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Design optimization of a viscoelastic dynamic vibration absorber using a modified fixed-points theory

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A viscoelastic dynamic vibration absorber (VDVA) is proposed for suppressing infrasonic vibrations of heavy structures because the traditional dynamic vibration absorber equipped with a viscous damper is not effective in suppressing low frequency vibrations. The proposed VDVA has an elastic spring and a viscoelastic damper with frequency dependent modulus and damping properties. The standard fixed-points theory cannot be applied to derive the optimum design parameters of the VDVA because both its stiffness and damping are frequency dependent. A modified fixed-points theory is therefore proposed to solve this problem. H_∞ design optimization of the proposed VDVA have been derived for the minimization of resonant vibration amplitude of a single degree-of-freedom system excited by harmonic forces or due to ground motions. The stiffness and damping of the proposed VDVA can be decoupled such that both of these two properties of the absorber can be tuned independently to their optimal values by following a specified procedure. The proposed VDVA with optimized design is tested numerically using two real commercial viscoelastic damping materials. It is found that the proposed viscoelastic absorber can provide much stronger vibration reduction effect than the conventional VDVA without the elastic spring.
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I. INTRODUCTION

Low frequency infrasonic vibrations at or below 20 Hz can come from many sources such as seismic activity, highway traffic, airports, HVAC systems of buildings, etc. These vibrations could travel over long distances from numerous constant sources through buildings and structures into the environment, and they produce problems to sensitive equipment[1] while they pass by. Slowly shaking motions of structures and buildings matching the natural frequency of human internal organs in the range of 5–8 Hz may cause human discomfort and health problems.[2,3] Since the inherent damping in many flexible structures is very low, vibration tends to continue for a long period of time unless steps are taken to absorb its energy.

Mounting a dynamic vibration absorber with proper design and location onto an oscillating structure can provide a robust solution to suppress its vibration and noise radiation. The traditional dynamic vibration absorber as illustrated in Fig. 1(a) is a passive device for reducing vibration of the primary system. It uses a viscous damping element to damp down the vibration of the primary system mostly at the pre-tuned frequency. This model in Fig. 1(a) is denoted as model A in this paper. Many research reports can be found in literature about the derivations of the optimum parameters of the traditional dynamic vibration absorber (DVA). Ormondroyd and Den Hartog[4] showed that the DVA has an optimum damping value for the minimization of the resonant amplitude response of single degree-of-freedom (SDOF) system. The optimum damping and optimum tuning frequency were derived by Brock[5] and Hahnkamn,[6] respectively. These formulations can be deduced using the fixed-points theory[7] which stated that there are two invariant points in the frequency spectra of the primary mass regardless of the amount of the viscous damping. The optimal frequency and damping ratios of the traditional DVA for the undamped SDOF primary system based on the fixed-points theory were very good approximation to the exact values derived by Nishihara and Asami.[8] Variant designs of DVA using viscous damper were considered and optimized using the fixed-points theory[9–13] for minimizing resonant vibrations of single and multiple degree-of-freedom systems. However, the commonly used viscous damper is not effective for damping low frequency infrasonic vibrations because the damping force is proportional to the vibration frequency and it is relatively small at low frequency. A very big viscous damper would be required to generate the damping force required in practice if the primary vibrating structure is heavy. On the other hand, the damping force of a viscoelastic damper is proportional to the vibration displacement instead and therefore it may be a better alternative of the viscous damper for suppressing low frequency vibrations.

The simplest way to construct a viscoelastic dynamic vibration absorber (VDVA) is to use a viscoelastic damper to provide both the resilient and the energy dissipating functions as illustrated in Fig. 1(b). This model in Fig. 1(b) is called model B in this paper. However, the complex modulus of viscoelastic material depends on both the vibration frequency and temperature of the material. These features of the viscoelastic materials cause the coupling of the stiffness and damping in the design of the VDVA.

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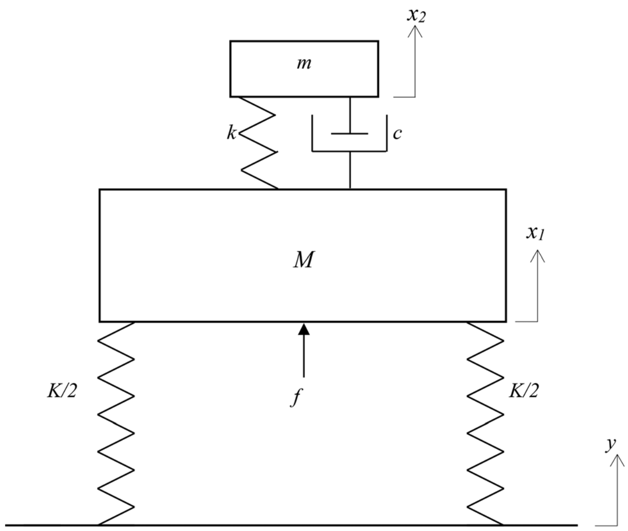

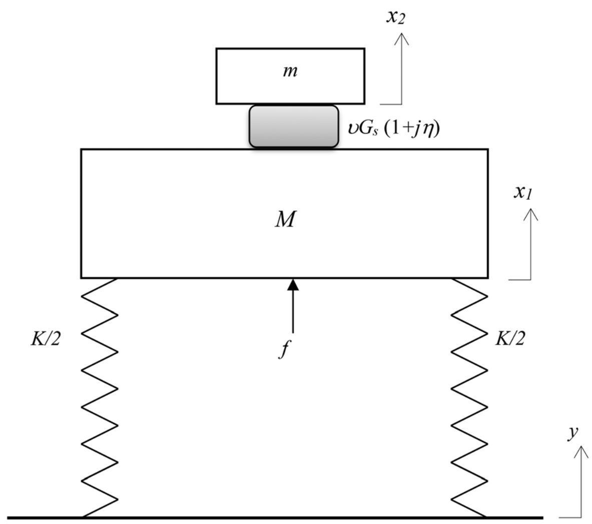

FIG. 1. A damped dynamic vibration absorber as an auxiliary mass-spring-damper system attached to a SDOF system (a) model A: traditional design of the absorber using a viscous damper. (b) Model B: dynamic vibration absorber using a viscoelastic damper for suppressing the vibration of the mass *M* excited by a harmonic force *f* or due to ground motion *y*.

Analysis of VDVA often requires an analytical feature of the rheological characteristics of the viscoelastic damper. There are different approaches to the analytical modeling of rheological behavior of linear viscoelastic materials. The classical approach is the mechanical model comprising a combination of linear springs and dashpots.[14–17] The mathematical model is expressed in form of Prony series known as the general Maxwell or Kelvin model. The fractional derivative model[18–21] has been applied to model the parametric models of viscoelastic materials by many researchers. Fractional derivatives can model the broadband behavior of viscoelastic materials with less parameters compared to the Prony series. In practice, the complex modulus of the damper material can be employed to the absorber structure whatever the configuration of the absorber is and the viscoelastic loss factor of the absorber can be obtained by direct measurement with a suitable experiment.[22]

Al-Rumaih[23] had designed a VDVA using a commercial viscoelastic damping material and he demonstrated the effectiveness of his design in suppressing resonant vibration of heavy structures. Espíndola *et al.*[24,25] proposed an optimal design theory based on the concept of equivalent generalized mass and damping parameters to optimize the parameters of one or multiple viscoelastic damping absorbers. Their method requires a numerical search of the anti-resonance frequency of the absorber in order to minimize the maximum vibration amplitude of the primary system. The genetic algorithm was explored by Xu *et al.*[26] to optimize parameters of multi-dimensional earthquake device composed of viscoelastic dampers. A direct performance-based design method was developed by Guo and Christopoulos[27] to study the seismic design of structures equipped with non-linear hysteretic material. However, there is no analytical method to derive an optimized design of the VDVA in order to minimize the resonant vibration of the primary vibrating system.

