The Kalman Decomposition for Linear Quantum Systems

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Abstract—**This paper studies the Kalman decomposition for linear quantum systems. Contrary to the classical case, the coordinate transformation used for the decomposition must belong to a specific class of transformations as a consequence of the laws of quantum mechanics. We propose a construction method for such transformations that put the system in a Kalman canonical form. Furthermore, we uncover an interesting structure for the obtained decomposition. In the case of passive systems, it is shown that there exist only controllable/observable and uncontrollable/unobservable subsystems. In the general case, controllable/unobservable and uncontrollable/observable subsystems may also be present, but their respective system variables must be conjugate variables of each other. This decomposition naturally exposes decoherence-free modes, quantum-nondemolition modes, quantum-mechanics-free subsystems, and back-action evasion measurements in the quantum system, which are useful resources for quantum information processing, and quantum measurements. The theory developed is applied to physical examples.**

Index Terms— **Linear quantum systems; controllability; observability; Kalman decomposition**

I. INTRODUCTION

Over the past few decades, great progress has been made in the theoretical investigation and experimental realization of controlled quantum systems. In particular, a multitude of control methods have been proposed and tested; see e.g. [43], [46], [5], [33], [1], [6], [55], [15], [32], [53], [58]. Linear quantum systems play a prominent role in these developments. In quantum optics, linear models are commonly used because they are often adequate approximations for more general dynamics. Furthermore, control problems for linear systems often enjoy analytical or computationally tractable solutions. In addition to their wide applications in quantum optics [8], [43], [22], [46], linear quantum systems have also found useful applications in many other quantum-mechanical systems, including circuit quantum electro-dynamical (circuit QED) systems [25], [18], cavity QED systems [4], quantum opto-mechanical systems [38], [23], [13], [6], [24], [48], [49], [29], atomic ensembles [35], [48], and quantum memories [14], [50].

Controllability and observability are two fundamental notions in modern control theory [59], [19], [3]. Roughly speak-ing, controllability describes the external input's ability to steer internal system states, while observability refers to the capability of reconstructing the state-space trajectories of a dynamical system based on its input–output data. Recently, these two fundamental notions have been investigated for linear quantum systems. In the study of optimal measurement-based linear quadratic Gaussian (LQG) control, Wiseman and Doherty showed the equivalence between detectability and stabilizability [45]. Yamamoto and Guta proved that con-trollability and observability are equivalent for passive linear quantum systems [12, Lemma 3.1] and they imply Hurwitz stability [12, Lemma 3.2]. Gough and Zhang showed the equivalence between controllability and observability holds for general (namely, not necessarily passive) linear quantum systems [11, Proposition 1]. Moreover, in the passive case, it is proved that Hurwitz stability, controllability and observability are all equivalent [11, Lemma 2]. The controllability and observability of passive linear quantum systems have been studied by Maalouf and Petersen [21]; using these notions the authors established a complex-domain bounded real lemma for passive linear quantum systems [21, Theorem 6.5]; see also [16], [17], [11]. Nurdin [28] studied model reduction for linear quantum systems based on controllability and observability decompositions; see also [31]. Interestingly, controllability and observability are closely related to the so-called decoherence-free subsystems (DFSs) [40], [41], [6], [48], [49], [11], and references therein. quantum-nondemolition (QND) variables [44], [39], [48], [49], and back-action evasion (BAE) mea-surements [38], [47], [29], [49], which are useful for quantum information processing [38], [6], [49], [58]. Of course, realis-tic quantum information processing applications will require going beyond linear quantum systems. Nevertheless, having the theoretical tools to identify all these useful resources in linear quantum systems is a necessary step in this direction.

