Affine Variational Inequalities on Normed Spaces

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Abstract: This paper studies infinite-dimensional affine variational inequalities (AVIs) on normed spaces. It is shown that infinite-dimensional quadratic programming problems and infinite-dimensional linear fractional vector optimization problems can be studied by using AVIs. We present two basic facts about infinite-dimensional AVIs: the Lagrange multiplier rule and the solution set decomposition.

Keywords: Infinite-dimensional affine variational inequality, Infinite-dimensional quadratic programming, Infinite-dimensional linear fractional vector optimization, Generalized polyhedral convex set, Solution set

1 Introduction

Affine variational inequalities (AVIs) in finite-dimensional Euclidean spaces, including *linear complementarity problems* (LCPs), have attracted a lot of attention from researchers worldwide during the last five decades. Solution existence and properties of the solution sets of LCPs (resp., of AVIs) have been discussed, e.g., in [1, Chapter 3] (resp., in [2] and [3, Chapter 6]). Solution stability and/or sensitivity of LCPs (resp., of AVIs) when the problem in question undergoes small perturbations have been studied, for example, in [1, Chapter 7] and [4–9] (resp., in [2], [3, Chapter 7], and [10–12]). The pioneering work of Dontchev and Rockafellar [13] and the subsequent studies in [14–22] show that the tools from variational analysis are very useful for investigating various stability properties of LCPs and AVIs . Meanwhile, global error bounds, local error bounds, and their applications to convergence analysis of iterative methods for solving LCPs and AVIs are presented in [23, 24]. We refer to the classical book of Cottle, Pang, and Stone [1, Chapters 4, 5] for a systematic presentation of the solution methods for LCPs.

The notion of *generalized polyhedral convex set* as defined by Bonnans and Shapiro [25, Definition 2.195] allows one to study not only generalized linear programming problems and quadratic programming problems under linear constraints on normed spaces [25, Chapter 2] but also infinite-dimensional AVIs on normed spaces.

This paper aims at establishing two basic facts about infinite-dimensional AVIs: the Lagrange multiplier rule and the solution set decomposition. We will see how the known results on finite-dimensional AVIs [2, 3] can be developed in the infinite-dimensional setting.

It is worthy to observe that infinite-dimensional quadratic programming problems and infinite-dimensional linear fractional vector optimization problems can be studied by using infinite-dimensional AVIs. But these notions only lead respectively to symmetric AVIs and anti-symmetric AVIs. Since a general AVI needs neither to be symmetric nor anti-symmetric, studies on infinite-dimensional AVIs are not confined to those serving as the first-order optimality conditions for quadratic programming and linear fractional vector optimization problems.

Note that very general necessary and sufficient optimality conditions for nonsmooth vector optimization problems can be found in the work by Jeyakumar and Yang [26]. For quadratic vector optimization problems under finitely many linear constraints on Banach spaces, necessary and sufficient optimality conditions have been obtained by Zheng and Yang [27, Proposition 4.1].

The rest of the paper has four sections. Quadratic programming and linear fractional vector optimization on normed spaces are discussed, respectively, in Sects. 2 and 3. Infinite-dimensional affine variational inequalities are introduced and studied in Sect. 4. Some concluding remarks are given in Sect. 5.

2 Quadratic Programming on Normed Spaces

Two fundamental optimization models leading to infinite-dimensional affine variational inequalities are investigated in this section and the next one. The first model is infinite-dimensional quadratic programming (see [25, Chapter 3]); the second one is infinite-dimensional linear fractional vector optimization (for the traditional finite-dimensional setting, see, e.g., [3, Chapter 8]).

From now on, if not otherwise stated, X is a normed space over the reals and X^* is the dual space of X. The value of x^* at $x \in X$ is denoted by $\langle x^*, x\rangle$.

Following [25, p. 193], we say that a function $\psi : X \times X \to \mathbb{R}$ is *bilinear* iff for any $x \in X$ the functions $\psi(., x)$ and $\psi(x, .)$ are linear. A bilinear function ψ is called *symmetric* iff $\psi(x, y) = \psi(y, x)$ for any $x, y \in X$. One says that a function $f : X \to \mathbb{R}$ is a *quadratic form* iff there is a symmetric bilinear function $\psi : X \times X \to \mathbb{R}$ such that $f(x) = \psi(x, x)$ for every $x \in X$. Since

$$\psi(x, y) = \frac{1}{4}\left(f(x + y) - f(x - y)\right), \tag{1}$$

the symmetric bilinear function ψ is uniquely defined via f. It is known [25, Proposition 3.71] that a quadratic form f is a convex function on X if and only if it is *non-negative*, i.e., $f(x) \geq 0$ for all $x \in X$.

The next proposition states an interesting but rather simple fact: *The Fréchet differentiability of a quadratic form is equivalent to the continuity of the corresponding bilinear function.* This kind of results should be known. But, having no related reference apart from [28, Example 2, pp. 36–37], we present here a detailed formulation of the fact and a proof to make our subsequent discussions as clear as possible.

Proposition 2.1. *Let ψ be the symmetric bilinear function corresponding to a quadratic form f defined on a normed space X. Then, the following properties are equivalent:*

(a) *f is Fréchet differentiable at a point $\bar{x} \in X$;*

(b) ψ *is continuous at* $(0,0) \in X \times X$*;*

(c) *There is a constant* $\beta \geq 0$ *such that* $|\psi(x,y)| \leq \beta||x||||y||$ *for all* $x, y \in X$*;*

(d) ψ *is continuous at any point* $(u,v) \in X \times X$*;*

(e) f *is Fréchet differentiable at any point* $u \in X$.

If one of these properties is valid, then the Fréchet derivative of f *at every point* $u \in X$ *is computed by the formula* $\nabla f(u) = 2\psi(u, .)$.

Proof Assuming the fulfillment of (a), we have that f is continuous at $\bar{x} \in X$. In order to obtain (b), given any vector sequences $\{x_k\}$ and $\{y_k\}$ in X converging to 0, we need to show that $\psi(x_k, y_k) \to 0$ as $k \to \infty$. For each k, from (1) it follows that

$$\psi(\bar{x} + x_k, y_k) = \frac{1}{4}\left[f(\bar{x} + x_k + y_k) - f(\bar{x} + x_k - y_k)\right]. \tag{2}$$

By the continuity of f at $\bar{x}$, one has $\lim\limits_{k\to\infty} f(\bar{x} + x_k + y_k) = f(\bar{x})$ and

$$\lim_{k\to\infty} f(\bar{x} + x_k - y_k) = f(\bar{x}).$$

So, (2) and the expression $\psi(\bar{x}+x_k, y_k) = \psi(\bar{x}, y_k)+\psi(x_k, y_k)$ yield the desired equality $\lim\limits_{k\to\infty} \psi(x_k, y_k) = 0$, if we can show that

$$\lim_{k\to\infty} \psi(\bar{x}, y_k) = 0. \tag{3}$$

We can deduce (3) from the differentiability of f at $\bar{x}$ as follows. Since f is differentiable at $\bar{x}$, it is Gâteaux differentiable at that point, i.e., for each $v \in X$ there directional derivative $f'(\bar{x}; v) := \lim\limits_{t\downarrow 0} \dfrac{f(\bar{x} + tv) - f(\bar{x})}{t}$ exists, and the functional $f'(\bar{x}; .) : X \to \mathbb{R}$ is linear and continuous. Therefore, from the relation

$$f(\bar{x} + tv) - f(\bar{x}) = 2t\psi(\bar{x}, v) + t^2\psi(v, v),$$

which is valid for any $t \in \mathbb{R}$, we deduce that $f'(\bar{x}; v) = 2\psi(\bar{x}, v)$ for all $v \in X$. In addition, the linear functional $\psi(\bar{x}, .)$ is continuous on X. In particular, $\psi(\bar{x}, .)$ is continuous at 0. This obviously yields (3). Thus, (a) implies (b).