In view of the coupling problem of the stiffness and damping of the traditional design of VDVA, a new design of the absorber which allows independent tuning of its stiffness and damping is proposed. It is named as model C in the following. The standard fixed-points theory cannot be applied to the VDVA because both its stiffness and damping are frequency dependent. A modified fixed-points theory is proposed for the derivation of the optimal parameters of the proposed VDVA for minimizing the resonant vibration amplitude of SDOF system. To the knowledge of the authors, this is the first research report on the H_∞ design optimization of a VDVA leading to analytical solutions of the optimum parameters of this type of vibration absorber.

The fixed-points theory used in the optimization of the traditional DVA (model A in the following) is briefly reviewed in Sec. II. The optimization method of Espíndola *et al.*[25] for VDVA applied to SDOF primary vibrating system (model B) is reviewed briefly in Sec. III. The mathematical model of the proposed VDVA (model C) is formulated in Sec. IV. Approximate "fixed points" are found in the frequency response spectra of the primary mass of the proposed VDVA. A modified fixed-points theory is proposed and applied to derive the optimal design parameters of the proposed VDVA for minimizing the resonant vibration amplitude of the primary vibrating system. The effectiveness of the proposed VDVA optimal design method is analyzed with two examples using real commercial viscoelastic damper materials.

II. THE TRADITIONAL DYNAMIC VIBRATION ABSORBER (MODEL A)

Figure 1(a) shows the schematic design of the traditional dynamic vibration absorber comprising a lumped mass, an elastic spring and a viscous damper. This vibration absorber denoted by model A is attached to a single degree-of-freedom undamped primary system. Vibration of mass M is excited by harmonic force $f = F\sin\omega t$ or due to ground motion $y = Y\sin\omega t$. The amplitude ratio $|H_A(\lambda)|$ can be derived as[5,7]

$$|H_A(\lambda)| = \left|\frac{X_1}{F/K}\right|_A = \left|\frac{X_1}{Y}\right|_A = \sqrt{\frac{(\gamma^2-\lambda^2)^2+(2\gamma\lambda\zeta)^2}{\left[(1-\lambda^2)(\gamma^2-\lambda^2)-\mu\gamma^2\lambda^2\right]^2+\left[2\gamma\lambda\zeta(1-\lambda^2-\mu\lambda^2)\right]^2}}, \tag{1}$$

where $\mu = m/M$, $\omega_a = \sqrt{k/m}$, $\omega_n = \sqrt{K/M}$, $\gamma = \omega_a/\omega_n$, $\lambda = \omega/\omega_n$, $\zeta = c/2\sqrt{mk}$, and X_1 is the vibration amplitude of the primary mass M.

The objective function of the H_∞ optimization is to minimize the maximum value of $|H_A(\lambda)|$. It may be expressed mathematically as

$$\max(|H_A(\lambda, \gamma_{\text{opt\_}A}, \zeta_{\text{opt\_}A})|) = \min(\max_{\gamma,\zeta} |H_A(\lambda)|). \tag{2}$$

The procedure to derive the optimum parameters of this absorber is based on the fixed-points theory by Den Hartog.[7] The optimum tuning frequency and damping ratios of this absorber can be written, respectively, as

$$\gamma_{\text{opt\_}A} = \frac{k/m}{K/M} = \frac{1}{1+\mu} \tag{3}$$

and

$$\zeta_{\text{opt\_}A} = \frac{c}{2\sqrt{mk}} = \sqrt{\frac{3\mu}{8(1+\mu)}}. \tag{4}$$

The approximate maximum amplitude ratio of the primary mass M derived by Den Hartog[7] is written as

$$|H_A(\lambda)|_{\max} = \sqrt{\frac{2+\mu}{\mu}}. \tag{5}$$

III. DYNAMIC VIBRATION ABSORBER EQUIPPED WITH A VISCOELASTIC DAMPER (MODEL B)

The viscoelastic vibration absorber denoted by model B has a lumped mass attached to the primary structure through a viscoelastic damper, as shown in Fig. 1(b). The stiffness of the viscoelastic damper is modeled by the complex stiffness written as

$$\begin{aligned} k_s(\omega) &= \upsilon G_c(\omega) = \upsilon(G_s(\omega) + iG_l(\omega)) \\ &= \upsilon G_s(\omega)(1 + i\eta(\omega)). \end{aligned} \tag{6}$$

In the above equation, ω is the frequency. υ is the geometric factor related to the shape of the viscoelastic damper. $G_c(\omega)$ is the complex elastic modulus of the viscoelastic material of the damper. $G_s(\omega)$ and $G_l(\omega)$ are the storage modulus and loss modulus, respectively, of the viscoelastic material. $\eta(\omega)$ is the loss factor defined as

$$\eta(\omega) = \frac{G_l(\omega)}{G_s(\omega)}. \tag{7}$$

The complex shear modulus depends on the vibration frequency and the temperature of the material. In the following analysis, steady state vibrations are assumed and the temperature effect to the viscoelastic damper is assumed constant in time as well. The moduli of the viscoelastic materials in the following analysis are therefore shown as functions of the vibration frequency only.

The vibration amplitude ratio $|H_B(\lambda)|$ of model B can be derived and written as[28]

$$|H_B(\lambda)| = \left|\frac{X_1}{F/K}\right|_B = \left|\frac{X_1}{Y}\right|_B = \frac{\sqrt{(\gamma^2-\lambda^2)^2+(\gamma^2)^2\eta^2}}{\sqrt{\left((1-\lambda^2)(\gamma^2-\lambda^2)-\mu\gamma^2\lambda^2\right)^2+\left[\gamma^2(1-\lambda^2-\mu\lambda^2)\right]^2\eta^2}}, \tag{8}$$

where $\mu = m/M$, $\omega_a = \sqrt{\upsilon G_s/m}$, $\omega_n = \sqrt{K/M}$, $\gamma = \omega_a/\omega_n$, and $\lambda = \omega/\omega_n$.

The purpose of the H_∞ optimization of the model B is to optimize the parameters of viscoelastic damper for minimizing the resonant vibration amplitude of mass M to the exciting force or ground motion. To apply the fixed-points theory to the VDVA, we may vary the loss factor η of the viscoelastic damper by changing the viscoelastic material used in the VDVA. However, the stiffness υG_s and hence γ will also change at any frequency λ after we change the viscoelastic material. Hence the fixed-points theory cannot be applied like in the case of model A before.

Espíndola *et al.*[25] reported a method to search numerically the optimum parameters of the viscoelastic damper for the minimization of maximum vibration amplitude of a VDVA illustrated as model B in Fig. 1(b). It may be expressed mathematically as

$$\max(|H_B(\lambda, \gamma_{\text{opt\_}B}, \upsilon_{\text{opt\_}B})|) = \min(\max_{\gamma,\nu} |H_B(\lambda)|). \tag{9}$$

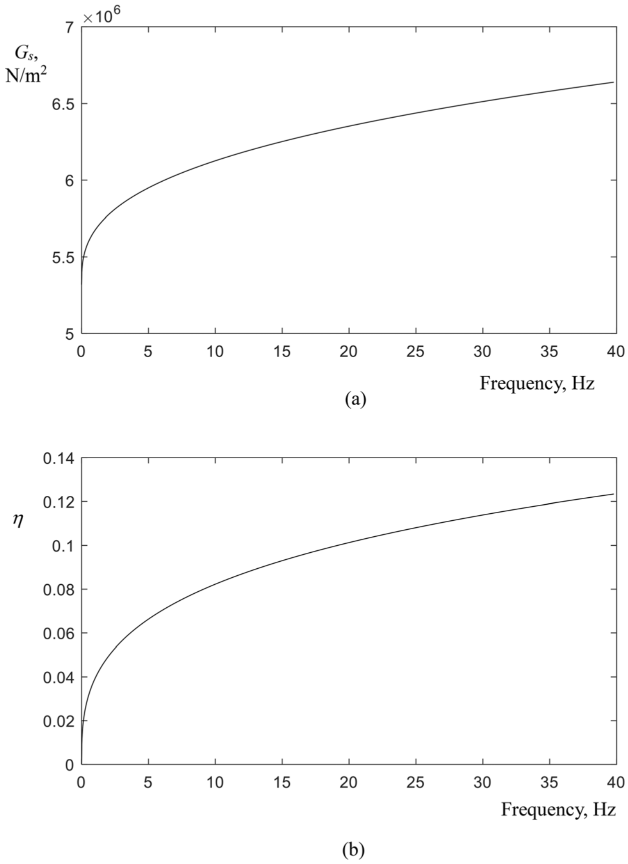

FIG. 2. Variation of the modulus G_s and loss factor η with frequency for the viscoelastic damper from Ref. 25.