Notwithstanding the above advances, a result corresponding to the classical Kalman decomposition (e.g., see [19, Chapter 2], [59, Chapter 3]) is still lacking for linear quantum systems. The critical issue is that, quantum-mechanical laws allow only specific types of coordinate transformations for linear quantum systems. More specifically, in the real quadrature operator representation where the two quadrature operators can be position and momentum operators respectively, the allowed transformations on quantum linear systems are orthogonal transformations for passive systems and symplectic transfor-mations for general (non-passive) systems. In the annihilation-creation operator representation, which is unitarily equiva-lent to the real quadrature operator representation, the al-lowed transformations are unitary transformations for passive

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systems and Bogoliubov transformations for general (non-passive) systems. It is not a priori obvious that transformations to a Kalman canonical form obtained by the standard methods of linear systems theory will satisfy these requirements for linear *quantum* systems. The main purpose of this work is to show that there do exist unitary, Bogoliubov and symplectic transformations, for the corresponding cases, that decompose linear quantum systems into controllable/observable (co), controllable/unobservable ($c\bar{o}$), uncontrollable/observable ($\bar{c}o$), and uncontrollable/unobservable ($\bar{c}\bar{o}$) subsystems. More specifically, in Section III, we study the Kalman decomposition for passive linear quantum systems. In particular, we show that in this case, the uncontrollable subspace is identical to the unobservable subspace, Theorem 3.1; we also give a characterization of these subspaces, Theorem 3.2. The general non-passive case is studied in Section IV. First, we construct the Kalman decomposition for general linear quantum systems in the annihilation-creation operator representation, Theorems 4.1 and 4.2. Then, we translate these theorems into the real quadrature operator representation for linear quantum systems, Theorems 4.3 and 4.4. As a by-product, the real quadrature operator representation of the Kalman canonical form of passive linear systems is given in Corollary 4.1. It is worth noting that the Kalman decomposition is achieved in a constructive way, as in the classical case. Moreover, all the transformations involved are unitary and thus the decomposition can be performed in a numerically stable way.

The Kalman decomposition of a linear quantum system proposed in this paper exhibits the following features: 1) The co and $\bar{c}\bar{o}$ subsystems are linear quantum systems in their own right, as is to be expected from a physics perspective; see Remark 4.4 for details. 2) The system variables of the $c\bar{o}$ subsystem are conjugate to those of the $\bar{c}o$ subsystem. This fact has already been noticed in [48]. An immediate consequence of this is that, a $c\bar{o}$ subsystem exists if and only if a $\bar{c}o$ subsystem does, and they always have the same dimension. Indeed, the question of how to handle the $c\bar{o}$ and $\bar{c}o$ subsystems properly is the major technicality involved in the quantum Kalman decomposition theory proposed in this work, see Lemmas 4.4-4.7. 3) The quantum-mechanical notions of Decoherence-Free subsystems (DFSs), Quantum Non-Demolition (QND) variables, Quantum Mechanics-Free subsystems (QMFS) and Back-Action Evasion (BAE) measurements, which are important in quantum information science and measurement theory, have natural connections with the subspace decomposition. In particular, the $\bar{c}\bar{o}$ subsystem of a linear quantum system (if it exists) is a DFS, and the $\bar{c}o$ subsystem (if it exists) is a QMFS, whose variables are QND variables; see Theorem 4.4, and Remarks 4.9 and 4.10.

The rest of the paper is organized as follows: In Section II, we briefly review linear quantum systems and several physical concepts. In Section III, we study the Kalman decomposition for passive linear quantum systems. The general case is studied in Section IV. The proposed methodology is applied to two physical systems in Section V. Section VI concludes the paper.