Now, suppose that (b) is valid. Then there is $\delta > 0$ such that $|\psi(u, v)| \leq 1$ for all $u, v \in X$ with $||u|| \leq \delta$ and $||v|| \leq \delta$. Given any $x, y \in X \setminus \{0\}$, by the last property we have

$$|\psi(x, y)| = \delta^{-2}||x||||y|||\psi\left(\delta\frac{x}{||x||}, \delta\frac{y}{||y||}\right)| \leq \beta||x||||y||,$$

where $\beta := \delta^{-2}$. Hence (c) is valid.

Next, assuming that (c) is fulfilled with some $\beta \geq 0$ and $(u, v) \in X \times X$ is given arbitrarily, to get (d) we need to show ψ is continuous at (u, v). For any $h_1, h_2 \in X$, since

$$\begin{aligned}|\psi(u + h_1, v + h_2) - \psi(u, v)| &= |\psi(u, h_2) + \psi(h_1, v) + \psi(h_1, h_2)| \\ &\leq \beta(||u||||h_2|| + ||v||||h_1|| + ||h_1||||h_2||),\end{aligned}$$

the value $|\psi(u+h_1, v+h_2)-\psi(u,v)|$ is smaller than a given constant $\varepsilon > 0$, provided that the norms of h_1 and h_2 are small enough. This means that ψ is continuous at (u, v).

Suppose now that (d) is valid and $u \in X$ is an arbitrary vector. By (d), $\psi(u, .)$ is a continuous linear functional and ψ is continuous at $(0, 0)$. As shown above, the latter implies the existence of $\beta > 0$ such that $|\psi(x, y)| \le \beta\|x\|\|y\|$ for all $x, y \in X$. Since

$$f(u + h) = f(u) + 2\psi(u, h) + \psi(h, h)$$

and $\psi(h, h) = o(\|h\|)$ because $\psi(h, h) \le \beta\|h\|^2$, we can assert that f is Fréchet differentiable at u and $\nabla f(u) = 2\psi(u, .)$.

Property (e) obviously implies (a). Hence the proof of the equivalence between (a)-(e) is complete. The second assertion has been established by our preceding arguments. □

Let f be a quadratic form on X with ψ being the corresponding symmetric bilinear function. Suppose that f is Fréchet differentiable at a point in X. Then f is differentiable at any point in X by the above proposition. Moreover, there exists a constant $\beta > 0$ satisfying the condition in (c). Then, the formula $Mx := 2\psi(x, .)$ defines a bounded linear operator mapping X to X^*. Namely, $\|Mx\| \le 2\beta\|x\|$ for all $x \in X$ or, the same, $\|M\| \le 2\beta$. We have seen that each differentiable quadratic form on X uniquely generates a bounded linear operator $M : X \to X^*$. In the sequel, we say that M is the *bounded linear operator associated with* f. The symmetry of ψ is equivalent to the requirement that

$$\langle Mx, y\rangle = \langle My, x\rangle \quad \forall x, y \in X. \tag{4}$$

Since the latter may not hold for an arbitrarily given operator $M : X \to X^*$, formula $\psi(x, y) = \langle Mx, y\rangle$ may give a non-symmetric bilinear function ψ.

In connection with Proposition 2.1, one may ask: *On an infinite-dimensional normed space, is there any discontinuous quadratic form?* To solve this question in the affirmative we can use the following standard construction.

Suppose that X is a normed space of infinite dimension. Let $\{e_\tau : \tau \in T\}$ be an algebraic basis of X, $\|e_\tau\| = 1$ for all $\tau \in T$. Select a countable subset $T_0 = \{\tau_k : k \in \mathbb{N}\}$ of T, where $\mathbb{N}$ denotes the set of non-negative integers. Put $\varphi(e_{\tau_k}) = k$ for all $k \in \mathbb{N}$, and $\varphi(e_\tau) = \alpha_\tau$ for all $\tau \notin T_0$, where the numbers $\alpha_\tau \in \mathbb{R}$ can be chosen arbitrarily. Every vector $x \in X$ admits a unique representation of the form $x = \sum\limits_{\tau\in T} \mu_\tau e_\tau$ where, except for finitely many indexes τ, one has $\mu_\tau = 0$. The formula $\varphi(x) = \sum\limits_{\tau\in T} \mu_\tau\varphi(e_\tau)$ defines an unbounded linear functional $\varphi : X \to \mathbb{R}$. Setting $\psi(x, y) = \varphi(x)\varphi(y)$ we obtain a discontinuous symmetric bilinear function. Then, in accordance with Proposition 2.1, the quadratic function $f(x) := \psi(x, x) = \varphi(x)^2$ is not Fréchet differentiable at any point in X.

A set $K \subset X$ is said to be a *polyhedral convex* (see [25, p. 133]where the simpler adjective "polyhedral" is used) iff it can be represented as the intersection of a finite number of closed half spaces, i.e., there exist $x_i^* \in X^*$ and $\alpha_i \in \mathbb{R}$, $i = 1, \ldots, m$, such that

$$K = \{x \in X \ : \ \langle x_i^*, x\rangle \le \alpha_i, \ i = 1, \ldots, m\}. \tag{5}$$

One says [25, p. 133] that a set $C \subset X$ is *generalized polyhedral convex* (or, simpler, generalized polyhedral) iff it can be represented as the intersection of a polyhedral convex set and a closed affine subspace of X. That is, there exist a closed affine subspace $L \subset X$ and $x_i^* \in X^*$, $\alpha_i \in \mathbb{R}$, $i = 1, \ldots, m$, such that

$$K = \{x \in L \ : \ \langle x_i^*, x\rangle \le \alpha_i,\ i = 1, \ldots, m\}. \tag{6}$$

If $L = a + L_0$, where L_0 is a closed linear subspace of X, then we put

$$L^\perp = \{x^* \in X^* \ : \ \langle x^*, v\rangle = 0 \ \ \forall v \in L_0\}.$$

Thus $L^\perp$ is the annihilator of L_0 in X^*.

