Quantum Projection Filtering for Open Quantum Systems*

Qing Gao† and Guofeng Zhang††

Abstract— **This paper presents an approximation quantum projection filtering strategy, aiming to reduce the computational cost in calculating the standard quantum filter equation in time. By using a differential geometric approach, the trajectory of the resulting quantum projection filter is constrained to evolve in a finite-dimensional differentiable manifold consisting of an exponential family of quantum density operators. A convenient design of the differentiable manifold is then developed through reduction of the local approximation errors, which allows simplification of the quantum projection filter equations. Finally, simulation results from a two-level quantum system example illustrate the approximation performance of the proposed filtering scheme. The proposed approach is expected to be of practical use in developing more efficient quantum control methods.**

Index Terms— **Quantum projection filtering; open quantum systems; quantum information geometry; differentiable manifold.**

I. INTRODUCTION

The fundamental postulates of quantum mechanics preclude simultaneous measurement of any two non-commuting observables in a single realization. In other words, any quantum measurement scheme can extract in principle only partial information from the observed quantum system, which makes any measurement based quantum feedback control problem necessarily one with partial observations ([10], [13], [17]). A quantum filter, like its classical counterpart optimal filter, recursively updates the information state of a quantum system undergoing continual measurements and provides the essential real-time knowledge that can be fed back to the quantum system through appropriately designed control actuators ([2], [3], [8]). In this way, a measurement based quantum feedback control problem can be converted into a control problem for an optimal quantum filter with the state information fully accessible, as in classical stochastic control theory. In this quantum control framework, the system and observations are often described by quantum stochastic differential equations, while the quantum filter is a classical *Itô* stochastic differential equation due to the fact that a laboratory measuring setup generates a classical stochastic signal. In order to implement the quantum feedback control setup, the quantum filtering equation has to be calculated in time, which tends to be computational expensive especially when the quantum system has a high dimension [15].

In order to make the implementation efficient, several approaches have been proposed in literature concerning the approximation of the quantum filter equation, to mention a few, see [7], [14]. In [7], an extended Kalman filtering approach was developed for a class of open quantum systems subject to continuous measurement, where a time-varying linearization is applied to the system dynamics and a Kalman filter is designed based on the linearized system. The proposed approach performs well for nearly linear quantum system. The authors in [14] proposed a numerical approach to reducing the computational burden associated with calculating quantum filter and used the approach to demonstrate a two-qubit feedback control scheme. It was shown by simulation studies that high approximation accuracy can be achieved even when a small number of integration steps is involved.

The main goal of this paper is to approximate the optimal quantum filter with a lower dimensional *quantum projection filter*, motivated by the pioneer works by Brigo, Hanzon and LeGland for classical filters ([5], [6]). The authors in [9] had applied the projection filtering approach to a highly nonlinear quantum model of a strongly coupled two-level atom in an optical cavity and the infinite-dimensional filter is reduced to a tractable set of equations. However, the approach in [9] requires exact prior knowledge of an invariant set of the filtering equation, which makes the proposed method applicable only in special situations. In this paper, we address the problem of quantum projection filtering for general open quantum systems subject to continuous observation, using differential-geometric methods in quantum information geometry theory. The solutions to the quantum filter are constrained to evolve within a lower-dimensional differentiable manifold consisting of an exponential family of quantum density operators, through a projection operation defined on the tangent space of this manifold. In other words, the resulting quantum trajectories reduce to a curve on the lower-dimensional manifold and the filter equation becomes a set of recursive equations satisfied by the corresponding coordinate system. A convenient design of this manifold is also given, aiming to reduce the local approximation errors.

This paper is organized as follows. Section II briefly introduces the theory of quantum filtering and its typical physical scenario. In Section III, we first introduce some mathematical foundations of quantum information geometry, based on which the equations of quantum projection filter are then derived. Section IV is devoted to simulation studies,

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where the approximation performance of the proposed filtering scheme is demonstrated using a spin-$\frac{1}{2}$ quantum system example. Section V concludes this paper.

II. Quantum Filter

We sketch the open quantum system model under consideration in this section. More detailed description can refer to several papers and books ([3], [4], [15], [17]) and the references therein. A cloud of atoms trapped in a cavity is in weak interaction with an external laser field that is initially in the vacuum state and spontaneously emits in all directions. One of the cavity mirrors, through which a forward mode of the electromagnetic scatters off, is made slightly leaky such that information about the atoms can be extracted using a homodyne detector. Let $\mathscr{H}_{\mathscr{S}}$ be the Hilbert space of the finite dimensional atomic system with $\dim(\mathscr{H}_{\mathscr{S}}) = n < \infty$. The probe laser field is modelled by the symmetric Fock space $\mathscr{E}$ that can be decomposed into the past and future components in the form of a tensor product $\mathscr{E} = \mathscr{E}_{t]} \otimes \mathscr{E}_{(t}$. The joint system state $\rho_0 = \pi_0 \otimes |\upsilon\rangle\langle\upsilon|$ is given by some quantum state π_0 on $\mathscr{H}_{\mathscr{S}}$ and the vacuum state $|\upsilon\rangle$.