Notation

1) x^* denotes the complex conjugate of a complex number x or the adjoint of an operator x. The commutator of two operators X and Y is defined as $[X, Y] \triangleq XY - YX$.
2) For a matrix $X = [x_{ij}]$ with number or operator entries, $X^\# = [x_{ij}^*]$, $X^\top = [x_{ji}]$ is the usual transpose, and $X^\dagger = (X^\#)^\top$. For a vector x, we define $\breve{x} = \begin{bmatrix} x \\ x^\# \end{bmatrix}$.
3) I_k is the identity matrix, and 0_k the zero matrix in $\mathbb{C}^{k \times k}$. δ_{ij} denotes the Kronecker delta symbol; i.e., $I_k = [\delta_{ij}]$. $\mathrm{Ker}\,(X)$, $\mathrm{Im}\,(X)$, and $\sigma\,(X)$ denote the null space, the range space, and the spectrum of a matrix X, respectively.
4) Let $J_k \triangleq \mathrm{diag}(I_k, -I_k)$. For a matrix $X \in \mathbb{C}^{2k \times 2r}$, define its $\flat$-adjoint by $X^\flat \triangleq J_r X^\dagger J_k$. The $\flat$-adjoint satisfies properties similar to the usual adjoint, namely $(x_1 A + x_2 B)^\flat = x_1^* A^\flat + x_2^* B^\flat$, $(AB)^\flat = B^\flat A^\flat$, and $(A^\flat)^\flat = A$.
5) Given two matrices $U, V \in \mathbb{C}^{k \times r}$, define $\Delta(U, V) \triangleq [U\ V; V^\#\ U^\#]$. A matrix with this structure will be called *doubled-up* [10]. It is immediate to see that the set of doubled-up matrices is closed under addition, multiplication and taking ($\flat$-) adjoints.
6) A matrix $T \in \mathbb{C}^{2k \times 2k}$ is called *Bogoliubov* if it is doubled-up and satisfies $T J_k T^\dagger = T^\dagger J_k T = J_k \Leftrightarrow TT^\flat = T^\flat T = I_{2k}$. The set of these matrices forms a complex non-compact Lie group known as the Bogoliubov group.
7) Let $\mathbb{J}_k \triangleq \begin{bmatrix} 0_k & I_k \\ -I_k & 0_k \end{bmatrix}$. For a matrix $X \in \mathbb{C}^{2k \times 2r}$, define its $\sharp$-*adjoint* $X^\sharp$ by $X^\sharp \triangleq -\mathbb{J}_r X^\dagger \mathbb{J}_k$. The $\sharp$-*adjoint* satisfies properties similar to the usual adjoint, namely $(x_1 A + x_2 B)^\sharp = x_1^* A^\sharp + x_2^* B^\sharp$, $(AB)^\sharp = B^\sharp A^\sharp$, and $(A^\sharp)^\sharp = A$.
8) A matrix $S \in \mathbb{C}^{2k \times 2k}$ is called *symplectic*, if it satisfies $S \mathbb{J}_k S^\dagger = S^\dagger \mathbb{J}_k S = \mathbb{J}_k \Leftrightarrow SS^\sharp = S^\sharp S = I_{2k}$. The set of these matrices forms a complex non-compact group known as the symplectic group. The subgroup of real symplectic matrices is one-to-one homomorphic to the Bogoliubov group.

II. LINEAR QUANTUM SYSTEMS

In this section, we briefly introduce linear quantum systems; more details can be found in, e.g., [30], [8], [51], [43], [46], [16], [9], [36], [55], [52]. The linear quantum system, as shown

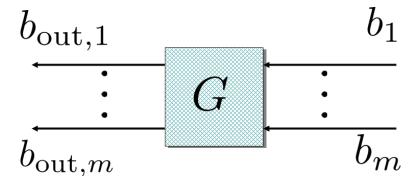Fig. 1. A linear quantum system.

in Fig. 1, is a collection of n quantum harmonic oscillators driven by m input boson fields. The mode of oscillator j, $j = 1, \ldots, n$, is described in terms of its annihilation operator $\boldsymbol{a}_j$, and its creation operator $\boldsymbol{a}_j^*$, the adjoint operator of $\boldsymbol{a}_j$. These are operators in an infinite-dimensional Hilbert space. The