Recently, a representation formula for generalized convex polyhedra in locally convex Hausdorff topological vector spaces has been obtained in [30]. Various applications of that formula to infinite-dimensional linear vector optimization problems have been given in [30, 31].

Definition 2.1. If f is a Fréchet differentiable quadratic form on X, $q \in X^*$ is a given vector, and $K \subset X$ is a polyhedral convex set, then the problem

$$\min\{f(x) + \langle q, x\rangle \ : \ x \in K\} \tag{7}$$

is called a *quadratic programming problem*, or a *quadratic program*.

Definition 2.2. If f is a Fréchet differentiable quadratic form on X, $q \in X^*$ is a given vector, and $K \subset X$ is a generalized polyhedral convex set, then (7) is said to be a *generalized quadratic programming problem*, or a *generalized quadratic program*.

The following two theorems formulate the Fermat rule for quadratic programs and generalized quadratic programs. We refer to [3, Theorem 3.1 and Proposition 5.1] for standard proof arguments of the necessity part. The proof of the sufficiency part will be given only for the first theorem, but it also works for the the second one.

Theorem 2.1. *Consider a quadratic program of the form* (7) *and suppose that $M : X \to X^*$ is the bounded linear operator associated with f. If $\bar{x}$ is a local solution of the program, then*

$$\langle M\bar{x} + q, x - \bar{x}\rangle \ge 0 \quad \ \forall x \in K. \tag{8}$$

Conversely, if f is non-negative on X, then (8) *is sufficient for $\bar{x}$ to be a global solution of* (7).

Proof (Sufficiency) Let $x \in K$ be given arbitrarily. By the convexity and the differentiability of the function $\widetilde{f}(x) := f(x) + \langle q, x\rangle$, from (8) we have

$$0 \le \langle M\bar{x} + q, x - \bar{x}\rangle = \langle \nabla\widetilde{f}(\bar{x}), x - \bar{x}\rangle \le \widetilde{f}(x) - \widetilde{f}(\bar{x})$$

for all $x \in K$. This shows that $\bar{x}$ is a global solution of (7). □

Theorem 2.2. *Consider a generalized quadratic program of the form* (7)*, where K is a generalized polyhedral convex set, and suppose that $M : X \to X^*$ is the bounded linear operator associated with f. If $\bar{x}$ is a local solution of the program, then* (8) *is valid. Conversely, if f is non-negative on X, then* (8) *is sufficient for $\bar{x}$ to be a global solution of* (7).

The above first-order necessary optimality conditions for quadratic programs can be put in the form involving Lagrange multipliers. The infinite-dimensional Farkas lemma [32] can be used for proving the Lagrange multipliers rule for quadratic programs, while the Moreau-Rockafellar theorem [28, Theorem 0.3.3 on pp. 47–50, and Theorem 1 on p. 200] is an additional suitable tool for getting that rule for generalized quadratic programs.

Theorem 2.3. (See [27] and also [25, Proposition 3.118 and Theorem 3.130] for a more general result) *Suppose that f is a quadratic form on X and M is the linear operator associated with f. If $\bar{x}$ is a local solution of* (7)*, where K is given by* (5)*, then there exist Lagrange multipliers $\lambda_1 \geq 0, \ldots, \lambda_m \geq 0$ such that*

$$M\bar{x} + q + \sum_{i=1}^{m} \lambda_i x_i^* = 0 \tag{9}$$

and $\lambda_i(\langle x_i^*, \bar{x}\rangle - \alpha_i) = 0$ for $i = 1, ..., m$. If f is non-negative on X, then the converse is also valid.

Proof Let $\bar{x}$ be a local solution of (7) and K be given by (5). Condition (8) can be rewritten equivalently as

$$0 \in M\bar{x} + q + N(\bar{x}; K), \tag{10}$$

where $N(\bar{x}; K) := \{x^* \in X^* \ : \ \langle x^*, x - \bar{x}\rangle \leq 0 \ \text{ for all } \ x \in K\}$ is the normal cone to K at $\bar{x}$. Put $I = \{1, \ldots, m\}$. The active index set $I(\bar{x})$ of the feasible point $\bar{x}$ is defined by $I(\bar{x}) := \{i \in I \ : \ \langle x_i^*, x\rangle = \alpha_i\}$. The *pseudo-face* $\mathcal{F}$ of K corresponding to the index set $I(\bar{x})$ is given by

$$\mathcal{F} = \{x \in X \ : \ \langle x_i^*, x\rangle = \alpha_i \ \ \forall i \in I(\bar{x}), \ \ \langle x_i^*, x\rangle < \alpha_i \ \ \forall i \in I \setminus I(\bar{x})\}. \tag{11}$$

It is clear that, for any $u \in \mathcal{F}$, $N(u; K) = N(u; K(\bar{x}))$ with

$$K(\bar{x}) := \{x \in X \ : \ \langle x_i^*, x\rangle \leq \alpha_i \ \ \forall i \in I(\bar{x})\}.$$

Hence, $x^* \in N(u; K)$ if and only if $\langle x^*, v\rangle \leq 0$ for every $v \in X$ satisfying $\langle x^*, v\rangle \leq 0$ for all $i \in I(\bar{x})$. By the Farkas lemma [26, Lemma 1], the latter is valid if and only if there exist multipliers $\lambda_i \geq 0$, $i \in I(\bar{x})$, such that $x^* = \sum\limits_{i \in I(\bar{x})} \lambda_i x_i^*$. Thus

$$N(u; K) = \Big\{x^* = \sum_{i \in I(\bar{x})} \lambda_i x_i^* \ : \ \lambda_i \geq 0 \ \ \forall i \in I(\bar{x})\Big\}.$$

This means that (10) holds if and only if there exist multipliers $\lambda_i \geq 0$, $i \in I(\bar{x})$, such that

$$M\bar{x} + q + \sum_{i \in I(\bar{x})} \lambda_i x_i^* = 0.$$

Therefore, choosing $\lambda_i = 0$ for all $i \in I \setminus I(\bar{x})$, we get a set of Lagrange multipliers $\lambda_i \geq 0$, $i = 1, ..., m$, such that (9) is satisfied and $\lambda_i(\langle x_i^*, \bar{x}\rangle - \alpha_i) = 0$ for every $i \in I$. The second assertion follows from the sufficiency part of Theorem 2.1. □

Theorem 2.4. *Let f and M be as in the preceding theorem. If $\bar{x}$ is a local solution of the generalized quadratic program with K being given by* (6)*, then there exist $\lambda_1 \geq 0, \ldots, \lambda_m \geq 0$ such that*

$$M\bar{x} + q + \sum_{i=1}^{m} \lambda_i x_i^* \in L^{\perp} \tag{12}$$

and $\lambda_i(\langle x_i^*, \bar{x}\rangle - \alpha_i) = 0$ for $i = 1, ..., m$. If f is non-negative on X, then the converse holds true.