The composite system composed of the atomic system and the field is assumed to be isolated. Then its temporal Heisenberg-picture evolution can be then described by a unitary operator $U(t)$ on the tensor product Hilbert space $\mathscr{H}_{\mathscr{S}} \otimes \mathscr{E}$, which satisfies the following Hudson-Parthasarathy quantum stochastic differential equations[1]:

$$dU(t) = \left\{ \left(-iH(t) - \frac{1}{2}L^\dagger L \right) dt + L dB^\dagger(t) - L^\dagger dB(t) \right\} U(t)$$

with the initial condition $U(0) = I$. Here $i = \sqrt{-1}$ is the imaginary unit, $H(t)$ is the system Hamiltonian, and the coupling strength operator L together with the field operator $b(t) = \dot{B}(t)$ models the system probe interaction. $Q(t) = B(t) + B^\dagger(t)$ is the real quadrature of the input laser field. A homodyne detector is used to continuously monitor the observable $Y(t)$, which generates a photocurrent that satisfies

$$dY(t) = U^\dagger(t)(L + L^\dagger)U(t)dt + dQ(t).$$

By using the past history of the observation process $Y(t)$, a quantum filter determines the least-mean-square sense optimal estimation of the atomic system state that satisfies the following quantum stochastic master equation [3]:

$$d\rho_t = \mathscr{L}^\dagger_{L,H}(\rho_t)dt + \mathscr{D}_L(\rho_t)dW(t), \tag{1}$$

with $\rho_0 = \pi_0$. Here $\mathscr{L}^\dagger_{L,H}$ is the adjoint Lindblad generator:

$$\mathscr{L}^\dagger_{L,H}(X) = -i[H,X] + LXL^\dagger - \frac{1}{2}(L^\dagger LX + XL^\dagger L),$$

$\mathscr{D}_L$ is defined by

$$\mathscr{D}_L(X) = LX + XL^\dagger - X\,\mathrm{Tr}(X(L+L^\dagger)),$$

and $dW(t) = dY(t) - \mathrm{Tr}(\rho_t(L+L^\dagger))dt$ is a classical Wiener process.

[1]We have assumed $\hbar$=1 by using atomic units in this paper.

Note that (1) is a classical stochastic differential equation that is driven by the classical photocurrent signal $Y(t)$ and can thus be implemented on a classical signal processor. Equation (1) is widely used in applications including quantum state estimation and quantum feedback control ([10], [13]), where in time calculation of (1) is essential. However, one has to calculate a system of recursive It$\hat{o}$ stochastic differential equations with the dimension $n^2 - 1$, in order to determine the conditional probability densities ρ_t defined on $\mathscr{H}_{\mathscr{S}}$. High computational cost will arise if the atomic system has a large number of energy levels. It is the main goal of this paper to reduce the dimension of the filtering equations while guaranteeing acceptable approximation performance.

III. Design of An Exponential Projection Quantum Filter

In this section, we propose a projection filtering approach to approximating the quantum filter equation in (1), using differential geometric methods in quantum information geometry theory. The basic idea of the projection filtering strategy is illustrated in Fig. 1. We consider to apply a projection operation to a space of quantum density operators and map the optimal quantum filter equation onto a fixed lower-dimensional submanifold. A natural basis will be derived for the tangent space in each point of this submanifold, and a local projection operation can be defined with respect to a quantum Fisher metric to map the infinitesimal increments generated by the quantum filter equation onto such tangent spaces. The resulting stochastic vector field on the submanifold then defines the dynamics of the approximation filter. In this paper, we consider to use a submanifold consisting of an exponential family of quantum density operators. It is noted that quantum density operators in the exponential form is useful in practice, examples being Gaussian states and general thermal states [12].

A. Some Preliminaries on Quantum Information Geometry

This subsection will be introducing some foundations of the quantum information geometry theory. Detailed formulation can be found in Chapter 7 from the book [1]. Denote the set of all self-adjoint operators on the Hilbert space $\mathscr{H}_{\mathscr{S}}$ by

$$\mathbb{A} = \{A | A = A^\dagger\}. \tag{2}$$

Subsequently, we focus on the geometry of the totality of nonnegative self-adjoint operators which is denoted by

$$\mathbb{Q} = \{\rho | \rho \geq 0, \rho \in \mathbb{A}\}. \tag{3}$$

$\mathbb{Q}$ is an open subset of $\mathbb{A}$ and hence is naturally regarded as a real manifold of dimension $\dim(\mathbb{Q}) = n^2$. Apparently, the tangent space at each point ρ to $\mathbb{Q}$, which is denoted by $\mathscr{T}_\rho(\mathbb{Q})$, is identified with $\mathbb{A}$.