A numerical solution of the optimized design of a VDVA (model B)[25] is presented in the following for illustration of their method. The complex stiffness of the viscoelastic damper of the VDVA as illustrated by model B in Fig. 1(b) is given by $k_s(\omega) = \upsilon G_c(\omega)$, where υ is the geometric factor of the viscoelastic damper of the absorber. The complex modulus of the viscoelastic damper used in the VDVA was expressed in terms of the four parameter model below,

$$G_c(\omega) = \frac{(5.32 \times 10^6) + (1.048 \times 10^8)(ib\omega)^{0.359}}{1 + (ib\omega)^{0.359}}, \tag{10}$$

where $b = 7.75 \times 10^{-7} \times 10^{-(10.1)(T-277.7)/(137+T-277.7)}$. ω and T are the vibration frequency and temperature of the viscoelastic damper in the absorber, respectively. T is assumed to be constant equals to 298 K. The modulus G_s and loss factor η functions of this viscoelastic damper from Ref. 25 are calculated according to Eq. (10) and plotted in Fig. 2 for reference. Mass ratio is 0.15 and the primary mass is 100 kg. Natural frequency ω_n equals to 30 Hz.

The optimum geometric factor υ_{opt_B} is searched numerically for the optimum stiffness leading to the optimum frequency ratio γ_{opt_B} such that the maximum vibration response of the primary mass is minimized. $|H_B(\lambda)|$ is calculated using Eq. (8) and plotted in Fig. 3 to show the optimization result. Double peaks of equal height of the vibration amplitude of model B can be observed in Fig. 3. With the same mass ratio, the frequency amplitude response of model A is calculated using Eqs. (1), (3), and (4) and plotted in Fig. 3 for comparison. It shows that model B with optimal design has a much higher resonant vibration amplitude response than model A in this case.

IV. THE PROPOSED DYNAMIC VIBRATION ABSORBER EQUIPPED WITH AN ELASTIC SPRING AND A VISCOELASTIC DAMPER (MODEL C)

The proposed viscoelastic vibration absorber denoted by the model C has a lumped mass connected through a viscoelastic damper and an elastic spring to the primary structure, as shown in Fig. 4. The purpose of adding an elastic spring to the VDVA is to adjust the total stiffness of the absorber without affecting the damping provided by the viscoelastic damper in the VDVA. Both the stiffness and damping of the proposed VDVA can be tuned independently by following a design procedure as described below. This design method

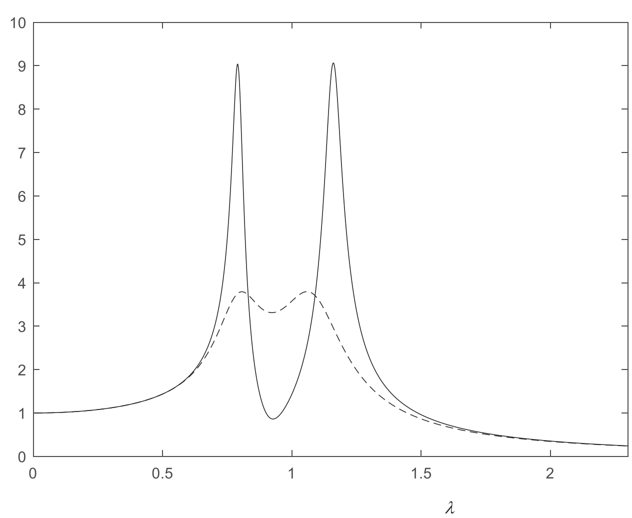

FIG. 3. Frequency response curves $|H_A(\lambda)|$ and $|H_B(\lambda)|$ of the primary mass M of the traditional DVA (model A, ----) optimized using the fixed-points theory, Ref. 7 and viscoelastic damping absorber (model B, ——) with $\mu = 0.15$, $M = 100$ kg, and $\omega_n = 30$ Hz optimized using the method of Ref. 25.

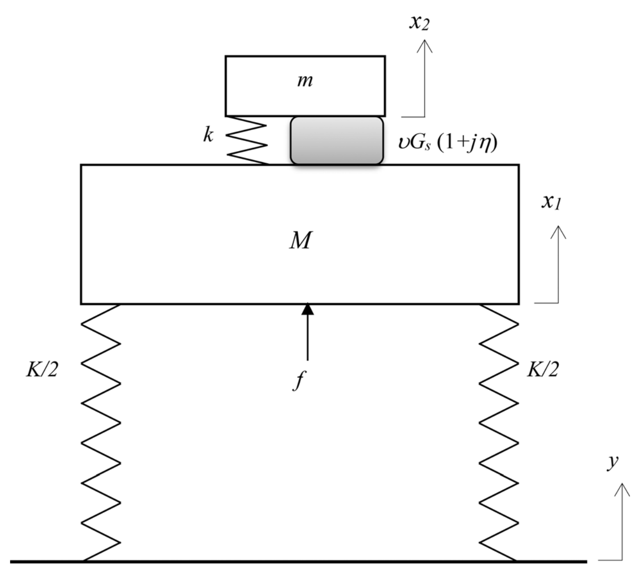

FIG. 4. Model C: the proposed dynamic vibration absorber using a viscoelastic damper and an elastic spring for suppressing the vibration of the mass M excited by a harmonic force f or due to ground motion y.

produces a decoupled design of the VDVA according to the axiomatic design theory.[29]

The governing equations of the proposed viscoelastic vibration absorber (model C) in the frequency domain may be written as follows.

Case 1: Vibration due to harmonic force excitation ($f = F\sin\omega t$, $y = 0$)

$$-MX_1\omega^2 + KX_1 + [k + \upsilon G_s(1+i\eta)](X_1 - X_2) = F(\omega), \tag{11a}$$

$$-mX_2\omega^2 + [k + \upsilon G_s(1+i\eta)](X_2 - X_1) = 0. \tag{11b}$$

Case 2: Vibration due to ground motion ($f = 0$, $y = Y\sin\omega t$)

$$-MX_1\omega^2 + K(X_1 - Y) + [k + \upsilon G_s(1+i\eta)](X_1 - X_2) = 0, \tag{12a}$$

$$-mX_2\omega^2 + [k + \upsilon G_s(1+i\eta)](X_2 - X_1) = 0, \tag{12b}$$

where X_1 and X_2 are the steady state vibration amplitudes of the masses M and m, respectively. Solving Eqs. (11a) and (11b) for case 1, and Eqs. (12a) and (12b) for case 2 yield

$$\text{case 1}: X_1 = \frac{\left(k + \upsilon G_s(1+i\eta) - m\omega^2\right)F}{\upsilon G_s(1+k+i\eta)(K - M\omega^2) - \left(K - M\omega^2 + \upsilon G_s(1+k+i\eta)\right)m\omega^2}, \tag{13a}$$

$$\text{case 2}: X_1 = \frac{\left(k + \upsilon G_s(1+i\eta) - m\omega^2\right)KY}{\upsilon G_s(1+k+i\eta)(K - M\omega^2) - \left(K - M\omega^2 + \upsilon G_s(1+k+i\eta)\right)m\omega^2}, \tag{13b}$$

$$\text{cases 1 and 2}: X_2 = \frac{\left(k + \upsilon G_s(1+i\eta)\right)X_1}{\upsilon G_s(1+k+i\eta) - m\omega^2}. \tag{13c}$$

The amplitude ratio of the primary system in both cases 1 and 2 may be written as

$$|H_C(\omega)| = \left|\frac{X_1}{F/K}\right|_C = \left|\frac{X_1}{Y}\right|_C = \sqrt{\frac{K^2(k + \upsilon G_s - m\omega^2)^2 + K^2(\upsilon G_s\eta)^2}{\left[(K - M\omega^2)(k + \upsilon G_s - m\omega^2) - km\omega^2\right]^2 + \left[\upsilon G_s\eta(K - M\omega^2 - m\omega^2)\right]^2}}. \tag{14}$$

Equation (14) may be expressed in the dimensionless form written as

$$|H_C(\lambda)| = \frac{\sqrt{\left(\gamma^2 - \lambda^2\right)^2 + \left(\gamma_o{}^2\right)^2\eta^2}}{\sqrt{\left((1-\lambda^2)\left(\gamma^2 - \lambda^2\right) - \mu\gamma^2\lambda^2\right)^2 + \left[\gamma_o{}^2\left(1 - \lambda^2 - \mu\lambda^2\right)\right]^2\eta^2}}, \tag{15}$$

where $\mu = m/M$, $\omega_a = \sqrt{(k + \upsilon G_s)/m}$, $\omega_n = \sqrt{K/M}$, $\gamma = \omega_a/\omega_n$, $\omega_{ao} = \sqrt{\upsilon G_s/m}$, $\gamma_o = \omega_{ao}/\omega_n$, and $\lambda = \omega/\omega_n$.

