Approximation Approaches for Inventory Systems with General Production/Ordering Cost Structures

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Abstract

The production/ordering cost structure is fundamental to determining an optimal inventory control policy. For example, it is well known that a base-stock policy is optimal for inventory systems with linear production costs, whereas an (s, S) policy is optimal if both linear and fixed costs exist. However, many of the cost structures that have arisen from the practice are quite complex and make the optimal policies too complicated for managers to implement. In this paper, we propose several easy-to-implement and efficient heuristic policies for inventory systems with general production costs, which include multiple linear pieces and fixed costs, suggesting a wide application to many practical problems that were previously difficult to solve. We establish the worst-case performance bounds on the proposed heuristic policies by using the concept of K-approximate convexity. Our extensive numerical studies, which are designed to reflect practical inventory control applications, evaluate the performance of the heuristic policies and show that the best heuristic policy we propose performs extremely well. We also try to provide explanations for the performance of different heuristic policies.

Key words: inventory control, general production/ordering cost, K-approximate convexity

1 Introduction

1.1 Motivation

The fundamental objective of inventory management is to characterize an optimal policy or design an efficient and easy-to-implement heuristic policy that reduces the mismatch between demand and supply, such that the total expected cost is minimized. One of the important factors affecting an optimal policy is the production cost structure. For example, it is well known that a base-stock policy is optimal for inventory systems that have only linear variable cost, whereas an (s, S) policy is optimal when considering both linear and fixed costs. However, in some real applications, firms face the challenge of complex production costs. To demonstrate this, we provide the following two examples.

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Motivation I. Labor cost is a key component of operation costs in the manufacturing industry, which is readily validated by the efforts made to reduce the labor cost. Numerous engineering and management innovations have attempted to enhance productivity and reduce the labor element in production costs. The driving force behind increased offshoring is the desire to seize cheap labor opportunities in developing countries. However, the labor cost is also a battlefield on which workers' rights are defended. The Fair Labor Standards Act (FLSA) of 1938 was a milestone in the protection of workers' benefits. The FLSA stipulates that "unless exempt, employees covered by the Act must receive overtime pay for hours worked over 40 in a workweek at a rate not less than one and a half their regular rates of pay."[1] Canada and South Korea have the same regulation regarding overtime pay as the U.S. In most European countries, the overtime rate is at least 1.25 times to twice the regular pay. In India, if a worker works more than nine hours in a day or more than forty-eight hours in a week, he or she receives overtime pay of at least twice the regular rate. China has two overtime rates. The overtime pay for working more than eight hours in a week day is at least 1.5 times the regular rate, and that for working during a weekend is at least double. Similarly, in Australia, daily workers get 1.5 times the regular rate for the first two overtime hours and twice the rate thereafter. Under all of these regulations, the labor cost is a piecewise linear convex function of the production quantity. The production cost also includes the raw material, energy, transportation, administrative costs, etc., which we refer to as the non-labor production cost. It is sufficient to consider the non-labor production cost as a piecewise linear concave function that reflects economies of scale. Integrating this non-labor production cost with a piecewise linear convex labor cost usually results in a general piecewise linear production cost, which does not necessarily follow any structure studied in the literature.

Motivation II. We consider a manufacturer who is equipped with two types of production equipment or technology and can choose either or both of them to produce the product in each period. Each unit of the product produced by either equipment type incurs both fixed and linear costs. Suppose that neither production equipment type dominates in the fixed and linear costs. Under such a cost structure, there is a threshold point such that if the production amount exceeds this point, the manufacturer uses the equipment with a higher fixed cost and a lower linear cost; otherwise, the other type of equipment is preferred. If one equipment type has a capacity, the cost structure is a piecewise linear function with two jump points, which is neither convex nor concave, and not even continuous. This structure can also be seen in inventory systems with multiple suppliers. For example, if a retailer has multiple suppliers with different fixed and linear costs,

[1] c.f. http://www.dol.gov/whd/overtime_pay.htm

it may face a non-convex, non-concave ordering cost that contains multiple jumps if the suppliers have the capacities. For any inventory model that considers a less-than-truckload shipping cost in the ordering cost, the ordering cost can also be modeled as a non-convex, non-concave piecewise linear function (c.f. Chan et al. 2002 and Li et al. 2004).

These practical applications have motivated us to study a classical periodic-review inventory system with a general cost structure. In each period, the firm must determine the production quantity to satisfy the stochastic demand. Any unsatisfied demand is fully backordered with a backlog cost, while the excess inventory is carried to the next period with a holding cost. Unlike the previous models in the literature, the production cost is a general piecewise linear function of the production quantity, which may not be convex or concave or even continuous. The firm aims to minimize the total expected cost over the entire planning horizon.

Convexity is a desirable property for establishing well-structured optimal policies for various inventory control problems that can be formulated as dynamic programs. Assuming that the cost-to-go functions are convex, the optimal policy can be fully characterized and easily implemented. Unfortunately, under a general piecewise linear production cost, the objective functions may not have this property, which makes the optimal policies too complicated to characterize. Even for cases where the structure of the optimal policies is identifiable, the daunting complexity of the optimal policies often limits their applicability in practice. To address this issue, we use a new tool named the K-approximate convexity. A function is K-approximate convex if it can be approximated by a convex function whose maximal distance to the original function is K. Applying this idea to replace non-convex functions with convex functions, we develop well-structured heuristic policies that can be implemented in practice. In particular, we propose three approximation approaches, each of which has a well-structured optimal policy. As the main difficulty arises from the non-convexity of the production cost, the first approach is to approximate the ordering cost function with a convex cost function. The second approach is to approximate the cost-to-go function, while the third is to approximate the sum of the expectation of the cost-to-go function and expected inventory holding/backlogging cost function.

When designing the heuristic policy, two fundamental issues must be addressed, including the computational complexity and policy effectiveness. For our proposed approximation approaches, we need to identify the complexity of obtaining a convex function that minimizes the maximal distance to the function that is approximated. We show this can be done by computing the convex envelope of the function. For a piecewise linear function, we provide a very efficient way of computing its convex envelope. For the effectiveness of heuristic policies, we establish their worst-case performance bounds, which are linear functions of K (the non-convexity level of the production cost) and quadratic functions of the number of periods. These performance bounds indicate that a slight non-convexity (a small K) of the production cost is acceptable because we can still get a well-structured heuristic policy with good performance. Surprisingly, extensive numerical experiments demonstrate that the second and third approaches perform extremely well and are close to optimal under various production cost structures.

Our main contribution is our proposal of efficient and easy-to-implement heuristic policies for inventory systems with a general production cost, which can be applied to many practical problems that were previously very hard to solve. The concept of K-approximate convexity provides us with a framework to construct these heuristic policies and establish their worst-case performance bounds.

The remainder of this paper is organized as follows. We review the related literature in the rest of this section. Section 2 describes the model and presents some results on K-approximate convexity. In Section 3, we first fully characterize the optimal policy for the single-period problem and then propose heuristic policies using the concept of K-approximate convexity. We present our numerical results in Section 4, and conclude the paper in Section 5.

1.2 Literature Review

The periodic-review, stochastic inventory control problem has been extensively studied since the 1950s. The classical model assumes that the variable production cost is a linear function of the production quantity. The corresponding optimal policy is a base-stock policy, due to the convexity of the objective function. It has long been recognized in the literature, e.g., Scarf (1963) and Porteus (1971), that the variable production cost can take a non-linear form. Efforts to address this issue have mainly focused on cases in which the production cost is either concave or convex. For inventory models with concave production costs, Porteus (1971, 1972) was among the first to prove the optimality of a generalized (s, S) policy under some conditions on the distribution of demand. Motivated by an inventory system with two suppliers, Fox et al. (2006) studied an inventory model in which the ordering cost was a piecewise linear concave function with two pieces. Zhang et al. (2012) extended this model to include a capacity constraint on the supplier with the lower unit ordering cost, which led to a non-convex, non-concave cost structure. As both Fox et al. (2006) and Zhang et al. (2012) assumed that the supplier with the lower unit ordering cost does not charge a fixed cost, the ordering cost was a continuous function of the order quantity. In contrast, our model can handle an ordering cost structure with multiple jumps caused by fixed costs and suppliers' capacities. Karlin (1960) pioneered the stream of research on inventory models with convex production costs. Henig et al. (1997) studied an inventory model with a piecewise linear convex ordering cost and used the corresponding total cost to derive the optimal volume in a supply contract. Lu and Song (2014) studied the optimal policy of an inventory system with a piecewise linear convex variable cost and a fixed cost K, which was a $\frac{K}{2}$-approximate convex function. Hence, the cost structure studied in this paper is more general than that of Lu and Song (2014). The heuristic algorithm developed in Lu and Song (2014) used the special structure of convex variable cost. However, the heuristic algorithm developed in this paper uses the idea of K-approximate convexity that allows us to solve problems with a more general cost structure. In some inventory models, although the variable replenishment cost is linear, fixed costs are incurred once the replenishment quantity in a period exceeds a certain level. This type of model is often referred to as an inventory model with quantity-dependent variable/fixed costs. Representative works include but are not limited to Lippman (1969), Iwaniec (1979), Chao and Zipkin (2008), Li et al. (2009), Huggins and Olsen (2010), and Caliskan-Demirag et al. (2012). Our model obviously generalizes all of the aforementioned inventory models. A special case of our model is the inventory control problem with a capacity constraint. Representative studies include Federgrun and Zipkin (1986), Shaoxiang and Lambrecht (1996), Aviv and Federgrun (1997, 2001), Gallego and Scheller-Wolf (2000), Özer and Wei (2004), Shaoxiang (2004), Huh et al. (2011), and Wang et al. (2012) among others.

The concept of K-approximate convexity was first introduced by Lu et al. (2016). This paper differs from Lu et al. (2016) in three major ways. First, it focuses on a different problem. Lu et al. (2016) studied the joint pricing and inventory control problem with incomplete demand information. In that problem, revenue function is not completely known and may be not concave, while the ordering cost is a linear function of the ordering quantity. This paper studies the inventory control problem where the production/ordering cost is a general piecewise linear function. Second, although the concept of K-approximate convexity is applied to solve both problems, the approximation approaches are quite different. In Lu et al. (2016), the heuristic policy was based on approximating a one-period revenue function by a convex function. In this paper, in addition to testing the heuristic policy of approximating a one-period cost function, which uses the same idea as that of Lu et al. (2016), we develop two new heuristic policies. One approximates the cost-to-go function. The other approximates the sum of the expectation of the cost-to-go function and expected inventory holding/backlogging cost function. We find that these two new heuristic policies perform much better than the heuristic policy of approximating a one-period cost function. Finally, the method of computing a convex approximation of a piecewise linear function in Lu et al. (2016) applies only to a continuous function, while the method in this paper can handle a discontinuous case.

2 Model Description and Preliminaries

In this section, we first describe our model in detail and then provide some preliminary results on K-approximate convexity that are used in our subsequent analysis.

2.1 Model Description

We consider a finite-horizon periodic-review stochastic inventory control problem with T periods. Let D_t denote the demand in period t, which is a discrete random variable. At the beginning of each period t, the firm must determine the production quantity to satisfy the stochastic demand. The production cost $c(z)$ in each period is a piecewise linear increasing function of the production quantity z with $c(0) = 0$ and breakpoints $0 = q_0 < q_1 < q_2 < \cdots < q_{n-1} < q_n = +\infty$, i.e., for $z \in (q_{i-1}, q_i]$,

$$c(z) = K_i + c_i z, \tag{1}$$

where $c_i \geq 0$, $i = 1, ..., n$.

We assume that $c(z)$ increases in z, which is equivalent to $K_1 \geq 0$ and $K_i + c_i q_i \leq K_{i+1} + c_{i+1} q_i$ for any $i = 1, ..., n-1$. Note that we allow $K_i + c_i q_i < K_{i+1} + c_{i+1} q_i$, which implies that fixed costs may exist at those breakpoints. For the inventory applications considered in this paper, the piecewise linear assumption is without loss of generality because in practice the inventory level and demand take only integers, which automatically leads to a piecewise linear cost structure. For other applications where the state can take continuous values, the cost function does not need to be piecewise linear. However, there are many ways of approximating a nonlinear function by a piecewise linear function (Lin et al. 2013).

At the end of each period, any unsatisfied demand is fully backlogged with a unit shortage cost $p \geq 0$, and the leftover inventory is carried to the next period with a unit holding cost $h \geq 0$. Let $H_t(y_t)$ denote the inventory holding and shortage costs in period t, which can then be expressed as $H_t(y_t) = \mathbb{E}[h(y_t - D_t)^+ + p(y_t - D_t)^-]$. The firm's objective is to determine a production policy that minimizes the total expected cost over the whole planning horizon.

Given the initial inventory level x_t in period t, let $V_t(x_t)$ denote the cost-to-go function at the beginning of period t, which represents the minimal expected costs incurred from period t to the end of the planning horizon if the firm acts optimally. Denote $\alpha \in [0, 1]$ as the discount factor for any period t. The Bellman equation states that the cost-to-go function $V_t(x_t)$ should satisfy

$$V_t(x_t) = \min_{y_t \geq x_t} \Big\{ c(y_t - x_t) + H_t(y_t) + \alpha \mathbb{E}[V_{t+1}(y_t - D_t)] \Big\}. \tag{2}$$

For simplicity, we assume that $V_{T+1}(x_{T+1}) = h_{T+1}x_{T+1}^+ + p_{T+1}x_{T+1}^-$ and that $-p_{T+1} \leq h_{T+1}$ to ensure the convexity of $V_{T+1}(x_{T+1})$. To avoid the trivial solution of producing an infinite amount in any period t, we also assume that $c_n + \sum_{i=0}^{T-t} \alpha^i h + \alpha^{T-t+1} h_{T+1} \geq 0$.

For problem (2), it is well known that the base-stock policy is optimal when $c(z)$ is a linear function, whereas the (s, S) policy is optimal if $c(z) = K1(z > 0) + cz$, where $1(\cdot)$ is an indicator function. Unfortunately, the optimal policy can become very complicated when $c(z)$ has multiple pieces presented by (1). First, the production region can be disconnected in the sense that a threshold below which it is optimal to produce and above which it is optimal not to produce may not exist. Second, the optimal produce-up-to level does not have to be a specific level or $x + q_i$. These complications are illustrated by the following example.

Example 1. Consider a two-period problem, i.e., $T = 2$. The production cost has three linear pieces with slopes $c_1 = 1.2489$, $c_2 = 1.6021$, and $c_3 = 1.5223$. The breakpoints are $q_1 = 118$ and $q_2 = 467$. The demand distribution follows $P(D_t = 0) = 0.3183$, $P(D_t = 1000) = 0.3062$, $P(D_t = 2000) = 0.2053$, and $P(D_t = 3000) = 0.1703$ for any $t = 1, 2$. In addition, $H_t(y_t) = \mathbb{E}[0.1567(y_t - D_t)^+ + 0.2194(y_t - D_t)^-]$ for $t = 1, 2$ and $V_{T+1}(x_{T+1}) = -0.0385x_{T+1}^+ + 1.9961x_{T+1}^-$.

The optimal policy in the first period is characterized as follows. For any $x_1 < 764$, the optimal produce-up-to level is 882. For any $x_1 \in [882, 1000)$, it is optimal to produce up to 1000. We should produce up to 1882 for $x_1 \in [1764, 1882)$. For any x_1 in the intervals $[764, 882)$ and $[1417, 1764)$, it is optimal to produce $q_1 = 118$. It is optimal not to produce if $x_1 \in [1000, 1417)$ or $x_1 \geq 1882$. $\square$

It can be observed from this example that there is no threshold point below which we must produce and above which we never produce, as it is optimal to produce in two disjointed intervals, i.e., $(-\infty, 1000)$ and $[1417, 1882)$. Even if the structure of the optimal policies is identifiable, the daunting complexity often limits their applicability in practice. These observations motivate us to focus on designing efficient and easy-to-implement heuristic policies.

2.2 K-Approximate Convexity

This subsection presents the definition and some properties of K-approximate convexity. We also construct an algorithm to obtain a convex approximation of a piecewise linear function that may not be continuous.

Lu et al. (2016) presented the following definition of K-approximate convexity and showed that it is a generalization of K-convexity.

Definition 2.1. *A function $f : S \mapsto \mathbb{R}$, where $S \subseteq \mathbb{R}$, is K-approximate convex (concave) if there exists a convex (concave) function $g : S \mapsto \mathbb{R}$ such that $\|f - g\|_\infty \equiv \sup_{x \in S} |f(x) - g(x)| \leq K$.*

In other words, a function is K-approximate convex if its distance in ℓ_∞ norm to some convex function is no greater than K. A natural question is how to obtain a convex approximation of a K-approximate convex function, such that the distance between the two functions in ℓ_∞ norm is bounded by K. Our main idea for obtaining such a convex approximation relies on the convex envelope of the function, which is defined as follows.

Definition 2.2. *For any function $f : S \mapsto \mathbb{R}$ where $S \subseteq \mathbb{R}$, the function $\underline{f}^* : S \mapsto \mathbb{R}$ is the convex envelope of f if*

$$\underline{f}^* = \sup\{\underline{f} : S \mapsto \mathbb{R} \mid \underline{f} \text{ is convex and } \underline{f} \leq f\}.$$

For any K-approximate convex function f, the following proposition shows that the maximum distance in ℓ_∞ norm between f and its convex envelope $\underline{f}^*$ is at most $2K$. As the convex envelope $\underline{f}^*$ is always below f, by shifting the convex envelope up by $\frac{1}{2}\|f - \underline{f}^*\|_\infty$, we can obtain a convex approximation of f whose distance from f is at most K.

Proposition 1. *For any K-approximate convex function $f : S \mapsto \mathbb{R}$ where $S \subseteq \mathbb{R}$, $\|f - \underline{f}^*\|_\infty \leq 2K$ and $\|f - \bar{f}\|_\infty \leq K$, where $\underline{f}^*$ is the convex envelope of f and $\bar{f} : S \mapsto \mathbb{R}$ is the convex function such that $\bar{f}(x) = \underline{f}^*(x) + \frac{1}{2}\|f - \underline{f}^*\|_\infty$ for any $x \in S$.*

Consider a piecewise linear function $W(x)$ with m pieces, where $m \geq 2$. Let $-\infty = x_0 < x_1 < x_2 < \cdots < x_{m-1} < x_m = +\infty$ denote the breakpoints defining the m pieces. For any $j \in \{0, 1, ..., m-1\}$, $W(x)$ is a linear function with slope b_j for all $x \in (x_j, x_{j+1})$. Thus, the interval (x_j, x_{j+1}) corresponds to piece j of $W(x)$ and b_j is referred to as the slope of piece j.

Algorithm 1. *Obtain a convex approximation $\bar{W}(x)$ of a piecewise linear K-approximate convex function $W(x)$.*

Step 1. For any $j \in \{1, ..., m-1\}$, define $y_j = \min\{W(x_j), \lim_{x \uparrow x_j} W(x), \lim_{x \downarrow x_j} W(x)\}$.

Step 2. Apply Andrew's monotone chain convex hull algorithm (c.f. Andrew 1979) to obtain the lower hull of the points in $P = \Big\{(x_j, y_j) : j \in \{1, ..., m-1\}\Big\}$. The lower hull of P is represented by the points in P on the lower hull in counter-clockwise order, i.e.,

$$\Big\{(x'_1, y'_1), (x'_2, y'_2), ..., (x'_{m'}, y'_{m'})\Big\} \subseteq P,$$

where m' is the number of points in P falling on its lower hull, $x_1 = x'_1 < x'_2 < \cdots < x'_{m'} = x_{m-1}$, $y_1 = y'_1$, and $y_{m-1} = y'_{m'}$.

Step 3. If $m = 2$, set $l = u = 1$. Otherwise, let $b'_j = \frac{y'_{j+1} - y'_j}{x'_{j+1} - x'_j}$ for any $j \in \{1, ..., m' - 1\}$,

$$l = \begin{cases} 1, & \text{if } b'_1 \geq b_0, \\ \max\left\{j \in \{2, ..., m'\} : b'_{j-1} < b_0\right\}, & \text{if } b'_1 < b_0, \end{cases}$$

$$u = \begin{cases} \min\left\{j \in \{1, ..., m' - 1\} : b'_j > b_{m-1}\right\}, & \text{if } b'_{m'-1} > b_{m-1}, \\ m', & \text{if } b'_{m'-1} \leq b_{m-1}. \end{cases}$$

Define

$$\underline{W}^*(x) = \begin{cases} y'_l + b_0(x - x'_l), & \text{for any } x \leq x'_l, \\ y'_j + b'_j(x - x'_j), & \text{for any } x \in (x'_j, x'_{j+1}], j \in \{l, l+1, ..., u-1\}, \\ y'_u + b_{m-1}(x - x'_u), & \text{for any } x > x'_u. \end{cases}$$

Step 4. Return

$$\bar{W}(x) = \underline{W}^*(x) + \frac{1}{2} \max\left\{\max\left\{W(x_j), \lim_{x \uparrow x_j} W(x), \lim_{x \downarrow x_j} W(x)\right\} - \underline{W}^*(x_j) : j \in \{1, ..., m-1\}\right\}.$$

Algorithm 1 can be interpreted as follows. Step 1 deals with the discontinuity of $W(x)$. Here, y_j represents the minimum of $W(x_j)$ and the values $W(x)$ converges to as x approaches x_j from the left and right. Obviously, $y_j = W(x_j)$ if $W(x)$ is continuous at x_j. Step 2 constructs the convex envelope of $W(x)$ for any $x \in (x_1, x_{m-1})$. For a finite set of points in a two-dimensional space, the lower hull of this set is the part of its convex hull visible from below, which runs from the leftmost point to the rightmost point in counter clockwise order. Note that the lower hull of P can be obtained by any convex hull algorithm in a two-dimensional space. Here, we choose the monotone chain algorithm (c.f. Andrew 1979) whose computational complexity is $O(m \log m)$ in general. However, as the points in P are sorted according to x_j, the monotone chain algorithm returns the lower hull of P in $O(m)$. Please note that by Algorithm 1, we can obtain the exact value of K, i.e., $K = \max\left\{\max\left\{W(x_j), \lim_{x \uparrow x_j} W(x), \lim_{x \downarrow x_j} W(x)\right\} - \underline{W}^*(x_j) : j \in \{1, ..., m-1\}\right\}$.

In Step 3, we extend the convex envelope of $W(x)$ from the domain (x_1, x_{m-1}) to the domain $(-\infty, +\infty)$. To obtain a convex approximation of $W(x)$, as shown in Proposition 1, we shift the convex envelope $\underline{W}^*(x)$ up by $\frac{1}{2}\|W - \underline{W}^*(x)\|_\infty$, which corresponds to the function $\bar{W}(x)$ in Step 4. Proposition 2 formally proves the validity of Algorithm 1 and analyzes its computational complexity.