Proof Suppose that $\bar{x}$ is a local solution of the generalized quadratic program, where K is described by (6). According to Theorem 2.2, (8) is valid. We can derive (12) from the latter in the following way. Put

$$K_1 = \{x \in X \ : \ \langle x_i^*, x\rangle \le \alpha_i, \ i = 1, \ldots, m\}.$$

Since $K = L \cap K_1$ and L is a generalized polyhedral convex set, while K_1 is a polyhedral convex set, by the intersection rule in [29, Theorem 4.10] we have

$$N(\bar{x}; K) = N(\bar{x}; L) + N(\bar{x}; K_1). \tag{13}$$

It is easy to see that $N(\bar{x}; L) = L^{\perp}$. Note that the normal cone $N(\bar{x}; K_1)$ has been computed in the proof of Theorem 2.3. Namely,

$$N(\bar{x}; K_1) = \Big\{x^* = \sum_{i \in I(\bar{x})} \lambda_i x_i^* \ : \ \lambda_i \ge 0 \ \ \forall i \in I(\bar{x})\Big\},$$

where the index set $I(\bar{x})$ remains the same as in the preceding proof. Combining these facts with the inclusion (10), we find a set of Lagrange multipliers satisfying (12) such that $\lambda_i(\langle x_i^*, \bar{x}\rangle - \alpha_i) = 0$ for $i = 1, ..., m$. □

Remark 2.1. For generalized polyhedral convex sets of the form (6), the notion of pseudo-face is given similarly as that for polyhedral convex sets (see (11)). Namely, the pseudo-face corresponding to an index set $I_1 \subset I$ is given by

$$\mathcal{F}_1 = \{x \in L \ : \ \langle x_i^*, x\rangle = \alpha_i \ \ \forall i \in I_1, \ \ \langle x_i^*, x\rangle < \alpha_i \ \ \forall i \in I \setminus I_1\}.$$

It is worthy to observe that the normal cone $N(u; \mathcal{F}_1)$ is the same for all $u \in \mathcal{F}_1$. This fact is crucial for our subsequent decomposition of the solution set of an infinite-dimensional affine variational inequality in Sect. 4.

3 Linear Fractional Vector Optimization on Normed Spaces

Linear fractional vector optimization (LFVO) is a remarkable segment of the theory of vector optimization (see, e.g., Choo and Atkins [33, 34], Steuer [35], Malivert [36], Malivert and Popovici [37], Hoa et al. [38, 39], Yen and Yao [40], and Yen [41]). LFVO problems appear in finance and production management; see [35]. In a LFVO problem, any point satisfying the first-oder necessary optimality condition is a solution. Up to now, LFVO has been considered only in a finite-dimensional setting. Here, following the scheme described in [3, Chapter 8], we present basic facts about LFVO problems in an infinite-dimensional setting.

Let $f_j : X \to \mathbb{R} \quad (j = 1, 2, \cdots, p)$ be *linear fractional functions*, that is

$$f_j(x) = \frac{\langle u_j^*, x\rangle + \beta_j}{\langle v_j^*, x\rangle + \gamma_j}$$

for some $u_j^* \in X^*, v_j^* \in X^*, \beta_j \in \mathbb{R}$, and $\gamma_j \in \mathbb{R}$. Let $K \subset X$ be a convex set. Suppose that all the functions f_j are well defined on K. Then, without loss of generality we can assume that

$\langle v_j^*, x\rangle + \gamma_j > 0$ for all $j \in \{1, \cdots, p\}$ and for all $x \in K$. Note that the vector function $f(x) := (f_1(x), \ldots, f_p(x))$ maps K into $\mathbb{R}^p$.

In the sequel, the inequality $y \leq y'$ (resp., $y < y'$) for $y = (y_1, \ldots, y_p)$ and $y = (y'_1, \ldots, y'_p)$ from $\mathbb{R}^p$ means that $y_i \leq y'_i$ (resp., $y_i < y'_i$) for $i = 1, \ldots, p$.

Consider the vector optimization problem

$$\min\{f(x)\ :\ x \in K\}. \tag{14}$$

If one cannot find any $y \in K$ such that $f(y) \leq f(x)$ and $f(y) \neq f(x)$, then $x \in K$ is called an *efficient solution* of (14). If there is no $y \in K$ such that $f(y) < f(x)$, then one says that $x \in K$ is a *weakly efficient solution* of (14). The *efficient solution set* and the *weakly efficient solution set* of (14) are denoted, respectively, by E and E^w. If K is a polyhedral convex, then (VP) is called a *linear fractional vector optimization problem* (LFVOP for brevity). In the case where K is a generalized polyhedral convex set, (VP) is said to be a *generalized linear fractional vector optimization problem* (g-LFVOP for brevity).

The following lemma is an analogue of a result in a finite-dimensional setting (see [36, 42] and also [3, Lemma 8.1]).

Lemma 3.1. *Let* $\varphi(x) = \dfrac{\langle u^*, x\rangle + \beta}{\langle v^*, x\rangle + \gamma}$ *be a linear fractional function satisfying the condition* $\langle v^*, x\rangle + \gamma \neq 0$ *for every* $x \in K$. *Then, one has*

$$\varphi(y) - \varphi(x) = \frac{\langle v^*, x\rangle + \gamma}{\langle v^*, y\rangle + \gamma}\langle \nabla\varphi(x), y - x\rangle, \tag{15}$$

for any $x, y \in K$, *where* $\nabla\varphi(x)$ *denotes the Fréchet derivative of* φ *at* x.

Proof The fact that φ is Fréchet differentiable at every point $x \in X$ satisfying $\langle v^*, x\rangle + \gamma \neq 0$ is clear. Now, for any $x, y \in K$, we observe that

$$\begin{aligned}
&\langle \nabla\varphi(x), y - x\rangle \\
&= \lim_{t\downarrow 0} \frac{1}{t}\left[\varphi(x + t(y - x)) - \varphi(x)\right] \\
&= \lim_{t\downarrow 0} \frac{1}{t}\left[\frac{\langle u^*, x + t(y - x)\rangle + \beta}{\langle v^*, x + t(y - x)\rangle + \gamma} - \frac{\langle u^*, x\rangle + \beta}{\langle v^*, x\rangle + \gamma}\right] \\
&= \frac{\langle u^*, y - x\rangle(\langle v^*, x\rangle + \gamma) - \langle v^*, y - x\rangle(\langle u^*, x\rangle + \beta)}{(\langle v^*, x\rangle + \gamma)^2}.
\end{aligned} \tag{16}$$