It can be observed by comparing Eq. (15) to Eq. (1) that the frequency response function (FRF) of the primary system of the VDVA (model C) is similar but not the same as the traditional DVA (model A). In the standard fixed-points theory, it states that all frequency response curves of the primary mass pass through two invariant points regardless of the amount of the damping. This is valid for viscous damping[7] and hysteretic damping[28] because both the stiffness and damping components of these absorbers are assumed constant with the vibration frequency. However, if a viscoelastic damper is used in the absorber then both the stiffness and damping of the absorber varies with the vibration frequency and therefore the standard procedure of the fixed-points theory cannot be applied.

The idea to make it possible to apply the fixed-points theory to this absorber is to keep the total stiffness, $k + \upsilon G_s$, of the proposed VDVA to be constant at frequency ω_n, i.e., $\gamma = 1$ at $\omega = \omega_n$, while varying the damping of the VDVA by changing the geometry factor υ instead of the loss factor η. A modified fixed-points theory is proposed and described in the following for the H_∞ optimization of model C. The first step is to check the existence of fixed points in the frequency response spectrum of the mass M in model C, a viscoelastic damper in Ref. 22 is chosen for the design of model C. The viscoelastic material used in the damper is 3 M-467 viscoelastic tape from the 3 M Ltd. Empirical functions of the stiffness and loss factor are given in Ref. 22, respectively, as $19838(4+f)^{0.7}$ N/m and $1.4e^{-0.175|\log(f/100)|^{1.5}}$ at 70 F, where f is the vibration frequency in Hz in the frequency range $0 \le f \le 10^4$ Hz. Variation of the stiffness $\upsilon_o G_s$ and loss factor η with frequency for this viscoelastic damper made with 3 M-467 are shown in Fig. 5 for reference. υ_o represents the reference geometric factor of this viscoelastic damper used in Ref. 22. Assuming the mass ratio $\mu = 0.15$, $M = 100\,\text{kg}$, and $\omega_n = 10\,\text{Hz}$, the relative geometric factor υ/υ_o of the VDVA can be chosen to be $m\omega_n{}^2/\upsilon_o G_{s\_\omega_n} = 0.47$ such that $\gamma_o = 1$ at $\omega = \omega_n$. $\upsilon_o G_{s\_\omega_n} = 125\,829\,\text{N/m}$ is the stiffness of the viscoelastic damper at ω_n.

In the first case, assume $k = 0$, the vibration amplitude response $|H_C(\lambda)|$ of the primary mass M is calculated according to Eq. (15) and plotted in Fig. 6. The idea to show the fixed points in the frequency spectrum of mass M is to maintain the total stiffness, $k + \upsilon G_s$, of the proposed VDVA to be constant at frequency ω_n, i.e., $\gamma = 1$ at $\omega = \omega_n$, while varying the damping of the VDVA by changing the geometric factor υ. In the second case, consider the reduction of υ by 20% leading to 20% reduction of the stiffness. This reduction of the stiffness $\upsilon G_{s\_\omega_n}$ at ω_n is compensated by the stiffness k of the added elastic spring, i.e., $k = 0.2\,\upsilon G_{s\_\omega_n}$. In the third case, υ is reduced by 40% and the corresponding reduction of stiffness is compensated by choosing $k = 0.4\,\upsilon G_{s\_\omega_n}$. In the fourth case, υ is reduced by 60% and choose $k = 0.6\,\upsilon G_{s\_\omega_n}$. The respective frequency response curves of the primary mass M for cases two to four above are calculated according to Eq. (15) and plotted also in Fig. 6 for comparison. There appears two intersecting points in the frequency spectra denoted by P and Q in Fig. 6. However, careful inspection of the intersections of the curves in Fig. 6 reveals that the intersections of the curves are just very close to the two points P and Q but not truly coincident at points P and Q. $\lambda_P \approx 0.86$ and $\lambda_Q \approx 1.13$ are found in this case. The two points P and Q are therefore named "pseudo fixed points" in the following. Points P and Q are not exactly fixed points like those described by the fixed-points theory. This is because γ of the viscoelastic damper varies with frequency and therefore γ can only be kept approximately constant at the two fixed points when we varies the geometric factor υ. $\gamma_P \approx 0.97$ and $\gamma_Q \approx 1.02$ are found in this case. Although we can keep the stiffness $k + \upsilon G_s$ constant at frequency $\gamma = 1$, $k + \upsilon G_s$ cannot be kept constant at the frequencies λ_P and λ_Q because the variations of υG_s at λ_P and λ_Q are not the same when the geometric factor υ is varied. It is found that in this case, $\upsilon_o G_{s\_P} \approx 116\,883\,\text{N/m}$ and $\upsilon_o G_{s\_Q} \approx 133\,898\,\text{N/m}$.

The response amplitude at the pseudo fixed points P and Q in Fig. 6 are found to be $|H_C|_P \approx 5.92$ and $|H_C|_Q \approx 2.05$, respectively. The variations of $|H_C|$ at the intersecting points among the four lines in Fig. 6 at those pseudo fixed points P and Q are found to be about 1%. Noting that γ_P and γ_Q are very close to 1, this illustrated that the principle of the fixed-points theory may be applied for the H_∞ optimization of the proposed VDVA (model C) if we assume γ is constant around the frequency ω_n.

The purpose of the H_∞ optimization of model C is to optimize its parameters including the geometric factor υ of the viscoelastic damper and the additional spring stiffness k for minimizing the resonant vibration response amplitude of the primary mass M to the excitation. It may be expressed mathematically as

$$\max(|H_C(\lambda, k_{\text{opt}\_C}, \upsilon_{\text{opt}\_C})|) = \min(\max_{k,\upsilon} |H_C(\lambda)|). \quad (16)$$

In order to apply the modified fixed-points theory as described before, we assume γ to be constant at the pseudo fixed points. This can be realized approximately by selecting

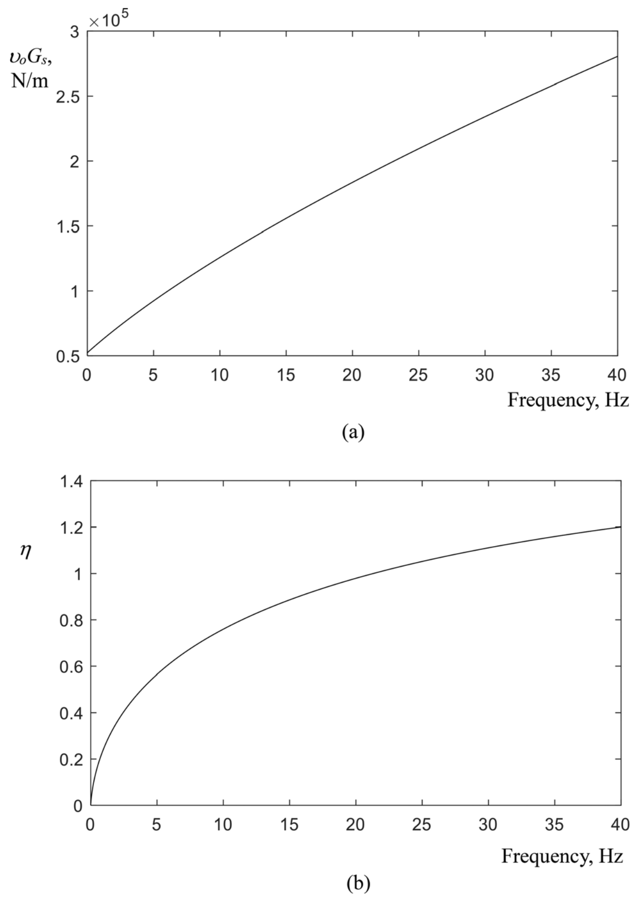

FIG. 5. Variation of (a) stiffness $\upsilon_o G_s$, and (b) loss factor η with frequency for the viscoelastic damper made with 3 M-467 from Ref. 22.