Proposition 2. *Suppose that $W(x)$ is K-approximate convex. Algorithm 1 returns $\bar{W}(x)$ in $O(m)$. Moreover, $\bar{W}(x)$ is a convex function of x and satisfies $\|W - \bar{W}\|_\infty \leq K$.*

Lu et al. (2016) proposed a linear programming formulation to obtain the convex approximation of a continuous piecewise linear function $W(x)$. However, the cost function defined by (1) in this paper may not be continuous, in which case it cannot be approximated by solving the linear programming formulation but can be approximated by using Algorithm 1. The computational complexity of Algorithm 1 is only $O(m)$, which is much lower than that of solving a linear programming problem with $O(m)$ decision variables and $O(m)$ constraints whose complexity is at least $O(m^{\frac{5}{2}})$.

The following proposition provides a preservation property of K-approximate convexity, which will be useful for proving the performance bound of the heuristic policy proposed in Section 3.

Proposition 3. *If $c : \mathbb{R}^+ \mapsto \mathbb{R}$ is K-approximate convex and $f : \mathbb{R} \mapsto \mathbb{R}$ is convex, then $g(x) = \min_{y \geq x}\{c(y-x) + f(y)\}$ is K-approximate convex.*

3 Optimality Analysis and Heuristic Policies

In this section, we first characterize the optimal policy for the single-period problem. This serves two purposes. First, the single-period model formulates the production control problem for a perishable product whose inventory cannot be carried over periods. Second, inspired by the structure of the single-period optimal policy, we then develop three practically implementable and efficient heuristic policies for the multi-period problem using the concept of K-approximate convexity, and compare their performance.

3.1 The Single-Period Model

Given any initial inventory level x, the single-period inventory problem can be formulated as

$$V(x) = \min_{z \geq 0}\{c(z) + H_t(x+z)\} = \min_{y \geq x}\{c(y-x) + H_t(y)\}. \tag{3}$$

Define

$$z^*(x) = \min\left\{\arg\min_{z \geq 0}\{c(z) + H_t(x+z)\}\right\} \quad \text{and} \quad y^*(x) = \min\left\{\arg\min_{y \geq x}\{c(y-x) + H_t(y)\}\right\},$$

i.e., for any given initial inventory level x, the optimal production quantity and optimal produce-up-to level are $z^*(x)$ and $y^*(x)$, respectively. Obviously, $y^*(x) = x + z^*(x)$. The following proposition shows that the optimal production quantity decreases with the initial inventory level.

Proposition 4. (Monotonicity of Optimal Order Quantity) *The optimal order quantity $z^*(x)$ decreases with x.*

Proposition 4 can be applied to develop a polynomial-time algorithm that solves the single-period problem.

Algorithm 2. *Solve the single-period problem* (3) *for any initial inventory level* $x \in (-\infty, +\infty)$.

Step 1. Define

$$S^i = \inf\left\{\arg \inf_{y\in(-\infty,+\infty)} \{c_i y + H_t(y)\}\right\}$$

and

$$f^i(x) = \begin{cases} K_i + c_i q_i + H_t(x+q_i), & \text{if } x < S^i - q_i, \\ K_i + c_i(S^i - x) + H_t(S^i), & \text{if } S^i - q_i \le x < S^i - q_{i-1}, \\ K_i + c_i q_{i-1} + H_t(x + q_{i-1}), & \text{if } x \ge S^i - q_{i-1}, \end{cases}$$

for any $i \in \{1, ..., n\}$. *Furthermore, set* $f^0(x) = H_t(x)$ *for any* x.

Step 2. Let $\hat{x}_{n+1} = -\infty$ *and* $\hat{x}_0 = \infty$. *For any* $i = n, ..., 1$, *define* $\hat{x}_i = \min\{\bar{x}_{i,0}, \bar{x}_{i,1}, ..., \bar{x}_{i,i-1}\}$, *where* $\bar{x}_{i,j} = \inf\{x \ge \hat{x}_{i+1} : f^j(x) \le f^i(x)\}$ *for all* $j \in \{0, 1, ..., i-1\}$. *Note that we set* $\bar{x}_{i,j} = \infty$ *if* $\{x \ge \hat{x}_{i+1} : f^j(x) \le f^i(x)\}$ *is an empty set.*

Step 3. Set $z^*(x) = S^n - x$ *for any* $x \in (-\infty, \hat{x}_n)$,

$$z^*(x) = \begin{cases} q_i, & \text{if } x \in [\hat{x}_{i+1}, \min\{\hat{x}_i, \max\{\hat{x}_{i+1}, S^i - q_i\}\}), \\ S^i - x, & \text{if } x \in [\min\{\hat{x}_i, \max\{\hat{x}_{i+1}, S^i - q_i\}\}, \hat{x}_i), \end{cases} \qquad \text{for all } i \in \{1, ..., n-1\},$$

and $z^*(x) = 0$ *for any* $x \in [\hat{x}_1, \infty)$.

For any $i \in \{0, 1, ..., n\}$, *set* $V(x) = f^i(x)$ *for any* $x \in [\hat{x}_{i+1}, \hat{x}_i)$.

The following proposition shows that Algorithm 2 solves the single-period problem.

Proposition 5. *Algorithm 2 solves the single-period problem* (3). *If* $H_t(x)$ *is a piecewise linear function with* m *pieces, the computational complexity of Algorithm 2 is* $O(mn^2)$.

Based on the values S^i and $\hat{x}_i$ returned by Algorithm 2, we can define

$$A = \big\{\hat{x}_i : i \in \{1, ..., n+1\}\big\} \cup \big\{\min\{\hat{x}_i, \max\{\hat{x}_{i+1}, S^i - q_i\}\} : i \in \{1, ..., n-1\}\big\}.$$

Let $l = |A| - 1 \le 2n - 1$. For any $j \in \{0, 1, ..., l\}$, we denote a_j to be the $(j+1)$th smallest element in A, i.e., $A = \{a_0, a_1, ..., a_l\}$ where $a_0 < a_1 < \cdots < a_l$. Note that $a_0 = \hat{x}_{n+1} = -\infty$ and $a_l = \hat{x}_1$. Also let $s = a_l$. We obtain the following theorem that characterizes the optimal policy for the single-period problem.

Theorem 1. *It is optimal to produce when $x < s$ and it is optimal not to produce when $x \geq s$. There are $l \leq 2n-1$ intervals $[a_j, a_{j+1})$, $j \in \{0, 1, ..., l\}$, with $-\infty = a_0 < a_1 < a_2 < \cdots < a_l = s$, such that for all $x \in [a_j, a_{j+1})$, it is optimal to produce either exactly q_i for some $i \in \{1, ..., n-1\}$ or up to S^i for some $i \in \{1, ..., n\}$. Furthermore, for any interval $[a_j, a_{j+1})$ where it is optimal to produce up to S^i for some i, the optimal production quantity $z^*(x) = S^i - x \in (q_{i-1}, q_i]$.*

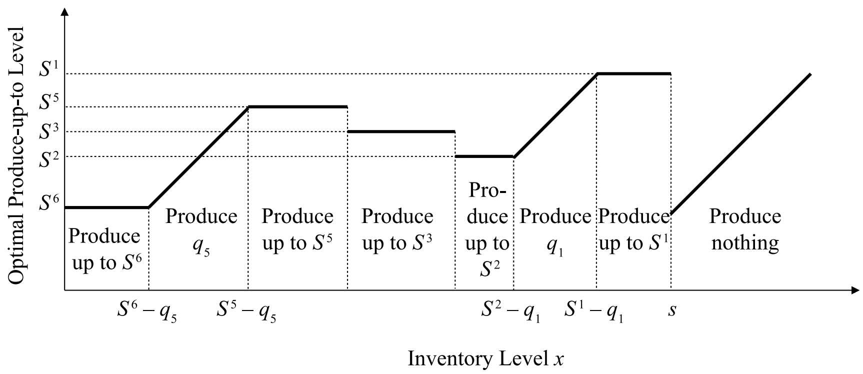

Figure 1: Optimal policy for the single-period problem (3)

Figure 1 illustrates the policy structure described in Theorem 1. The policy has a threshold s to determine whether to produce. The set over which it is optimal to produce can be divided into at most $2n-1$ intervals by the parameters a_j. In each of these intervals, the optimal production decision is to produce either up to some S^i or exactly q_i. Therefore, the optimal policy is a state-independent policy in the sense that it can be determined by the parameters $\hat{x}_i$ and $S^i - q_i$ defined in Algorithm 2. This nice structure only relies on the convexity of $H_t(y)$.

An important observation here is that both Algorithm 2 and Theorem 1 are applicable to capacitated inventory models. To see this, note that if c_n is sufficiently large, as $H_t(y)$ is a piecewise linear convex function, $c_n y + H_t(y)$ increases with y and hence $S^n = -\infty$. For example, when $H_t(y) = \mathbb{E}[h(y - D_t)^+ + p(y - D_t)^-]$, we have $S^n = -\infty$ as long as $c_n \geq p$. As $S^n = -\infty$, the case of producing up to S^n can never exist, and hence Theorem 1 implies that for any initial inventory level x, there exists some $i \in \{1, ..., n-1\}$ such that we should either produce exactly q_i or produce up to S^i with the corresponding production quantity in $(q_{i-1}, q_i]$. Therefore, the optimal production quantity will never exceed the capacity q_{n-1}. Consequently, such a policy is optimal for the corresponding capacitated problem with a production capacity q_{n-1}. In fact, Algorithm 2 returns the optimal policy for the counterpart with capacity q_{n-1} as long as we force $\hat{x}_n = -\infty$ in Step 2.

Proposition 4 implies that the optimal production quantity $z^*(x)$ decreases in the initial inventory. However, the optimal produce-up-to level $y^*(x)$ may not have any monotonicity property. The following proposition presents a monotonic property of $y^*(x)$ under a certain production cost structure.

Proposition 6. *If there exists a constant $q > 0$ such that $c(z)$ is concave for $z \in [0, q]$ and $c(z)$ is convex for $z \in [q, +\infty)$, then there exists a point $v \leq s$ such that $y^*(x)$ increases for any $x \in (-\infty, v)$ and decreases for any $x \in [v, s)$.*

3.1.1 A Special Case: Two Pieces

In this subsection, we characterize the optimal policy when $n = 2$. This subsection serves two purposes. First, many applications fall into the case with $n = 2$. Hence, a clear characterization of the optimal policy for this special case can benefit those applications. Second, it can help us to better understand the structure of the optimal policy based on Theorem 1.

Definition 3.1. *Given K_1, K_2, c_1 and c_2, we define*

$$
\begin{aligned}
s^1 &= \min\{x : c_1 x + H_t(x) \leq K_1 + c_1 S^1 + H_t(S^1)\};\\
s^2 &= \min\{x : H_t(x) - H_t(x + q_1) \leq K_1 + c_1 q_1\};\\
s^3 &= \min\{x : c_2 x + H_t(x + q_1) \leq K_2 + c_2 S^2 - c_1 q_1 - K_1 + H_t(S^2)\};\\
s^4 &= \min\{x : c_2 x + H_t(x) \leq K_2 + c_2 S^2 + H_t(S^2)\};\\
s^5 &= \min\{x : (c_2 - c_1)x + H_t(S^1) \leq K_2 + c_2 S^2 - K_1 - c_1 S^1 + H_t(S^2)\}.
\end{aligned}
$$

The following lemma partially identifies the relationships between these threshold points, which are helpful for characterizing the optimal policy.

Lemma 1. *(i) If $S^1 - q_1 < s^1$, then $s^2 \geq S^1 - q_1$. (ii) $s^2 \leq s^1$. (iii) $s^3 \leq S^2 - q_1$. (iv) If $S^2 - q_1 > s^2$, then $s^4 \leq S^2 - q_1$.*

Theorem 2. *Define $s^6 = \min\{S^2 - q_1, \max\{s^4, s^1\}\}$, $s^7 = \min\{\max\{s^5, S^1 - q_1\}, S^2 - q_1\}$, and $s^8 = \min\{s^1, \max\{s^5, S^1 - q_1\}\}$. The optimal policy structure of the single-period problem is characterized by Table 1.*

Theorem 2 implies that there are only four possible strategies: order nothing, order up to S^1, order exactly q_1, and order up to S^2. Once S^j, $j = 1, 2$, and s^i, $i = 1, ..., 5$, are computed, the optimal strategy is completely determined.

Table 1: Optimal Policy Structure

	Order nothing	Order up-to S^1	Order exactly q_1	Order up-to S^2
$S^1 - q_1 < s^1$ & $S^2 \le S^1$	$x \ge s^1$	$S^1 - q_1 \le x < s^1$	$s^3 \le x < S^1 - q_1$	$x < s^3$
$S^1 - q_1 < s^1$ & $s^1 + q_1 \ge S^2 > S^1$	$x \ge s^1$	$s^7 \le x < s^1$	$\min\{s^3, S^1 - q_1\} \le x < S^1 - q_1$	$S^1 - q_1 \le x < s^7$, $x < \min\{s^3, S^1 - q_1\}$
$S^1 - q_1 < s^1$ & $S^2 > s^1 + q_1$	$x \ge s^6$	$s^8 \le x < s^1$	$\min\{s^3, S^1 - q_1\} \le x < S^1 - q_1$	$s^1 \le x < s^6$, $S^1 - q_1 \le x < s^8$, $x < \min\{s^3, S^1 - q_1\}$
$S^1 - q_1 \ge s^1$ & $S^2 - q_1 \le s^2$	$x \ge s^2$	empty	$s^3 \le x < s^2$	$x < s^3$
$S^1 - q_1 \ge s^1$ & $S^2 - q_1 > s^2$	$x \ge \max\{s^4, s^2\}$	empty	$\min\{s^3, s^2\} \le x < s^2$	$s^2 \le x < \max\{s^4, s^2\}$, $x < \min\{s^3, s^2\}$

3.2 Heuristic Algorithms for the Multi-period Model

In this subsection, we propose three heuristic policies for the multi-period problem. The first, named as the cost-to-go function approximation (CTGA for short), is mainly inspired by the fact that the well-structured, single-period optimal policy only depends on the convexity of the inventory holding and the shortage cost function $H_t(y)$. By applying K-approximate convexity to approximate the cost-to-go function, the heuristic policy is practically implementable because its structure is the same as that of the single-period optimal policy illustrated in Figure 1. The second is named as the cost-to-go function expectation approximation approach (CTGEA), which is inspired by the same spirit of the CTGA approach. However, unlike the CTGA approach, which directly approximates the cost-to-go function, the CTGEA approach approximates its expectation. The third, named as the ordering-cost function approximation (OCA), is inspired by the fact that if the ordering cost in every period is a convex function, then so is the cost-to-go function. Hence, by applying K-approximate convexity to approximate the ordering cost function, we can have a well-structured heuristic policy that shares the same structure as that of the optimal policy for inventory systems with a piecewise linear convex ordering cost, as shown in Bensoussan et al. (1983). For all heuristic policies, K-approximate convexity allows us to derive their worst-case performance bounds.

3.2.1 Cost-to-go Function Approximation Approach

Recall that the cost-to-go function $V_{t+1}(x_{t+1})$ in (2) may not be convex, which complicates the optimal policy for the multi-period problem. To restore the nice properties of the optimal policy for the single-period problem, we use a convex function $\bar{W}_{t+1}(x_{t+1})$ to approximate $V_{t+1}(x_{t+1})$ and solve the following optimization problem:

$$W_t(x_t) = \min_{y_t \geq x_t} \Big\{ c(y_t - x_t) + H_t(y_t) + \alpha \mathbb{E}[\bar{W}_{t+1}(y_t - D_t)] \Big\}. \tag{4}$$

For any period t, the structure of the optimal policy for (4) is the same as that of the single-period optimal policy because $H_t(y) + \alpha \mathbb{E}[\bar{W}_{t+1}(y_t - D_t)]$ is a convex function. The key step in the CTGA approach is to obtain a convex approximation $\bar{W}_t(x_t)$ of $W_t(x_t)$.

When $t = T+1$, we choose $\bar{W}_{T+1}(x_{T+1}) = V_{T+1}(x_{T+1})$ because $V_{T+1}(x_{T+1})$ is a convex function. As $\bar{W}_{t+1}(x_{t+1})$ is convex, Proposition 3 ensures that if $c(z)$ is K-approximate convex, then so is $W_t(x_t)$. According to Proposition 2, a convex function $\bar{W}_t(x_t)$ such that $\|W_t - \bar{W}_t\|_\infty \leq K$ can be obtained by Algorithm 1. For each period t, the CTGA approach consists of two steps, summarized in Algorithm 3.

Algorithm 3. (CTGA Approach) *Obtain a heuristic policy for the multi-period inventory control model* (2).

Step 0. Define $\bar{W}_{T+1}(x_{T+1}) = V_{T+1}(x_{T+1})$ *for any* x_{T+1}.

Step 1. For any $t = T, ..., 1$,

Step 1.1. solve model (4) *by Algorithm 2 to get the heuristic policy and* $W_t(x_t)$;

Step 1.2. apply Algorithm 1 to get the convex approximation $\bar{W}_t(x_t)$ *of* $W_t(x_t)$.

Remark 1. *Comparing Algorithm 3 with the exact dynamic programming algorithm, we find that the computational complexity to obtain* $\bar{W}_t(x_t)$ *by Algorithm 1 is linear in the number of pieces of* $W_t(x_t)$. *Model* (4) *is much easier to solve than model* (2) *because* $\bar{W}_{t+1}(x)$ *is convex, whereas* $V_{t+1}(x)$ *is an arbitrary piecewise linear function. Most importantly, we can create a well-structured heuristic policy whose structure is the same as that shown in Figure 1, whereas the optimal policy is not practically implementable due to its complexity.*

Remark 2. *This heuristic is optimal for the single-period problem, as* $\bar{W}_{T+1}(x_{T+1}) = V_{T+1}(x_{T+1})$. *If the production cost* $c(z)$ *is convex, it is straightforward to show that* $V_t(x_t) = W_t(x_t) = \bar{W}_t(x_t)$ *for any* t *because model* (4) *implies that* $W_t(x_t)$ *must be a convex function if* $W_{t+1}(x_{t+1})$ *is a convex function. Thus, the heuristic policy is also optimal for the multi-period problem with a convex production cost.*

Given any inventory level x_t at the beginning of period t, let $\bar{V}_t(x_t)$ denote the total expected cost from period t to $T+1$ if the heuristic policy computed by Algorithm 3 is used. The following theorem provides a worst-case bound on the performance of the CTGA approach.

Theorem 3. *If $c(z)$ is K-approximate convex, $\bar{V}_t(x_t) \le V_t(x_t) + 2K\sum_{i=1}^{T-t} i\alpha^i$ for any $x_t \in \mathbb{R}$ and $t \in \{1, ..., T\}$.*

Remark 3. *This performance bound depends only on the number of periods T, the discount factor α, and the parameter K measuring the non-convexity of $c(z)$. It is independent of the inventory holding and shortage costs. Hence, the heuristic policy is very close to optimal when the inventory holding cost or shortage cost is large. This bound does not blow up when T goes to infinity. For any $\alpha \in [0, 1)$, $\sum_{i=1}^{T} i\alpha^i = \frac{\alpha-\alpha^{T+1}}{(1-\alpha)^2} - \frac{T\alpha^{T+1}}{1-\alpha}$. Hence, $\lim_{T\to\infty}\sum_{i=1}^{T} i\alpha^i = \frac{\alpha}{(1-\alpha)^2}$, which implies that $\frac{2\alpha K}{(1-\alpha)^2}$ is the worst-case performance bound of the heuristic policy for the infinite horizon problem.*

Theorem 3 provides a theoretical worst-case performance bound for the heuristic policy. Note that the complexity of Algorithm 1, which constructs the K-approximate function of $W_t(x_t)$, heavily depends on the number of pieces of $W_t(x_t)$. Suppose that the random demand is bounded by a maximum demand $\bar{D}$. The following proposition implies that the number of pieces linearly increases in the period length.

Proposition 7. *Recall that $V_{T+1}(x_{T+1}) = h_{T+1}x_{T+1}^+ + p_{T+1}x_{T+1}^-$. Let $h_t = h + \alpha h_{t+1}$ and $p_t = \min\{c_n, p + \alpha p_{t+1}\}$ for any $t = 1, ..., T$. For any $t = 1, ..., T$,*

$$W_t(x_t) = \begin{cases} p_t(-(T-t+1)\bar{B} - x_t) + W_t(-(T-t+1)\bar{B}) & \text{if } x_t \le -(T-t+1)\bar{B} \\ h_t(x_t - (T-t+1)\bar{D}) + W_t((T-t+1)\bar{D}) & \text{if } x_t \ge (T-t+1)\bar{D}, \end{cases}$$

where $\bar{B}$ is a finite positive number defined in the proof.

Proposition 7 shows that $W_t(x_t)$ is one linear piece when $x_t \le -(T-t+1)\bar{B}$ or $x_t \ge (T-t+1)\bar{D}$. The explanation behind this insight is as follows. First, when the inventory is high enough to satisfy the maximum possible demands in the remaining periods., i.e., $x_t \ge (T-t+1)\bar{D}$, it is optimal to order nothing from period t onward, which leads to the linear form of $V_t(x_t)$. Second, when the inventory is low enough, it is always optimal to order to satisfy the aggregated backlogged demand by period t. Thus, adding one unit of inventory in this case saves a unit shortage cost. As the convex hull approximation, $W_t(x_t)$ behaves consistently with $V_t(x_t)$ in both cases. In other words, all but two linear pieces of $W_t(x_t)$ are contained in the interval $[-(T-t+1)\bar{B}, (T-t+1)\bar{D}]$. In practice, the domain of $W_t(x_t)$ only takes integers because the demand and inventory level are integral values. Hence, Proposition 7 implies that the number of pieces of $W_t(x_t)$ is bounded by $(\bar{B}+\bar{D})(T-t+1)$, which is a linear function of $T-t$.

3.2.2 Cost-to-go Function Expectation Approximation Approach

In this subsection, we introduce another heuristic policy: the cost-to-go function expectation approximation approach (CTGEA). Instead of approximating the cost-to-go function, the CTGEA approach approximates the sum of the expectation of the cost-to-go function and expected inventory holding/backlogging cost function. This is an improvement of the CTGA approach developed in the previous section. The idea is motivated by the following example.

Example 2. Consider a two-period problem, i.e., $T = 2$. Referring to the cost structure in (1), we set the production cost to have three linear pieces with slopes $c_1 = 1.2$, $c_2 = 1.53$, and $c_3 = 1.5$. The breakpoints are $q_1 = 30$ and $q_2 = 50$. The demand distribution follows a discrete normal distribution whose cumulative distribution function is defined as $\Phi(\cdot, \mu, \sigma)$ with a mean of μ and a standard deviation of σ. We set $\mu = 30$ and $\sigma^2 = 10$. The demand takes a value z from an integer set of $\{0, 1, ..., 200\}$ with the probability of $\Phi(z+1, \mu, \sigma) - \Phi(z, \mu, \sigma)$. In addition, we set $\alpha = 0.9$, $H_t(y_t) = \mathbb{E}[0.2(y_t - D_t)^+ + 0.9(y_t - D_t)^-]$ for $t = 1, 2$, and $V_{T+1}(x_{T+1}) = 1.5x^-_{T+1}$.