When a tangent vector $X \in \mathscr{T}_\rho(\mathbb{Q})$ is considered as an element of $\mathscr{T}_\rho(\mathbb{Q})$ by this identification, we denote it by $X^{(m)}$ and call it the *m-representation* of X. When a coordinate system $[\varepsilon^i]$ is given on $\mathbb{Q}$ so that each state is parameterised

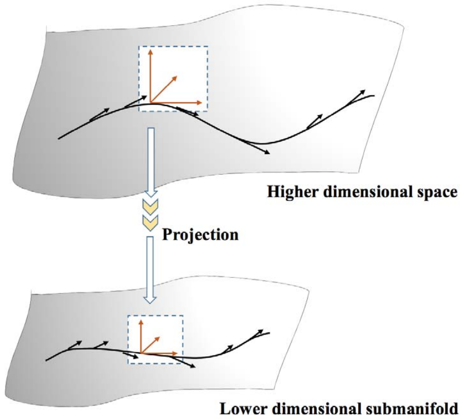

Fig. 1. Cartoon illustrating the basic ideas of projection filtering ([5], [6]). The solid black line in the upper half of the figure represents a quantum trajectory which flows within a higher dimensional space. The vector field (the black arrows) along the curve represents the temporal filter dynamics. The tangent vectors at each point live in a linear space called tangent vector space. In the bottom half of the figure, we build a lower dimensional submanifold which is embedded in the higher dimensional space. We aim to make the solutions to the optimal filter remain in the finite dimensional submanifold. This can be done if we project the linear tangent vector space of the higher dimensional space to the tangent vector space of the finite dimensional one, and make the two solid lines start from the same initial point.

as ρ_ε, the $m-$representation of the natural basis vector of the tangent vector space is identified with

$$(\partial_i)^{(m)} = \partial_i, \tag{4}$$

where $\partial_i := \partial\rho/\partial\varepsilon^i$. Assuming $\{\partial_i\}$ are linear independent, then we have

$$\mathscr{T}_\rho(\mathbb{Q}) = \mathrm{Span}\{\partial_i\}. \tag{5}$$

A differentiable manifold is not naturally endowed with an inner product structure. We need to add to the manifold a Riemannian structure. To be specific, we define a Riemannian metric on $\mathbb{Q}$. The *Bogoliubov inner product* is employed to define the inner product $\{\ll,\gg_\rho, \rho\in\mathbb{Q}\}$ on $\mathbb{A}$ [1]:

$$\ll A,B \gg_\rho = \int_0^1 \mathrm{Tr}(\rho^\lambda A\rho^{1-\lambda}B)d\lambda, \forall A,B\in\mathbb{A}. \tag{6}$$

Based on this inner product, we define another useful representation called $e-representation$ of a tangent vector $X\in\mathscr{T}_\rho(\mathbb{Q})$ as the self-adjoint operator $X^{(e)}\in\mathbb{A}$ satisfying

$$\ll X^{(e)}, A \gg_\rho = \mathrm{Tr}(X^{(m)}A), \forall A\in\mathbb{A}. \tag{7}$$

Using the $e-$representation defined above, we define an inner product $\langle,\rangle$ on $\mathscr{T}_\rho(\mathbb{Q})$ by

$$\begin{aligned}\langle X,Y\rangle_\rho &= \ll X^{(e)}, Y^{(e)} \gg_\rho \\ &= \mathrm{Tr}(X^{(m)}Y^{(e)}), \forall X,Y\in\mathscr{T}_\rho(\mathbb{Q}).\end{aligned} \tag{8}$$

Then $g=\langle,\rangle$ forms a Riemman metric on $\mathbb{Q}$ which may be regarded as a quantum version of the Fisher metric. The components of this metric are given by

$$g_{ij} = \langle\partial_i,\partial_j\rangle_\rho = \mathrm{Tr}(\partial_i^{(m)}\partial_j^{(e)}). \tag{9}$$

B. Design of the Quantum Projection Filter

Section III-A has demonstrated the differential geometric structure of the quantum state space. The main aim of this subsection will be deriving the quantum projection filter equation. We start from the unnormalized version of the quantum filter equation in (1):

$$d\bar\rho_t = \mathscr{L}^\dagger_{L,H}(\bar\rho_t)dt + \left(L\bar\rho_t + \bar\rho_t L^\dagger\right)dY(t), \tag{10}$$

where $\bar\rho_t$ is the unnormalized information state corresponding to ρ_t such that $\rho_t = \bar\rho_t/\mathrm{Tr}(\bar\rho_t)$. $\bar\rho_t$ is nonnegative and self-adjoint, and is initially set to be $\bar\rho_0 = \rho_0 = \pi_0$.