operators $\boldsymbol{a}_j, \boldsymbol{a}_k^*$ satisfy the *canonical commutation relations* $[\boldsymbol{a}_j(t),\ \boldsymbol{a}_k(t)] = 0$, $[\boldsymbol{a}_j^*(t),\ \boldsymbol{a}_k^*(t)] = 0$, and $[\boldsymbol{a}_j(t),\ \boldsymbol{a}_k^*(t)] = \delta_{jk}$, $\forall j,k = 1,\ldots n, \forall t \in \mathbb{R}^+$. Let $\boldsymbol{a} = [\boldsymbol{a}_1 \ \cdots \ \boldsymbol{a}_n]^\top$. The system Hamiltonian $\boldsymbol{H}$ is given by $\boldsymbol{H} = (1/2)\breve{\boldsymbol{a}}^\dagger \Omega \breve{\boldsymbol{a}}$, where $\breve{\boldsymbol{a}} = [\boldsymbol{a}^\top \ (\boldsymbol{a}^{\#})^\top]^\top$, and $\Omega = \Delta(\Omega_-, \Omega_+) \in \mathbb{C}^{2n \times 2n}$ is a Hermitian matrix with $\Omega_-, \Omega_+ \in \mathbb{C}^{n \times n}$. The coupling of the system to the input fields is described by the operator $\boldsymbol{L} = [C_-\ C_+]\breve{\boldsymbol{a}}$, with $C_-, C_+ \in \mathbb{C}^{m \times n}$. The input boson field k, $k = 1,\ldots,m$, is described in terms of an annihilation operator $\boldsymbol{b}_k(t)$ and a creation operator $\boldsymbol{b}_k^*(t)$, the adjoint operator of $\boldsymbol{b}_k(t)$. These are operators on a symmetric Fock space (a special kind of infinite-dimensional Hilbert space). The operators $\boldsymbol{b}_k(t)$ and $\boldsymbol{b}_k^*(t)$ satisfy the singular commutation relations $[\boldsymbol{b}_j(t),\ \boldsymbol{b}_k(r)] = 0$, $[\boldsymbol{b}_j^*(t),\ \boldsymbol{b}_k^*(r)] = 0$, and $[\boldsymbol{b}_j(t),\ \boldsymbol{b}_k^*(r)] = \delta_{jk}\delta(t - r)$, $\forall j,k = 1,\ldots,m, \forall t, r \in \mathbb{R}$. Let $\boldsymbol{b}(t) = [\boldsymbol{b}_1(t) \ \cdots \ \boldsymbol{b}_m(t)]^\top$ and $\breve{\boldsymbol{b}}(t) = [\boldsymbol{b}(t)^\top \ (\boldsymbol{b}(t)^{\#})^\top]^\top$.

The dynamics of the open linear quantum system in Fig. 1 is described by the following quantum stochastic differential equations (QSDEs)

$$\dot{\breve{\boldsymbol{a}}}(t) = \mathcal{A}\breve{\boldsymbol{a}}(t) + \mathcal{B}\breve{\boldsymbol{b}}(t), \tag{1}$$
$$\breve{\boldsymbol{b}}_{\text{out}}(t) = \mathcal{C}\breve{\boldsymbol{a}}(t) + \mathcal{D}\breve{\boldsymbol{b}}(t), \quad t \geq 0, \tag{2}$$

where the system matrices are given by

$$\mathcal{D} = I_{2m},\ \mathcal{C} = \Delta(C_-,\ C_+),\ \mathcal{B} = -\mathcal{C}^\flat,\ \text{and}$$
$$\mathcal{A} = -\imath J_n \Omega - \frac{1}{2}\mathcal{C}^\flat \mathcal{C}. \tag{3}$$

An equivalent way to characterize the structure of (3) (given that all matrices are doubled-up) is by the following *physical realizability* conditions [16], [26], [34], [55]:

$$\mathcal{A} + \mathcal{A}^\flat + \mathcal{B}\mathcal{B}^\flat = 0,\ \mathcal{B} = -\mathcal{C}^\flat. \tag{4}$$

It can be shown that [7], the above forms of system matrices are the only ones with the property that the temporal evolution of (1)-(2) preserves the fundamental commutation relations

$$[\breve{\boldsymbol{a}}(t),\ \breve{\boldsymbol{a}}^\dagger(t)] = [\breve{\boldsymbol{a}}(0),\ \breve{\boldsymbol{a}}^\dagger(0)], \quad t \geq 0,$$
$$[\breve{\boldsymbol{a}}(t), \breve{\boldsymbol{b}}_{\text{out}}^\dagger(r)] = 0,\ 0 \leq r < t.$$

Only under the condition that the above physical realizability conditions are satisfied, do the QSDEs (1)-(2) represent the dynamics of a linear quantum system that can be practically implemented, say, with optical devices, [20], [27], [56].

A very important issue for the purpose of this work is the kind of coordinate transformations $\breve{\boldsymbol{a}}_{\text{new}} = T\breve{\boldsymbol{a}}$ allowed in the QSDEs (1)-(2). It is straightforward to show that the form of (3) is preserved (with $\mathcal{C}_{\text{new}} = \mathcal{C}T^{-1}$ and $\Omega_{\text{new}} = (T^{-1})^\dagger \Omega T^{-1}$) only if T is Bogoliubov. This is a system-theoretic re-statement of the quantum mechanical requirement that T must be Bogoliubov so that the new annihilation and creation operators also satisfy the canonical commutation relations. It is this additional constraint on the allowed coordinate transformations of linear quantum systems that forces us to re-examine the classical method for constructing the Kalman decomposition for such systems.