In this example, the objective function $V_1(x_1)$ is not convex. Actually, one can check that the difference $V_1(x_1 + 1) - V_1(x_1)$ is not always increasing in x_1, for example, $V_1(-25) - V_1(-26) = -1.5 > V_1(-24) - V_1(-25) = -1.53$. However, after taking the expectation, the function $\mathbb{E}[V_1(y_1 - D_1)]$ is indeed convex. □

In this example, $V_1(x_1)$ is not convex. However, $\mathbb{E}[V_1(y_1 - D_1)]$ is convex. This observation confirms that the expectation under some demand distributions may smooth the objective function in the sense that the gap with the convex approximation function becomes smaller. Proposition 1 in Lu et al.(2016) showed that if $f(x)$ is K-approximate convex, then $E[f(x-D)]$ is K-approximate convex for any random variable D. Therefore, smoothing by expectation at least does not make the function become non-convex if it does not make the function more convex.

Because $H_t(y_t) + \alpha\mathbb{E}[V_{t+1}(y_t - D_t)]$ can be more convex than $\mathbb{E}[V_{t+1}(y_t - D_t)]$, we use a convex function $\widehat{R}_{t+1}(y_t)$ to approximate $H_t(y_t) + \alpha\mathbb{E}[V_{t+1}(y_t - D_t)]$, and solve the following optimization problem:

$$R_t(x_t) = \min_{y_t \geq x_t} \Big\{ c(y_t - x_t) + \widehat{R}_{t+1}(y_t) \Big\}. \quad (5)$$

The structure of the heuristic policy achieved by solving (5) is the same as that of the single-period optimal policy illustrated by Figure 1 because $\widehat{R}_{t+1}(y_t)$ is a convex function. This heuristic is summarized in the following algorithm.

Algorithm 4. (CTGEA Approach) *Obtain a heuristic policy for the multi-period inventory control model* (2).

Step 0. Define $\widehat{R}_{T+1}(y_T) = H_T(y_T) + \alpha\mathbb{E}[V_{T+1}(y_T - D_T)]$ *for any* y_T.

Step 1. For any $t = T, ..., 1$,

Step 1.1. solve model (5) *by Algorithm 2 to get the heuristic policy and* $R_t(x_t)$;

Step 1.2. apply Algorithm 1 to get the convex approximation $\widehat{R}_t(y_{t-1})$ *of*

$$H_{t-1}(y_{t-1}) + \alpha\mathbb{E}[R_t(y_{t-1} - D_{t-1})].$$

Remark 4. *As K-approximation convexity can be preserved by expectation (Proposition 1 in Lu et al. 2016), Theorem 3 also holds for the CTGEA approach.*

3.2.3 Ordering Cost Approximation Approach

If $c(z)$ is convex, then the cost-to-go function $V_{t+1}(x_{t+1})$ in (2) is also convex, which can lead to a well-structured optimal policy (see Bensoussan et al. 1983). This motivates us to use a convex function $\bar{c}(z)$ to approximate $c(z)$. This approach is named as the ordering cost approximation approach (OCA). The optimization problem can be rewritten as

$$U_t(x_t) = \min_{y_t \geq x_t} \Big\{ \bar{c}(y_t - x_t) + H_t(y_t) + \alpha\mathbb{E}[U_{t+1}(y_t - D_t)] \Big\}, \tag{6}$$

with $U_{T+1}(x_{T+1}) = V_{T+1}(x_{T+1})$ for any x_{T+1}.

We summarize this heuristic policy in Algorithm 5.

Algorithm 5. (OCA Approach) *Obtain a heuristic policy for the multi-period inventory control model* (2).

Step 0. Apply Algorithm 1 to get the convex approximation $\bar{c}(z)$ *of* $c(z)$.

Step 1. Define $U_{T+1}(x_{T+1}) = V_{T+1}(x_{T+1})$ *for any* x_{T+1}.

Step 2. For any $t = T, ..., 1$, *recursively solve model (6) by Algorithm 2 to get the heuristic policy and* $U_t(x_t)$.

Given any inventory level x_t at the beginning of period t, let $\widehat{V}_t(x_t)$ denote the total expected cost from period t to $T+1$ if the heuristic policy computed by Algorithm 5 is used. The following theorem provides a worst-case bound on the performance of the OCA approach.

Theorem 4. *If* $c(z)$ *is K-approximate convex,* $\widehat{V}_t(x_t) \leq V_t(x_t) + 2K\sum_{i=1}^{T-t+1} i\alpha^i$ *for any* $x_t \in \mathbb{R}$ *and* $t \in \{1, ..., T\}$.

Remark 5. *The worst-case bound of the OCA approach is slightly worse than that of the CTGA and CTGEA approaches presented by Theorem 3. In particular, the OCA approach incurs an additional performance gap in the last period, i.e., period T. Under the CTGA and CTGEA approaches, the function $\bar{W}_{T+1}(x_{T+1}) = V_{T+1}(x_{T+1})$ is already convex and hence the two heuristics are optimal in period T. However, under the OCA approach, replacing $c(z)$ with $\bar{c}(z)$ results in a performance gap.*

4 Numerical Study

In this section, we present a set of numerical experiments to test the effectiveness of the three proposed heuristic policies. The experiments are exercised in two practical settings. In the first setting, the firm faces a non-convex and non-concave production cost. In the second setting, the firm has two suppliers with different costs and capacities.

4.1 Non-convex and Non-concave Production Cost

We analyze the weekly labor cost as a function of the weekly production quantity. Suppose that the workers get a weekly or monthly salary and receive overtime pay if they are asked to work overtime. Let m denote the number of products a worker can produce in an hour, which can be calculated, for instance, from the productivity of the machine and the number of machines a worker can operate simultaneously. Using m and the number of workers a company employs, we can determine the number of units that the company can produce in a week without overtime work, i.e., in 40 hours, which is referred to as q_1. Note that there is a maximum amount of time that a worker can work in a day. Consequently, even with overtime work, there is a limit to the quantity that the company can produce during the weekdays in a week, which is denoted as q_2. Similarly, we can obtain a weekly production capacity, q_3, the maximum amount that can be produced in a week when the workers work overtime on both the weekdays and weekends. Let l denote the labor cost per unit, which can be calculated from the salary, the quantity m, and the working hours without overtime work in a week. If the weekly production quantity is no greater than q_1, the workers do not need to work overtime and the company does not need to pay the workers any extra money besides their regular salary. The marginal labor cost to produce one more unit is thus zero. We consider a regulation under which a worker must get an overtime rate that is at least 1.5 times the regular pay if he or she works more than 8 hours on a weekday, and at least 2 times the regular pay if he or she works on the weekend. Hence, when the production quantity exceeds q_1 but is no greater than q_2, the workers must work overtime on weekdays and receive 1.5 times the regular pay. The marginal labor cost to produce one more unit is thus $1.5l$. If the production quantity is greater than q_2, the workers must work on weekends and the marginal labor cost increases to $2l$. In summary, the weekly labor cost that depends on the production quantity is

$$l(z) = \begin{cases} 0 & \text{if } z \leq q_1, \\ 1.5l(z - q_1)^+ & \text{if } q_1 < z \leq q_2, \\ 1.5l(q_2 - q_1)^+ + 2l(z - q_2)^+ & \text{if } q_2 < z \leq q_3, \end{cases}$$

which is a piecewise linear convex function of the production quantity.

We generate l uniformly in $[0.4, 0.8]$ and q_1 uniformly in $[1000, 2000]$. We set q_2 to $1.3q_1$ and q_3 to $1.6q_1$, respectively. To capture economies of scale in the non-labor production cost, we uniformly generate two independent random variables β_q and β_c in $[0.5, 1.5]$ and $[0.6, 0.8]$, respectively. The non-labor production cost is $1 - l$ per unit if the production quantity is less than $\beta_q q_1$, and it decreases to $\beta_c(1 - l)$ if the production quantity exceeds $\beta_q q_1$. This results in a piecewise linear concave non-labor production cost. Summing up the convex labor cost and concave non-labor cost, the production cost $c(z)$ is thus a non-convex, non-concave function with four linear pieces, which is illustrated in Figure 2. The production capacity is $1.6q_1$.

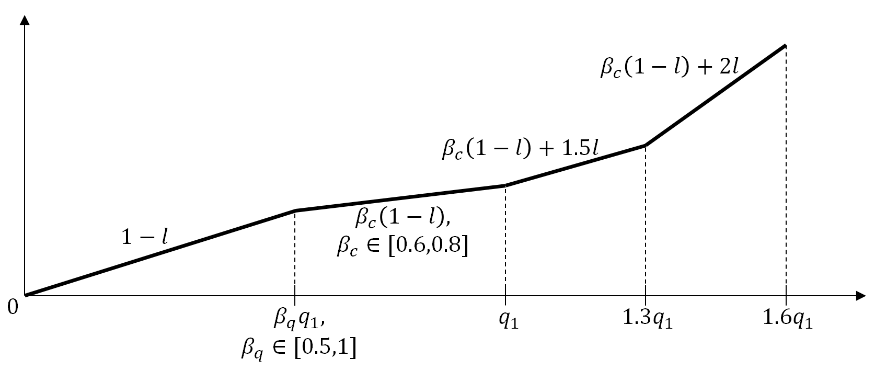

Figure 2: Variable production costs

We set $T = 10$ and $\alpha = 0.9$. We assume that the demand in each period t is identically and independently distributed and that the support of D_t is set to $\{500, 1000, 1500, 2000, 2500, 3000\}$. To verify whether the performance of heuristic policies is sensitive to demand distribution, we test three different types of demand distributions and generate 100 instances for each type. We report the average and worst performance of different policies over these 100 instances.

- The first demand distribution has a randomly generated probability mass function. For any $i = 1, 2, ..., 6$,

$$P(D_t = 500i) = \frac{U_i}{\sum_{i'=1}^{6} U_{i'}}, \tag{7}$$

where U_i are i.i.d random variables uniformly generated in [0,1].

- The second demand distribution is a uniform distribution in $\{500, 1000, ..., 3000\}$, i.e., $P(D_t = 500i) = 1/6$ for any $i = 1, 2, ..., 6$.
- The third demand distribution is a discrete normal distribution. We generate μ uniformly in $[1500, 2000]$ and σ uniformly in $[\mu/4, \mu/3]$. The demand distribution is set to $P(D_t = 500) = \Phi(750, \mu, \sigma)$, $P(D_t = 500i) = \Phi(500i + 250, \mu, \sigma) - \Phi(500i - 250, \mu, \sigma)$ for $i = 2, ..., 5$, and $P(D_t = 3000) = 1 - \Phi(2750, \mu, \sigma)$, where $\Phi(\cdot, \mu, \sigma)$ denotes the cumulative distribution function of a normal distribution with a mean of μ and a standard deviation of σ.

For each period t, we choose $H_t(y_t) = \mathbb{E}[h(y_t - D_t)^+ + p(y_t - D_t)^-]$, where the unit inventory holding cost h and the unit shortage cost p are uniformly generated in $[0.02, 0.2]$. The cost incurred at the end of the planning horizon is $V_{T+1}(x_{T+1}) = p_{T+1}x_{T+1}^-$, where p_{T+1}, the cost to fulfill a backlogged demand at the end of the planning horizon, is generated uniformly in $[1.4, 2.2]$.

For the purpose of comparison, we also consider a heuristic policy that assumes that the unit production cost is $c(1.6q_1)/1.6q_1$ and the production capacity is $1.6q_1$ for any period t. That is, this heuristic linearizes the production cost (OCLA for short). The resulting policy is a capacitated base-stock policy.

Let V_t^H denote the total expected cost from period t to $T + 1$ under a specific heuristic policy. As the maximum demand in each period is 3000 and the time horizon is 10 periods, we restrict the state region on $[-3 \times 10^4, 3 \times 10^4]$ in the numerical experiments. In particular, we use $\sup_{x_t \in [-3\times10^4, 3\times10^4]} \left\{V_t^H(x_t)/V_t(x_t) - 1\right\}$ to measure the performance of the corresponding heuristic policy for a problem with $T - t + 1$ periods. As $T = 10$, $t = 1$ implies that there are 10 periods, whereas $t = 9$ implies that there are 2 periods.

Table 2 displays the average and worst performance of different heuristic policies over 100 randomly generated instances for each demand distribution. It can be observed that the performance of CTGEA approach is always bounded by 1.94%, which implies that its cost is at most 101.94% of the optimal cost. Its average performance over all the tested instances in Table 2 is 100.02% of the optimal cost. The worst and average performance of the CTGA approach over all of tested instances are 103.71% and 100.27% of the optimal cost, respectively. For the OCA approach, the average performance over all of the tested instances is 109.34%, while the worst-case performance is around 182.51%. Finally, the OCLA approach is on average 126.56% of the optimal cost, and the worst case exceeds 218% of the optimal cost.

These results indicate that both the CTGEA and CTGA approaches are close to optimal and significantly better than the OCA and OCLA approaches. Moreover, as expected, the CTGEA

Table 2: Performance of three heuristics and base-stock heuristics for general production cost (%)

Distribution by (7)		$t=1$	$t=2$	$t=3$	$t=4$	$t=5$	$t=6$	$t=7$	$t=8$	$t=9$	$t=10$
CTGEA	Average	0.01	0.01	0.01	0.01	0.01	0.03	0.03	0.05	0.09	0
	Worst	0.06	0.07	0.08	0.11	0.10	0.15	0.30	0.84	1.94	0
CTGA	Average	0.14	0.16	0.17	0.20	0.22	0.24	0.28	0.38	0.41	0
	Worst	0.55	0.59	0.66	0.72	0.95	1.31	1.37	2.09	2.74	0
OCA	Average	8.72	8.47	8.21	8.03	8.11	8.12	8.73	9.17	10.61	14.01
	Worst	66.21	65.05	63.10	60.87	58.93	55.26	48.34	46.02	44.17	62.73
OCLA	Average	25.83	26.33	28.10	27.84	28.42	29.11	30.25	30.28	28.29	29.41
	Worst	70.26	71.81	73.57	72.35	74.36	78.62	81.37	83.62	81.98	118.13
Uniform distribution		$t=1$	$t=2$	$t=3$	$t=4$	$t=5$	$t=6$	$t=7$	$t=8$	$t=9$	$t=10$
CTGEA	Average	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.02	0.03	0
	Worst	0.03	0.03	0.03	0.05	0.05	0.06	0.06	0.18	0.45	0
CTGA	Average	0.14	0.14	0.15	0.16	0.18	0.20	0.24	0.27	0.21	0
	Worst	0.52	0.57	0.59	0.62	0.86	1.03	1.20	1.44	1.57	0
OCA	Average	9.51	9.32	9.27	9.19	9.01	8.82	9.34	9.83	11.75	15.05
	Worst	46.17	49.27	51.69	55.94	62.03	63.86	67.15	68.68	67.26	82.51
OCLA	Average	26.08	25.81	25.98	26.57	27.13	28.81	29.10	29.37	28.95	29.21
	Worst	74.87	75.24	74.68	75.25	75.89	76.07	76.12	76.58	76.83	95.53
Discrete normal distribution		$t=1$	$t=2$	$t=3$	$t=4$	$t=5$	$t=6$	$t=7$	$t=8$	$t=9$	$t=10$
CTGEA	Average	0.01	0.02	0.02	0.02	0.02	0.03	0.04	0.06	0.12	0
	Worst	0.09	0.13	0.16	0.22	0.27	0.32	0.49	0.78	1.26	0
CTGA	Average	0.29	0.31	0.32	0.34	0.42	0.48	0.53	0.66	0.73	0
	Worst	1.38	1.56	1.47	1.51	1.64	1.77	1.74	2.48	3.71	0
OCA	Average	8.37	8.46	8.35	8.29	8.21	8.36	8.49	8.62	9.50	10.19
	Worst	67.52	67.21	65.90	64.05	61.24	58.11	55.25	53.03	51.18	54.74
OCLA	Average	21.08	21.51	22.09	22.43	23.89	24.69	26.15	27.27	23.92	23.04
	Worst	69.33	70.14	71.07	71.85	70.63	73.91	79.55	72.67	77.35	76.84

approach performs a bit better than the CTGA approach because smoothing by expectation can make the objective function more convex. Finally, the performance of both the CTGEA and CTGA approaches is consistent for all three types of distribution, i.e., not sensitive to demand distribution.

4.2 Two-supplier Case

In this subsection, we conduct a set of comprehensive numerical experiments for the two-supplier case in which the firm faces two suppliers with different fixed costs, linear ordering costs, and capacities. For supplier one, we assume that the variable cost $c_1 = 1$, capacity $Q_1 = 1000$, and fixed cost K_1 is uniformly generated in $[50, 100]$. We assume that supplier two has unlimited capacity and a higher fixed cost, but a lower variable cost $c_2 = 0.8$.

In each period, the firm must determine which supplier to choose and how many units of product to order from each supplier. To cover different cost structures, we assume that the fixed cost of supplier two K_2 takes the values of 100, 200, 300, 500, 800, and 1000. Figure 3 depicts the ordering cost structures when $K_2 = 300$ and 500. It can be seen that when $K_2 = 300$, the cost includes a fixed cost and a variable concave ordering cost. However, when $K_2 = 500$, the cost has two jumps that lead to a neither concave nor convex discontinuous cost structure. All of the other parameters are generated in the same way as those in Section 4.1.

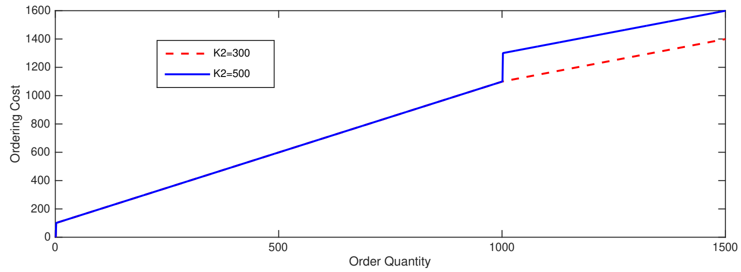

Figure 3: Ordering Costs when $K_1 = 100$ and $K_2 = 300$ and 500

Table 3: Performance of three heuristics for the two-supplier case (%)

Distribution by (7)		$t=1$	$t=2$	$t=3$	$t=4$	$t=5$	$t=6$	$t=7$	$t=8$	$t=9$	$t=10$
CTGEA	Average	0.48	0.45	0.52	0.57	0.61	0.64	0.66	0.42	0.30	0
	Worst	3.77	4. 12	4.75	4.91	5.16	4.83	5.06	5.38	6.15	0
CTGA	Average	1.20	1.19	1.15	1.17	1.22	1.25	1.20	1.13	0.71	0
	Worst	11.39	11.51	11.78	12.05	12.42	11.04	10.36	10.06	8.61	0
OCA	Average	8.84	9.01	8.92	9.16	9.55	10.14	10.38	11.47	12.33	23.57
	Worst	47.61	46.32	45.19	43.26	41.43	39.58	36.17	40.22	66.20	89.83
Uniform distribution		$t=1$	$t=2$	$t=3$	$t=4$	$t=5$	$t=6$	$t=7$	$t=8$	$t=9$	$t=10$
CTGEA	Average	0.45	0.47	0.48	0.47	0.52	0.55	0.61	0.43	0.29	0
	Worst	2.55	2.47	2.61	2.83	3.28	3.16	2.60	3.74	3.26	0
CTGA	Average	1.41	1.39	1.33	1.35	1.37	1.43	1.31	1.24	0.71	0
	Worst	9.23	9.35	8.86	9.03	8.71	8.74	8.43	7.51	6.32	0
OCA	Average	9.42	9.51	9.59	9.60	9.82	10.11	10.73	11.02	11.57	21.33
	Worst	48.05	47.17	45.81	44.46	42.48	39.13	36.21	31.30	35.10	58.85
Discrete normal distribution		$t=1$	$t=2$	$t=3$	$t=4$	$t=5$	$t=6$	$t=7$	$t=8$	$t=9$	$t=10$
CTGEA	Average	0.51	0.52	0.55	0.60	0.66	0.71	0.73	0.58	0.50	0
	Worst	3.28	3.48	3.91	4.15	4.36	4.23	3.37	4.82	6.95	0
CTGA	Average	1.21	1.20	1.24	1.25	1.27	1.31	1.30	1.27	0.89	0
	Worst	9.78	9.42	9.33	9.58	10.25	10.10	9.92	9.63	8.82	0
OCA	Average	11.26	11.33	11.52	11.84	12.05	12.19	12.73	13.21	13.94	28.49
	Worst	56.18	54.90	53.67	52.28	49.36	46.81	41.95	37.23	44.35	73.55

Table 3 presents the average and worst performance of three proposed heuristic policies in tested instances. It can be seen that the overall average (worst) performance of the CTGEA approach is within 100.48% (106.95%); that of the CTGA approach is within 101.09% (112.42%); and that of the OCA approach is 112.15% (189.83%). Overall, the CTGEA approach outperforms both the CTGA and OCA approaches. Note that the average and worst performance reported in each entry is not only over 100 randomly generated instances, as in the previous section, but also over 6 different values of the fixed cost of supplier two, K_2. Hence, for each period t and demand distribution, those reported values are the average and worst of 600 instances.