In the subsequent analysis, we will focus on the unnormalized filter equation (10) since its linear form is easier to manipulate compared with the nonlinear filter equations in (1). However, it is worth mentioning that in order to illustrate this dynamical equation using a differential manifold structure, we must interpret the above system using Stratonovich integral theory because It$\hat{o}$'s rule is incompatible with manifold structure [5]. We have the following result.

Lemma 1. The It$\hat{o}$ quantum stochastic differential equation in (10) is equivalent to the following Stratonovich quantum stochastic differential equation:

$$d\bar\rho_t = (-i[H,\bar\rho_t] - \mathscr{S}_L(\bar\rho_t))dt + \left(L\bar\rho_t + \bar\rho_t L^\dagger\right)\circ dY(t). \tag{11}$$

where $\mathscr{S}_L(\bar\rho_t) = \frac{(L+L^\dagger)L\bar\rho_t + \bar\rho_t L^\dagger(L+L^\dagger)}{2}$.

Now we design the quantum projection filter following the scheme illustrated in Fig. 1. It follows from (11) that the two terms $-i[H,\bar\rho_t] - \mathscr{S}_L(\bar\rho_t)$ and $L\bar\rho_t + \bar\rho_t L^\dagger$ on the right hand side of the equation are vectors in $\mathbb{A}$. The submanifold is designed to be a C^∞ manifold consisting of an exponential family of unnormalized quantum density operators.

$$\mathbb{S} = \{\bar\rho_\theta\} = \left\{e^{\sum_{i=1}^m \theta_i A_i}\right\}, \tag{12}$$

where the submanifold operators $A_i, i\in\{1,2,...,m\}\in\mathbb{A}$ are self-adjoint and predesigned. We suppose that the entire submanifold $\mathbb{S}$ can be covered by a single coordinate chart $(\mathbb{S}, \theta = (\theta_1,...,\theta_m)\in\Theta)$, where Θ is an open subset of $\Re^m$. Then we have $\dim\{\mathbb{S}\} = m$.

According to the chain rule in Stratonovich stochastic calculus, we have

$$d\bar\rho_\theta = \sum_{i=1}^m \bar\partial_i \circ d\theta_i, \tag{13}$$

where $\bar\partial_i := \partial\bar\rho_\theta/\partial\theta_i$. Assuming the set $\{\bar\partial_1,...,\bar\partial_m\}$ is linearly independent, then this set forms an $m-$representation of the natural basis of $\mathscr{T}_{\bar\rho_\theta}(\mathbb{S})$, i.e., the tangent vector space at each point $\bar\rho_\theta$ to $\mathbb{S}$. We have

$$\mathscr{T}_{\bar\rho_\theta}(\mathbb{S}) = \mathrm{Span}\{\bar\partial_i\}, i=1,...,m. \tag{14}$$

A useful formula for density operators of the exponential form (12) is [16]

$$\frac{\partial\bar\rho_\theta}{\partial\theta_i} = \int_0^1 \bar\rho_\theta^\lambda \frac{\partial_i \log\bar\rho_\theta}{\partial\theta_i}\bar\rho_\theta^{1-\lambda}d\lambda. \tag{15}$$

It then follows directly from (6) and (7) that $\bar{\partial}_i^{(e)} = A_i$. Thus the components of the quantum Fisher metric are given by

$$g_{ij}(\theta) = \ll \bar{\partial}_i^{(e)}, \bar{\partial}_j^{(e)} \gg_{\bar{\rho}_\theta} = \int_0^1 \mathrm{Tr}(\bar{\rho}_\theta^{\lambda} A_i \bar{\rho}_\theta^{1-\lambda} A_j) d\lambda. \quad (16)$$

The quantum Fisher information matrix is an $m \times m$ dimensional real matrix given by $G(\theta) = (g_{ij}(\theta))$. Then an orthogonal projection operation Π_θ can be defined for every $\theta \in \Theta$ as follows:

$$\begin{aligned} \mathbb{A} \quad & \longrightarrow \mathscr{T}_{\bar{\rho}_\theta}(\mathbb{S}) \\ \nu \quad & \longmapsto \sum_{i=1}^m \sum_{j=1}^m g^{ij}(\theta) \left\langle \nu, \bar{\partial}_j \right\rangle_{\bar{\rho}_\theta} \bar{\partial}_i, \end{aligned} \quad (17)$$

where the matrix $\left(g^{ij}(\theta)\right)$ is the inverse of the quantum information matrix $G(\theta)$.