Linear quantum systems that do not require an external source of energy for their operation are called *passive*. For this important class of systems, $C_+ = 0$ and $\Omega_+ = 0$. This results

in the QSDEs for system and field annihilation operators to decouple from those for the creation operators of either type. Then, a description of the system in terms of annihilation operators only is possible. The QSDEs for a passive linear quantum system are (e.g., see [11, Sec. 3.1]),

$$\dot{\boldsymbol{a}}(t) = \mathcal{A}\boldsymbol{a}(t) + \mathcal{B}\boldsymbol{b}(t), \tag{5}$$
$$\boldsymbol{b}_{\text{out}}(t) = \mathcal{C}\boldsymbol{a}(t) + \mathcal{D}\boldsymbol{b}(t), \tag{6}$$

where

$$\mathcal{A} = -\imath\Omega_- - \frac{1}{2}C_-^\dagger C_-,\ \mathcal{B} = -C_-^\dagger,\ \mathcal{C} = C_-,\ \mathcal{D} = I_m \tag{7}$$

(although we use the same symbols for the system matrices in the passive and the general cases, it should be clear from the context which case we are referring to). An equivalent way to characterize the structure of (7), is by the physical realizability conditions

$$\mathcal{A} + \mathcal{A}^\dagger + \mathcal{B}\mathcal{B}^\dagger = 0,\ \mathcal{B} = -\mathcal{C}^\dagger. \tag{8}$$

The restriction that the allowed coordinate transformations of a general linear quantum system must be Bogoliubov reduces in the passive case to the requirement that the allowed coordinate transformations of a passive linear quantum system must be unitary. This can be deduced from the result for the general case, or directly from (7).

So far, we have used the so-called complex annihilation-creation operator representation to describe the linear quantum system (1)-(2). There is another useful representation of this system, the so-called real quadrature operator representation [54, Sec. II.E]. It can be obtained from the annihilation-creation operator representation through the following transformations:

$$\begin{bmatrix} \boldsymbol{q} \\ \boldsymbol{p} \end{bmatrix} \equiv \boldsymbol{x} \triangleq V_n\breve{\boldsymbol{a}},$$
$$\begin{bmatrix} \boldsymbol{q}_{\text{in}} \\ \boldsymbol{p}_{\text{in}} \end{bmatrix} \equiv \boldsymbol{u} \triangleq V_m\breve{\boldsymbol{b}},\ \begin{bmatrix} \boldsymbol{q}_{\text{out}} \\ \boldsymbol{p}_{\text{out}} \end{bmatrix} \equiv \boldsymbol{y} \triangleq V_m\breve{\boldsymbol{b}}_{\text{out}}, \tag{9}$$

where the unitary matrices V are defined by

$$V_k \triangleq \frac{1}{\sqrt{2}} \begin{bmatrix} I_k & I_k \\ -\imath I_k & \imath I_k \end{bmatrix}.$$

The operators $\boldsymbol{q}_i$ and $\boldsymbol{p}_i$, $i = 1,\ldots,n$, of the real quadrature operator representation are called *conjugate* variables, and they are self-adjoint operators, that is, observables. Moreover, they satisfy the canonical commutation relations $[\boldsymbol{q}_j(t), \boldsymbol{q}_k(t)] = 0$, $[\boldsymbol{p}_j(t), \boldsymbol{p}_k(t)] = 0$, and $[\boldsymbol{q}_j(t), \boldsymbol{p}_k(t)] = \imath\delta_{jk}$, $\forall j,k = 1,\ldots,n$, $\forall t \in \mathbb{R}$. The QSDEs that describe the dynamics of the linear quantum system in Fig. 1 in the real quadrature operator representation are the following:

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}, \tag{10}$$
$$\boldsymbol{y} = C\boldsymbol{x} + D\boldsymbol{u}, \tag{11}$$

where

$$D = V_m\mathcal{D}V_m^\dagger = I_{2m},\ C = V_m\mathcal{C}V_n^\dagger,$$
$$B = V_n\mathcal{B}V_m^\dagger = -C^\sharp,$$
$$A = V_n\mathcal{A}V_n^\dagger = \mathbb{J}H - \frac{1}{2}C^\sharp C. \tag{12}$$