We next illustrate how the performance of the CTGEA and CTGA approaches depends on the

Table 4: Performance of Algorithms 3 and 4 with respect to K (%)

K-Approximation			$t=1$	$t=2$	$t=3$	$t=4$	$t=5$	$t=6$	$t=7$	$t=8$	$t=9$	$t=10$
K=50	CTGEA	Average	0.09	0.09	0.10	0.11	0.12	0.13	0.14	0.08	0.01	0
		Worst	0.85	0.87	0.89	0.89	0.94	1.03	1.10	0.74	0.38	0
	CTGA	Average	0.28	0.27	0.27	0.26	0.26	0.29	0.32	0.39	0.02	0
		Worst	1.05	0.98	0.94	0.92	0.99	1.01	1.09	1.15	0.61	0
K=100	CTGEA	Average	0.18	0.19	0.20	0.23	0.24	0.28	0.33	0.25	0.03	0
		Worst	1.29	1.34	1.47	1.55	1.62	1.85	1.58	1.07	0.71	0
	CTGA	Average	0.82	0.72	0.73	0.76	0.78	0.81	0.77	1.03	0.11	0
		Worst	2.14	2.06	1.93	1.89	2.23	2.37	2.19	2.26	1.30	0
K=150	CTGEA	Average	0.28	0.30	0.33	0.35	0.39	0.41	0.59	0.35	0.06	0
		Worst	1.71	1.68	1.83	1.95	2.04	2.17	2.42	1.70	1.05	0
	CTGA	Average	1.06	1.04	1.01	0.99	1.18	0.98	1.02	1.21	0.22	0
		Worst	3.34	3.28	3.13	3.09	3.02	2.98	3.11	3.27	2.24	0
K=250	CTGEA	Average	0.45	0.48	0.52	0.55	0.68	0.76	0.82	0.41	0.13	0
		Worst	2.57	2.68	2.72	3.10	3.36	3.48	2.75	1.85	1.33	0
	CTGA	Average	1.51	1.48	1.45	1.41	1.45	1.50	1.58	1.40	0.90	0
		Worst	4.83	4.55	4.62	4.66	4.81	5.36	5.25	4.58	3.61	0
K=400	CTGEA	Average	0.92	1.01	1.01	1.06	1.12	1.15	1.09	0.74	0.62	0
		Worst	3.58	3.63	4.05	4.66	4.71	4.70	3.92	3.80	3.20	0
	CTGA	Average	2.15	2.13	2.11	2.05	2.17	2.27	2.15	1.60	1.35	0
		Worst	8.60	8.04	7.56	7.63	8.02	7.94	7.83	7.22	5.83	0
K=500	CTGEA	Average	1.04	1.12	1.15	1.20	1.29	1.33	1.16	1.10	1.35	0
		Worst	4.52	4.81	5.20	5.55	5.87	5.48	5.51	6.03	6.95	0
	CTGA	Average	1.91	1.88	1.85	1.89	2.10	2.15	1.89	1.62	1.87	0
		Worst	11.41	11.96	12.14	12.42	12.33	12.07	10.92	9.66	8.83	0

approximate value K with respect to the ordering cost. For our tested instances, the value of K happens to be $\frac{K_2}{2}$. Hence, K takes the values of 50, 100, 150, 250, 400, and 500. As we can see from Table 4, both the CTGEA and CTGA approaches perform slightly worse when K increases, which is expected. It should be noted that $K = 500$ ($K_2 = 1000$) is already a very large value in this application because the maximal demand is 3000 and the unit ordering costs of the two suppliers are 1 and 0.8, respectively. The overall performance of the two approaches is very impressive, perhaps because the minimal solutions of two functions may be close to each other even if the maximum distance K between these two functions is large. Our way of approximating a function by its lower convex envelope makes this scenario likely to happen.

4.3 Comparison of Approximation Approaches

In this section, we try to provide some explanations for why the CTGA and CTGEA approaches perform much better than the OCA approach. We make two important observations and establish one proposition.

Observation 4.1. *The heuristic policies of the CTGA and CTGEA approaches are optimal for the single-period problem, while the heuristic policy of the OCA approach is not optimal.*

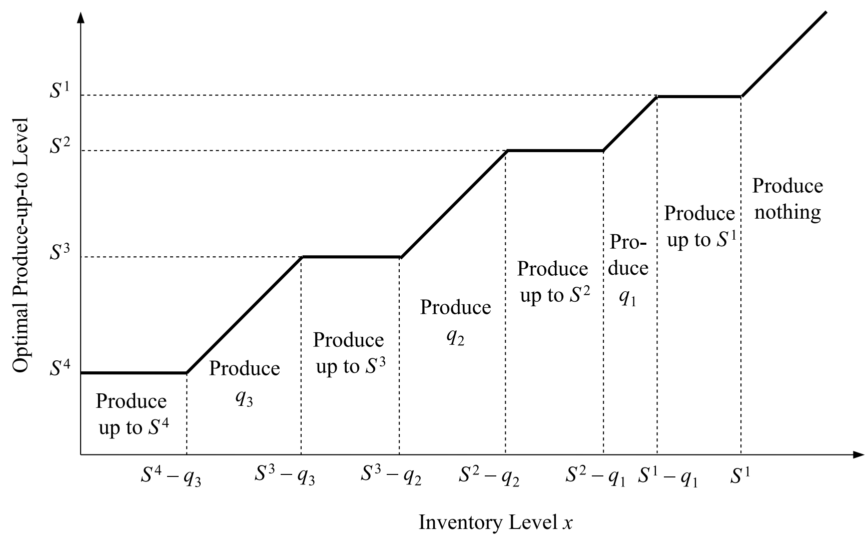

Figure 4: Structure of the heuristic policy by OCA approach

This observation indicates that the OCA approach loses to the CTGA and CTGEA approaches in the beginning because it is not even optimal for the single period problem.

Observation 4.2. *The structure of the heuristic policies of the CTGA and CTGEA approaches illustrated in Figure 1 is more flexible than the structure of the heuristic policy of the OCA approach illustrated in Figure 4, which makes the heuristic policies of the CTGA and CTGEA approaches closer to optimal.*

The OCA approach leads to a piecewise linear convex cost structure. Bensoussan et al. (1983) showed that the structure of the inventory control policy is that illustrated in Figure 4, where the produce-up-to level is a continuously increasing function of the initial inventory level. However, we know that this property does not hold for the optimal policy. This makes the heuristic policy of the OCA approach restrictive and hence not perform well. In contrast, Figure 1 shows that the produce-up-to level of the CTGA and CTGEA approaches can be up and down and even have jumps. This flexibility can make it closer to the optimal policy. For example, for the inventory control problems in Porteus(1971, 1972), the optimal policies shared the same structure as that illustrated in Figure 1.

Observation 4.3. *If the production/ordering cost is piecewise linear convex, all three of the heuristic policies of the CTGA, CTGEA, and OCA approaches are optimal.*

If the production cost $c(z)$ in (2) is convex, it is easy to show that the cost-to-go function $V_t(x)$ is convex, which immediately implies that all three of the heuristics are optimal. One may wonder what would happen if the production cost is piecewise linear concave. For the stochastic demand case, it is very difficult to compare their performance. However, under deterministic demand, we are able to show in the following proposition that the CTGA and CTGEA approaches are always better than the OCA approach.

Consider a T-period inventory control problem where the demand D_t in any period t is deterministic. We assume that the production cost $c(z)$ is a piecewise linear concave increasing function for any $z \geq 0$. Note that K_1 may be strictly positive, implying that it is possible to have a positive fixed ordering cost. Suppose that a shortage is not allowed and hence $x_t \geq 0$ for any period t. In this case, the inventory holding cost in period t and the cost incurred at the end of the planning horizon are specified by $H_t(y_t) = h(y_t - D_t)$ for any $y_t \geq D_t$ and $V_{T+1}(x_{T+1}) = h_{T+1}x_{T+1}$ for any $x_{T+1} \geq 0$, respectively. Such an inventory control model is referred to as the dynamic lot-sizing problem with a concave production cost.

Proposition 8. *In the dynamic lot-sizing problem with a concave production cost, for any $x_t \geq 0$ and $t \in \{1, ..., T\}$, the costs under the CTGA and CTGEA approaches are both equal to $\bar{V}_t(x_t)$ and are less than the cost under the OCA approach $\widehat{V}_t(x_t)$, i.e., $\bar{V}_t(x_t) \leq \widehat{V}_t(x_t)$.*

5 Conclusion

We study an inventory control problem with general piecewise linear production costs. We fully characterize the optimal policy for the single-period problem and propose several practically implementable and close-to-optimal heuristic policies for the multi-period problem. The worst-case performance bounds of the heuristic policies are established by applying the concept of K-approximate convexity. We test these heuristic policies on a set of practical applications that were previously difficult to solve and observe excellent performance. Overall, numerical experiments show that CTGEA approach outperforms CTGA and OCA approaches. For a lost-sales model, our CTGEA algorithm still works because we can always get a convex approximation of $\mathbb{E}[R_t((y_{t-1}-D_{t-1})^+)]$. However, the worst-case performance bound in Theorem 3 does not hold any more because $\mathbb{E}[R_t((y_{t-1}-D_{t-1})^+)]$ may not be K-approximate convex even if $R_t(x)$ is K-approximate convex.

Given the generality of the cost structure, our model may find many other applications in real-world problems. Specifically, our method can be applied to solve dynamic programming problems where the cost-to-go function is not convex, whereas convexity is necessary to have a well-structured policy to implement in practice. These problems include a finite horizon joint pricing and inventory control problem with nonlinear ordering cost and a stochastic cash balance problem with fixed costs, among others.

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A Online Appendix

Proof of Proposition 1. The K-approximate convexity of f implies that there exists a convex function $g : S \mapsto \mathbb{R}$ such that $\|f - g\|_\infty \leq K$. Define $h : S \mapsto \mathbb{R}$ such that $h(x) = g(x) - K$ for any $x \in S$. Note that $\|f - g\|_\infty \leq K$ implies $g(x) \in [f(x) - K, f(x) + K]$ and hence $h(x) \in [f(x) - 2K, f(x)]$ for any $x \in S$. As h is a convex function, the definition of convex envelope yields $f(x) \geq \underline{f}^*(x) \geq h(x) \geq f(x) - 2K$ for any $x \in S$. Therefore, we have $\|f - \underline{f}^*\|_\infty \leq 2K$.

For any $x \in S$ such that $f(x) - \underline{f}^*(x) \leq \frac{1}{2}\|f - \underline{f}^*\|_\infty$, we have $0 \leq f(x) - \underline{f}^*(x) \leq \frac{1}{2}\|f - \underline{f}^*\|_\infty$ and hence

$$f(x) - \bar{f}(x) = f(x) - \underline{f}^*(x) - \frac{1}{2}\|f - \underline{f}^*\|_\infty \in \left[-\frac{1}{2}\|f - \underline{f}^*\|_\infty, 0\right].$$

Similarly, for any $x \in S$ such that $f(x) - \underline{f}^*(x) > \frac{1}{2}\|f - \underline{f}^*\|_\infty$, we can obtain $\frac{1}{2}\|f - \underline{f}^*\|_\infty < f(x) - \underline{f}^*(x) \leq \|f - \underline{f}^*\|_\infty$ and

$$f(x) - \bar{f}(x) = f(x) - \underline{f}^*(x) - \frac{1}{2}\|f - \underline{f}^*\|_\infty \in \left(0, \frac{1}{2}\|f - \underline{f}^*\|_\infty\right].$$

Consequently, $\|f - \bar{f}\|_\infty \leq \frac{1}{2}\|f - \underline{f}^*\|_\infty \leq K$. □

Proof of Proposition 2. We first show that if $W(x)$ is a K-approximate convex function whose convex envelope is $w(x)$, then $\partial_- w(x^*) \geq b_0$ for any $x^* \leq x_1$ and $\partial_+ w(x^*) \leq b_{m-1}$ for any $x^* \geq b_{m-1}$, where $\partial_- w(x)$ and $\partial_+ w(x)$ denote the left and right derivatives of $w(x)$ at x, respectively.

As $w(x)$ is a convex function of x, $\partial_- w(x)$ and $\partial_+ w(x)$ are well-defined for any $x \in (-\infty, +\infty)$. Assume for contradiction that $\partial_- w(x^*) < b_0$ for some $x^* \leq x_1$. For any $x < x^* \leq x_1$, we have $W(x) = \lim_{x' \uparrow x^*} W(x') + b_0(x - x^*)$ by the definition of $W(x)$. Also, $w(x) \geq w(x^*) + \partial_- w(x^*) \cdot (x - x^*)$ by the convexity of $w(x)$. Therefore,

$$\begin{aligned}
w(x) - W(x) &\geq w(x^*) - \lim_{x' \uparrow x^*} W(x') + \Big(\partial_- w(x^*) - b_0\Big)(x - x^*) \\
= \lim_{x' \uparrow x^*} \Big(w(x') - W(x')\Big) &+ \Big(\partial_- w(x^*) - b_0\Big)(x - x^*) \geq -2K + \Big(\partial_- w(x^*) - b_0\Big)(x - x^*),
\end{aligned}$$

where the first equality holds because the convexity of $w(x)$ yields its continuity at x^* and the second inequality follows from $\|W - w\|_\infty \leq 2K$ shown in Proposition 1. It follows that $w(x) - W(x) > 0$ for any $x < x^* - \frac{2K}{b_0 - \partial_- w(x^*)} \leq x^*$, which contradicts the fact that $w(x)$ is the convex envelope of $W(x)$.

Similarly, we can also show that $\partial_+ w(x^*) \leq b_{m-1}$ for any $x^* \geq x_{m-1}$. Furthermore, because the convexity of $w(x)$ yields $\partial_- w(x_1) \leq \partial_+ w(x_{m-1})$, $\partial_- w(x_1) \geq b_0$ and $\partial_+ w(x_{m-1}) \leq b_{m-1}$ imply $b_0 \leq b_{m-1}$, which will be used in the subsequent proof.

Next, we prove that Algorithm 1 returns a well-defined convex function $\bar{W}(x)$ in $O(m)$. Suppose that $m > 2$. Then we have $x'_1 = x_1 < x_{m-1} = x_{m'}$ in Step 2, which implies $m' \geq 2$. Consequently, b'_j for any $j \in \{1, ..., m'-1\}$, l, and u in Step 3 are all well-defined. In this case, we can show $l \leq u$ as follows.

- Suppose that $b'_1 \geq b_0$, which yields $l = 1$. As $u \in \{1, ..., m'\}$, it is straightforward that $l \leq u$.
- Suppose that $b'_1 < b_0$. Recall that $b_0 \leq b_{m-1}$. The definition of l yields $b'_{l-1} < b_0 \leq b_{m-1}$. Note that b'_j, where $j \in \{1, ..., m'-1\}$, corresponds to the slope of piece j of the lower hull of P, which is a piecewise linear convex function. Therefore, b'_j is increasing in j, and hence $b'_j \leq b'_{l-1} \leq b_{m-1}$ for any $j \leq l-1$. If $l = m'$, then $u = m' = l$. Otherwise, the definition of u immediately implies $u > l-1$, i.e., $l \leq u$.

Also note that $l = u = 1$ when $m = 2$. Therefore, l and u in Step 3 always satisfy $l \leq u$. Consider the function $\underline{W}^*(x)$ defined in Step 3.

- If $l = u$, then
$$\underline{W}^*(x) = \begin{cases} y'_l + b_0(x - x'_l), & \text{for any } x \leq x'_l, \\ y'_l + b_{m-1}(x - x'_l), & \text{for any } x > x'_l, \end{cases}$$
which is a piecewise linear continuous and convex function as $b_0 \leq b_{m-1}$.
- Suppose that $l < u$. $l, u \in \{1, ..., m'\}$ implies $l < m'$ and $u > 1$, i.e., $1 \leq l \leq u-1 \leq m'-1$. The definitions of b'_j for all $j \in \{1, ..., m'\}$ imply that $\underline{W}^*(x)$ is a piecewise linear continuous function. According to the definitions of l and u, we obtain $b'_l \geq b_0$ and $b'_{u-1} \leq b_{m-1}$. Recall that $b_0 \leq b_{m-1}$ and b'_j is increasing in j. It follows that $\underline{W}^*(x)$ is convex in x.

As a result, $\underline{W}^*(x)$ in Step 3 and hence $\bar{W}(x)$ in Step 4 are both well-defined piecewise linear continuous and convex functions. Furthermore, because the points in P are sorted according to x_j, the monotone chain algorithm obtains the lower hull of P in $O(m)$ (c.f. Andrew 1979). It is straightforward that the computational complexity of Algorithm 1 is also $O(m)$.

Finally, we show that $\|W - \bar{W}\|_\infty \leq K$ through the following three parts.

Part 1 shows that $\underline{W}^*(x) \geq w(x)$ for any $x \in (-\infty, +\infty)$, where $w(x)$ is the convex envelope of $W(x)$. The convexity of $w(x)$ implies that $w(x)$ is continuous at any $x \in (-\infty, +\infty)$. Combining with $w(x) \leq W(x)$ for any $x \in (-\infty, +\infty)$, we have $w(x_j) \leq W(x_j)$, $w(x_j) = \lim_{x \uparrow x_j} w(x) \leq \lim_{x \uparrow x_j} W(x)$, and $w(x_j) = \lim_{x \downarrow x_j} w(x) \leq \lim_{x \downarrow x_j} W(x)$ for any $j \in \{1, ..., m-1\}$. The definition of y_j in Step 1 yields $w(x_j) \leq y_j$ for any $j \in \{1, ..., m-1\}$.

Note that the lower hull of P can be represented by the piecewise linear continuous and convex function $L_P(x)$ with the domain $[x_1, x_{m-1}] = [x'_1, x'_{m'}]$, where

$$L_P(x) = \begin{cases} y'_1, & \text{if } x = x'_1, \\ y'_j + b'_j(x - x'_j), & \text{for any } x \in (x'_j, x'_{j+1}], j \in \{1, ..., m' - 1\}. \end{cases}$$

Let the function $U_P(x)$ with the domain $[x_1, x_{m-1}]$ be the upper hull of P, i.e., the part of P's convex hull visible from above, which runs from the leftmost point to the rightmost point in clockwise order. Then the convex hull of P can be represented as

$$\mathcal{H}_P = \Big\{(x, y) : x \in [x_1, x_{m-1}], y \in [L_P(x), U_P(x)]\Big\}. \tag{8}$$

Obviously, $U_P(x)$ is a concave function of $x \in [x_1, x_{m-1}]$. As $P \subseteq \mathcal{H}_P$, we have $y_j \leq U_P(x_j)$ for any $j \in \{1, ..., m-1\}$. Recall that $w(x)$ is convex in x and $w(x_j) \leq y_j$ for any $j \in \{1, ..., m-1\}$. Therefore, we can define the following convex set

$$\tilde{\mathcal{H}}_P = \Big\{(x, y) : x \in [x_1, x_{m-1}], y \in [w(x), U_P(x)]\Big\}.$$

such that $P \subseteq \tilde{\mathcal{H}}_P$. Note that the convex hull $\mathcal{H}_P$ of P is the minimum convex set containing all the points in P. It is straightforward that $\mathcal{H}_P \subseteq \tilde{\mathcal{H}}_P$, which, by (8) and the definition of $\tilde{\mathcal{H}}_P$, immediately yields $L_P(x) \geq w(x)$ for any $x \in [x_1, x_{m-1}]$. Applying this property, we can show that $\underline{W}^*(x) \geq w(x)$ for any $x \in (-\infty, +\infty)$ by considering the following three cases.

- Consider any $x \in [x'_l, x'_u] \subseteq [x'_1, x'_{m'}] = [x_1, x_{m-1}]$. According to the definition of $\underline{W}^*(x)$ in Step 4, it is straightforward that $\underline{W}^*(x) = L_P(x) \geq w(x)$.
- For any $x \leq x'_l$, we have $\underline{W}^*(x) = L_P(x'_l) + b_0(x - x'_l)$. Recall that $\partial_- w(x) \geq b_0$ for any $x \leq x_1$. According to the convexity of $w(x)$, $\partial_- w(x)$ is increasing in x, which implies $\partial_- w(x) \geq b_0$ for any $x \in (-\infty, +\infty)$. Consequently, $\underline{W}^*(x) - w(x)$ is decreasing in x for any $x \leq x'_l$. Applying $\underline{W}^*(x'_l) = L_P(x'_l) \geq w(x'_l)$ shown in the previous case, we have $\underline{W}^*(x) \geq w(x)$ for any $x \leq x'_l$.
- Similar to the case with $x \leq x'_l$, we can show that $\underline{W}^*(x) \geq w(x)$ for any $x \geq x'_u$.

Part 2 shows that $\underline{W}^*(x) \leq W(x)$ for any $x \in (-\infty, +\infty)$. The following two cases prove $L_P(x) \leq W(x)$ for any $x \in [x_1, x_{m-1}]$.

- For any $j \in \{1, ..., m-1\}$, the definition of y_j implies $y_j \leq W(x_j)$. As $\mathcal{H}_P$ is the convex hull of P, we have $(x_j, y_j) \in \mathcal{H}_P$ and so (8) yields $L_P(x_j) \leq y_j \leq W(x_j)$.

- Consider any $x \in (x_j, x_{j+1})$ where $j \in \{1, ..., m-2\}$, i.e., $x = \lambda x_j + (1-\lambda)x_{j+1}$ for some $\lambda \in (0,1)$. Then

$$W(x) = \lambda \lim_{x' \downarrow x_j} W(x') + (1-\lambda) \lim_{x' \uparrow x_{j+1}} W(x') \geq \lambda y_j + (1-\lambda) y_{j+1}.$$

  Note that $(x_j, y_j), (x_{j+1}, y_{j+1}) \in \mathcal{H}_P$ and $\mathcal{H}_P$ is a convex set. We obtain $(x, \lambda y_j + (1-\lambda) y_{j+1}) \in \mathcal{H}_P$. Again, (8) shows that $L_P(x) \leq \lambda y_j + (1-\lambda) y_{j+1} \leq W(x)$.

Now consider $\underline{W}^*(x)$ and $L_P(x)$ for any $x \in [x_1, x_{m-1}]$. Note that $\underline{W}^*(x) = L_P(x)$ for any $x \in [x'_l, x'_u] \subseteq [x'_1, x'_{m'}] = [x_1, x_{m-1}]$. If $l > 1$, the definition of l yields $b'_j < b_0$ for any $j \in \{1, ..., l-1\}$. Recall that both $\underline{W}^*(x)$ and $L_P(x)$ are continuous piecewise linear functions, whose slopes in the interval $[x'_j, x'_{j+1}]$ are b_0 and b'_j respectively for any $j \in \{1, ..., l-1\}$. As $\underline{W}^*(x'_l) = L_P(x'_l)$ and $b'_j < b_0$, we have $\underline{W}^*(x) \leq L_P(x)$ for any $x \in [x'_1, x'_l] = [x_1, x'_l]$. Symmetrically, it can be shown that $\underline{W}^*(x) \leq L_P(x)$ for any $x \in [x'_u, x'_{m'}] = [x'_u, x_{m-1}]$. Combining with $L_P(x) \leq W(x)$ for any $x \in [x_1, x_{m-1}]$, we obtain $\underline{W}^*(x) \leq L_P(x) \leq W(x)$ for any $x \in [x_1, x_{m-1}]$.

For any $x < x_1$, we have $W(x) = \lim_{x' \uparrow x_1} W(x') + b_0(x - x_1)$ and

$$\underline{W}^*(x) = \underline{W}^*(x_1) + b_0(x - x_1) \leq L_P(x_1) + b_0(x - x_1) = y_1 + b_0(x - x_1),$$

where the second equality is obtained because $L_P(x_1) = L_P(x'_1) = y'_1 = y_1$. The definition of y_1 yields $y_1 \leq \lim_{x' \uparrow x_1} W(x')$, and hence $\underline{W}^*(x) \leq W(x)$ for any $x < x_1$. Similarly, we can prove that $\underline{W}^*(x) \leq W(x)$ for any $x > x_{m-1}$, which completes the proof of Part 2.