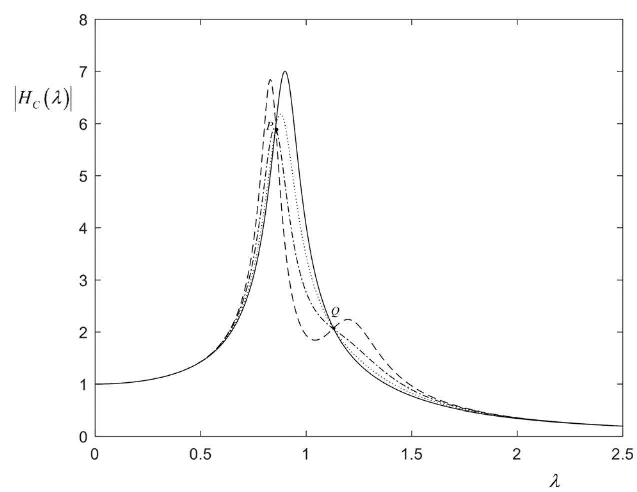

FIG. 6. Frequency response curves $|H_C(\lambda)|$ using the proposed damping absorber (model C) with $\mu = 0.15$, $M = 100\,\text{kg}$, and $\omega_n = 10\,\text{Hz}$. $\gamma = 1$ at $\lambda = 1$ with $k = 0$ (—); $k = 0.2\upsilon G_{s\_\omega_n}$ (···); $k = 0.4\upsilon G_{s\_\omega_n}$ (-·-), and $k = 0.6\upsilon G_{s\_\omega_n}$ (- - -). Pseudo fixed points P and Q are marked with •. The viscoelastic damper material used in model C has stiffness and loss factor as shown in Fig. 5.

k such that $k + \upsilon G_{s\_\omega_n}$ is a constant at the frequency ω_n. Equation (15) may be rewritten as

$$|H_C(\lambda)| = \sqrt{\frac{A + B\upsilon^2}{C + D\upsilon^2}}, \tag{17}$$

where $A = (\gamma^2 - \lambda^2)^2$, $B = G_s{}^2\eta^2/(m^2\omega_n{}^2)$, $C = [(1-\lambda^2)(\gamma^2 - \lambda^2) - \mu\gamma^2\lambda^2]^2$, and $D = (1 - \lambda^2 - \mu\lambda^2)^2 G_s{}^2\eta^2/(m^2\omega_n{}^2)$.

It is known that the corresponding frequencies λ_P and λ_Q of the pseudo fixed points are quite close to each other if the mass ratio μ is small and this is the case for applications in suppressing vibrations of heavy structures. If we assume the difference of the modulus of the VDVA at the two pseudo fixed points is negligible then we may apply the proposed modified fixed-points theory in the optimal design of the proposed VDVA. To find these two points analytically, we consider the frequency response curves of Eq. (17) for $\upsilon = 0$ and $\upsilon = \infty$. The curves for $\upsilon = 0$ and $\upsilon = \infty$ and other real values of υ would "pass through" the points P and Q as shown in Fig. 6. This may be expressed mathematically as

$$\frac{A}{C} = \frac{B}{D} = \frac{A + B\upsilon^2}{C + D\upsilon^2}. \tag{18}$$

Substituting $\upsilon = 0$ into Eq. (17), we may write

$$|H_C(\lambda)|_{\upsilon=0} = \left|\frac{\gamma^2 - \lambda^2}{(1-\lambda^2)(\gamma^2 - \lambda^2) - \mu\gamma^2\lambda^2}\right| = \left(\frac{A}{C}\right)^{1/2}. \tag{19}$$

Substituting $\upsilon = \infty$ into Eq. (17), we may write

$$|H_C(\lambda)|_{\upsilon=\infty} = \left|\frac{1}{1 - \lambda^2 - \mu\lambda^2}\right| = \left(\frac{B}{D}\right)^{1/2}. \tag{20}$$

Using Eqs. (18), (19), and (20), we may write

$$\left(\frac{\gamma^2 - \lambda^2}{(1-\lambda^2)(\gamma^2 - \lambda^2) - \mu\gamma^2\lambda^2}\right)^2 = \left(\frac{1}{1 - \lambda^2 - \mu\lambda^2}\right)^2. \tag{21}$$

Taking square root of Eq. (21) and consider the responses at $\upsilon = 0$ and $\upsilon = \infty$ at opposite phases,[28] we may write

$$\frac{\gamma^2 - \lambda^2}{(1-\lambda^2)(\gamma^2 - \lambda^2) - \mu\gamma^2\lambda^2} = \frac{-1}{1 - \lambda^2 - \mu\lambda^2}. \tag{22}$$

Equation (22) may be simplified as

$$(2 + \mu)\lambda^4 - 2\lambda^2(1 + \gamma^2 + \mu\gamma^2) + 2\gamma^2 = 0. \tag{23}$$

The two roots of Eq. (23) expressed as λ_P and λ_Q may be written, respectively, as

$$\lambda_P = \sqrt{\frac{1 + (1+\mu)\gamma_P{}^2 - \sqrt{1 - 2\gamma_P{}^2 + (1+\mu)^2\gamma_P{}^4}}{2 + \mu}} \tag{24}$$

and

$$\lambda_Q = \sqrt{\frac{1 + (1+\mu)\gamma_Q{}^2 + \sqrt{1 - 2\gamma_Q{}^2 + (1+\mu)^2\gamma_Q{}^4}}{2 + \mu}}. \tag{25}$$

λ_P and λ_Q are the non-dimensionless frequencies of the fixed points. λ_P and λ_Q in Eqs. (24) and (25) have real positive values if μ and γ are also real positive values. This shows that Eq. (23) has real solutions of λ_P and λ_Q after μ and γ are chosen for the VDVA. The frequency response amplitudes of mass M at λ_P and λ_Q can be derived by substituting Eqs. (24) and (25), respectively, into Eq. (17) and written, respectively, as

$$|H_C(\lambda_P)| = \left|\frac{1}{1 - \lambda_P{}^2 - \mu\lambda_P{}^2}\right| = \frac{2 + \mu}{1 - \gamma_P{}^2(1+\mu)^2 + (1+\mu)\sqrt{1 - 2\gamma_P{}^2 + (1+\mu)^2\gamma_P{}^4}} \tag{26}$$

and

$$|H_C(\lambda_Q)| = \left|\frac{1}{1 - \lambda_Q^2 - \mu\lambda_Q^2}\right| = -\frac{2 + \mu}{1 - \gamma_Q{}^2(1+\mu)^2 - (1+\mu)\sqrt{1 - 2\gamma_Q{}^2 + (1+\mu)^2\gamma_Q{}^4}}. \tag{27}$$

Since λ_P and λ_Q are closed to 1 when the mass ratio μ is small, γ_P and γ_Q may be assumed to be equal. The optimum tuning frequency can be derived by equating $|H_C(\lambda_P)|$ and $|H_C(\lambda_Q)|$ using Eqs. (26) and (27) and simplified as

$$\gamma_{\text{opt\_}C} = \frac{1}{1 + \mu}. \tag{28}$$

Equations (3) and (28) shows that the optimum tuning frequencies of the traditional DVA (model A) and the viscoelastic DVA (model C) are the same. The frequency ratios of the pseudo fixed points, λ_P and λ_Q can be obtained by substituting Eq. (28) into Eqs. (24) and (25) and written as

$$\lambda_P^2 = \frac{\sqrt{2 + \mu} - \sqrt{\mu}}{(1 + \mu)\sqrt{2 + \mu}}, \tag{29}$$

$$\lambda_Q^2 = \frac{\sqrt{2 + \mu} + \sqrt{\mu}}{(1 + \mu)\sqrt{2 + \mu}}. \tag{30}$$