Hence,

$$\begin{aligned}
&\frac{\langle v^*, x\rangle + \gamma}{\langle v^*, y\rangle + \gamma}\langle \nabla\varphi(x), y - x\rangle \\
&= \frac{\langle u^*, y - x\rangle(\langle v^*, x\rangle + \gamma) - \langle v^*, y - x\rangle(\langle u^*, x\rangle + \beta)}{(\langle v^*, x\rangle + \gamma)(\langle v^*, y\rangle + \gamma)} \\
&= \frac{(\langle u^*, y\rangle + \beta)(\langle v^*, x\rangle + \gamma) - (\langle u^*, x\rangle + \beta)(\langle v^*, y\rangle + \gamma)}{(\langle v^*, x\rangle + \gamma)(\langle v^*, y\rangle + \gamma)} \\
&= \frac{\langle u^*, y\rangle + \beta}{\langle v^*, y\rangle + \gamma} - \frac{\langle u^*, x\rangle + \beta}{\langle v^*, x\rangle + \gamma} \\
&= \varphi(y) - \varphi(x).
\end{aligned}$$

So the equality (15) is valid. $\square$

Remark 3.1. The above proof is valid for any pair $x, y \in X$ belonging to the effective domain of φ. Now, suppose that $\langle v^*, x\rangle + \gamma \neq 0$. Substituting $y = x + \tau v$, where $v \in X$ is an arbitrary vector and $\tau > 0$ is chosen as small as $\langle v^*, y\rangle + \gamma \neq 0$, into (16) yields

$$\langle \nabla\varphi(x), \tau v\rangle = \tau\frac{\langle u^*, v\rangle(\langle v^*, x\rangle + \gamma) - \langle v^*, v\rangle(\langle u^*, x\rangle + \beta)}{(\langle v^*, x\rangle + \gamma)^2}.$$

Hence,

$$\langle \nabla\varphi(x), v\rangle = \frac{\langle u^*, v\rangle(\langle v^*, x\rangle + \gamma) - \langle v^*, v\rangle(\langle u^*, x\rangle + \beta)}{(\langle v^*, x\rangle + \gamma)^2} \quad \forall v \in X.$$

We have thus obtained a formula for the Fréchet derivative of φ at x.

For any $x, y \in K$ with $x \neq y$, consider two points from the line segment $[x, y]$:

$$z_t = x + t(y - x), \quad z_{t'} = x + t'(y - x) \quad (t \in [0, 1],\ t' \in [0, t[).$$

By (15) we see that

(i) If $\langle \nabla\varphi(x), y - x\rangle > 0$, then $\varphi(z_{t'}) < \varphi(z_t)$ for every $t' \in [0, t[$;

(ii) If $\langle \nabla\varphi(x), y - x\rangle < 0$, then $\varphi(z_{t'}) > \varphi(z_t)$ for every $t' \in [0, t[$;

(iii) If $\langle \nabla\varphi(x), y - x\rangle = 0$, then $\varphi(z_{t'}) = \varphi(z_t)$ for every $t' \in [0, t[$.

This shows that φ is monotonic on every line segment or ray contained in K.

Put $\Sigma = \{\xi = (\xi_1, \dots, \xi_p) \in \mathbb{R}^p_+ \ :\ \sum_{j=1}^{p} \xi_j = 1\}$ and notice that the relative interior of Σ is given by

$$\mathrm{ri}\Sigma = \{\xi \in \mathbb{R}^p_+ \ :\ \sum_{j=1}^{p} \xi_j = 1,\ \xi_j > 0 \ \text{ for all } \ j\}.$$

From now on, if not otherwise stated, we consider the vector optimization problem (14) with K being given by (6). If one puts $I(x) = \{i \ :\ \langle x_i^*, x\rangle = \alpha_i\}$ for every $x \in K$, then necessary and sufficient optimality conditions for (14) can be formulated as follows.

Theorem 3.1. *Let $x \in K$. The following assertions hold:*

(a) *$x \in E$ if and only if there exists $\xi = (\xi_1, \dots, \xi_p) \in \mathrm{ri}\Sigma$ such that*

$$\Big\langle \sum_{j=1}^{m} \xi_j\Big[(\langle v_j^*, x\rangle + \gamma_j)u_j^* - (\langle u_j^*, x\rangle + \beta_j)v_j^*\Big], y - x\Big\rangle \geq 0, \quad \forall y \in K. \tag{17}$$

(b) *$x \in E^w$ if and only if there exists $\xi = (\xi_1, \dots, \xi_p) \in \Sigma$ such that* (17) *is fulfilled.*

(c) *If K is given by* (5)*, then* (17) *is satisfied if and only if there exist Lagrange multipliers $\lambda_1 \geq 0, \dots, \lambda_m \geq 0$ such that*

$$\sum_{j=1}^{m} \xi_j \Big[(\langle v_j^*, x\rangle + \gamma_j)u_j^* - (\langle u_j^*, x\rangle + \beta_j)v_j^*\Big] + \sum_{i\in I(x)} \lambda_i x_i^* = 0.$$

(d) *If K is given by* (6)*, then* (17) *is satisfied if and only if there exist Lagrange multipliers $\lambda_1 \geq 0, \ldots, \lambda_m \geq 0$ such that*

$$\sum_{j=1}^{m} \xi_j \left[(\langle v_j^*, x\rangle + \gamma_j) u_j^* - (\langle u_j^*, x\rangle + \beta_j) v_j^* \right] + \sum_{i \in I(x)} \lambda_i x_i^* \in L^{\perp}.$$

Proof (a) First, let us show that $x \in E$ if and only if

$$Q_x(K - x) \cap (-\mathbb{R}_+^p) = \{0\}, \tag{18}$$

where $Q_x : X \to \mathbb{R}^p$ is a bounded linear operator defined by the formal writing

$$Q_x = \begin{pmatrix} (\langle v_1^*, x\rangle + \gamma_1) u_1^* - (\langle u_1^*, x\rangle + \beta_1) v_1^* \\ \vdots \\ (\langle v_p^*, x\rangle + \gamma_p) u_p^* - (\langle u_p^*, x\rangle + \beta_p) v_p^* \end{pmatrix}$$

and $Q_x(K - x) = \{Q_x(y - x) \ : \ y \in K\}$, where

$$Q_x u := \begin{pmatrix} \langle (\langle v_1^*, x\rangle + \gamma_1) u_1^* - (\langle u_1^*, x\rangle + \beta_1) v_1^*, u \rangle \\ \vdots \\ \langle (\langle v_p^*, x\rangle + \gamma_p) u_p^* - (\langle u_p^*, x\rangle + \beta_p) v_p^*, u \rangle \end{pmatrix}$$

for every $u \in X$. Indeed, $x \notin E$ if and only if there exist $y \in K$ and j_0 with

$$f_j(y) \leq f_j(x) \ \ \forall j \in \{1, \ldots, p\}, \quad f_{j_0}(y) < f_{j_0}(x).$$

Applying formula (15) to the functions f_j, we can equivalently rewrite these conditions as

$$\left\langle \nabla f_j(x), y - x \right\rangle \leq 0 \ \ \forall j \in \{1, \ldots, p\}, \quad \left\langle \nabla f_{j_0}(x), y - x \right\rangle < 0. \tag{19}$$

