end{array}$$
$$\begin{pmatrix} \mathbf{0} & 2e_{ij}^{*T} & \mathbf{0} & -2e_{ij}^{*T} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & e_{ij}^{*T} + e_{ik}^{*T} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix},$$

which is obviously linearly independent. Similar to the induction in the proof of Theorem 3.9, it can be verified that $\mathrm{rank}(\bar{R}_w^*) = \mathrm{rank}(R_w^*) = 2n - 3$. It follows that $\mathrm{rank}(\bar{R}_w^{*T} R_w^*) = 2n - 3$. Although numerical experiments show that a suitable K can always be chosen, it is still difficult to determine the existence of K rigorously, and so this is the topic of ongoing research endeavors.

5. A simulation example. In this section, we present a numerical example to illustrate the effectiveness of the main results.

Consider a group of six autonomous agents moving in the plane. We try to stabilize these agents to form a regular hexagon with edge lengths equal to 2. The formation graph $\mathcal{G}_f$ and the sensing graph $\mathcal{G}_s$ are identically set to be the path graph $\mathcal{G}$ in Figure 2(b). $(\mathcal{G}, p)$ is obviously not rigid but infinitesimally weakly rigid with $\mathcal{T}_{\mathcal{G}}^* = \{(1,2,6), (2,1,3), (3,2,4), (4,3,5), (i,j,j), (i,j) \in \mathcal{E}, i > j\}$. Because the sensing graph is not rigid, the distance-based formation strategies are inapplicable. Now given a configuration $p^* = (2, 0, 4, 0, 5, \sqrt{3}, 4, 2\sqrt{3}, 2,\ 2\sqrt{3}, 1, \sqrt{3})^T$ which forms the target formation shape, let each agent implement the control law (17). In fact, if we set $K = I$, the eigenvalues of J^* are 45.9712, 40.4991, 32.7903, 24.0000, 15.8549, 10.0916, 5.6563, 1.4093, -0.2727, 0, 0, 0. This does not satisfy our condition in Theorem 4.5. Now we employ a gain matrix $K = \mathrm{diag}(K_1, \ldots, K_6)$ with $K_1 = \mathrm{diag}(0.3, -0.04)$, $K_2 = \mathrm{diag}(0.15, 1.34)$, $K_3 = \mathrm{diag}(0.23, 1.09)$, $K_4 = \mathrm{diag}(1.32,\ 0.34)$, $K_5 = \mathrm{diag}(1.32, 0.21)$, $K_6 = \mathrm{diag}(-0.45, 0.42)$. Then the eigenvalues of J^* become 48.9899, 36.7915, 12.6938, 8.1539, 3.7883, 2.7087, 1.7132, $0.1053 + 0.1757i$, $0.1053 - 0.1757i$, 0, 0, 0. By Theorem 4.5, the desired formation shape can be formed locally exponentially, which is consistent with the result shown in Figure 3.

6. Conclusion. We presented a weak rigidity theory which allows us to recognize a framework in arbitrarily dimensional spaces by fewer edges than the distance-based rigidity theory. The main idea is to determine the framework by constraining pairwise inner products of relative displacements in the framework, which actually utilizes additional subtended angle information not used in distance-based rigidity theory. We showed that weak rigidity is a condition milder than distance rigidity for a framework and derived a necessary and sufficient graphical condition for infinitesimal weak rigidity in the plane. The proposed graphical condition can easily verify infinitesimal weak rigidity of a framework without examining rank of the rigidity matrix, whereas no graphical conditions for infinitesimal rigidity exist in the literature. Two novel distributed formation control schemes via weak rigidity theory were also proposed. Our control strategies only require local relative displacement measurements and thus are distributed and communication-free. In particular, for the nongradient-based control law, local exponential stability of formation was obtained under a weakly rigid sensing graph. That is, our control law requires less information flowed in the network compared to the distance-based formation strategy and thus reduces costs and can be efficient in a more demanding environment. Future work includes (1) the design of the gain matrix to stabilize the nongradient-based formation system, and (2) preservation of weak rigidity of the formation during agents' motion.

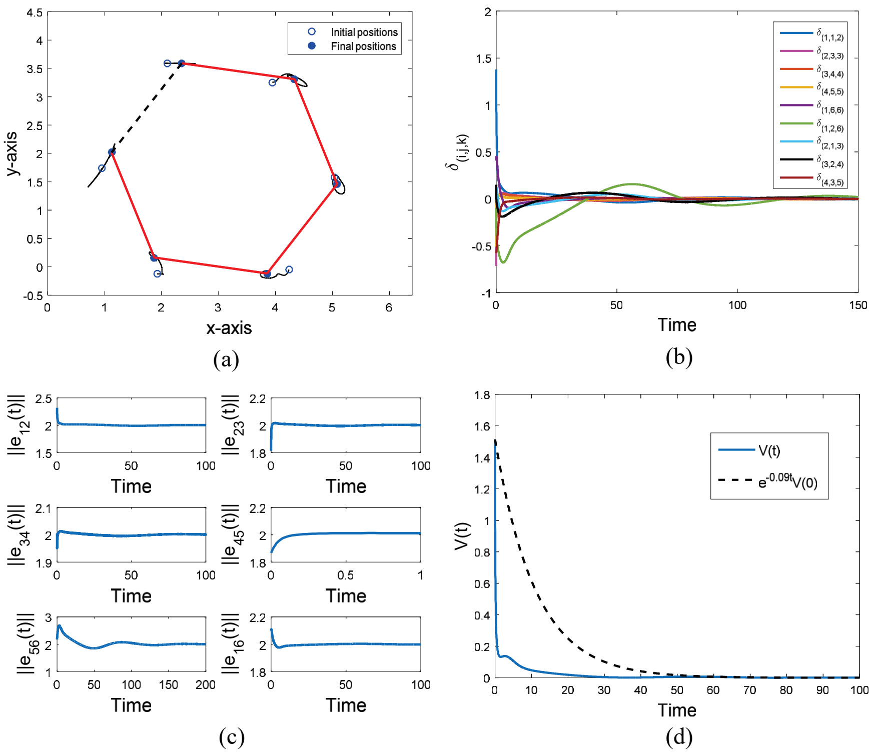

FIG. 3. (a) *The agents with initial positions in a neighborhood of* p^* *asymptotically converge into another point in* $\mathscr{E}$. (b) $\delta_{(i,j,k)}$ *asymptotically vanishes to zero,* $(i,j,k) \in \mathcal{T}^*_{\mathcal{G}}$. (c) *The length of each edge asymptotically converges to* 2. (d) *The cost function* $V = \sum_{i \in \mathcal{V}} V_i$ *with* V_i *in* (16) *vanishes to zero exponentially.*

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