The response amplitude of the mass M at the pseudo fixed points can be derived by substituting Eq. (29) into Eq. (26), and Eq. (30) into Eq. (27) and written as

$$|H_C|_{P,Q} = \sqrt{\frac{2+\mu}{\mu}}. \quad (31)$$

To determine the optimum geometric factor υ of the viscoelastic damper such that points P and Q become the maximum points on the response curve of mass M, we consider zero slopes at the two pseudo fixed points, P and Q. We may therefore write

$$\left.\frac{\partial |H_C(\lambda)|^2}{\partial \lambda^2}\right|_{\lambda=\lambda_P,\lambda_Q} = 0. \quad (32)$$

Rewrite Eq. (15) as

$$|H_C(\lambda)|^2 = \frac{S}{T}, \quad (33)$$

where

$$S = (\gamma^2 - \lambda^2)^2 + ({\gamma_o}^2)^2\eta^2 \quad (34)$$

and

$$T = ((1-\lambda^2)(\gamma^2-\lambda^2) - \mu\gamma^2\lambda^2)^2 + \left[{\gamma_o}^2(1-\lambda^2-\mu\lambda^2)\right]^2\eta^2. \quad (35)$$

If $\partial|H_C(\lambda)|^2/\partial\lambda^2 = 0$, then we may write

$$\frac{\partial}{\partial\lambda^2}\left(\frac{S}{T}\right) = \left(\frac{S'T - ST'}{T^2}\right) = 0, \quad (36)$$

where $S' = \partial S/\partial\lambda^2$ and $T' = \partial T/\partial\lambda^2$.

Using Eq. (36), we may write

$$S'T - ST' = 0. \quad (37)$$

Differentiating Eqs. (34) and (35) with respect to λ^2 and then substitute them back to Eq. (37), we may write

$$X{\gamma_o}^4 + Y = 0, \quad (38)$$

where

$$X = \eta^2(1-\lambda^2-\mu\lambda^2)(1+\mu) \quad (39)$$

and

$$Y = -(\gamma^2-\lambda^2)(1-\lambda^2-\mu\lambda^2)^2 + (1-2\lambda^2+\gamma^2+\mu\gamma^2)\left[(1-\lambda^2)(\gamma^2-\lambda^2)-\mu\lambda^2\gamma^2\right]. \quad (40)$$

The optimum geometric factor $\upsilon_{\text{opt\_}C}$ at the pseudo fixed points P and Q may be derived using Eqs. (28)–(30) and (38)–(40) and written, respectively, as

$${\gamma_o}^4{\eta_P}^2 = -\left.\frac{Y\eta^2}{X}\right|_{\lambda^2={\lambda_P}^2,\,\gamma=\gamma_{\text{opt\_}C}} = \frac{1}{(1+\mu)^4}\left[\frac{3\mu}{2} + \frac{\mu^2}{2(2+\mu)} - 4\mu\sqrt{\frac{\mu}{2+\mu}}\right] \quad (41)$$

and

$${\gamma_o}^4{\eta_Q}^2 = -\left.\frac{Y\eta^2}{X}\right|_{\lambda^2={\lambda_Q}^2,\,\gamma=\gamma_{\text{opt\_}C}} = \frac{1}{(1+\mu)^4}\left[\frac{3\mu}{2} + \frac{\mu^2}{2(2+\mu)} + 4\mu\sqrt{\frac{\mu}{2+\mu}}\right]. \quad (42)$$

Taking the average of ${\eta_P}^2$ and ${\eta_Q}^2$, and approximating η_P and η_Q by η_{ω_n}, where η_{ω_n} is the damping of the viscoelastic damper at ω_n, we may write

$${\gamma_o}^2\eta_{\omega_n} = \frac{1}{(1+\mu)^2}\sqrt{\frac{3\mu}{2} + \frac{\mu^2}{2(2+\mu)}}. \quad (43)$$

Recalling ${\gamma_o}^2 = \upsilon G_s/m\,{\omega_n}^2$ and approximating G_{s_P} and G_{s_Q} by $G_{s\_\omega_n}$, we may write

$$\frac{\upsilon G_{s\_\omega_n}\eta_{\omega_n}}{m\,{\omega_n}^2} = \frac{1}{(1+\mu)^2}\sqrt{\frac{3\mu}{2} + \frac{\mu^2}{2(2+\mu)}}. \quad (44)$$

Using Eq. (44), the optimal geometric factor may be written as

$$\upsilon_{\text{opt\_}C} = \frac{m\,{\omega_n}^2}{G_{s\_\omega_n}\eta_{\omega_n}(1+\mu)^2}\sqrt{\frac{3\mu}{2} + \frac{\mu^2}{2(2+\mu)}}. \quad (45)$$

Having the optimal geometric factor $\upsilon_{\text{opt\_}C}$ the designer can determine the shape and size of the viscoelastic spring of the VDVA leading to stiffness equals to $\upsilon_{\text{opt\_}C}\, G_{s\_\omega_n}$ at $\omega = \omega_n$. Using Eq. (28) and the definition of γ, we may write

$$\gamma_{\text{opt\_}C} = \sqrt{\frac{k_{\text{opt}} + \upsilon_{\text{opt\_}C}\, G_{s\_\omega_n}}{m\,{\omega_n}^2}} = \frac{1}{1+\mu}. \quad (46)$$

Substituting $\upsilon_{\text{opt\_}C}$ from Eq. (45) into Eq. (46), the optimum stiffness k_{opt} for the elastic spring in model C can then be derived and written as

$$k_{\text{opt}} = \frac{m\,{\omega_n}^2}{(1+\mu)^2}\left[1 - \frac{1}{\eta_{\omega_n}}\sqrt{\frac{3\mu}{2} + \frac{\mu^2}{2(2+\mu)}}\right]. \quad (47)$$

If the mass ratio μ is small like 0.1 or less, the higher order term of μ in Eqs. (45) and (47) may be neglected and the optimum geometric factor $\upsilon_{\text{opt\_}C}$ and stiffness k_{opt} may be approximately written, respectively, as

$$\upsilon_{\text{opt\_}C} = \frac{m\,{\omega_n}^2}{G_{s\_\omega_n}\eta_{\omega_n}(1+\mu)^2}\sqrt{\frac{3\mu}{2}} \quad (48)$$

and

$$k_{\text{opt}} = \frac{m\,{\omega_n}^2}{(1+\mu)^2}\left(1 - \frac{1}{\eta_{\omega_n}}\sqrt{\frac{3\mu}{2}}\right). \quad (49)$$

The resonant amplitude response of the mass M of model C with optimum geometric factor υ_{opt_C} of the VDVA and the additional stiffness k_{opt} may be approximately written using Eqs. (31) as

$$|H_C|_{max} = \sqrt{\frac{2+\mu}{\mu}}. \tag{50}$$

Comparing Eq. (5) with Eq. (50) shows that the approximate maximum vibration amplitude ratios of the optimized traditional DVA (model A) and the optimized VDVA (model C) are the same.

Equation (49) shows that the additional stiffness k_{opt} is a positive value if the following condition applies:

$$k_{opt} > 0 \Rightarrow \eta_{\omega_n} > \sqrt{\frac{3\mu}{2}}. \tag{51}$$

To avoid k_{opt} to become negative, we should select a viscoelastic damper material with relatively high damping. For the sake of illustration, two examples are given in the following to demonstrate how to apply the proposed optimization method in the proposed VDVA design. The optimized design of model C is tested numerically using two real viscoelastic dampers. The first one is the viscoelastic damper considered in the previous example with measurement data provided in Ref. 22 and its variations of stiffness and loss factor with frequency are plotted in Fig. 5 for reference. Mass ratio μ is 0.15 and the primary mass M is 100 kg. Natural frequency of primary system ω_n is 30 Hz. The optimum relative geometric factor $\upsilon_{opt\_C}/\upsilon_o$ is determined according to Eq. (48) to be 0.735 and the stiffness of the elastic spring k_{opt} of model C is determined according to Eq. (49) to be 230 940 N/m. The frequency response function of the primary mass M is calculated using Eq. (15) and plotted in Fig. 7 for illustration of the proposed optimization method. Double peaks of the frequency amplitude response of model C can be observed in Fig. 7 with $|H_C|_{max} = 3.72$. Using the same mass ratio, the frequency amplitude response of model A is calculated using Eqs. (1), (3), and (4) and plotted in Fig. 7 for comparison. It is observed that the peak response of model C is slightly lower than that of model A and the double peaks of model C are not of the same height. These deviations are due to the approximations taken in the derivation of the optimal values of the geometric factor υ_{opt_C} and the elastic spring constant k_{opt}. This shows that the proposed optimization method is useful to derive an approximate optimal design of the proposed VDVA.