Part 3 shows that $\|W - \bar{W}\|_\infty \leq K$. According Parts 1 and 2, we have $w(x) \leq \underline{W}^*(x) \leq W(x)$ for any $x \in (-\infty, +\infty)$. Recall that $\underline{W}^*(x)$ is convex in x and $w(x)$ is the convex envelope of $W(x)$. It is straightforward that $w(x) = \underline{W}^*(x)$, i.e., $\underline{W}^*(x)$ is the convex envelope of $W(x)$. Note that $\underline{W}^*(x)$ is a piecewise linear function whose breakpoints are a subset of those of $W(x)$. Consequently, the distance in ℓ_∞ norm between $W(x)$ and $\underline{W}^*(x)$ is measured by

$$\max \left\{ \max \left\{ W(x_j), \lim_{x \uparrow x_j} W(x), \lim_{x \downarrow x_j} W(x) \right\} - \underline{W}^*(x_j) : j \in \{1, ..., m-1\} \right\}.$$

Proposition 1 immediately implies $\|W - \bar{W}\|_\infty \leq K$. □

The following lemma (given in Lu et al. 2016) is useful for proving Proposition 3.

Lemma 2. *For any $f, g : S \mapsto \mathbb{R}$, if $\|f - g\|_\infty \leq K$, then*
(a) $|\min_{x \in X} f(x) - \min_{x \in X} g(x)| \leq K$ for any $X \subseteq S$;
(b) $g(x_f) - \min_{x \in X} g(x) \leq 2K$ for any $X \subseteq S$, where $x_f \in \arg\min_{x \in X} f(x)$.

Proof of Proposition 3. As c is K-approximate convex, there exists a convex function $\bar{c}$ such that $\|c - \bar{c}\|_\infty \leq K$. Define $\bar{g}(x) = \min_{y \geq x}\{\bar{c}(y - x) + f(y)\}$. For any $x_1, x_2 \in \mathbb{R}$, we define $y_i^* \in \min_{y \geq x}\{\bar{c}(y - x_i) + f(y)\}$ for $i = 1, 2$. Then for any $\lambda \in [0, 1]$,

$$\begin{aligned}\bar{g}(\lambda x_1 + (1-\lambda)x_2) &\leq \bar{c}(\lambda y_1^* + (1-\lambda)y_2^* - (\lambda x_1 + (1-\lambda)x_2)) + f(\lambda y_1^* + (1-\lambda)y_2^*)\\ &\leq \lambda\bar{c}(y_1^* - x_1) + (1-\lambda)\bar{c}(y_2^* - x_2) + \lambda f(y_1^*) + (1-\lambda)f(y_2^*)\\ &= \lambda\bar{g}(x_1) + (1-\lambda)\bar{g}(x_2),\end{aligned}$$

where the first inequality follows from $\lambda y_1^* + (1-\lambda)y_2^* \geq \lambda x_1 + (1-\lambda)x_2$, and the second inequality is yielded by the convexity of $\bar{c}$ and f. Hence, $\bar{g}$ is convex.

For any fixed x, $\|c - \bar{c}\|_\infty \leq K$ yields $|c(y-x) + f(y) - \bar{c}(y-x) - f(y)| = |c(y-x) - \bar{c}(y-x)| \leq K$ for any $y \geq x$. Applying Lemma 2 (a), we have $|\min_{y \geq x}\{c(y-x) + f(y)\} - \min_{y \geq x}\{c(y-x) + \bar{f}(y)\}| = |g(x) - \bar{g}(x)| \leq K$. As x is arbitrary, we have $\|g - \bar{g}\|_\infty \leq K$. Because $\bar{g}$ is convex, g is K-approximate convex. □

Proof of Proposition 4. Let $J(x, z) = c(z) + H_t(x + z)$. Then $z^*(x) = \min\{\arg\min_{z \geq 0}\{J(x, z)\}\}$. As $H_t(x)$ is a convex function, $J(x, z)$ is supermodular in x and z. As the constraint is independent of x, we have by Theorem 8.1 in Porteus (2002) that $z^*(x)$ is decreasing in x. □

To prove Proposition 5, we first show that $\hat{x}_i$ defined in Step 2 of Algorithm 2 satisfies the following inequality.

Lemma 3. $\hat{x}_{i+1} \leq \hat{x}_i \leq \max\{\hat{x}_{i+1}, S^i - q_{i-1}\}$ *for any* $i \in \{1, ..., n\}$.

Proof of Lemma 3. Applying the convexity of $H_t(y)$, it is straightforward that $f^i(x)$ defined in Step 1 of Algorithm 2 satisfies

$$f^i(x) = \min_{z \in [q_{i-1}, q_i]}\{K_i + c_i z + H_t(x + z)\} \qquad \text{for all } i \in \{1, ..., n\}. \tag{9}$$

For any $x \geq S^i - q_{i-1}$, we have

$$f^i(x) = K_i + c_i q_{i-1} + H_t(x + q_{i-1}) \geq K_{i-1} + c_{i-1}q_{i-1} + H_t(x + q_{i-1}) \geq f^{i-1}(x),$$

where the two inequalities follows from the monotonicity of $c(z)$ and (9), respectively. The definitions of $\bar{x}_{i,i-1}$ and $\hat{x}_i$ imply

$$\hat{x}_i \leq \bar{x}_{i,i-1} = \inf\{x \geq \hat{x}_{i+1} : f^{i-1}(x) \leq f^i(x)\} \leq \max\{\hat{x}_{i+1}, S^i - q_{i-1}\}.$$

In addition, $\hat{x}_i = \min\{\bar{x}_{i,0}, \bar{x}_{i,1}, ..., \bar{x}_{i,i-1}\} \geq \hat{x}_{i+1}$ as $\bar{x}_{i,j} \geq \hat{x}_{i+1}$ for all $j \in \{0, 1, ..., i-1\}$ by definition. □

Proof of Proposition 5. According to the convexity of $H_t(y)$, for any $i \in \{1, ..., n\}$, $f^i(x)$ in Step 1 of Algorithm 2 satisfies (9). Recall that $c(z) = K_i + c_i z$ for any $z \in (q_{i-1}, q_i]$ and $c(q_{i-1}) \leq K_i + c_i q_{i-1}$. The single-period problem (3) is equivalent to

$$V(x) = \min_{i \in \{0,1,\ldots,n\}} \{f^i(x)\} \qquad \text{and} \qquad z^*(x) = \min_{i \in \{0,1,\ldots,n\}} \{z^i(x) : V(x) = f^i(x)\}, \tag{10}$$

where $z^0(x) = 0$ for any x and, due to the convexity of $H_t(y)$,

$$z^i(x) = \min\left\{\arg\min_{z \in [q_{i-1}, q_i]} \{K_i + c_i z + H_t(x+z)\}\right\} = \begin{cases} q_i, & \text{if } x < S^i - q_i, \\ S^i - x, & \text{if } S^i - q_i \leq x < S^i - q_{i-1}, \\ q_{i-1}, & \text{if } x \geq S^i - q_{i-1}, \end{cases} \tag{11}$$

for any $i \in \{1, ..., n\}$. Also note that $f^i(x)$, $i \in \{0, 1, ..., n\}$, is continuous in $(-\infty, \infty)$ because $H_t(x)$ is convex and so continuous in $(-\infty, \infty)$.

Consider $\hat{x}_i$, $i \in \{0, 1, ..., n+1\}$, defined in Step 2. We would show by induction on i that $z^*(x) \leq q_i$ for all $x \in [\hat{x}_{i+1}, \infty) \setminus \{-\infty\}$ and $i \in \{0, 1, ..., n\}$. As $q_n = \infty$, this is trivially true when $i = n$. Now consider any $i \in \{0, 1, ..., n-1\}$ and suppose that $z^*(x) \leq q_{i+1}$ for all $x \in [\hat{x}_{i+2}, \infty) \setminus \{-\infty\}$. As $\hat{x}_{i+1} \geq \hat{x}_{i+2}$, for all $x \in [\hat{x}_{i+1}, \infty) \setminus \{-\infty\}$, we have $z^*(x) \leq q_{i+1}$ and, by (10),

$$V(x) = \min_{j \in \{0,1,\ldots,i+1\}} \{f^j(x)\} \qquad \text{and} \qquad z^*(x) = \min_{j \in \{0,1,\ldots,i+1\}} \{z^j(x) : V(x) = f^j(x)\}.$$

Assume for contradiction that $z^*(u) \in (q_i, q_{i+1}]$ for some $u \in [\hat{x}_{i+1}, \infty) \setminus \{-\infty\}$. According to Proposition 4, for all $x \in [\hat{x}_{i+1}, u] \setminus \{-\infty\}$, $z^*(x) \in (q_i, q_{i+1}]$ and hence $z^*(x) = z^{i+1}(x) > q_i \geq q_j \geq z^j(x)$ for all $j \in \{0, 1, ..., i\}$. Therefore, $f^j(x) > f^{i+1}(x)$ for any $j \in \{0, 1, ..., i\}$ and $x \in [\hat{x}_{i+1}, u] \setminus \{-\infty\}$. Consider the following cases.

- Suppose that $\hat{x}_{i+1} > -\infty$. According to the definition of $\hat{x}_{i+1}$, there exists some $j^* \in \{0, 1, ..., i\}$ such that $\hat{x}_{i+1} = \bar{x}_{i+1,j^*} = \inf\{x \geq \hat{x}_{i+2} : f^{j^*}(x) \leq f^{i+1}(x)\} > -\infty$. As $f^{j^*}(x)$ and $f^{i+1}(x)$ are both continuous functions, $f^{j^*}(\hat{x}_{i+1}) \leq f^{i+1}(\hat{x}_{i+1})$, which contradicts $f^j(x) > f^{i+1}(x)$ for any $j = 0, 1, ..., i$ and $x \in [\hat{x}_{i+1}, u] \setminus \{-\infty\} = [\hat{x}_{i+1}, u]$.

- Suppose that $\hat{x}_{i+1} = -\infty$, which implies $\hat{x}_{i+2} = -\infty$ as $\hat{x}_{i+2} \leq \hat{x}_{i+1}$. Similarly, there exists some $j^* \in \{0, 1, ..., i\}$ such that $\hat{x}_{i+1} = \bar{x}_{i+1,j^*} = \inf\{x : f^{j^*}(x) \leq f^{i+1}(x)\} = -\infty$. Recall that $f^{j^*}(x) > f^{i+1}(x)$ for any $x \in [\hat{x}_{i+1}, u] \setminus \{-\infty\} = (-\infty, u]$. We have $\inf\{x : f^{j^*}(x) \leq f^{i+1}(x)\} \geq u > -\infty$, which results in a contradiction.

As a result, we conclude that for any $i \in \{0, 1, ..., n\}$, $z^*(x) \leq q_i$ for all $x \in [\hat{x}_{i+1}, \infty) \setminus \{-\infty\}$.

Next, we would like to show that

$$V(x) = f^i(x) \quad \text{and} \quad z^*(x) = z^i(x) \qquad \text{for any } x \in [\hat{x}_{i+1}, \hat{x}_i) \setminus \{-\infty\} \text{ and } i \in \{0, 1, ..., n\}. \tag{12}$$

Note that $z^*(x) \le q_i$ for any $x \in [\hat{x}_{i+1}, \hat{x}_i) \setminus \{-\infty\} \subseteq [\hat{x}_{i+1}, \infty) \setminus \{-\infty\}$, implying

$$V(x) = \min_{j=0,1,\ldots,i} \{f^j(x)\} \qquad \text{and} \qquad z^*(x) = \min_{j=0,1,\ldots,i} \{z^j(x) : V(x) = f^j(x)\}.$$

(12) is trivially true when $i = 0$, which yields $V(x)$ and $z^*(x)$ for $x \in [\hat{x}_1, \hat{x}_0) \setminus \{-\infty\} = [\hat{x}_1, \infty) \setminus \{-\infty\}$ obtained in Step 3 of Algorithm 2.

Consider any $i \in \{1, ..., n\}$. For any $j \in \{0, 1, ..., i-1\}$, according to the definition of $\hat{x}_i$, $x \in [\hat{x}_{i+1}, \hat{x}_i) \setminus \{-\infty\}$ yields $x < \bar{x}_{i,j} = \inf\{x \ge \hat{x}_{i+1} : f^j(x) \le f^i(x)\}$ and hence $f^j(x) > f^i(x)$, which implies (12). According to Lemma 3, for any $x \in [\hat{x}_{i+1}, \hat{x}_i) \setminus \{-\infty\}$ and $i \in \{1, ..., n\}$, we have $\hat{x}_{i+1} \le x < \hat{x}_i \le \max\{\hat{x}_{i+1}, S^i - q_{i-1}\} = S^i - q_{i-1}$. To see why $\max\{\hat{x}_{i+1}, S^i - q_{i-1}\} = S^i - q_{i-1}$, note that $\max\{\hat{x}_{i+1}, S^i - q_{i-1}\} = \hat{x}_{i+1}$ would result in the contradiction that $\hat{x}_{i+1} < \hat{x}_{i+1}$. Furthermore, we have $q_n = \infty$, implying that $S^n - q^n = -\infty = \hat{x}_{n+1}$. Combining with the definition of $z^i(x)$ in (11), it is straightforward to show than $V(x)$ and $z^*(x)$ for $x \in [\hat{x}_{i+1}, \hat{x}_i) \setminus \{-\infty\}$ and $i \in \{1, ..., n\}$ can be computed in Step 3 of Algorithm 2.

To see the computational complexity of Algorithm 2, note that $f^i(x)$ is also piecewise linear function with $O(m)$ number of pieces. Steps 1 and 3 can be both completed in $O(mn)$. Step 2 requires $O(mn^2)$ operations as $\bar{x}_{i,j}$ for any i, j can be computed in $O(m)$. □

Proof of Theorem 1. According to Step 3 of Algorithm 2, $z^*(x)$ is computed as follows.

- For any $x \in (-\infty, \hat{x}_n)$, $z^*(x) = S^n - x$. Note that $-\infty = \hat{x}_{n+1} < x < \hat{x}_n$. According to Lemma 3, $\hat{x}_n \le \max\{\hat{x}_{n+1}, S^n - q_{n-1}\}$. If $\max\{\hat{x}_{n+1}, S^n - q_{n-1}\} = \hat{x}_{n+1}$, we obtain the contradiction $\hat{x}_{n+1} < x < \hat{x}_n \le \hat{x}_{n+1}$. Thus, $\max\{\hat{x}_{n+1}, S^n - q_{n-1}\} = S^n - q_{n-1}$ and $x < \hat{x}_n \le S^n - q_{n-1}$, implying $z^*(x) = S^n - x > q_{n-1} > 0$. Also note that $z^*(x) \le q_n = \infty$.

- For any $x \in [\hat{x}_{i+1}, \min\{\hat{x}_i, \max\{\hat{x}_{i+1}, S^i - q_i\}\})$ where $i \in \{1, ..., n-1\}$, $z^*(x) = q_i > 0$.

- For any $x \in [\min\{\hat{x}_i, \max\{\hat{x}_{i+1}, S^i - q_i\}\}, \hat{x}_i)$ where $i \in \{1, ..., n-1\}$, $z^*(x) = S^i - x$. Applying Lemma 3 and an argument similar to that in the first case, we have $S^i - q_i \le \max\{\hat{x}_{i+1}, S^i - q_i\} = \min\{\hat{x}_i, \max\{\hat{x}_{i+1}, S^i - q_i\}\} \le x < \hat{x}_i \le \max\{\hat{x}_{i+1}, S^i - q_{i-1}\} = S^i - q_{i-1}$. Thus, $z^*(x) = S^i - x \in (q_{i-1}, q_i] \subseteq (0, \infty)$.

- For any $x \in [\hat{x}_1, \infty)$, $z^*(x) = 0$.

Note that $-\infty = \hat{x}_{n+1} \leq \hat{x}_n \leq \cdots \leq \hat{x}_1 \leq \hat{x}_0 = \infty$. The result follows immediately from the definitions of $a_0, a_1, ..., a_l$ and $s = a_l = \hat{x}_1$. □

Proof of Proposition 6. Let $v = \sup\{x : z^*(x) \geq q\} \leq s$. Proposition 4 implies that $z^*(x) \geq q$ for any $x < v$ and $z^*(x) < q$ for any $x \in (v, s)$. According to Theorem 1, $z^*(x)$ is right continuous. Therefore, we have $z^*(v) \leq q$. In other words, $y^*(x) \geq x + q$ for any $x \in (-\infty, v)$ and $x < y^*(x) \leq x + q$ for any $x \in [v, s)$.

For any $x \in (-\infty, v)$, as $y^*(x) \geq x + q$, $y^*(x)$ should be the solution of the following problem:

$$y^*(x) = \min\left\{\arg\min_{y \geq x+q}\{c(y-x) + H_t(y)\}\right\}.$$

The convexity of $c(z)$ in $[q, \infty)$ implies that the function $c(y - x) + H_t(y)$ is a submodular function of x and y defined on the lattice $\{(x, y) : y \geq x + q\}$. The feasible decision set is ascending in x. By Theorem 8.1 in Porteus (2002), we have that $y^*(x)$ is increasing for any $x \in (-\infty, v)$.

For any $z \in (0, q)$, let $c'(z)$ denote the left derivative of $c(z)$ at z. Define $K(z)$ such that $c(z) = K(z)+c'(z)z$. According to the concavity of $c(z) \in [0, q]$, it is straightforward that $K(z) \geq 0$ and $c(\tilde{z}) \leq K(z) + c'(z)\tilde{z}$ for any $\tilde{z} \in [0, q]$. In addition, for any $z \in (0, q)$, define

$$S(z) = \inf\left\{\arg\inf_{y \in (-\infty, +\infty)}\{c'(z)y + H_t(y)\}\right\}.$$

First, we show that $S(z^*(x)) > x$ for any $x \in (v, s)$. Recall that $z^*(x) \in (0, q)$ for any $x \in (v, s)$. Assume for contradiction that $S(z^*(x)) \leq x$. Because $c'(z^*(x))y + H_t(y)$ is a convex function, the definition of $S(z^*(x))$ implies that $c'(z^*(x))x + H_t(x) \leq c'(z^*(x))y^*(x) + H_t(y^*(x))$ as $S(z^*(x)) \leq x < y^*(x)$. Therefore,

$$\begin{aligned}
H_t(x) &\leq c'(z^*(x))y^*(x) + H_t(y^*(x)) - c'(z^*(x))x \\
&\leq K(z^*(x)) + c'(z^*(x))y^*(x) + H_t(y^*(x)) - c'(z^*(x))x \\
&= K(z^*(x)) + c'(z^*(x))(y^*(x) - x) + H_t(y^*(x)) \\
&= c(y^*(x) - x) + H_t(y^*(x)).
\end{aligned}$$

where the second inequality follows from $K(z^*(x)) \geq 0$ and the last equality is yielded by $c(z) = K(z)+c'(z)z$ and $z^*(x) = y^*(x)-x$. This inequality implies that it is better not to order at x, which contradicts the definition of $y^*(x)$ as the minimum optimal order-up-to level and the property that $y^*(x) > x$ for any $x < s$.

Next, we show that $S(z^*(x)) < x+q$ for any $x \in (v, s)$. Again, we assume for contradiction that $S(z^*(x)) \geq x + q$. According to the convexity of $c'(z^*(x))y + H_t(y)$, as $S(z^*(x))$ is the minimum global minimizer of this function, the function is strictly decreasing in $(-\infty, S(z^*(x))]$, and hence

$c'(z^*(x))(x+q) + H_t(x+q) < c'(z^*(x))y^*(x) + H_t(y^*(x))$ as $S(z^*(x)) \geq x+q > y^*(x)$. Therefore, applying the concavity of $c(q)$ in $[0, q]$, we obtain

$$\begin{aligned} c(q) + H_t(x+q) &\leq K(z^*(x)) + c'(z^*(x))q + H_t(x+q) \\ &= K(z^*(x)) + c'(z^*(x))(x+q) + H_t(x+q) - c'(z^*(x))x \\ &< K(z^*(x)) + c'(z^*(x))y^*(x) + H_t(y^*(x)) - c'(z^*(x))x \\ &= c(y^*(x) - x) + H_t(y^*(x)), \end{aligned}$$

which contradicts that the optimal order quantity is $z^*(x) < q$.

Assume for contradiction that $y^*(x) < S(z^*(x))$. Since $S(z^*(x)) \in (x, x+q)$, the concavity of $c(z)$ in $[0, q]$ yields

$$\begin{aligned} c(S(z^*(x)) - x) + H_t(S(z^*(x))) &\leq K(z^*(x)) + c'(z^*(x))(S(z^*(x)) - x) + H_t(S(z^*(x))) \\ &= K(z^*(x)) + c'(z^*(x))S(z^*(x)) + H_t(S(z^*(x))) - c'(z^*(x))x \\ &< K(z^*(x)) + c'(z^*(x))y^*(x) + H_t(y^*(x)) - c'(z^*(x))x \\ &= c(y^*(x) - x) + H_t(y^*(x)), \end{aligned}$$

where the second inequality follows from the definition of $S(z^*(x))$. Similarly, if $y^*(x) > S(z^*(x))$ we can show that $c(S(z^*(x)) - x) + H_t(S(z^*(x))) \leq c(y^*(x) - x) + H_t(y^*(x))$. As a result, we have $y^*(x) = S(z^*(x))$.

Note that $c'(z)$ is decreasing for $z \in (0, q)$, and hence it is straightforward that $S(z)$ is increasing for $z \in (0, q)$. As $z^*(x) \in (0, q)$ is decreasing for $x \in (v, s)$, the property that $y^*(x) = S(z^*(x))$ yields that $y^*(x)$ is decreasing for $x \in (v, s)$. Also note that $y^*(x)$ is right continuous by Theorem 1. Therefore, $y^*(x)$ is decreasing for $x \in [v, s)$. □

Proof of Lemma 1. (i) From Definition 3.1, we know that s^1 is the minimal number that satisfies $c_1 x + H_t(x) \leq K_1 + c_1 S^1 + H_t(S^1)$. Hence, the relationship of $S^1 - q_1 < s^1$ implies that $S^1 - q_1$ does not satisfy the above equality. That is, we have $c_1(S^1 - q_1) + H_t(S^1 - q_1) > K_1 + c_1 S^1 + H_t(S^1)$, i.e., $H_t(S^1 - q_1) - H_t(S^1) > K_1 + c_1 q_1$, which implies that $s^2 \geq S^1 - q_1$ by the definition of s^2.

(ii) It follows from the definition of s^1 that

$$c_1 s^1 + H_t(s^1) \leq K_1 + c_1 S^1 + H_t(S^1). \tag{13}$$

As S^1 is a minimizer of $c_1 y + H_t(y)$, we have

$$c_1 S^1 + H_t(S^1) \leq c_1(s^1 + q_1) + H_t(s^1 + q_1). \tag{14}$$

Therefore, we have that

$$\begin{aligned} H_t(s^1) - H_t(s^1 + q_1) &\leq H_t(s^1) - c_1 S^1 - H_t(S^1) + c_1(s^1 + q_1) \\ &\leq K_1 + c_1 S^1 + H_t(S^1) - c_1 S^1 - H_t(S^1) + c_1 q_1 \\ &= K_1 + c_1 q_1, \end{aligned} \tag{15}$$

where the first inequality follows from (14), and the second one follows from (13).