Since

$$\left\langle \nabla f_j(x), y - x \right\rangle = \frac{\langle (\langle v_j^*, x\rangle + \gamma_j) u_j^* - (\langle u_j^*, x\rangle + \beta_j) v_j^*, y - x \rangle}{(\langle v_j^*, x\rangle + \gamma_j)^2}$$

by (16), the inequalities system (19) can be transformed to

$$\begin{cases} \langle (\langle v_j^*, x\rangle + \gamma_j) u_j^* - (\langle u_j^*, x\rangle + \beta_j) v_j^*, y - x \rangle \leq 0 \ \ \forall j \in \{1, \ldots, p\}, \\ \left\langle (\langle v_{j_0}^*, x\rangle + \gamma_{j_0}) u_{j_0}^* - (\langle u_{j_0}^*, x\rangle + \beta_{j_0}) v_{j_0}^*, y - x \right\rangle < 0. \end{cases}$$

Hence, $x \notin E$ iff there exists $y \in K$ with $Q_x(y - x) \in -\mathbb{R}_+^p \setminus \{0\}$. This establishes the criterion (18) for a point $x \in K$ to belong to E.

Next, since K is a generalized polyhedral convex set given by (6). Fixing any $x \in K$, one has

$$K - x = \{z \in L - x \ : \ \langle x_i^*, z\rangle \leq \alpha_i - \langle x_i^*, x\rangle, \ i = 1, \ldots, m\}. \tag{20}$$

As $L-x$ is a closed linear subspace of X, formula (20) shows that $K-x$ is a generalized polyhedral convex set. Therefore, invoking [30, Theorem 2.7] we find $u_1,\dots,u_k \in K$, $v_1,\dots,v_\ell \in X$, and a closed linear subspace $X_0 \subset X$ such that

$$\begin{aligned} K-x = \Big\{ \sum_{i=1}^{k} \theta_i u_i + \sum_{j=1}^{\ell} \mu_j v_j \ : \theta_i \geq 0,\ \forall i = 1,\dots,k,& \\ \sum_{i=1}^{k} \theta_i = 1,\ \mu_j \geq 0,\ \forall j = 1,\dots,\ell \Big\} + X_0.& \end{aligned} \tag{21}$$

From (21) and the linearity of $Q-x$ it follows that

$$\begin{aligned} &Q_x(K-x) \\ &= \Big\{ \sum_{i=1}^{k} \theta_i Q_x(u_i) + \sum_{j=1}^{\ell} \mu_j Q_x(v_j) \ : \ \theta_i \geq 0,\ \forall i = 1,\dots,k, \\ &\qquad \sum_{i=1}^{k} \theta_i = 1,\ \mu_j \geq 0,\ \forall j = 1,\dots,\ell \Big\} + Q_x(X_0). \end{aligned}$$

Using this formula and [43, Theorem 19.1] we see at once that $D := Q_x(K-x)$ is a polyhedral convex set in $\mathbb{R}^p$. Then, by [43, Corollary 19.7.1] we can assert that $\mathcal{K}_D := \{tw \ : \ t \geq 0,\ w \in D\}$ is a polyhedral convex cone. In particular, $\mathcal{K}_D$ is a closed convex cone. It is clear that (18) yields $\mathcal{K}_D \cap (-\mathbb{R}^p_+) = \{0\}$. Setting

$$\mathcal{K}_D^+ = \{z \in \mathbb{R}^p \ : \ \langle z, v\rangle \geq 0 \quad \forall v \in \mathcal{K}_D\},$$

we have $\mathcal{K}_D^+ \cap \mathrm{int}\mathbb{R}^p_+ \neq \emptyset$. Indeed, if $\mathcal{K}_D^+ \cap \mathrm{int}\mathbb{R}^p_+ = \emptyset$ then, by the separation theorem, there exists $\xi \in \mathbb{R}^p \setminus \{0\}$ such that

$$\langle \xi, w\rangle \leq 0 \leq \langle \xi, z\rangle \quad \forall w \in \mathrm{int}\mathbb{R}^p_+,\ \forall z \in \mathcal{K}_D^+.$$

This yields $\xi \in -\mathbb{R}^p_+$ and $\xi \in (\mathcal{K}_D^+)^+ = \mathcal{K}_D$. So we get $\xi \in \mathcal{K}_D \cap (-\mathbb{R}^p_+) = \{0\}$, a contradiction.

Select any $\widetilde{\xi} \in \mathcal{K}_D^+ \cap \mathrm{int}\mathbb{R}^p_+$ and put $\xi = \widetilde{\xi}/(\widetilde{\xi}_1 + \cdots + \widetilde{\xi}_m)$. It is clear that $\xi \in \mathcal{K}_D^+ \cap \mathrm{ri}\Sigma$. Since $\langle \xi, v\rangle \geq 0$ for every $v \in \mathcal{K}_D$, we have $\langle \xi, Q_x(y-x)\rangle \geq 0$ for every $x \in K$. Hence (17) is valid.

(b) Arguing similarly as above, we can show that $x \in E^w$ if and only if

$$Q_x(K-x) \cap (-\mathrm{int}\mathbb{R}^p_+) = \emptyset.$$

Then, by the separation theorem we can find a multiplier $\lambda = (\lambda_1,\dots,\lambda_m) \in \Sigma$ satisfying (17).

For getting (c) and (d) from (a) and (b), respectively, it suffices to apply the infinite-dimensional Farkas lemma as it was done in the proofs of Theorems 2.3 and 2.4. $\square$

Remark 3.2. Theorem 3.1 is an extension of the corresponding results of Choo and Atkins [33] and Malivert [36] to the general normed space setting. The above proof is based on the proof scheme in [3] and a new result of Luan and Yen [30] on generalized polyhedral convex sets.

4 Infinite-Dimensional AVIs

In this section, we first define the notions of affine variational inequality and generalized affine variational inequality, and then we clarify the decomposition structure of their solution sets. Connections of the new concepts with the two classes of optimization problems considered in the preceding sections will be discussed in detail.

Definition 4.1. If $M : X \to X^*$ is a bounded linear operator, $q \in X^*$ a vector, and $K \subset X$ a polyhedral convex set, then the problem of finding a vector $\bar{x} \in K$ satisfying

$$\langle M\bar{x} + q, x - \bar{x}\rangle \geq 0 \quad \forall x \in K \tag{22}$$

is called the *affine variational inequality* (AVI for brevity) defined by the data set $\{M, q, K\}$.

Definition 4.2. If $M : X \to X^*$ is a bounded linear operator, $q \in X^*$ a vector, and $K \subset X$ a generalized polyhedral convex set, then the problem of finding a vector $\bar{x} \in K$ satisfying (22) is called the *generalized affine variational inequality* (g-AVI for brevity) defined by the data set $\{M, q, K\}$.