Comparing Fig. 7 with Fig. 3 we can see that the mass M of the optimized model C in this example has a much smaller resonant amplitude than model B. This shows that model C may be a better design of VDVA than model B. In order to check the accuracy of the $\upsilon_{opt\_C}/\upsilon_o$ and k_{opt_C} using the proposed method, a numerical search of the minimum of $|H_C|_{max}$ with k and υ/υ_o as variables have been carried out in MATLAB to determine the exact minimum of $|H_C|_{max}$ in this example. The contours of $|H_C|_{max}$ are plotted in Fig. 8 and the minimum of $|H_C|_{max}$ from the numerical search result is 3.57. The "exact" k_{opt_C} and $\upsilon_{opt\_C}/\upsilon_o$ found in the numerical search are 238 000 N/m and 0.74, respectively. It shows that the error of the minimum of $|H_C|_{max}$ using the proposed optimization method is about 4%.

The second viscoelastic material considered for the proposed VDVA is Sorbothane which is commonly used for damping structural vibrations. Its modulus is given in form of a Prony series written as[23]

$$G_s(\omega) = 4.121\times10^5 + (9.1718\times10^5)\times\left(\frac{0.6281\times0.02^2\omega^2}{1+0.02^2\omega^2} + \frac{0.1908\times0.1819^2\omega^2}{1+0.1819^2\omega^2}\right)\ \mathrm{N/m^2}, \tag{52}$$

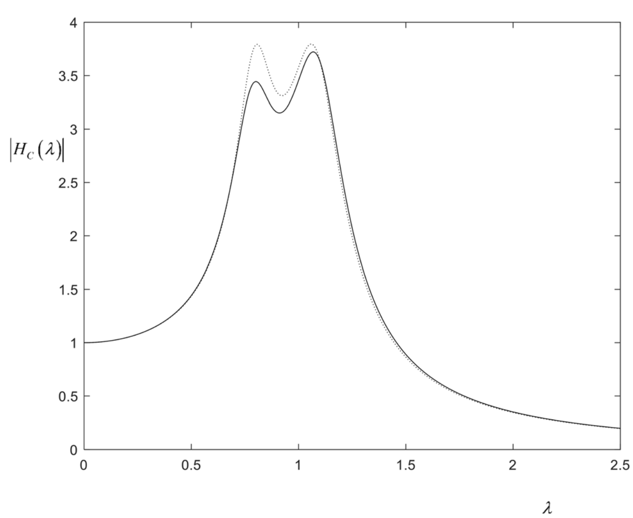

FIG. 7. Frequency response curves of the primary mass M of the proposed optimized VDVA, $|H_C(\lambda)|$(model C, —), and the traditional DVA, $|H_A(\lambda)|$(model A, - - -). $\mu = 0.15$, $M = 100$ kg, and $\omega_n = 30$ Hz. The viscoelastic damper used in model C has stiffness and loss factor as shown in Fig. 5.

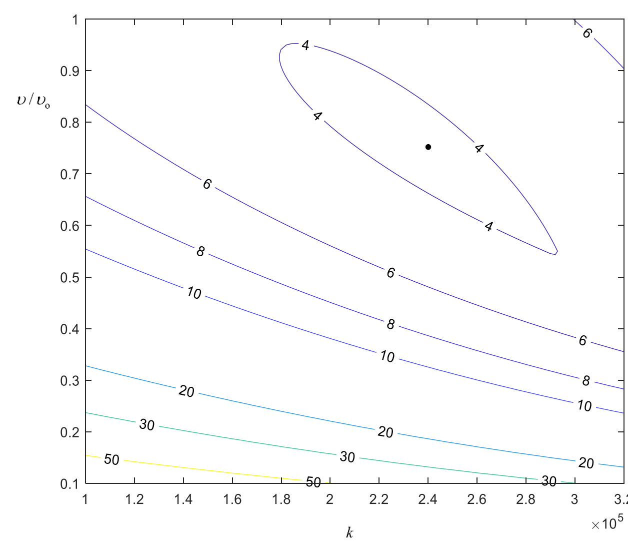

FIG. 8. (Color online) Contours of $|H_C|_{max}$ of model C with different values of the added stiffness k and geometric factor υ. $\mu = 0.15$, $M = 100$ kg, and $\omega_n = 30$ Hz. The viscoelastic damper material used in model C has stiffness and loss factor as shown in Fig. 5. The minimum of $|H_C|_{max}$ is marked with •.

$$G_l(\omega) = (9.1718 \times 10^5)\left(\frac{0.6281 \times 0.02\omega}{1 + 0.02^2\omega^2} + \frac{0.1908 \times 0.1819\omega}{1 + 0.1819^2\omega^2}\right) \text{N/m}^2, \quad (53)$$

and

$$\eta(\omega) = G_s(\omega)/G_l(\omega). \quad (54)$$

Variation of the modulus G_s and loss factor η with frequency for Sorbothane from Ref. 23 are calculated using Eqs. (52)–(54) and plotted in Fig. 9 for reference. Mass ratio μ is 0.02 and the primary mass M is 100 kg. Natural frequency of primary system ω_n is 10 Hz. The optimum geometric factor υ_{opt_C} is determined according to Eq. (48) to be 0.0045 m and the stiffness of the elastic spring k_{opt_C} of model C is determined according to Eq. (49) to be 3399 N/m.

The frequency response function of the primary mass M is calculated using Eq. (15) and plotted in Fig. 10 for illustration of the proposed optimization method. Double peaks of same height of the vibration amplitude of model C can be observed in Fig. 10. With the same mass ratio, the frequency amplitude response of model A is calculated using Eqs. (1), (3), and (4) and plotted in Fig. 10 for comparison. The frequency amplitude response of the optimal design of model B in this case is searched numerically according to Eq. (9) and also plotted in Fig. 10 for comparison. It can be found that $|H_B|_{\max} = 15.5$ which is much larger than $|H_A|_{\max}$ and $|H_C|_{\max}$. The theoretical dimensionless resonant response can be determined using Eq. (52) as $|H_C|_{\max} = 10.0$ while the maximum value of the curve for model C in Fig. 10 is found to be 9.77. In order to check the accuracy of the υ_{opt_C} and k_{opt_C} using the proposed method in this case, a numerical search of the minimum of $|H_C|_{\max}$ with k and υ as variables have been carried out in MATLAB to determine the exact minimum of $|H_C|_{\max}$ in this example. The contours of $|H_C|_{\max}$ are ploted in Fig. 11 and the minimum of $|H_C|_{\max}$ from the numerical search result is 9.75. The "exact" k_{opt_C} and υ_{opt_C} found in the numerical search are 3560 N/m and 0.0043, respectively. It shows that the error of the minimum of