The matrix H in Eq. (12) is defined by $H \triangleq V_n \Omega V_n^\dagger$ (hence, $\boldsymbol{H} = (1/2)\boldsymbol{x}^\top H \boldsymbol{x}$), and is real symmetric. In the above, the useful identities

$$V_k J_k V_k^\dagger = \imath \mathbb{J}_k, \quad V_k X^\flat V_j^\dagger = (V_j X V_k^\dagger)^\sharp,$$

for $X \in \mathbb{C}^{j \times k}$, were used. The matrices A, B, C, D, and H are all real due to the fact that $V_k X V_j^\dagger$ is real if and only if $X \in \mathbb{C}^{2k \times 2j}$ is doubled-up.

In the real quadrature operator representation, the physical realizability conditions (4) take the form

$$A + A^\sharp + BB^\sharp = 0, \ B = -C^\sharp.$$

Finally, the only coordinate transformations that preserve the structure of (12) are real symplectic. This can be deduced from the fact that only Bogoliubov transformations preserve the structure of (3), and that $S = V_n T V_n^\dagger$ is real symplectic if and only if T is Bogoliubov. Finally, since S is symplectic, it preserves the commutation relations.

We end this section by introducing some important notions from quantum information science and quantum measurement theory. We will show later that these notions are naturally exposed by the Kalman decomposition of linear quantum systems. We begin with two well-known notions in linear systems theory.

The controllability and observability matrices for the linear quantum system (1)-(2) are defined respectively by (e.g., see [48, Sec. III-B] and [11, Proposition 2])

$$\begin{aligned}
C_G &\triangleq \begin{bmatrix} \mathcal{B} & \mathcal{A}\mathcal{B} & \cdots & \mathcal{A}^{2n-1}\mathcal{B} \end{bmatrix}, \\
O_G &\triangleq \begin{bmatrix} \mathcal{C} \\ \mathcal{C}\mathcal{A} \\ \vdots \\ \mathcal{C}\mathcal{A}^{2n-1} \end{bmatrix}.
\end{aligned} \quad (13)$$

$\mathrm{Im}(C_G)$ and $\mathrm{Ker}\,(O_G)$ are the controllable and unobservable subspaces of the space of system variables $\mathbb{C}^{2n}$. We define the uncontrollable and observable subspaces to be their *orthogonal complements* in $\mathbb{C}^{2n}$, that is $\mathrm{Ker}(C_G^\dagger)$, and $\mathrm{Im}(O_G^\dagger)$, respectively.

Definition 2.1: The linear span of the system variables related to the uncontrollable/unobservable subspace of a linear quantum system is called its *decoherence-free subsystem (DFS)*.

Decoherence-free subsystems for linear quantum systems have recently been studied in e.g., [40], [41], [6], [48], [49], [11], and references therein.

Definition 2.2: An observable F is called a continuous-time *quantum-nondemolition (QND) variable* if

$$[F(t_1), F(t_2)] = 0 \quad (14)$$

for all time instants $t_1, t_2 \in \mathbb{R}^+$.

The physical meaning of Eq. (14) is that F may be measured an arbitrary number of times (in fact, continuously) during the evolution of the quantum system, with no quantum limit on the predictability of these measurements [44], [39], [49].

A natural extension of the notion of a QND variable is the following concept [39].

Definition 2.3: The span of a set of observables F_i, $i = 1, \ldots, r$, is called a *quantum mechanics-free subsystem (QMFS)* if

$$[F_i(t_1), F_j(t_2)] = 0 \quad (15)$$

for all time instants $t_1, t_2 \in \mathbb{R}^+$, and $i, j = 1, \ldots, r$.

The transfer function of the linear system (10)-(11) is

$$\Xi_{\boldsymbol{u} \to \boldsymbol{y}}(s) \triangleq D - C(sI - A)^{-1}B.$$

This transfer function relates the overall input $\boldsymbol{u}$ to the overall output $\boldsymbol{y}$. However, in many applications, we are interested in a particular subvector $\boldsymbol{u}'$ of the input vector $\boldsymbol{u}$ and a particular subvector $\boldsymbol{y}'$ of the output vector $\boldsymbol{y}$. This motivates us to introduce the following concept.