By the definition of s^2, the inequality (15) implies $s^2 \leq s^1$.

(iii) As $c(z)$ increases with z, we have $K_2 + c_2 q_1 \geq K_1 + c_1 q_1$. Therefore, we can obtain $c_2(S^2 - q_1) + H_t(S^2) \leq K_2 + c_2 S^2 - c_1 q_1 - K_1 + H_t(S^2)$. By the definition of s^3, $s^3 \leq S^2 - q_1$.

(iv) As $S^2 - q_1 > s^2$, it follows by the definition of s^2 that $H_t(S^2 - q_1) - H_t(S^2) \leq K_1 + c_1 q_1$ which implies that $c_2(S^2 - q_1) + H_t(S^2 - q_1) \leq K_1 + c_1 q_1 + H_t(S^2) + c_2(S^2 - q_1) \leq K_2 + c_2 q_1 + H_t(S^2) + c_2(S^2 - q_1) = K_2 + c_2 S^2 + H_t(S^2)$, where the second inequality follows from the fact that $K_2 + c_2 q_1 \geq K_1 + c_1 q_1$. Thus, $s^4 \leq S^2 - q_1$ by the definition of s^4. □

Define $\Pi_1(x) = \min_{x+q_1 \geq y \geq x} K_1 * 1_{\{y>x\}} + c_1(y-x) + H_t(y)$ and $\Pi_2(x) = \min_{y \geq x+q_1} K_2 + c_2(y-x) + H_t(y)$. We first note that when $n = 2$, the optimization problem (3) becomes

$$\min \{\Pi_1(x), \Pi_2(x)\}. \tag{16}$$

The following lemma presents the expressions of $\Pi_1(x)$ and $\Pi_2(x)$.

Lemma 4. *(i) If $S^1 - q_1 < s^1$, then*

$$\Pi_1(x) = \begin{cases} H_t(x), & \text{if } x \geq s^1, \\ K_1 + c_1(S^1 - x) + H_t(S^1) & \text{if } S^1 - q_1 \leq x < s^1. \\ K_1 + c_1 q_1 + H_t(x + q_1) & \text{if } x < S^1 - q_1. \end{cases} \tag{17}$$

If $S^1 - q_1 \geq s^1$, then

$$\Pi_1(x) = \begin{cases} H_t(x), & \text{if } x \geq s^2, \\ K_1 + c_1 q_1 + H_t(x + q_1) & \text{if } x < s^2. \end{cases} \tag{18}$$

(ii) $\Pi_2(x)$ can be expressed as:

$$\Pi_2(x) = \begin{cases} K_2 + c_2 q_1 + H_t(x + q_1), & \text{if } x \geq S^2 - q_1, \\ K_2 + c_2(S^2 - x) + H_t(S^2) & \text{if } x < S^2 - q_1. \end{cases} \tag{19}$$

(iii) For any $x > S^2 - q_1$, $\Pi_1(x) \leq \Pi_2(x)$.

Proof. (i) When $x \geq s^1$, for any $y > x$, we have

$$K_1 + c_1 y + H_t(y) \geq K_1 + c_1 S^1 + H_t(S^1)$$
$$\geq c_1 x + H_t(x), \tag{20}$$

where the first inequality follows from the definition of S^1 and the second from the definition of s^1 and $x \geq s^1$.

Equation (20) implies that when $x \geq s^1$, it is optimal to order nothing. We next first consider the case of $S^1 - q_1 < s^1$. When $S^1 - q_1 \leq x < s^1$, it is clear that ordering up-to S^1 is optimal. When $x < S^1 - q_1$, it is worthy noting that if placing an order, then it should order exactly q_1. Thus, we only need to compare two decisions: ordering nothing or exactly q_1. Lemma 1 (i) indicates that $s^2 \geq S^1 - q_1$. Therefore, by the definition of s^2, for any $x < S^1 - q_1 \leq s^2$, ordering exactly q_1 is better.

We now consider the case of $S^1 - q_1 \geq s^1$. Clearly, for any $x < s^1$, it is optimal to either order nothing or exactly q_1. Note that Lemma 1 (ii) indicates that $s^2 \leq s^1$. When $s^2 \leq x < s^1$, again by the definition of s^2, it is optimal not to order. Similarly, when $x < s^2$, it is optimal to order exactly q_1.

(ii) The results immediately follow from the convexity of $H_t(x)$.

(iii) As $c(z)$ increases with z, we have $K_2 + c_2 q_1 \geq K_1 + c_1 q_1$. Therefore, for any $x > S^2 - q_1$, $\Pi_2(x) = K_2 + c_2 q_1 + H_t(x + q_1) \geq K_1 + c_1 q_1 + H_t(x + q_1) \geq \Pi_1(x)$, where the first equality follows from (ii), and the second inequality follows from the definition of $\Pi_1(x)$. □

Proof of Theorem 2. The policy structure in Table 1 is equivalent to the following description.

(i) If $S^1 - q_1 < s^1$ and $S^2 \leq S^1$, it is optimal to

  (a) order nothing when $x \geq s^1$;
  (b) order up-to S^1 when $S^1 - q_1 \leq x < s^1$;
  (c) order exactly q_1 when $s^3 \leq x < S^1 q_1$;
  (d) order up-to S^2 when $x < s^3$;

(ii) If $S^1 - q_1 < s^1$ and $s^1 + q_1 \geq S^2 > S^1$, it is optimal to

  (a) order nothing when $x \geq s^1$;
  (b) order up-to S^1 when $\min\{\max\{s^5, S^1 - q_1\}, S^2 - q_1\} \leq x < s^1$;

(c) order up-to S^2 when $S^1 - q_1 \leq x < \min\{\max\{s^5, S^1 - q_1\}, S^2 - q_1\}$;

(d) order exactly q_1 when $\min\{s^3, S^1 - q_1\} \leq x < S^1 - q_1$;

(e) order up-to S^2 when $x < \min\{s^3, S^1 - q_1\}$;

(iii) If $S^1 - q_1 < s^1$ and $S^2 > s^1 + q_1$, it is optimal to

(a) order nothing when $x \geq \min\{S^2 - q_1, \max\{s^4, s^1\}\}$;

(b) order up-to S^2 when $s^1 \leq x < \min\{S^2 - q_1, \max\{s^4, s^1\}\}$;

(c) order up-to S^1 when $\min\{s^1, \max\{s^5, S^1 - q_1\}\} \leq x < s^1$;

(d) order up-to S^2 when $S^1 - q_1 \leq x < \min\{s^1, \max\{s^5, S^1 - q_1\}\}$;

(e) order exactly q_1 when $\min\{s^3, S^1 - q_1\} \leq x < S^1 - q_1$;

(f) order up-to S^2 when $x < \min\{s^3, S^1 - q_1\}$;

(iv) If $S^1 - q_1 \geq s^1$ and $S^2 - q_1 \leq s^2$, it is optimal to

(a) order nothing when $x \geq s^2$;

(b) order exactly q_1 when $s^3 \leq x < s^2$;

(c) order up-to S^2 when $x < s^3$;

(v) If $S^1 - q_1 \geq s^1$ and $S^2 - q_1 > s^2$, it is optimal to

(a) order nothing when $x \geq \max\{s^4, s^2\}$;

(b) order up-to S^2 when $s^2 \leq x < \max\{s^4, s^2\}$

(c) order exactly q_1 when $\min\{s^3, s^2\} \leq x < s^2$

(d) order up-to S^2 when $x < \min\{s^3, s^2\}$.

We now prove the results from (i) to (v).

(i) We prove the cases (a)-(d).

(a) For any $x \geq s^1 > S^1 - q_1 \geq S^2 - q_1$, Lemma 4 (iii) implies that we only need to optimize $\Pi_1(x)$. Lemma 4 (i) shows that it is optimal to order nothing for $x \geq s^1$.

(b) For any $s^1 > x \geq S^1 - q_1 \geq S^2 - q_1$, Lemma 4 (iii) implies that we only need to optimize $\Pi_1(x)$. Lemma 4 (i) shows that it is optimal to order up-to S^1 for any $S^1 - q_1 \leq x < s^1$.

(c,d) Similarly, for any $S^2 - q_1 \leq x < S^1 - q_1$, we only need to consider $\Pi_1(x)$, and Lemma 4 (i) shows that it is optimal to order exactly q_1. For any $x < S^2 - q_1$, Lemma 4 (i) and (ii) implies $\Pi_1(x) = K_1 + c_1 q_1 + H_t(x + q_1)$ and $\Pi_2(x) = K_2 + c_2(S^2 - x) + H_t(S^2)$. By the definition of s^3 and $s^3 \leq S^2 - q_1$ of Lemma 1 (iii), we know that it is optimal to order exactly q_1 if $s^3 \leq x < S^2 - q_1$ and up-to S^2 if $x < s^3$. Hence, we can obtain cases (c) and (d).

(ii) We prove cases (a)-(e).

(a) For any $x \geq s^1 > S^2 - q_1 \geq S^1 - q_1$, Lemma 4 (iii) implies that we only need to optimize $\Pi_1(x)$. Lemma 4 (i) shows that it is optimal to order nothing for $x \geq s^1$.

(b,c) For any $S^1 - q_1 \leq x < s^1$, Equation (17) shows that $\Pi_1(x) = K_1 + c(S^1 - x) + H_t(S^1)$, i.e., it is optimal to order up to S^1. Hence, for any $S^1 - q_1 < S^2 - q_1 \leq x < s^1$, Lemma 4 (iii) implies that it is optimal to order up-to S^1. For any $S^1 - q_1 \leq x < S^2 - q_1$, Equation (19) shows that $\Pi_2(x) = K_2 + c_2(S^2 - x) + H_t(S^2)$. By the definition of s^5, it is optimal to order up-to S^1 when $\min\{\max\{s^5, S^1 - q_1\}, S^2 - q_1\} \leq x < S^2 - q_1$ and order up-to S^2 when $S^1 - q_1 \leq x < \min\{\max\{s^5, S^1 - q_1\}, S^2 - q_1\}$.

(d,e) For any $x < S^1 - q_1$, Lemma 4 implies that $\Pi_1(x) = K_1 + c_1 q_1 + H_t(x + q_1)$ and $\Pi_2(x) = K_2 + c_2(S^2 - x) + H_t(S^2)$ because $x < S^2 - q_1$ by $S^2 > S^1$. Hence, by the definition of s^3, we can show that it is optimal to order exactly q_1 when $\min\{s^3, S^1 - q_1\} \leq x < S^1 - q_1$ and order up-to S^2 when $x < \min\{s^3, S^1 - q_1\}$.

(iii) We prove cases (a)-(f).

(a,b) For any $x \geq s^1$, Equation (17) implies that $\Pi_1(x) = H_t(x)$. If $x \geq S^2 - q_1$, Lemma 4 (iii) states that $\Pi_1(x) \leq \Pi_2(x)$, which implies that it is optimal to order nothing for any $x \geq S^2 - q_1$ because $S^2 - q_1 > s^1$. For $s^1 \leq x < S^2 - q_1$, $\Pi_2(x) = K_2 + c_2(S^2 - x) + H_t(S^2)$. The definition of s^4 implies that $\Pi_1(x) \leq \Pi_2(x)$ if and only if $x \geq s^4$. Hence, it is optimal to order nothing for $\min\{S^2 - q_1, \max\{s^4, s^1\}\} \leq x < S^2 - q_1$, and it is optimal to order up-to S^2 for $s^1 \leq x < \min\{S^2 - q_1, \max\{s^4, s^1\}\}$, which proves (a) and (b).

(c,d) For any $S^1 - q_1 \leq x < s^1$, Lemma 4 implies that $\Pi_1(x) = K_1 + c_1(S^1 - x) + H_t(S^1)$ and $\Pi_2(x) = K_2 + c_2(S^2 - x) + H_t(S^2)$ because $s^1 < S^2 - q_1$. By the definition of s^5, $\Pi_1 \leq \Pi_2(x)$ if and only if $x \geq s^5$. Thus, it is optimal to order up-to S^1 if $\min\{s^1, \max\{s^5, S^1 - q_1\}\} \leq x < s^1$, and order up-to S^2 if $S^1 - q_1 \leq x < \min\{s^1, \max\{s^5, S^1 - q_1\}\}$.

(e,f) For any $x < S^1 - q_1$, Lemma 4 implies that $\Pi_1(x) = K_1 + c_1 q_1 + H_t(x + q_1)$ and $\Pi_2(x) = K_2 + c_2(S^2 - x) + H_t(S^2)$. The definition of s^3 implies that $\Pi_1(x) \leq \Pi_2(x)$ if and only if $x \geq s^3$. Thus, it is optimal to order exactly q_1 when $\min\{s^3, S^1 - q_1\} \leq x < S^1 - q_1$, and order up-to S^2 when $x < \min\{s^3, S^1 - q_1\}$.

(iv) We prove the cases (a)-(c).

(a) For any $x \geq s^2 \geq S^2 - q_1$, Lemma 4 (iii) implies that we only need to optimize $\Pi_1(x)$. Hence, Equation (18) implies that it is optimal to order nothing when $x \geq s^2$.

(b) Lemma 1 (iii) shows that $s^3 \leq S^2 - q_1 \leq s^2$. We first consider $S^2 - q_1 \leq x < s^2$ and then $s^3 \leq x < S^2 - q_1$. For $S^2 - q_1 \leq x < s^2$, Lemma 4 (iii) implies that we only need to optimize $\Pi_1(x)$, and Equation (18) implies that it is optimal to order exactly q_1. For $s^3 \leq x < S^2 - q_1$, Lemma 4 shows that $\Pi_1(x) = K_1 + c_1 q_1 + H_t(x + q_1)$ and $\Pi_2(x) = K_2 + c_2(S^2 - x) + H_t(S^2)$. Hence, by the definition of s^3, it is optimal to order exactly q_1 when $s^3 \leq x < S^2 - q_1$.

(c) For any $x < s^3 \leq S^2 - q_1 \leq s^2$, Lemma 4 shows that $\Pi_1(x) = K_1 + c_1 q_1 + H_t(x + q_1)$ and $\Pi_2(x) = K_2 + c_2(S^2 - x) + H_t(S^2)$. By the definition of s^3, it is optimal to order up-to S^2.

(v) We prove the cases (a)-(d).

(a) Note that $S^2 - q_1 > s^2$ and Lemma 1 (iv) implies that $S^2 - q_1 \geq s^4$. Hence, $\max\{s^4, s^2\} \leq S^2 - q_1$. For any $x \geq S^2 - q_1$, Lemma 4 (iii) implies that we only need to optimize $\Pi_1(x)$ and Equation (18) implies that it is optimal to order nothing. For $\max\{s^4, s^2\} \leq x < S^2 - q_1$, Lemma 4 implies that $\Pi_1(x) = H_t(x)$ and $\Pi_2(x) = K_2 + c_2(S^2 - x) + H_t(S^2)$. The definition of s^4 implies that $\Pi_1(x) \leq \Pi_2(x)$ if and only if $x \geq s^4$. Hence, it is optimal to order nothing when $\max\{s^4, s^2\} \leq x < S^2 - q_1$.

(b) Lemma 1 (iv) implies that $S^2 - q_1 \geq s^4$. Hence, for any $s^2 \leq x < s^4 \leq S^2 - q_1$, Lemma 4 implies that $\Pi_1(x) = H_t(x)$ and $\Pi_2(x) = K_2 + c_2(S^2 - x) + H_t(S^2)$. By the definition of s^4, we know that $\Pi_1(x) > \Pi_2(x)$ and hence it is optimal to order up-to S^2 for any $s^2 \leq x < s^4$. If $s^4 \leq s^2$, (b) does not exist.

(c,d) For any $x < s^2 \leq S^2 - q_1$, Lemma 4 implies that $\Pi_1(x) = K_1 + c_1 q_1 + H_t(x + q_1)$ and $\Pi_2(x) = K_2 + c_2(S^2 - x) + H_t(S^2)$. The definition of s^3 immediately implies (c) and (d). □

Proof of Theorem 3. We first show that $\|V_t - W_t\|_\infty \leq \sum_{i=1}^{T-t} \alpha^i K$ and $\|V_t - \bar{W}_t\|_\infty \leq \sum_{i=0}^{T-t} \alpha^i K$ for any $t = 1, ..., T$. Note that $\bar{W}_{T+1}(x_{T+1}) = V_{T+1}(x_{T+1})$. (4) yields that $W_T(x_T) = V_T(x_T)$ for any x_T. Proposition 3 shows that $W_T(x_T)$ is K-approximate convex, and hence $\|W_T - \bar{W}_T\| \le K$ by Proposition 2. Therefore, both results hold when $t = T$.

Suppose that $\|V_{t+1} - \bar{W}_{t+1}\|_\infty \le \sum_{i=0}^{T-t-1} \alpha^i K$ for some $t = 1, ..., T-1$. Consider any $x_t \in \mathbb{R}$. For all $y_t \ge x_t$,

$$\begin{aligned}\Big|\alpha\mathbb{E}[V_{t+1}(y_t - D_t)] - \alpha\mathbb{E}[\bar{W}_{t+1}(y_t - D_t)]\Big| &\le \alpha E\Big[\big|V_{t+1}(y_t - D_t) - \bar{W}_{t+1}(y_t - D_t)\big|\Big] \\ &\le \alpha \sum_{i=0}^{T-t-1} \alpha^i K = \sum_{i=1}^{T-t} \alpha^i K,\end{aligned} \tag{21}$$

which yields $|V_t(x_t) - W_t(x_t)| \le \sum_{i=1}^{T-t} \alpha^i K$ by Lemma 2 (a). As x_t is arbitrary, we obtain $\|V_t - W_t\|_\infty \le \sum_{i=1}^{T-t} \alpha^i K$. Also note that $W_t(x_t)$ is K-approximate convex by Proposition 3 and hence $\|W_t - \bar{W}_t\|_\infty \le K$ by Proposition 2. Therefore, for any $x_t \in \mathbb{R}$,

$$|V_t(x_t) - \bar{W}_t(x_t)| \le |V_t(x_t) - W_t(x_t)| + |W_t(x_t) - \bar{W}_t(x_t)| \le \sum_{i=1}^{T-t} \alpha^i K + K = \sum_{i=0}^{T-t} \alpha^i K.$$

$\bar{W}_{T+1}(x_{T+1}) = \bar{V}_{T+1}(x_{T+1})$ yields $\bar{V}_T(x_T) = V_T(x_T)$, which implies that Theorem 3 holds when $t = T$. Suppose that $\bar{V}_{t+1}(x_{t+1}) \le V_{t+1}(x_{t+1}) + 2K \sum_{i=1}^{T-t-1} i\alpha^i$ for some $t = 1, ..., T-1$. Consider any $x_t \in \mathbb{R}$ and

$$\bar{y}_t(x_t) \in \arg\min_{y_t \ge x_t} \Big\{c(y_t - x_t) + H_t(y_t) + \alpha\mathbb{E}[\bar{W}_{t+1}(y_t - D_t)]\Big\}.$$

Then

$$\begin{aligned}\bar{V}_t(x_t) &= c(\bar{y}_t(x_t) - x_t) + H_t(\bar{y}_t(x_t)) + \alpha\mathbb{E}[\bar{V}_{t+1}(\bar{y}_t(x_t) - D_t)] \\ &\le c(\bar{y}_t(x_t) - x_t) + H_t(\bar{y}_t(x_t)) + \alpha\mathbb{E}[V_{t+1}(\bar{y}_t(x_t) - D_t)] + 2K \sum_{i=1}^{T-t-1} i\alpha^{i+1}.\end{aligned}$$

Applying Lemma 2 (b), (21) implies

$$c(\bar{y}_t(x_t) - x_t) + H_t(\bar{y}_t(x_t)) + \alpha\mathbb{E}[V_{t+1}(\bar{y}_t(x_t) - D_t)] \le V_t(x_t) + 2\sum_{i=1}^{T-t} \alpha^i K.$$

Therefore, we obtain

$$\bar{V}_t(x_t) \le V_t(x_t) + 2\sum_{i=1}^{T-t} \alpha^i K + 2K \sum_{i=1}^{T-t-1} i\alpha^{i+1} = V_t(x_t) + 2K \sum_{i=1}^{T-t} i\alpha^i. \qquad \square$$

Proof of Proposition 7. First, note that $h_t = \sum_{i=0}^{T-t} \alpha^i h + \alpha^{T-t+1} h_{T+1}$ for any $t = 1, ..., T$. Recall that h_{T+1} satisfies $c_n + \sum_{i=0}^{T-t} \alpha^i h + \alpha^{T-t+1} h_{T+1} \ge 0$, i.e., $c_n + h_t \ge 0$ for any $t = 1, ..., T$. Moreover, $-p_{T+1} \le h_{T+1}$. Assume by induction that $-p_{t+1} \le h_{t+1}$ for some $t = 1, ..., T$. As $p, h \ge 0$, we have $-p - \alpha p_{t+1} \le h + \alpha h_{t+1} = h_t$. Combining with $c_n + h_t \ge 0$, it is straightforward that $-p_t = -\min\{c_n, p + \alpha p_{t+1}\} \le h_t$.

Next, we define $\dot{p}$ and $\ddot{p}$, where $\dot{p} > c_n$, $\ddot{p} < +\infty$, and $p + \alpha p_{t+1} \in [\dot{p}, \ddot{p}] \cup (-\infty, c_n]$ for any $t = 1, ..., T$, based on the following cases.

- If $p + \alpha p_{T+1} > c_n$ and $p + \alpha c_n \leq c_n$, then $p_T = c_n$ and $p_t = p + \alpha p_{t+1} \leq c_n$ for any $t = 1, ..., T-1$. Let $\dot{p} = \ddot{p} = p + \alpha p_{T+1} > c_n$. For any $t = 1, ..., T$, $p_{t+1} \in \{p_{T+1}\} \cup (-\infty, c_n]$ and hence $p + \alpha p_{t+1} \in [\dot{p}, \ddot{p}] \cup (-\infty, c_n]$.

- If $p + \alpha p_{T+1} > c_n$ and $p + \alpha c_n > c_n$, then $p_t = c_n$ for any $t = 1, ..., T$. Let $\dot{p} = p + \alpha \min\{c_n, p_{T+1}\} > c_n$ and $\ddot{p} = p + \alpha \max\{c_n, p_{T+1}\} > c_n$. For any $t = 1, ..., T$, $p_{t+1} \in \{p_{T+1}, c_n\}$ and hence $p + \alpha p_{t+1} \in [\dot{p}, \ddot{p}]$.