Now we are ready to formulate the projection quantum filter. Consider a curve in $\mathbb{S}$ around the point $\bar{\rho}_\theta$ to be of the form $t \mapsto \bar{\rho}_{\theta_t}$. This corresponds to a real curve $\gamma : t \mapsto \theta_t$ in Θ around the real vector θ, though the coordinate chart $(\mathbb{S}, \theta)$. Let us consider that (11) starts from the initial condition $\pi_0 = \bar{\rho}_{\theta_0}$ for some $\theta_0 \in \Theta$. The unnormalized quantum projection filter is then defined as the following quantum stochastic differential equation on the m-dimensional differentiable manifold $\mathbb{S}$:

$$d\bar{\rho}_{\theta_t} = \Pi_{\theta_t}\left(-i[H, \bar{\rho}_{\theta_t}] - \mathscr{S}_L(\bar{\rho}_{\theta_t})\right) dt + \Pi_{\theta_t}\left(L\bar{\rho}_{\theta_t} + \bar{\rho}_{\theta_t} L^\dagger\right) \circ dY(t). \quad (18)$$

From the definition of the manifold $\mathbb{S}$ in (12), the projection quantum filter can be equivalently written using the equations satisfied by the real curve γ in Θ. Denote $\theta_t = (\theta_1(t), ..., \theta_m(t))'$. An explicit form of the curve equations is given in the following theorem.

Theorem 1. The real curve $\gamma : t \mapsto \theta_t$ satisfies the following recursive stochastic differential equations:

$$d\theta_t = G(\theta_t)^{-1} \{\Xi(\theta_t) dt + \Gamma(\theta_t) \circ dY(t)\}, \quad (19)$$

where $\Xi(\theta_t)$ and $\Gamma(\theta_t)$ are both $m-$dimensional column vector of real functions on θ_t. The jth element of them are given by

$$\Xi_j(\theta_t) = \mathrm{Tr}\left\{\bar{\rho}_{\theta_t}\left(i[H, A_j] - \frac{A_j(L+L^\dagger)L + L^\dagger(L+L^\dagger)A_j}{2}\right)\right\},$$

and

$$\Gamma_j(\theta_t) = \mathrm{Tr}(\bar{\rho}_{\theta_t}(A_j L + L^\dagger A_j)),$$

respectively.

Proof. Submitting the chain rule (13) into (18) yields

$$\begin{aligned} d\bar{\rho}_{\theta_t} &= \sum_{i=1}^m \bar{\partial}_i \circ d\theta_i(t) \\ &= \sum_{i=1}^m \sum_{j=1}^m g^{ij}(\theta) \mathrm{Tr}((i[\bar{\rho}_{\theta_t}, H] - \mathscr{S}_L(\bar{\rho}_{\theta_t}))A_j) \bar{\partial}_i dt \quad (20) \\ &\quad + \sum_{i=1}^m \sum_{j=1}^m g^{ij}(\theta) \mathrm{Tr}((L\bar{\rho}_{\theta_t} + \bar{\rho}_{\theta_t} L^\dagger) A_j) \bar{\partial}_i \circ dY(t) \\ &= \sum_{i=1}^m \sum_{j=1}^m g^{ij}(\theta) \mathrm{Tr}(\bar{\rho}_{\theta_t}(i[H, A_j] - \mathscr{S}_L^\dagger(A_j))) dt \bar{\partial}_i \\ &\quad + \sum_{i=1}^m \sum_{j=1}^m g^{ij}(\theta) \mathrm{Tr}(\bar{\rho}_{\theta_t}(A_j L + L^\dagger A_j)) \circ dY(t) \bar{\partial}_i, \quad (21) \end{aligned}$$

from which (19) can be concluded. □

The stochastic differential equation (19) combined with the equation (13) determines the unnormalized projection quantum density operator. In this paper, (19) is called the *unnormalized projection quantum filter*. The approximation quantum information state $\tilde{\rho}_t$ can be then simply obtained as $\tilde{\rho}_t = \bar{\rho}_{\theta_t} / \mathrm{Tr}(\bar{\rho}_{\theta_t})$. It can be observed that only a system of stochastic differential equation with the dimension m is needed to be calculated in order to determine $\tilde{\rho}_t$. Thus the computational cost would be reduced significantly if the number m is chosen to be small.

C. Design of the Submanifold

The design procedure in Subsection III-B requires a predesign of the submanifold operators $A_i, i = 1, ..., m$. A convenient design of these self-adjoint operators will be given in this subsection.