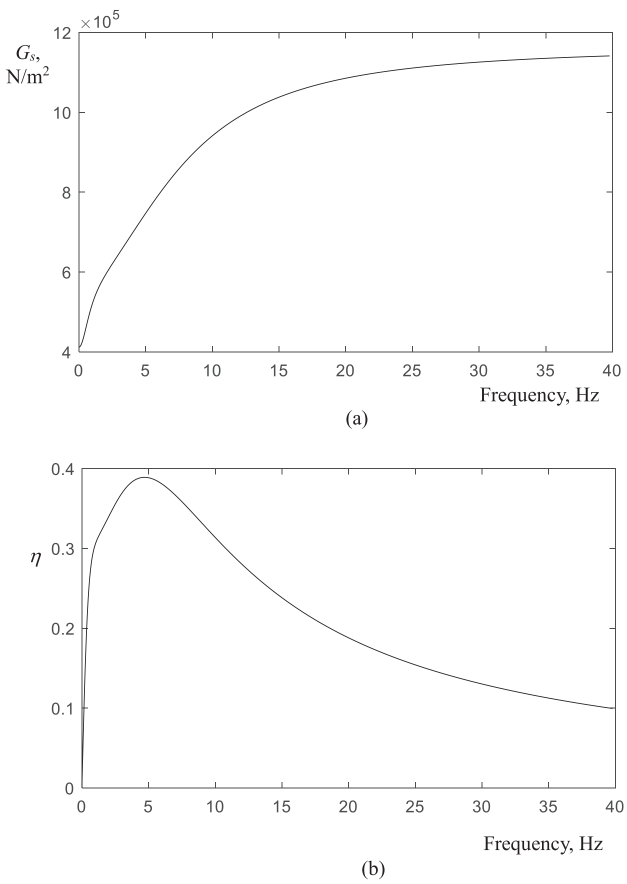

FIG. 9. Variation of (a) the modulus G_s and (b) loss factor η with frequency for Sorbothane from Ref. 23.

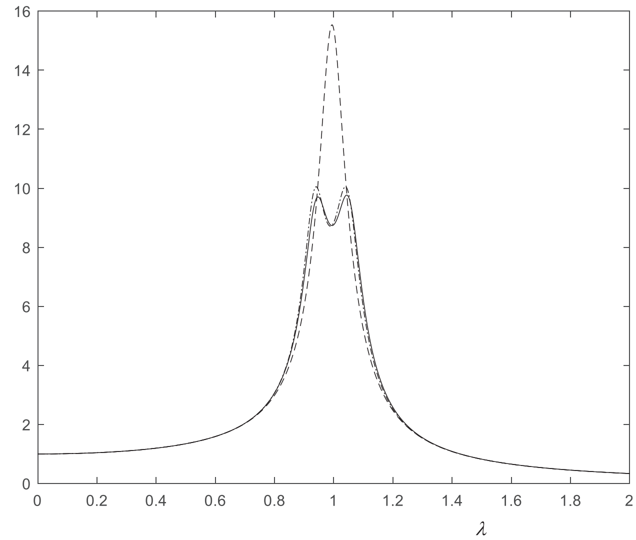

FIG. 10. Frequency response curves of the primary mass M with the optimized traditional DVA, $|H_A(\lambda)|$(model A, –·–), and the optimized model B, $|H_B(\lambda)|$(model B, - - -), and the proposed optimized VDVA, $|H_C(\lambda)|$(model C, —). $\mu = 0.02$, $M = 100$ kg, and $\omega_n = 10$ Hz. The viscoelastic damper material used in model B and model C has modulus and loss factor as shown in Fig. 9.

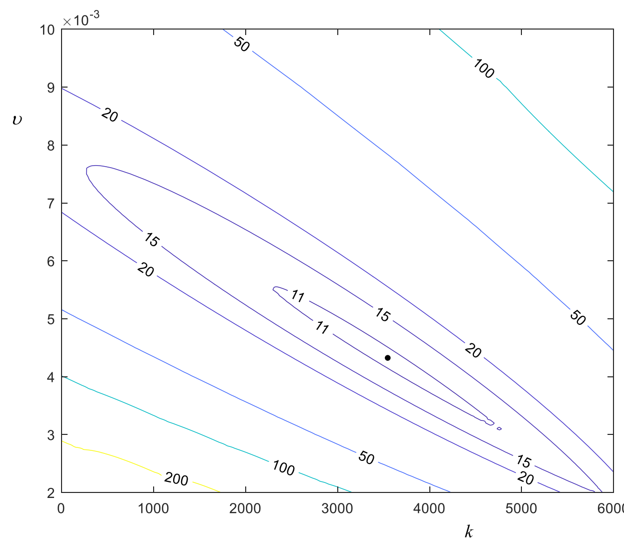

FIG. 11. (Color online) Contours of $|H_C|_{\max}$ of model C with different values of the added stiffness k and geometric factor υ. $\mu = 0.02$, $M = 100$ kg, and $\omega_n = 10$ Hz. The viscoelastic damper material used in model C is has modulus and loss factor as shown in Fig. 9. The minimum of $|H_C|_{\max}$ is marked with •.

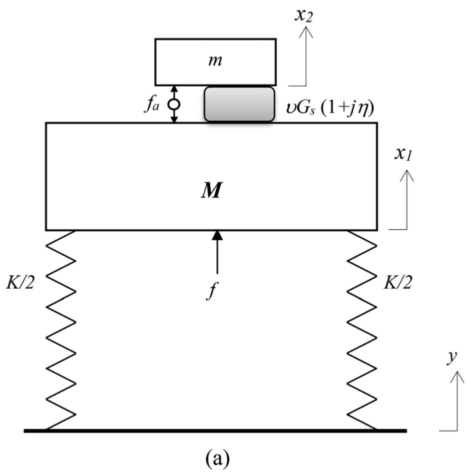

FIG. 12. Possible forms of model C in Fig. 4 when k is negative. (a) The spring is replaced by an actuator to provide the "negative" spring force. (b) The spring is replaced by a passive device having negative stiffness.

$|H_C|_{max}$ using the proposed optimization method is about 0.2%.

Both the theoretical prediction from Eq. (52) and the numerical tests on the maximum vibration amplitude response of the primary mass M of model C show that model C can be used as an alternative design of the traditional DVA (model A). Model C is especially useful for the suppression of the resonant infrasonic vibration because the viscous damper force in model A is proportional to the vibration frequency and it is relatively small at low vibration frequency. Model C is also suitable to be applied on heavy structures such as tall buildings or bridges because the mass ratio μ in those applications are usually small so it is easier to satisfy Eq. (51) such that the additional stiffness k required would not become negative.

If one applies the proposed optimization method to the numerical example in Sec. III using the viscoelastic damper from Ref. 25, then k would be found to be a negative value. To solve this problem, a viscoelastic damper with higher damping can be used such that Eq. (51) can be satisfied. Another solution to this problem is to replace the elastic spring in model C by an actuator to apply a force f_a equal to $k(x_1 - x_2)$ to masses M and m as illustrated in Fig. 12(a) or by a structure with negative stiffness[30] equal to $-|k|$ as illustrated in Fig. 12(b). The major advantage of using the design of model C instead of model B is that the resonant vibration of the mass M would be reduced significantly. Using the numerical example in Sec. III considered in Ref. 25 with modulus and loss factor as shown in Fig. 2, the maximum dimensionless response amplitude of the primary mass M as shown in Fig. 3 is $|H_B|_{max} \approx 9$. If model C is used in that example, then following the proposed optimization method we can derive $|H_C|_{max} = \sqrt{(2+\mu)/\mu} = 3.8$ leading to about 58% reduction of the resonant vibration amplitude of the primary system.

V. CONCLUSION

A VDVA is proposed for suppressing infrasonic vibrations of heavy mechanical or civil structures. The stiffness and damping of the proposed VDVA can be decoupled such that both of these two properties of the absorber can be tuned independently to their optimal values by following a specified procedure. The standard fixed-points theory cannot be applied to the VDVA with frequency dependent stiffness and damping. A modified fixed-points theory is therefore proposed to solve this problem. H_∞ design optimization of the proposed VDVA have been derived analytically for the minimization of resonant vibration of a SDOF system excited by harmonic forces or due to ground motions. Simple analytical expressions of the optimal additional stiffness and geometric factor of the proposed VDVA are derived using the modified fixed-points theory. The proposed VDVA with optimized design is tested numerically using two real commercial viscoelastic damping materials. It is found that the proposed viscoelastic absorber can provide much stronger vibration reduction effect than the conventional VDVA without the elastic spring. The proposed optimal design methodology of dynamic vibration absorber may help engineers to suppress infrasonic vibrations of heavy structures and the proposed VDVA may be considered as an alternative design of the traditional DVA as well.

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