- Suppose that $p + \alpha p_{T+1} \leq c_n$ and $p \leq (1-\alpha) \max\{c_n, p_{T+1}\}$. For any $\tau = 0, 1, ...$, let

$$\rho_\tau = \sum_{i=0}^{\tau} \alpha^i p + \alpha^{\tau+1} p_{T+1} = \frac{1-\alpha^{\tau+1}}{1-\alpha} p + \alpha^{\tau+1} p_{T+1} = (1-\alpha^{\tau+1}) \left(\frac{p}{1-\alpha} - p_{T+1} \right) + p_{T+1}. \quad (22)$$

  Alternatively, we have $\rho_\tau = p + \alpha \rho_{\tau-1}$ with $\rho_{-1} = p_{T+1}$. $p \leq (1-\alpha) \max\{c_n, p_{T+1}\}$ implies

$$\rho_\tau \leq (1-\alpha^{\tau+1}) (\max\{c_n, p_{T+1}\} - p_{T+1}) + p_{T+1} \leq \max\{c_n, p_{T+1}\}.$$

  Also note that

$$p + \alpha \max\{c_n, p_{T+1}\} = \begin{cases} p + \alpha p_{T+1} \leq c_n & \text{if } c_n \leq p_{T+1} \\ p + \alpha c_n \leq (1-\alpha) \max\{c_n, p_{T+1}\} + \alpha c_n = c_n & \text{if } c_n > p_{T+1}. \end{cases}$$

  It is straightforward to verify that $p_t = \rho_{T-t} \leq c_n$ for any $t = 1, ..., T$. Consequently, we can set $\dot{p} = +\infty$ and $\ddot{p} = c_n$.

- Suppose that $p + \alpha p_{T+1} \leq c_n$ and $p > (1-\alpha) \max\{c_n, p_{T+1}\}$. Obviously, ρ_τ in (22) is increasing in $\tau = 0, 1, ...$, $\rho_0 = p + \alpha p_{T+1} \leq c_n$, and $\lim_{\tau \to \infty} \rho_\tau = p/(1-\alpha) > c_n$. Therefore, there exists $\tau^* \in \{1, 2, ...\}$ such that $\rho_{\tau^*} > c_n$ and $\rho_\tau \leq c_n$ for any $\tau = 0, 1, ..., \tau^* - 1$. For any $t = T - \tau^* + 1, ..., T$, we have $T - t \leq \tau^* - 1$ and hence $p_t = \rho_{T-t} \leq c_n$. When $t = T - \tau^*$, $p_t = \min\{c_n, p + \alpha \rho_{\tau^*-1}\} = \min\{c_n, \rho_{\tau^*}\} = c_n$. For any $t = 1, ..., T - \tau^* - 1$, $p_t = \min\{c_n, p + \alpha c_n\} = c_n$. Note that $p + \alpha p_t > c_n$ only if $p_t \in \{\rho_{\tau^*-1}, c_n\}$. Thus, let $\dot{p} = \min\{\rho_{\tau^*}, p + \alpha c_n\} > c_n$ and $\ddot{p} = \max\{\rho_{\tau^*}, p + \alpha c_n\} > c_n$. We have $p + \alpha p_{t+1} \in [\dot{p}, \ddot{p}] \cup (-\infty, c_n]$.

Furthermore, let

$$\bar{B} = \max\left\{ q_{n-1}, \frac{K_n}{\dot{p} - c_n}, \max_{i=1,...,n-1} \left\{ \frac{(\ddot{p} - c_i) q_i - K_i + K_n}{\dot{p} - c_n} \right\} \right\} \geq 0,$$

which is finite as $\dot{p} > c_n$ and $\ddot{p} < +\infty$.

For some $t = 1, ..., T$, assume for induction that

$$\bar{W}_{t+1}(x) = \begin{cases} p_{t+1}(-(T-t)\bar{B} - x) + \bar{W}_{t+1}(-(T-t)\bar{B}) & \text{if } x \le -(T-t)\bar{B} \\ h_{t+1}(x - (T-t)\bar{D}) + \bar{W}_{t+1}((T-t)\bar{D}) & \text{if } x \ge (T-t)\bar{D}, \end{cases}$$

which obviously holds when $t = T$ since $\bar{W}_{T+1}(x) = V_{T+1}(x) = h_{T+1}x^+ + p_{T+1}x^-$.

Consider the optimization problem (4). Note that $K_1 \ge 0$ and $K_i + c_i q_i \le K_{i+1} + c_{i+1} q_i$ for any $i = 1, ..., n-1$. The definition of $c(z)$ yields $W_t(x) = \min\{f_t^i(x) : i = 0, 1, ..., n\}$, where

$$f_t^0(x) = \mathbb{E}[H_t(x - D_t)^+ + p(x - D_t)^-] + \alpha\mathbb{E}[\bar{W}_{t+1}(x - D_t)];$$

$$f_t^i(x) = \min_{z \in [q_{i-1}, q_i]} \left\{K_i + c_i z + f_t^0(x + z)\right\} \text{ for all } i = 1, ..., n.$$

For any $x \le -(T-t)\bar{B}$ and $D \in [0, \bar{D}]$, $H_t(x-D)^+ + p(x-D)^- = p(D-x)$ and $\bar{W}_{t+1}(x-D) = p_{t+1}(-(T-t)\bar{B} - x + D) + \bar{W}_{t+1}(-(T-t)\bar{B})$. Therefore, for any $x \le -(T-t)\bar{B}$,

$$\begin{aligned} f_t^0(x) &= \mathbb{E}[p(D_t - x)] + \alpha\mathbb{E}[p_{t+1}(-(T-t)\bar{B} - x + D_t) + \bar{W}_{t+1}(-(T-t)\bar{B})] \\ &= -(p + \alpha p_{t+1})x + (p + \alpha p_{t+1})\mathbb{E}[D_t] + \alpha\bar{W}_{t+1}(-(T-t)\bar{B}) - \alpha p_{t+1}(T-t)\bar{B}. \end{aligned}$$

Similarly, for any $x \ge (T-t+1)\bar{D}$,

$$\begin{aligned} f_t^0(x) &= \mathbb{E}[H_t(x - D_t)] + \alpha\mathbb{E}[h_{t+1}(x - D_t - (T-t)\bar{D}) + \bar{W}_{t+1}((T-t)\bar{D})] \\ &= (h + \alpha h_{t+1})x - (h + \alpha h_{t+1})\mathbb{E}[D_t] + \alpha\bar{W}_{t+1}((T-t)\bar{D}) - \alpha h_{t+1}(T-t)\bar{D}. \end{aligned}$$

Thus, we obtain

$$f_t^0(x) = \begin{cases} (p + \alpha p_{t+1})(-(T-t+1)\bar{B} - x) + f_t^0(-(T-t+1)\bar{B}) & \text{if } x \le -(T-t)\bar{B} \\ (h + \alpha h_{t+1})(x - (T-t+1)\bar{D}) + f_t^0((T-t+1)\bar{D}) & \text{if } x \ge (T-t+1)\bar{D}. \end{cases} \tag{23}$$

For any $i = 1, ..., n-1$, as $\bar{B} \ge q_{n-1} \ge q_i$, it is straightforward that

$$\begin{aligned} f_t^i(x) &= \min_{z \in [q_{i-1}, q_i]} \left\{K_i + c_i z + f_t^0(x + z)\right\} \\ &= \begin{cases} f_t^0(x) + \min_{z \in \{q_{i-1}, q_i\}}\{K_i + (c_i - p - \alpha p_{t+1})z\} & \text{if } x \le -(T-t+1)\bar{B} \\ f_t^0(x) + \min_{z \in \{q_{i-1}, q_i\}}\{K_i + (c_i + h + \alpha h_{t+1})z\} & \text{if } x \ge (T-t+1)\bar{D}. \end{cases} \end{aligned}$$

Let

$$S_t^n = \inf\left\{\arg\inf_{y \in (-\infty, +\infty)} \left\{c_n y + f_t^0(y)\right\}\right\}.$$

Note that $f_t^0(y)$ is convex as $\bar{W}_{t+1}(x)$ is convex. Recall that $c_n + h + \alpha h_{t+1} = c_n + h_t \ge 0$. (23) implies $S_t^n = -\infty$ if $c_n - p - \alpha p_{t+1} \ge 0$ and $S_t^n \in [-(T-t)\bar{B}, (T-t+1)\bar{D}]$ otherwise. Consequently,

$$\begin{aligned} f_t^n(x) &= \min_{z \ge q_{n-1}} \left\{K_n + c_n z + f_t^0(x + z)\right\} \\ &= \begin{cases} f_t^0(x) + K_n + (c_n - p - \alpha p_{t+1})q_{n-1} & \text{if } x \le -(T-t+1)\bar{B} \text{ and } c_n - p - \alpha p_{t+1} \ge 0 \\ -c_n x + K_n + c_n S_t^n + f_t^0(S_t^n) & \text{if } x \le -(T-t+1)\bar{B} \text{ and } c_n - p - \alpha p_{t+1} < 0 \\ f_t^0(x) + K_n + (c_n + h + \alpha h_{t+1})q_{n-1} & \text{if } x \ge (T-t+1)\bar{D}. \end{cases} \end{aligned}$$

Now we can determine $W_t(x)$ for the following cases.

- Suppose that $x \le -(T-t+1)\bar{B}$ and $c_n - p - \alpha p_{t+1} \ge 0$. Then

$$\begin{aligned}
W_t(x) &= \min\{f_t^i(x) : i = 0,1,...,n\} \\
&= \min\left\{\begin{array}{c} f_t^0(x), \ \min\limits_{i=1,\dots,n-1}\left\{f_t^0(x) + \min\limits_{z\in\{q_{i-1},q_i\}}\{K_i + (c_i - p - \alpha p_{t+1})z\}\right\}, \\ f_t^0(x) + K_n + (c_n - p - \alpha p_{t+1})q_{n-1}) \end{array}\right\} \\
&= \min\left\{\begin{array}{c} f_t^0(x), \ \min\limits_{i=1,\dots,n-1}\left\{f_t^0(x) + K_i + (c_i - p - \alpha p_{t+1})q_{i-1}\}\right\}, \\ \min\limits_{i=1,\dots,n-1}\left\{f_t^0(x) + K_i + (c_i - p - \alpha p_{t+1})q_i\}\right\}, \\ f_t^0(x) + K_n + (c_n - p - \alpha p_{t+1})q_{n-1}) \end{array}\right\} \\
&= \min\left\{\begin{array}{c} f_t^0(x), \ \min\limits_{i=0,\dots,n-2}\left\{f_t^0(x) + K_{i+1} + (c_{i+1} - p - \alpha p_{t+1})q_i\}\right\}, \\ \min\limits_{i=1,\dots,n-1}\left\{f_t^0(x) + K_i + (c_i - p - \alpha p_{t+1})q_i\}\right\}, \\ f_t^0(x) + K_n + (c_n - p - \alpha p_{t+1})q_{n-1}) \end{array}\right\} \\
&= \min\left\{f_t^0(x), \ \min_{i=1,\dots,n-1}\left\{f_t^0(x) + K_i + (c_i - p - \alpha p_{t+1})q_i\}\right\}\right\} \\
&= f_t^0(x) + \min\left\{0, \ \min_{i=1,\dots,n-1}\left\{K_i + (c_i - p - \alpha p_{t+1})q_i\}\right\}\right\},
\end{aligned}$$

where the second last equality is obtained from $K_1 \ge 0$ and $K_i + c_i q_i \le K_{i+1} + c_{i+1} q_i$ for $i = 1, ..., n-1$.

- Suppose that $x \le -(T-t+1)\bar{B}$ and $c_n - p - \alpha p_{t+1} < 0$. Similar to the previous case, we have

$$\begin{aligned}
\min\{f_t^i(x) : i = 0,1,...,n-1\} &= f_t^0(x) + \min\left\{0, \ \min_{i=1,\dots,n-1}\left\{K_i + (c_i - p - \alpha p_{t+1})q_i\}\right\}\right\} \\
&\ge f_t^0(x) + \min\left\{0, \ \min_{i=1,\dots,n-1}\left\{K_i + (c_i - \ddot{p})q_i\}\right\}\right\},
\end{aligned}$$

where the inequality follows from $p + \alpha p_{t+1} \in [\dot{p}, \ddot{p}] \cup (-\infty, c_n]$ and $c_n - p - \alpha p_{t+1} < 0$. Recall that

$$\begin{aligned}
f_t^n(x) &= -c_n x + K_n + c_n S_t^n + f_t^0(S_t^n) \le -c_n x + K_n + c_n(-(T-t)\bar{B}) + f_t^0(-(T-t)\bar{B}) \\
&= -c_n x + K_n + c_n(-(T-t)\bar{B}) + f_t^0(x) - (p + \alpha p_{t+1})(-(T-t)\bar{B} - x) \\
&= f_t^0(x) + K_n + (c_n - p - \alpha p_{t+1})(-(T-t)\bar{B} - x) \le f_t^0(x) + K_n + (c_n - p - \alpha p_{t+1})\bar{B} \\
&\le f_t^0(x) + K_n + (c_n - \dot{p})\bar{B} \\
&\le f_t^0(x) + K_n + (c_n - \dot{p}) \max\left\{\frac{K_n}{\dot{p} - c_n}, \max_{i=1,\dots,n-1}\left\{\frac{(\ddot{p} - c_i)q_i - K_i + K_n}{\dot{p} - c_n}\right\}\right\} \\
&= f_t^0(x) + K_n - \max\left\{K_n, \max_{i=1,\dots,n-1}\left\{(\ddot{p} - c_i)q_i - K_i + K_n\right\}\right\} \\
&= f_t^0(x) + \min\left\{0, \min_{i=1,\dots,n-1}\left\{K_i + (c_i - \ddot{p})q_i\right\}\right\},
\end{aligned}$$

where the first inequality is obtained from the definition of S_t^n, the second equality is yielded by (23), the second and third inequalities follow from $x \le -(T-t+1)\bar{B}$, $c_n - p - \alpha p_{t+1} < 0$, and $p+\alpha p_{t+1} \in [\dot{p}, \ddot{p}] \cup (-\infty, c_n]$, and the last inequality is a result of $\dot{p} > c_n$ and the definition of $\bar{B}$. Thus, we obtain

$$f_t^n(x) \le \min\{f_t^i(x) : i = 0, 1, ..., n-1\}$$

and hence

$$W_t(x) = \min\{f_t^i(x) : i = 0, 1, ..., n\} = f_t^n(x) = -c_n x + K_n + c_n S_t^n + f_t^0(S_t^n).$$

- Suppose that $x \ge (T-t+1)\bar{D}$. Similar to the previous two cases, we have

$$\begin{aligned} W_t(x) &= \min\{f_t^i(x) : i = 0, 1, ..., n\} \\ &= \min \left\{ \begin{array}{c} f_t^0(x), \min_{i=1,...,n-1} \left\{ f_t^0(x) + \min_{z \in \{q_{i-1}, q_i\}} \{K_i + (c_i + h + \alpha h_{t+1}) z\} \right\}, \\ f_t^0(x) + K_n + (c_n + h + \alpha h_{t+1}) q_{n-1}) \end{array} \right\} \\ &= f_t^0(x) + \min \left\{ 0, \min_{i=1,...,n-1} \left\{ K_i + (c_i + h + \alpha h_{t+1}) q_i \right\} \right\}. \end{aligned}$$

Based on these results, (23) implies that Proposition 7 holds for $W_t(x)$.

Recall that $-p_t \le h_t$. According to Algorithm 1, we know that $\bar{W}_t(x_t)$ and $W_t(x_t)$ have the same slope when $x_t \le -(T-t+1)\bar{B}$ or $x_t \ge (T-t+1)\bar{D}$. This completes the induction proof. □

Proof of Theorem 4. We first show that $\|V_t - U_t\|_\infty \le \sum_{i=1}^{T-t+1} \alpha^i K$ for any $t = 1, ..., T$. Note that $U_{T+1}(x_{T+1}) = V_{T+1}(x_{T+1})$. As $\bar{c}(z)$ is a K-approximation function of $c(z)$, $\|V_T - U_T\|_\infty \le K$. Therefore, the result holds when $t = T$.

Suppose that $\|V_{t+1} - U_{t+1}\|_\infty \le \sum_{i=0}^{T-t} \alpha^i K$ for some $t = 1, ..., T-1$. Consider any $x_t \in \mathbb{R}$. For all $y_t \ge x_t$,

$$\begin{aligned} \Big| V_t(x_t) - U_t(x_t) \Big| &\le \Big| \bar{c}(y_t - x_t) - c(y_t - x_t) \Big| + \Big| \alpha \mathbb{E}[V_{t+1}(y_t - D_t)] - \alpha \mathbb{E}[U_{t+1}(y_t - D_t)] \Big| \\ &\le K + \alpha \sum_{i=0}^{T-t} \alpha^i K = \sum_{i=1}^{T-t+1} \alpha^i K. \end{aligned} \tag{24}$$

Consider $t = T$. For any $x_t \in \mathbb{R}$, we define

$$\hat{y}_T(x_T) \in \arg\min_{y_T \ge x_T} \Big\{ \bar{c}(y_T - x_T) + H_T(y_T) + \alpha \mathbb{E}[U_{T+1}(y_T - D_T)] \Big\}.$$

Note that $U_{T+1} = V_{T+1} = \widehat{V}_{T+1}$. By Lemma 2 (b), we have

$$\widehat{V}_T(x_T) \le V_t(x_T) + 2K.$$

Therefore, Theorem 4 holds when $t = T$. Suppose that the result also holds for period $t+1$, i.e., $\widehat{V}_{t+1}(x_{t+1}) \le V_{t+1}(x_{t+1}) + 2K\sum_{i=1}^{T-t} i\alpha^i$. Next, we prove that the result holds for period t. For any $x_t \in \mathbb{R}$, we define

$$\hat{y}_t(x_t) \in \arg\min_{y_t \ge x_t} \Big\{ \bar{c}(y_t - x_t) + H_t(y_t) + \alpha\mathbb{E}[U_{t+1}(y_t - D_t)] \Big\}.$$

Then, we have

$$\begin{aligned} \widehat{V}_t(x_t) &= c(\hat{y}_t(x_t) - x_t) + H_t(\hat{y}_t(x_t)) + \alpha\mathbb{E}[\widehat{V}_{t+1}(\hat{y}_t(x_t) - D_t)] \\ &\le c(\hat{y}_t(x_t) - x_t) + H_t(\hat{y}_t(x_t)) + \alpha\mathbb{E}[V_{t+1}(\hat{y}_t(x_t) - D_t)] + 2K\sum_{i=1}^{T-t} i\alpha^{i+1}. \end{aligned}$$

Applying Lemma 2 (b), (24) implies

$$c(\hat{y}_t(x_t) - x_t) + H_t(\hat{y}_t(x_t)) + \alpha\mathbb{E}[V_{t+1}(\hat{y}_t(x_t) - D_t)] \le V_t(x_t) + 2\sum_{i=1}^{T-t+1} \alpha^i K.$$

Therefore, we obtain

$$\widehat{V}_t(x_t) \le V_t(x_t) + 2K\sum_{i=1}^{T-t+1} \alpha^i + 2K\sum_{i=1}^{T-t} i\alpha^{i+1} = V_t(x_t) + 2K\sum_{i=1}^{T-t+1} i\alpha^i. \qquad \square$$

Proof of Proposition 8. First, we show that the CTGA and CTGEA approaches, i.e., Algorithms 3 and 4 are equivalent. Obviously, $\widehat{R}_{T+1}(y) = H_T(y) + \alpha V_{T+1}(y - D_T) = H_T(y) + \alpha\bar{W}_{T+1}(y - D_T)$ for all $y \ge D_T$. For any $t \in \{1, ..., T\}$, assume by induction that $\widehat{R}_{t+1}(y) = H_t(y) + \alpha\bar{W}_{t+1}(y - D_t)$ for all $y \ge D_t$. Then, for any $x \ge 0$,

$$R_t(x) = \min_{y \ge \max\{x, D_t\}} \Big\{ c(y-x) + \widehat{R}_{t+1}(y) \Big\} = \min_{y \ge \max\{x, D_t\}} \Big\{ c(y-x) + H_t(y) + \alpha\bar{W}_{t+1}(y - D_t) \Big\} = W_t(x).$$

Let $\underline{R}_t^*$ and $\underline{W}_t^*$ denote the convex envelope of $H_{t-1}(y) + \alpha R_t(y - D_{t-1})$ for $y \ge D_{t-1}$ and $W_t(x)$ for $x \ge 0$, respectively. Applying the definition of convex envelope, $R_t(x) = W_t(x)$ for all $x \ge 0$, and $H_{t-1}(y) = h(y - D_{t-1})$ for all $y \ge D_{t-1}$, it is straightforward to show that $\underline{R}_t^*(y) = H_{t-1}(y) + \alpha\underline{W}_t^*(y - D_{t-1})$ for all $y \ge D_{t-1}$. $\widehat{R}_t(y) = H_{t-1}(y) + \alpha\bar{W}_t(y - D_{t-1})$ for all $y \ge D_{t-1}$ follows immediately from

$$\begin{aligned} \bar{W}_t(x) &= \underline{W}_t^*(x) + \frac{1}{2}\sup_{x \ge 0} \big| W_t(x) - \underline{W}_t^*(x) \big| && \text{for all } x \ge 0, \\ \widehat{R}_t(y) &= \underline{R}_t^*(y) + \frac{1}{2}\sup_{y \ge D_{t-1}} \big| \big(H_{t-1}(y) + \alpha R_t(y - D_{t-1})\big) - \underline{R}_t^*(y) \big| && \text{for all } y \ge D_{t-1}. \end{aligned}$$

This completes the induction proof showing the equivalence between Algorithms 3 and 4.

The remaining part of the proof compares the CTGA and OCA approaches, i.e., Algorithms 3 and 5. We start by considering Algorithm 5. Recall that $\bar{c}(z)$ represents the convex approximation of $c(z)$ obtained in Step 0 of Algorithm 5. As $c(z)$ is concave, it is straightforward that there exists a constant $\bar{K}_1$ such that $\bar{c}(z) = \bar{K}_1 + c_n z$ (refer to the definition of $c(z)$ in (1)). Let $\widehat{y}_t(x)$ denote the order-up-to level of the OCA approach (see Algorithm 5). We first show that $\widehat{y}_t(x) = \max\{x, D_t\}$ for any $x \geq 0$ and $t \in \{1, ..., T\}$. This can be proved by assuming for induction that

(H1) $U_t(x)$ is a convex function in $[0, \infty)$ with $c_n + h + \alpha\partial_+ U_t(0) \geq 0$, where $\partial_+ f(x)$ denotes the right derivative of f at x.

Note that $U_{T+1}(x) = V_{T+1}(x) = h_{T+1}x$ for any $x \geq 0$, implying $c_n + h + \alpha\partial_+ U_{T+1}(0) = c_n + h + \alpha h_{T+1}$. As h_{T+1} is assumed to satisfy $c_n + h + \alpha h_{T+1} \geq 0$ so as to ensure the existence of a finite optimal order-up-to level, (H1) holds when $t = T + 1$.