Remark 4.1. The above notions of AVI and g-AVI are just, in fact, two special cases of the concept of *variational inequality* studied in many books and papers. For instance, in [44, Chapter 3], instead of our affine operator $x \mapsto Mx + q$ one considers a mapping $F : K \to X^*$ and instead of our polyhedral convex set $K \subset X$ (resp., generalized polyhedral convex set $K \subset X$) one considers any closed convex set $K \subset X$. Note that, here we do not require the reflexiveness of X and monotonicity of the operator $x \mapsto Mx + q$ on K.

Remark 4.2. In the case $X = \mathbb{R}^n$, the definition of g-AVI reduces to that of AVI. This model has been studied intensively by many authors; see, e.g., [2, 3] and the references therein. If, in addition, K is the non-negative octant $\mathbb{R}^n_+$ in $\mathbb{R}^n$, then (22) is a *linear complementarity problem* [1], which can be rewritten as

$$M\bar{x} + q \geq 0, \quad \bar{x} \geq 0, \quad \langle M\bar{x} + q, \bar{x}\rangle = 0. \tag{23}$$

Different studies on the problem (23) have been mentioned in Sect. 1.

Definition 4.3. If $\langle Mx, y\rangle = \langle My, x\rangle$ for every pair $(x, y) \in X \times X$, then one says that the generalized affine variational inequality (4.1), with K being a generalized polyhedral convex set, is *symmetric*.

Based on the important research of Luo and Tseng [24], we think that *symmetric AVIs and g-AVIs deserve a special attention*.

Proposition 4.1. *The generalized affine variational inequality corresponding to a generalized quadratic programming problem is symmetric.*

Proof The assertion is immediate from Theorem 2.2 and formula (4). □

Definition 4.4. If $\langle Mx, y\rangle = -\langle My, x\rangle$ for every pair $(x, y) \in X \times X$, then one says that the generalized affine variational inequality (4.1), with K being a generalized polyhedral convex set, is *anti-symmetric*.

In [45], it has been noted that linear fractional vector optimization problems lead to anti-symmetric (or skew-symmetric) affine variational inequalities. Hence, anti-symmetric AVIs (resp., anti-symmetric g-AVIs) also form interesting class of AVIs (resp., of g-AVIs).

Proposition 4.2. *The generalized affine variational inequalities corresponding to a generalized linear fractional vector optimization problem are anti-symmetric.*

Proof The assertion can be deduced from Theorem 3.1 as follows. For each $\xi \in \Sigma$, denote the bounded linear operator from X to X^* corresponding to the affine variational inequality (17) by M_ξ and the solution set of the latter by $\mathrm{Sol(VI}_\xi)$. Then one has

$$M_\xi x = \sum_{j=1}^{m} \xi_j \left[\langle v_j^*, x\rangle u_j^* - \langle u_j^*, x\rangle v_j^*\right] \quad \forall x \in X.$$

So, for every $x \in X$ one can notice that

$$\langle M_\xi x, x\rangle = \sum_{j=1}^{m} \xi_j \left[\langle v_j^*, x\rangle\langle u_j^*, x\rangle - \langle u_j^*, x\rangle\langle v_j^*, x\rangle\right] = 0.$$

Given any pair $(y, z) \in X \times X$, combing the last property with the evident equality

$$\langle M_\xi(y - z), y - z\rangle = \langle M_\xi y, y\rangle - \langle M_\xi y, z\rangle - \langle M_\xi z, y\rangle + \langle M_\xi z, z\rangle,$$

one obtains $\langle M_\xi y, z\rangle = -\langle M_\xi z, y\rangle$ for every pair $(x, y) \in X \times X$. The anti-symmetric property of M_ξ has been established for every $\xi \in \Sigma$. By assertion (a) of Theorem 3.1, $E = \bigcup\limits_{\xi \in \mathrm{ri}\Sigma} \mathrm{Sol(VI}_\xi)$. Hence the efficient solution set of (14) is the union of the solution sets of a family of anti-symmetric AVIs. Similarly, the assertion (b) of Theorem 3.1 implies that $E^w = \bigcup\limits_{\xi \in \Sigma} \mathrm{Sol(VI}_\xi)$. Thus, the weakly efficient solution set of (14) is also the union of the solution sets of a family of anti-symmetric AVIs. □

Theorem 4.1. *A vector $\bar{x} \in K$ is a solution the generalized affine variational inequality problem* (22) *with K being given by* (6) *if and only if there exist Lagrange multipliers $\lambda_1 \geq 0, \ldots, \lambda_m \geq 0$ such that*

$$M\bar{x} + q + \sum_{i=1}^{m} \lambda_i x_i^* \in L^\perp \tag{24}$$

and $\lambda_i(\langle x_i^*, \bar{x}\rangle - \alpha_i) = 0$ for $i = 1, ..., m$.

Proof The necessity can be obtained by repeating the arguments given in the proof of Theorem 2.4. Let us prove the sufficiency by direct verification. Suppose that $\bar{x} \in K$ and there exist Lagrange multipliers $\lambda_1 \geq 0, \ldots, \lambda_m \geq 0$ satisfying (24) with $\lambda_i(\langle x_i^*, \bar{x}\rangle - \alpha_i) = 0$ for $i = 1, ..., m$.

To show that $\bar{x}$ is a solution the g-AVI problem (22) with K being given by (6), we take any $x \in K$ and use (24) to represent

$$M\bar{x} + q = -\sum_{i=1}^{m} \lambda_i x_i^* + v^*$$

for some $v^* \in L^{\perp}$. Since both vectors x and $\bar{x}$ belong to L, one has $x - \bar{x} \in L_0$. Therefore,

$$\begin{aligned}\langle M\bar{x} + q, x - \bar{x}\rangle &= \Big\langle -\sum_{i=1}^{m} \lambda_i x_i^* + v^*, x - \bar{x}\Big\rangle \\ &= -\Big\langle \sum_{i=1}^{m} \lambda_i x_i^*, x - \bar{x}\Big\rangle \\ &= -\sum_{i\in I(\bar{x})}^{m} \lambda_i[\langle x_i^*, x\rangle - \alpha_i] \geq 0.\end{aligned}$$

We have thus shown that $\bar{x}$ is a solution of (22). □

Remark 4.3. Theorem 4.1 extends a result of Gowda and Pang [2, p. 834] (see also [10, Theorem 5.3]) on finite-dimensional AVIs to the general normed spaces setting.

Theorem 4.2. *The solution set of the generalized affine variational inequality problem* (22) *with K being given by* (6) *is the union of finitely many generalized polyhedral convex sets.*

Proof First, let us show that the solution set of (22), which is denoted by $\mathcal{S}$, is closed. From the Definition 4.2 it is clear that $\bar{x} \in \mathcal{S}$ if and only if $\bar{x} \in \bigcap_{x\in K} K_x$, where $K_x := \{u \in K : \langle Mu + q, x - u\rangle \geq 0\}$. For each $x \in K$, since the function $u \mapsto \langle Mu + q, x - u\rangle$ is continuous, we see that K_x is a closed set. As $\mathcal{S} = \bigcap_{x\in K} K_x$, this implies that $\mathcal{S}$ is closed.