In fact, the proposed approximation scheme in Subsection III-B is implemented through two steps. First, the right-hand side of (11) is evaluated at the current projection filter quantum density operator $\bar{\rho}_{\theta(t)}$ on $\mathbb{S}$, instead of the true density operator $\bar{\rho}_t$. However, the right-hand side vectors $-i[H, \bar{\rho}_t] - \mathscr{S}_L(\bar{\rho}_t)$ and $L\bar{\rho}_t + \bar{\rho}_t L^\dagger$ will generally make the solution leave the manifold $\mathbb{S}$. Thus a second approximation is made by projecting these vector fields onto the linear tangent vector space $\mathscr{T}_{\bar{\rho}_\theta}(\mathbb{S})$. In this section, we will present a design of the submanifold $\mathbb{S}$ by concerning the local error for the quantum projection filter occurring in the *second* approximation step at time t.

Following the similar idea as in [5], we define at each point $\bar{\rho}_{\theta_t}$ the prediction residual as

$$\mathfrak{P}(\bar{\rho}_{\theta_t}) = \Pi_{\theta_t}(i[H, \bar{\rho}_{\theta_t}]) - i[H, \bar{\rho}_{\theta_t}], \quad (22)$$

and the correction residuals as

$$\mathfrak{C}_1(\bar{\rho}_{\theta_t}) = \Pi_{\theta_t}(\mathscr{S}_L(\bar{\rho}_{\theta_t})) - \mathscr{S}_L(\bar{\rho}_{\theta_t}) \quad (23)$$

and

$$\mathfrak{C}_2(\bar{\rho}_{\theta_t}) = L\bar{\rho}_{\theta_t} + \bar{\rho}_{\theta_t} L^\dagger - \Pi_{\theta_t}(L\bar{\rho}_{\theta_t} + \bar{\rho}_{\theta_t} L^\dagger), \quad (24)$$

respectively.

We will first show below that the submanifold can be designed in such a way that the correction residual $\mathfrak{C}_2$ is identically zero for all $t \geq 0$, which allows us to estimate the local error from the projection in terms of a single residual operator $\mathfrak{P} + \mathfrak{C}_1$.

For any given real vector θ_t, denote $\Lambda = \sum_{i=1}^m \theta_i A_i$. Define the adjoint operator ad_Λ by iterated commutators:

$$ad_\Lambda^0(X) = X, \text{ and } ad_\Lambda^j(X) = [\Lambda, ad_\Lambda^{j-1}(X)], j \geq 1. \quad (25)$$

Theorem 2. The correction residual $\mathfrak{C}_2 \equiv 0$ for all $t \geq 0$ if there exists a set of scalar functions $\{\xi_i(\theta_t), i = 1, ..., m\}$ such that

$$L + f(ad_\Lambda)(L^\dagger) = \sum_{i=1}^{m} \xi_i(\theta_t) h(ad_\Lambda)(A_i), \tag{26}$$

where the generating functions f and h are given by

$$f(t) = e^t \text{ and } h(t) = \frac{e^t - 1}{t}.$$

Proof. By using the nested-commutator relation

$$\begin{aligned}\bar{\rho}_\theta L^\dagger \bar{\rho}_\theta^{-1} &= e^\Lambda L^\dagger e^\Lambda \\ &= L^\dagger + [\Lambda, L^\dagger] + \frac{1}{2!}[\Lambda, [\Lambda, L^\dagger]] + ... \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} ad_\Lambda^k (L^\dagger) = f(ad_\Lambda)(L^\dagger). \end{aligned} \tag{27}$$

Similarly,

$$\begin{aligned}\frac{\partial \bar{\rho}_\theta}{\partial \theta_i} &= \int_0^1 \bar{\rho}_\theta^\lambda A_i \bar{\rho}_\theta^{1-\lambda} d\lambda \\ &= \int_0^1 ds \left(A_i + s[\Lambda, A_i] + \frac{s^2}{2!}[\Lambda, [\Lambda, A_i]] + ... \right) \bar{\rho}_\theta \\ &= \left(A_i + \frac{1}{2!}[\Lambda, A_i] + \frac{1}{3!}[\Lambda, [\Lambda, A_i]] + ... \right) \bar{\rho}_\theta \\ &= \sum_{k=0}^{\infty} \frac{1}{(k+1)!} ad_\Lambda^k (A_i) \bar{\rho}_\theta = h(ad_\Lambda)(A_i) \bar{\rho}_\theta. \end{aligned} \tag{28}$$

Thus (26) implies that $L\bar{\rho}_{\theta_t} + \bar{\rho}_{\theta_t} L^\dagger = \sum_{i=1}^{m} \xi_i(\theta_t) \bar{\partial}_i$, which means that the vector $L\bar{\rho}_{\theta_t} + \bar{\rho}_{\theta_t} L^\dagger$ already lie in the tangent vector space $\mathscr{T}_{\bar{\rho}_\theta}(\mathbb{S})$. It then follows from the definition of the projection operation in (17) that $\mathfrak{C}_2 \equiv 0$. □

The Lie algebraic equation (28) provides a general guideline for the submanifold design. For a special class of open quantum systems, we have the following result.