For any $t \in \{1, ..., T\}$, suppose that $U_{t+1}(x)$ satisfies the induction hypothesis (H1). Let $\widehat{y}_t(x)$ correspond to an optimal solution of

$$U_t(x) = \min_{y \geq \max\{x, D_t\}} \Big\{ \bar{K}_1 + c_n(y - x) + h(y - D_t) + \alpha U_{t+1}(y - D_t) \Big\}. \tag{25}$$

Consider the objective function in the above optimization problem. (H1) for U_{t+1} implies that it is a convex function of y and its right derivative at $y = D_t$ is $c_n + h + \alpha\partial_+ U_{t+1}(0) \geq 0$. Therefore, it is nondecreasing for any $y \geq D_t$ and we have $\widehat{y}_t(x) = \max\{x, D_t\}$. This result yields

$$U_t(x) = K_1 + c_n(D_t - x)^+ + h(x - D_t)^+ + \alpha U_{t+1}((x - D_t)^+),$$

and hence

$$\partial_+ U_t(x) = \begin{cases} -c_n, & \text{if } 0 \leq x < D_t; \\ h + \alpha\partial_+ U_{t+1}(x - D_t), & \text{if } x \geq D_t. \end{cases}$$

If $D_t > 0$, then $\partial_+ U_t(0) = -c_n$. If $D_t = 0$, then $\partial_+ U_t(0) = h + \alpha\partial_+ U_{t+1}(0) \geq -c_n$, where the inequality follows from (H1) for U_{t+1}. Since $h \geq 0$ and $\alpha \in [0, 1]$, $c_n + h + \alpha\partial_+ U_t(0) \geq c_n + h - \alpha c_n \geq 0$. Furthermore, the convexity of $U_{t+1}(x)$ in (H1) and the definition of $U_t(x)$ in (25) implies the convexity of $U_t(x)$, which completes the proof of (H1) for U_t. As a result, we obtain $\widehat{y}_t(x) = \max\{x, D_t\}$ and hence

$$\widehat{V}_t(x) = c(\max\{x, D_t\} - x) + h(\max\{x, D_t\} - D_t) + \alpha\widehat{V}_{t+1}(\max\{x, D_t\} - D_t), \tag{26}$$

for any $x \geq 0$ and $t \in \{1, ..., T\}$.

To show that $\bar{V}_t(x) \leq \widehat{V}_t(x)$ for any $x \in \mathbb{R}$ and $t \in \{1, ..., T\}$, consider the following induction hypotheses.

(H2) $\bar{V}_t(x) \leq \widehat{V}_t(x)$ for any $x \geq 0$.

(H3) $\bar{W}_t(x)$ is a piecewise linear, continuous, and convex function in $[0,\infty)$. It has a finite number of breakpoints denoted by the set $\bar{\mathcal{B}}_t \subseteq [0, \sum_{i=t}^T D_i]$ with $\min \bar{\mathcal{B}}_t = 0$. Furthermore, $\partial_+ \bar{W}_t(\sum_{i=t}^T D_i) = \sum_{i=0}^{T-t} \alpha^i h + \alpha^{T-t+1} h_{T+1}$.

(H4) There exists a constant ν_t such that $\bar{W}_t(x) = \bar{V}_t(x) + \nu_t$ for any $x \in \bar{\mathcal{B}}_t$ and $\bar{W}_t(x) \le \bar{V}_t(x) + \nu_t$ for any $x \ge 0$.

(H5) $c(x) \ge \bar{W}_t(0) - \bar{W}_t(x)$ for any $x \in \bar{\mathcal{B}}_t$.

(H2–5) hold for $t = T+1$ because $\bar{V}_{T+1}(x) = \widehat{V}_{T+1}(x) = \bar{W}_{T+1}(x) = V_{T+1}(x) = h_{T+1}x$ for any $x \ge 0$ and $\bar{\mathcal{B}}_t = \{0\}$. For any $t \in \{1, ..., T\}$, suppose that $\bar{V}_{t+1}$, $\widehat{V}_{t+1}$, and $\bar{W}_{t+1}$ possess the properties in (H2–5). For any $x \ge 0$, the heuristic policy $\bar{y}_t(x)$ obtained by Algorithm 3 is the minimum optimal solution to

$$W_t(x) = \min_{y \ge \max\{x, D_t\}} \Big\{ c(y-x) + h(y-D_t) + \alpha \bar{W}_{t+1}(y-D_t) \Big\}. \tag{27}$$

For any $i \in \{1, ..., n\}$, define

$$f_t^i(y) = c_i y + h(y-D_t) + \alpha \bar{W}_{t+1}(y-D_t) \qquad \text{and} \qquad S_t^i = \min \left\{ \arg\min_{y \ge D_t} f_t^i(y) \right\}.$$

According to (H3) for $\bar{W}_{t+1}$, f_t^i is a piecewise linear convex function, its set of breakpoints is

$$\mathcal{B}_t^0 = \{x + D_t : x \in \bar{\mathcal{B}}_{t+1}\} \subseteq \left[D_t, \sum_{i=t}^T D_i \right], \tag{28}$$

and the slope of its last piece is

$$\begin{aligned} \partial_+ f_t^i \left(\sum_{i'=t}^T D_{i'} \right) &= c_i + h + \alpha \partial_+ \bar{W}_{t+1} \left(\sum_{i'=t+1}^T D_{i'} \right) = c_i + h + \alpha \left(\sum_{i'=0}^{T-(t+1)} \alpha^{i'} h + \alpha^{T-(t+1)+1} h_{T+1} \right) \\ &= c_i + \sum_{i'=0}^{T-t} \alpha^{i'} h + \alpha^{T-t+1} h_{T+1} \ge c_n + \sum_{i'=0}^{T-t} \alpha^{i'} h + \alpha^{T-t+1} h_{T+1} \ge 0, \end{aligned}$$

where the first inequality, i.e., $c_i \ge c_n$, is obtained by the concavity of c and the second inequality corresponds to the assumption on h_{T+1} ensuring the existence of a finite optimal order-up-to level. Consequently, we have $S_t^i \in \mathcal{B}_t^0$.

As shown in Theorem 1, there exists s_t such that $\bar{y}_t(x) > x$ if $x \in [0, s_t)$ and $\bar{y}_t(x) = x$ if $x \in [s_t, \infty)$. Furthermore, the set $[0, s_t)$ can be divided into at most $m_t \le 2n-1$ intervals $[a_t^j, a_t^{j+1})$, $j = 0, 1, \ldots, m_t - 1$ with $0 = a_t^0 < a_t^1 < a_t^2 < \cdots < a_t^{m_t-1} < a_t^{m_t} = s_t$. For any $j = 0, ..., m_t - 1$, either there exists some i such that $\bar{y}_t(x) = x + q_i$ for all $x \in [a_t^j, a_t^{j+1})$ or there exists some i such that $\bar{y}_t(x) = S_t^i$ for all $x \in [a_t^j, a_t^{j+1})$. According to Porteus (1971), $\bar{y}_t(x)$ is nonincreasing in x for any $x \in [0, s_t)$. As $x + q_i$ is strictly increasing in x, we know that for any $j = 0, ..., m_t - 1$, there exists some i such that $\bar{y}_t(x) = S_t^i$ for all $x \in [a_t^j, a_t^{j+1})$, which also implies that $m_t \le n$. In other words, $\bar{y}_t(x)$ for all $x \in [0, \infty)$ is characterized as follows:

$$\bar{y}_t(x) = \begin{cases} \bar{S}_t^j > x, & \text{if } x \in [a_t^{j-1}, a_t^j) \text{ for some } j = 1, ..., m_t, \\ x, & \text{if } x \in [a_t^{m_t}, \infty), \end{cases} \tag{29}$$

where $0 = a_t^0 < a_t^1 < a_t^2 < \cdots < a_t^{m_t - 1} < a_t^{m_t} = s_t$, $\bar{S}_t^j \in \mathcal{B}_t^0$ for all $j = 1, ..., m_t$, and $m_t \in \{0, 1, ..., n\}$.

As $\bar{y}_t(x) \ge D_t$ for all $x \ge 0$, we have $s_t \ge D_t$. Assume for contradiction that $s_t > D_t$. (29) implies $D_t < \bar{y}_t(D_t) \in \{\bar{S}_t^j : j = 1, ..., m_t\} \subseteq \mathcal{B}_t^0$. As $\bar{y}_t(D_t)$ is the minimum optimal solution to (27), we have

$$c(\bar{y}_t(D_t) - D_t) + h(\bar{y}_t(D_t) - D_t) + \alpha \bar{W}_{t+1}(\bar{y}_t(D_t) - D_t) < \alpha \bar{W}_{t+1}(0),$$

i.e.,

$$\alpha \Big(\bar{W}_{t+1}(0) - \bar{W}_{t+1}(\bar{y}_t(D_t) - D_t) \Big) > c(\bar{y}_t(D_t) - D_t) + h(\bar{y}_t(D_t) - D_t) \ge c(\bar{y}_t(D_t) - D_t),$$

which leads to $\bar{W}_{t+1}(0) - \bar{W}_{t+1}(\bar{y}_t(D_t) - D_t) > c(\bar{y}_t(D_t) - D_t)$ as $\alpha \in [0, 1]$. According to the definition of $\mathcal{B}_t^0$, $\bar{y}_t(D_t) \in \mathcal{B}_t^0$ yields $\bar{y}_t(D_t) - D_t \in \bar{\mathcal{B}}_{t+1}$. Therefore, (H5) for $\bar{W}_{t+1}$ implies $c(\bar{y}_t(D_t) - D_t) \le \bar{W}_{t+1}(0) - \bar{W}_{t+1}(\bar{y}_t(D_t) - D_t)$, which results in a contradiction. As a result, we have $s_t = a_t^{m_t} = D_t$.

Next, we show that (H2) holds for $\bar{V}_t$ and $\widehat{V}_t$. For any $x \in [a_t^{m_t}, \infty) = [D_t, \infty)$, $\bar{y}_t(x) = x$ and

$$\bar{V}_t(x) = h(x - D_t) + \alpha \bar{V}_{t+1}(x - D_t) \le h(x - D_t) + \alpha \widehat{V}_{t+1}(x - D_t) = \widehat{V}_t(x),$$

where the inequality follows from (H2) for $\bar{V}_{t+1}$ and $\widehat{V}_{t+1}$ and the second equality is yielded by (26). Consider $x \in [a_t^{j-1}, a_t^j)$ for some $j = 1, ..., m_t$. We have $\bar{y}_t(x) = \bar{S}_t^j \in \mathcal{B}_t^0$, which, by the definition of $\mathcal{B}_t^0$ in (28), implies $\bar{S}_t^j - D_t \in \bar{\mathcal{B}}_{t+1}$ and so $\bar{V}_{t+1}(\bar{S}_t^j - D_t) = \bar{W}_{t+1}(\bar{S}_t^j - D_t) + \nu_{t+1}$ by (H4) for $\bar{W}_{t+1}$ and $\bar{V}_{t+1}$. Therefore, we obtain

$$\begin{aligned} \bar{V}_t(x) &= c(\bar{S}_t^j - x) + h(\bar{S}_t^j - D_t) + \alpha \bar{V}_{t+1}(\bar{S}_t^j - D_t) \\ &= c(\bar{S}_t^j - x) + h(\bar{S}_t^j - D_t) + \alpha \bar{W}_{t+1}(\bar{S}_t^j - D_t) + \alpha \nu_{t+1} \\ &\le c(\max\{x, D_t\} - x) + h(\max\{x, D_t\} - D_t) + \alpha \bar{W}_{t+1}(\max\{x, D_t\} - D_t) + \alpha \nu_{t+1} \\ &\le c(\max\{x, D_t\} - x) + h(\max\{x, D_t\} - D_t) + \alpha \bar{V}_{t+1}(\max\{x, D_t\} - D_t) \\ &\le c(\max\{x, D_t\} - x) + h(\max\{x, D_t\} - D_t) + \alpha \widehat{V}_{t+1}(\max\{x, D_t\} - D_t) = \widehat{V}_t(x), \end{aligned}$$

where the first inequality is obtained because $\bar{y}_t(x) = \bar{S}_t^j$ is an optimal solution to (27), the second inequality follows from (H4) for $\bar{W}_{t+1}$ and $\bar{V}_{t+1}$, the last inequality follows from (H2) for $\bar{V}_{t+1}$ and $\widehat{V}_{t+1}$, and the last equality follows from (26). This completes the proof of (H2) for $\bar{V}_t$ and $\widehat{V}_t$.

Now consider W_t. (27) and (29) yield

$$W_t(x) = \begin{cases} c(\bar{S}_t^j - x) + h(\bar{S}_t^j - D_t) + \alpha \bar{W}_{t+1}(\bar{S}_t^j - D_t), & \text{if } x \in [a_t^{j-1}, a_t^j) \text{ for some } j = 1, ..., m_t, \\ h(x - D_t) + \alpha \bar{W}_{t+1}(x - D_t), & \text{if } x \in [a_t^{m_t}, \infty) = [D_t, \infty). \end{cases} \tag{30}$$

As both c and $\bar{W}_{t+1}$ are piecewise linear, W_t is obviously piecewise linear with a finite number of breakpoints. Let $\mathcal{B}_t$ denote the set of the breakpoints for W_t. It is trivial that $\min \mathcal{B}_t = 0$. Also note that

$$\max \mathcal{B}_t = \max\{D_t, \max \mathcal{B}_t^0\} \leq \sum_{i=t}^{T} D_i,$$

as $\mathcal{B}_t^0 \subseteq [D_t, \sum_{i=t}^T D_i]$. This implies that the slope of the last piece of W_t is

$$\begin{aligned} \partial_+ \bar{W}_t \left(\sum_{i=t}^{T} D_i \right) &= h + \alpha \partial_+ \bar{W}_{t+1} \left(\sum_{i=t+1}^{T} D_i \right) = h + \alpha \left(\sum_{i=0}^{T-(t+1)} \alpha^i h + \alpha^{T-(t+1)+1} h_{T+1} \right) \\ &= \sum_{i=0}^{T-t} \alpha^i h + \alpha^{T-t+1} h_{T+1} \geq 0, \end{aligned}$$

where the second equality follows from (H3) for $\bar{W}_{t+1}$.

Furthermore, for any $j = 1, ..., m_t$, $W_t(x)$ is continuous in $[a_t^{j-1}, a_t^j)$ because $\bar{y}_t(x) = \bar{S}_t^j > x$ for all $x \in [a_t^{j-1}, a_t^j)$ and the concavity of c implies its continuity in $(0, \infty)$. The continuity of $\bar{W}_{t+1}$ also suggests that W_t is continuous in $[a_t^{m_t}, \infty)$. Consequently, W_t is right continuous in $[0, \infty)$. According to Algorithm 3, $\bar{W}_t(x) = \underline{W}_t^*(x) + \nu_t'$ for all $x \in [0, \infty)$, where $\underline{W}_t^*$ is the convex envelope of W_t and ν_t' is a constant. The properties of the convex envelope imply that $\underline{W}_t^*$ satisfies (H3) due to the piecewise linearity and the right continuity of W_t; so does $\bar{W}_t$.

To prove (H4) and (H5) for $\bar{W}_t$ and $\bar{V}_t$, we first prove $\underline{W}_t^*(x) = W_t(x)$ for all $x \in \bar{\mathcal{B}}_t$, where $\bar{\mathcal{B}}_t$ represents the set of breakpoints for both $\bar{W}_t$ and $\underline{W}_t^*(x)$.

- Suppose that $a_t^j \in \bar{\mathcal{B}}_t$ for some $j = 1, ..., m_t$. As $\bar{y}_t(x) = \bar{S}_t^j \geq \max\{x, D_t\}$ for all $x \in [a_t^{j-1}, a_t^j)$, we have $\bar{S}_t^j \geq \max\{a_t^j, D_t\}$ and the definition of W_t in (27) yields

$$\begin{aligned} W_t(a_t^j) &\leq c(\bar{S}_t^j - a_t^j) + h(\bar{S}_t^j - D_t) + \alpha \bar{W}_{t+1}(\bar{S}_t^j - D_t) \\ &\leq \lim_{z \downarrow \bar{S}_t^j - a_t^j} c(z) + h(\bar{S}_t^j - D_t) + \alpha \bar{W}_{t+1}(\bar{S}_t^j - D_t) \\ &= \lim_{x \uparrow a_t^j} c(\bar{S}_t^j - x) + h(\bar{S}_t^j - D_t) + \alpha \bar{W}_{t+1}(\bar{S}_t^j - D_t) = \lim_{x \uparrow a_t^j} W_t(x), \end{aligned}$$

  where the second inequality is obtained from the property of c, while the last equality follows from the value of W_t in (30) and the continuity of $\bar{W}_{t+1}$ by (H3). As $a_t^j \in \bar{\mathcal{B}}_t \setminus \{0\}$, Algorithm

1 implies

$$\underline{W}_t^*(a_t^j) = \min\left\{W_t(a_t^j), \lim_{x\uparrow a_t^j} W_t(x), \lim_{x\downarrow a_t^j} W_t(x)\right\} = \min\left\{W_t(a_t^j), \lim_{x\uparrow a_t^j} W_t(x)\right\} = W_t(a_t^j),$$

where the second equality is yielded by the right continuity of W_t.

- $a_t^0 = 0 \in \bar{\mathcal{B}}_t$ and, by the right continuity of W_t, $\underline{W}_t^*(0) = \min\{W_t(0), \lim_{x'\downarrow x} W_t(x')\} = W_t(0)$.

- For any $x \in \bar{\mathcal{B}}_t$ such that $x \in (a_t^{j-1}, a_t^j)$ for some $j = 1, ..., m_t$ or $x > a_t^{m_t}$,

$$\underline{W}_t^*(x) = \min\left\{W_t(x), \lim_{x'\uparrow x} W_t(x'), \lim_{x'\downarrow x} W_t(x')\right\} = W_t(x),$$

where the two equalities follow from Algorithm 1 and the continuity of W_t in the corresponding interval, respectively.

To prove (H4) for $\bar{W}_t$ and $\bar{V}_t$, let $\nu_t = \alpha\nu_{t+1} + \nu_t'$. For any $x \in [0, \infty)$,

$$\begin{aligned}\bar{W}_t(x) &= \underline{W}_t^*(x) + \nu_t' \le W_t(x) + \nu_t' = c(\bar{y}_t(x) - x) + h(\bar{y}_t(x) - D_t) + \alpha\bar{W}_{t+1}(\bar{y}_t(x) - D_t) + \nu_t' \\ &\le c(\bar{y}_t(x) - x) + h(\bar{y}_t(x) - D_t) + \alpha\bar{V}_{t+1}(\bar{y}_t(x) - D_t) + \alpha\nu_{t+1} + \nu_t' = \bar{V}_t(x) + \nu_t,\end{aligned}$$

where the two inequalities follows respectively from the definition of the convex envelope and (H4) for $\bar{W}_{t+1}$ and $\bar{V}_{t+1}$. For any $x \in \bar{\mathcal{B}}_t$, we show $\bar{W}_t(x) = \bar{V}_t(x) + \nu_t$, which is equivalent to $\underline{W}_t^*(x) = \bar{V}_t(x) + \alpha\nu_{t+1}$ since $\bar{W}_t(x) = \underline{W}_t^*(x) + \nu_t'$, through the following cases.

- Suppose that $x \in [0, a_t^{m_t})$ and $x \in \bar{\mathcal{B}}_t$. We obtain

$$\underline{W}_t^*(x) = W_t(x) = c(\bar{y}_t(x) - x) + h(\bar{y}_t(x) - D_t) + \alpha\bar{W}_{t+1}(\bar{y}_t(x) - D_t).$$

$x \in [0, a_t^{m_t})$ implies $\bar{y}_t(x) \in \{\bar{S}_t^j : j = 1, ...m_t\} \subseteq \mathcal{B}_t^0$. The definition of $\mathcal{B}_t^0$ in (28) shows $\bar{y}_t(x) - D_t \in \bar{\mathcal{B}}_{t+1}$. According to (H4) for $\bar{W}_{t+1}$ and $\bar{V}_{t+1}$, we have $\bar{W}_{t+1}(\bar{y}_t(x) - D_t) = \bar{V}_{t+1}(\bar{y}_t(x) - D_t) + \nu_{t+1}$ and hence

$$\underline{W}_t^*(x) = c(\bar{y}_t(x) - x) + h(\bar{y}_t(x) - D_t) + \alpha\bar{V}_{t+1}(\bar{y}_t(x) - D_t) + \alpha\nu_{t+1} = \bar{V}_t(x) + \alpha\nu_{t+1}.$$

- Suppose that $x \ge a_t^{m_t}$ and $x \in \bar{\mathcal{B}}_t$. We have $\underline{W}_t^*(x) = W_t(x)$ and the value of W_t in (30) yields

$$\underline{W}_t^*(x) = W_t(x) = h(x - D_t) + \alpha\bar{W}_{t+1}(x - D_t).$$

If $x = a_t^{m_t} = D_t$, $x - D_t = 0 \in \bar{\mathcal{B}}_{t+1}$, which is the set of breakpoints of $\bar{W}_{t+1}$. If $x > D_t$, Algorithm 1 implies that x is a breakpoint of W_t, so we also have $x - D_t \in \bar{\mathcal{B}}_{t+1}$ by the value of W_t in (30). (H4) for $\bar{W}_{t+1}$ and $\bar{V}_{t+1}$ then yields

$$\underline{W}_t^*(x) = h(x - D_t) + \alpha\bar{V}_{t+1}(x - D_t) + \alpha\nu_{t+1} = \bar{V}_t(x) + \alpha\nu_{t+1},$$

where the second equality follows from $\bar{y}_t(x) = x$ for $x \ge a_t^{m_t}$ shown in (29).

For any $x \geq 0$, we have $\bar{y}_t(x) \geq \max\{x, D_t\} \geq \max\{0, D_t\}$. Therefore, the definition of W_t in (27) yields

$$W_t(0) \leq c(\bar{y}_t(x)) + h(\bar{y}_t(x) - D_t) + \alpha \bar{W}_{t+1}(\bar{y}_t(x) - D_t).$$

Also note that

$$W_t(x) = c(\bar{y}_t(x) - x) + h(\bar{y}_t(x) - D_t) + \alpha \bar{W}_{t+1}(\bar{y}_t(x) - D_t).$$

We obtain

$$W_t(0) - W_t(x) \leq c(\bar{y}_t(x)) - c(\bar{y}_t(x) - a_t^j) \leq c(x),$$

where the second inequality follows from the concavity of c and $c(0) = 0$. Note that $\bar{W}_t(x) = \underline{W}_t^*(x) + \nu_t'$ for all $x \geq 0$, $\underline{W}_t^*(x) = W_t(x)$ for all $x \in \bar{\mathcal{B}}_t$, and $0 \in \bar{\mathcal{B}}_t$. Thus, for all $x \in \bar{\mathcal{B}}_t$, we have

$$\bar{W}_t(0) - \bar{W}_t(x) = \underline{W}_t^*(0) - \underline{W}_t^*(x) = W_t(0) - W_t(x) \leq c(x),$$

which proves (H5) for $\bar{W}_t$. $\square$