Now, put $I = \{1, \dots, m\}$. By Theorem 4.1, $x \in X$ is a solution of (22) if and only if there exist $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m$ such that

$$\begin{cases} Mx + q + \sum_{i=1}^{m} \lambda_i x_i^* \in L^{\perp}, \\ \langle x_i^*, x\rangle \leq \alpha_i \ \ \forall i \in I, \ \ \lambda_i \geq 0 \ \ \forall i \in I, \\ \lambda_i(\langle x_i^*, x\rangle - \alpha_i) = 0 \ \ \forall i \in I. \end{cases} \tag{25}$$

Given a point $x \in \mathcal{S}$, we put $I_0 = \{i \in I : \langle x_i^*, x\rangle = \alpha_i\}$ and

$$I_1 = \{i \in I : \langle x_i^*, x\rangle < \alpha_i\}.$$

Then $I = I_0 \cup I_1$ and $I_0 \cap I_1 = \emptyset$. Let $\lambda \in \mathbb{R}^m$ be a Lagrange multiplier corresponding to x. From the last line of (25) it follows that $\lambda_i = 0$ for every $i \in I_1$. So the pair (x, λ) satisfies the system

$$\begin{cases} Mx + q + \sum_{i=1}^{m} \lambda_i x_i^* \in L^{\perp}, \\ \langle x_i^*, x\rangle = \alpha_i \ \ \forall i \in I_0, \ \ \lambda_i \geq 0 \ \ \forall i \in I_0, \\ \langle x_i^*, x\rangle \leq \alpha_i \ \ \forall i \in I_1, \ \ \lambda_i = 0 \ \ \forall i \in I_1. \end{cases} \tag{26}$$

It is clear that the formula $\Psi(x,\lambda) = Mx + \sum_{i=1}^{m} \lambda_i x_i^*$ defines a bounded linear operator $\Psi : X \times \mathbb{R}^m \to X^*$. As $L^\perp$ is a closed linear subspace of X^*, the set

$$\mathcal{L} := (\Psi(.) + q)^{-1}(L^\perp) = \Big\{(x,\lambda) \in X \times \mathbb{R}^m \ : \ Mx + q + \sum_{i=1}^{m} \lambda_i x_i^* \in L^\perp\Big\}$$

is a closed affine subspace of $X \times \mathbb{R}^m$. Hence, denoting by Q_{I_0} the set of all (x,λ) satisfying (26), we observe that Q_{I_0} is the solution set of the system

$$\begin{cases} (x,\lambda) \in \mathcal{L}, \\ \langle x_i^*, x\rangle = \alpha_i \ \ \forall i \in I_0, \ \ \lambda_i \geq 0 \ \ \forall i \in I_0, \\ \langle x_i^*, x\rangle \leq \alpha_i \ \ \forall i \in I_1, \ \ \lambda_i = 0 \ \ \forall i \in I_1. \end{cases}$$

This implies that Q_{I_0} is a polyhedral convex set in $Z := X \times \mathbb{R}^m$. Hence, according to [30, Theorem 2.7], there exist we $z_1, \ldots, z_k \in Q_{I_0}$, $w_1, \ldots, w_\ell \in Z$, and a closed linear subspace $Z_0 \subset Z$ such that

$$\begin{aligned} Q_{I_0} = \Big\{ \sum_{i=1}^{k} \theta_i z_i + \sum_{j=1}^{\ell} \mu_j w_j \ : \theta_i \geq 0, \ \forall i = 1, \ldots, k, \\ \sum_{i=1}^{k} \theta_i = 1, \ \mu_j \geq 0, \ \forall j = 1, \ldots, \ell \Big\} + Z_0. \end{aligned} \tag{27}$$

In the above notation, we have

$$\mathcal{S} = \bigcup_{I_0 \subset I} \mathrm{Pr}_X(Q_{I_0}), \tag{28}$$

where $\mathrm{Pr}_X(x,\lambda) := x$ is the natural projection of $Z = X \times \mathbb{R}^m$ onto X. Setting $S_{I_0} = \mathrm{Pr}_X(Q_{I_0})$, we deduce from (27) that

$$\begin{aligned} S_{I_0} = \Big\{ \sum_{i=1}^{k} \theta_i \mathrm{Pr}_X(z_i) + \sum_{j=1}^{\ell} \mu_j \mathrm{Pr}_X(w_j) \ : \theta_i \geq 0, \ \forall i = 1, \ldots, k, \\ \sum_{i=1}^{k} \theta_i = 1, \ \mu_j \geq 0, \ \forall j = 1, \ldots, \ell \Big\} \\ + \mathrm{Pr}_X(Z_0). \end{aligned} \tag{29}$$

Since S_{I_0} is a subset of $\mathcal{S}$ and the latter is closed, from (28) and (29) it follows that

$$\mathcal{S} = \bigcup_{I_0 \subset I} \overline{S}_{I_0}, \tag{30}$$

where

$$\begin{aligned} \overline{S}_{I_0} := \Big\{ \sum_{i=1}^{k} \theta_i \mathrm{Pr}_X(z_i) + \sum_{j=1}^{\ell} \mu_j \mathrm{Pr}_X(w_j) \ : \theta_i \geq 0, \ \forall i = 1, \ldots, k, \\ \sum_{i=1}^{k} \theta_i = 1, \ \mu_j \geq 0, \ \forall j = 1, \ldots, \ell \Big\} \\ + \mathrm{cl}(\mathrm{Pr}_X(Z_0)) \end{aligned} \tag{31}$$

with $\mathrm{cl}(\mathrm{Pr}_X(Z_0))$ standing for the closure of the linear subspace $\mathrm{Pr}_X(Z_0)$ of X. Using again the characterization of generalized polyhedral convex sets in [30, Theorem 2.7], by (31) we see that $\overline{S}_{I_0}$ is a generalized polyhedral convex set. Hence, (30) shows that $\mathcal{S}$ is the union of finitely many generalized polyhedral convex sets. $\square$

Remark 4.4. Theorem 4.2 extends a well-known result about the decomposed structure of the solution sets of finite-dimensional AVIs (see, e.g., [3, Theorem 5.4]).

5 Conclusions

We have established the Lagrange multiplier rule and the solution set decomposition for infinite-dimensional AVIs on normed spaces and showed that the latter provide an effective tool for studying infinite-dimensional quadratic programs and infinite-dimensional linear fractional vector optimization problems.

Solution existence theorems, solution stability, and local error bounds for infinite-dimensional AVIs, similar to those which have been obtained in [2, 3, 24] for finite-dimensional AVIs, deserve further investigations.

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