Theorem 3. Suppose the coupling operator L is self-adjoint and admits a spectral decomposition $L = \sum_{i=1}^{n_0} \lambda_i P_{L_i}$, where $n_0 \leq n$ is the number of the nonzero eigenvalues of L. The correction residuals $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are both identically zero for all $t \geq 0$, if the submanifold has a dimension $m = n_0$ and the submanifold operators $\{A_i\}$ are designed as

$$A_i = P_{L_i}, i = 1, ..., n. \tag{29}$$

Moreover, the unnormalized projection quantum filter reduces to

$$d\theta_t = G(\theta_t)^{-1} \operatorname{Tr}(i\bar{\rho}_{\theta_t}[H, A_j])dt - 2\alpha dt + 2\beta dY(t), \tag{30}$$

where $\alpha = (\lambda_1^2, ..., \lambda_{n_0}^2)'$ and $\beta = (\lambda_1, ..., \lambda_{n_0})'$.

Proof. Theorem 3 follows from Theorems 1 and 2, and the proof is omitted here due to space limitation. □

IV. Numerical Simulations

In this section, simulation results from a spin-$\frac{1}{2}$ system example is used to illustrate the performance of the proposed approximation filtering method. With the spin-$\frac{1}{2}$ system, the Hilbert space of the system is given by $\mathscr{H}_\mathscr{S} = \mathbb{C}^2$ and the quantum density matrix can be represented as

$$\rho_t = \frac{1}{2}(I_2 + x_t \sigma_x + y_t \sigma_y + z_t \sigma_z), \tag{31}$$

where $(x_t, y_t, z_t) = (\operatorname{Tr}(\rho_t \sigma_x), \operatorname{Tr}(\rho_t \sigma_y), \operatorname{Tr}(\rho_t \sigma_z)) \in \mathbb{R}^3$ is the Bloch vector of ρ_t, and $\sigma_x, \sigma_y, \sigma_z$ are Pauli matrices described by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{ and } \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The representation described in (31) forms an isomorphism between the state space of a two-level quantum system and the state space of Bloch vectors which is described by $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$.

Following the similar experimental settings as in [10], where the spin-$\frac{1}{2}$ system interacts with a laser field along the $z-$axis, the coupling operator $L = \sqrt{\mu}\sigma_z$, where $\mu > 0$ represents the coupling strength of the interaction between the atomic system and the field. Let the system Hamiltonian be $H = \frac{\omega_z}{2}\sigma_y$, where ω_z is the two-level pulsation. A straightforward computation yields

$$\begin{aligned} L\rho_t L^\dagger - \frac{1}{2}L^\dagger L \rho_t - \frac{1}{2}\rho_t L^\dagger L &= \mu(-x_t \sigma_t - y_t \sigma_y) \\ -i[H, \rho_t] &= \frac{\omega_z}{2}(z_t \sigma_x - x_t \sigma_z) \\ \sqrt{\mu}(-x_t z_t \sigma_x - y_t z_t \sigma_y + (1 - z_t^2)\sigma_z) &= \mathscr{D}_L(\rho_t) \end{aligned}$$

Then the filter equation in (1) can be equivalently written as

$$\begin{cases} dx_t = (-2\mu x_t + \omega_z z_t)dt - 2\sqrt{\mu} x_t z_t dW(t) \\ dy_t = -2\mu y_t dt - 2\sqrt{\mu} y_t z_t dW(t) \\ dz_t = -\omega_z x_t dt + 2\sqrt{\mu}(1 - z_t^2) dW(t) \end{cases}$$

Since L is self-adjoint, Theorem 3 can be used to design the submanifold. According to Theorem 3, the two submanifold operators can be given by

$$A_1 = P_{L_1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } A_2 = P_{L_2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

respectively, where P_{L_1} and P_{L_2} are the two projection operators of L though spectral decomposition. As a result, the unnormalized quantum projection filter becomes that in (30) with $\lambda_1 = \sqrt{\mu}$ and $\lambda_2 = -\sqrt{\mu}$.

In the simulation, the photocurrent is simulated from $dY(t) = \operatorname{Tr}(\rho_t(L + L^\dagger))dt + dW(t)$ and is used to drive the unnormalized quantum projection filter. Monte Carlo simulations have been conducted by using the discretization approach as in [11]. The simulation parameters used are as follows: the simulation interval $t \in [0, T]$ with $T = 5$, the normally distributed variance is $\delta t = T/N_0$ with $N_0 = 2^{12}$, and the step size is chosen to be $\Delta t = 2\delta t$. Let the initial quantum density matrix be given by a Bloch vector $(x_0, y_0, z_0) = (-0.5, 0.5, 0)$, $\mu = 1$, and $\omega_z = 0.2$.

The performance of the proposed approximation filtering scheme is demonstrated by comparing the probabilities that the atomic system is in the excited state, calculated using the quantum filter equation and the quantum projection filter equation, respectively. A number of simulations have been

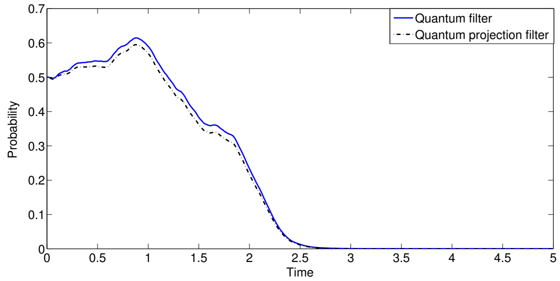

Fig. 2. Trajectories of the quantum filter and the quantum projection filter respectively

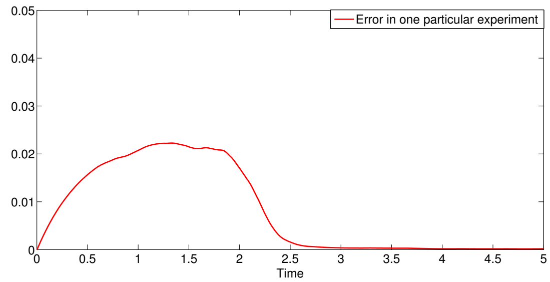

Fig. 3. Approximation error between the quantum filter and the quantum projection filter

conducted and it is found that the two probabilities are very close along the time. Simulation results from one particular experiment are presented in Fig. 2 and Fig. 3, respectively.

V. Conclusions

In this paper, a quantum projection filtering strategy is developed for open quantum systems subject to continuous homodyne detection. The quantum information geometry theory is used as the basic tool for deriving the quantum projection filter equation. Simulation results from a two-level quantum system suggest that the quantum projection filter is able to approximate the quantum filter with high accuracy. Further research topics include analysis of the approximation error bounds and extension of the approach to infinite dimensional quantum systems case.

APPENDIX

Proof of Lemma 1. Let $t_0 < t_1 < t_2 \ldots < t_p < T$ be a partition of any time interval $[t_0, T]$ and let the positive integer p be big enough. A direct discretization of the filter equation (10) yields

$$\begin{aligned}\bar{\rho}_{t_{i+1}} \simeq \bar{\rho}_{t_i} + (-i[H,\bar{\rho}_t] - \mathscr{L}_L(\bar{\rho}_t))\,\Delta t_i \\ + \left(L\bar{\rho}_t + \bar{\rho}_t L^\dagger\right)\Delta Y(t_i), i = 0, 1, \ldots, p-1, \qquad (32)\end{aligned}$$

where $\Delta t_i = t_{i+1} - t_i$ and $\Delta Y(t_i) = Y(t_{i+1}) - Y(t_i)$.

It is noted that $Y(t)$ is a classical Wiener process. Thus, when $p \to \infty$, one has $\Delta Y(t_i)\Delta Y(t_i) = \Delta t_i$ and $\Delta Y(t_i)\Delta t_i = 0$. From the definition of Stratonovich integral and (32), one has

$$\begin{aligned}&(s)\int_{t_0}^{T} L\bar{\rho}_t + \bar{\rho}_t L^\dagger \circ dY(t)\\
&= \lim_{p\to\infty}\sum_{k=0}^{p}\frac{L(\bar{\rho}_{t_{k+1}} + \bar{\rho}_{t_k}) + (\bar{\rho}_{t_{k+1}} + \bar{\rho}_{t_k})L^\dagger}{2}\Delta Y(t_k)\\
&= (I)\int_{t_0}^{T} L\bar{\rho}_t + \bar{\rho}_t L^\dagger dY(t)\\
&\quad + \lim_{p\to\infty}\sum_{k=0}^{p}\frac{L(L\bar{\rho}_{t_k} + \bar{\rho}_{t_k}L^\dagger) + (L\bar{\rho}_{t_{k+1}} + \bar{\rho}_{t_k}L^\dagger)L^\dagger}{2}\Delta t(t_k)\\
&= (I)\int_{t_0}^{T} L\bar{\rho}_t + \bar{\rho}_t L^\dagger dY(t)\\
&\quad + \frac{1}{2}\int_{t_0}^{T} LL\bar{\rho}_t + \bar{\rho}_t L^\dagger L^\dagger + 2L\bar{\rho}_t L^\dagger dt. \qquad (33)\end{aligned}$$

Lemma 1 can be obtained by submitting (33) into (10). □

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