Definition 2.4: For the linear quantum system (10)-(11), let $\Xi_{\boldsymbol{u}' \to \boldsymbol{y}'}(s)$ be the transfer function from a subvector $\boldsymbol{u}'$ of the input vector $\boldsymbol{u}$ and a subvector $\boldsymbol{y}'$ of the output vector $\boldsymbol{y}$. We say that system (10)-(11) realizes the *back-action evasion (BAE) measurement* of the output $\boldsymbol{y}'$ with respect to the input $\boldsymbol{u}'$ if $\Xi_{\boldsymbol{u}' \to \boldsymbol{y}'}(s) = 0$ for all s.

More discussions on BAE measurements can be found in, e.g., [44], [38], [47], [29], [49] and the references therein.

We shall see that all of these notions emerge naturally from the study of the Kalman decomposition of a linear quantum system, see Remarks 4.9 and 4.10.

III. THE KALMAN DECOMPOSITION FOR PASSIVE LINEAR QUANTUM SYSTEMS

In this section, we study the Kalman decomposition for passive linear quantum systems. First, we show that their uncontrollable subspace is identical to their unobservable subspace.

Let us define the controllability and observability matrices of system (5)-(6), respectively, by

$$\begin{aligned}
C_G &\triangleq \begin{bmatrix} \mathcal{B} & \mathcal{A}\mathcal{B} & \cdots & \mathcal{A}^{n-1}\mathcal{B} \end{bmatrix}, \\
O_G &\triangleq \begin{bmatrix} \mathcal{C} \\ \mathcal{C}\mathcal{A} \\ \vdots \\ \mathcal{C}\mathcal{A}^{n-1} \end{bmatrix}.
\end{aligned}$$

$\mathrm{Im}(C_G)$ and $\mathrm{Ker}\,(O_G)$ are the controllable and unobservable subspaces of the space of system variables $\mathbb{C}^n$. As in the general case, we define the uncontrollable and observable subspaces to be their *orthogonal complements* in $\mathbb{C}^n$, that is $\mathrm{Ker}(C_G^\dagger)$, and $\mathrm{Im}(O_G^\dagger)$, respectively. We have the following theorem.

Theorem 3.1: The uncontrollable and unobservable subspaces of the passive linear quantum system (5)-(6) are identical. That is,

$$\mathrm{Ker}(C_G^\dagger) = \mathrm{Ker}\,(O_G). \quad (16)$$

Proof: Let us define the auxiliary matrices

$$\begin{aligned}
C_s &\triangleq \begin{bmatrix} \mathcal{B} & (-\imath \Omega_-)\mathcal{B} & \cdots & (-\imath \Omega_-)^{n-1}\mathcal{B} \end{bmatrix}, \\
O_s &\triangleq \begin{bmatrix} \mathcal{C} \\ \mathcal{C}(-\imath \Omega_-) \\ \vdots \\ \mathcal{C}(-\imath \Omega_-)^{n-1} \end{bmatrix}.
\end{aligned}$$

It can be readily shown that

$$C_s^\dagger = - \begin{bmatrix} I_m & & & \\ & -I_m & & \\ & & \ddots & \\ & & & (-1)^{n-1} I_m \end{bmatrix} O_s.$$

Thus, we have

$$\mathrm{Ker}(C_s^\dagger) = \mathrm{Ker}(O_s). \qquad (17)$$

Now, we show that

$$\mathrm{Ker}\,(O_G) = \mathrm{Ker}(O_s). \qquad (18)$$

Let $\mu \in \mathrm{Ker}(O_s)$. Then, $C_-(\Omega_-)^j \mu = 0, \quad j = 0, 1, \ldots$. As a result, $C_-(-i\Omega_- - \frac{1}{2}C_-^\dagger C_-)^j \mu = 0, \quad j = 0, 1, \ldots$. That is, $\mu \in \mathrm{Ker}(O_G)$. Hence, $\mathrm{Ker}(O_s) \subset \mathrm{Ker}(O_G)$. The fact that, $\mathrm{Ker}(O_G) \subset \mathrm{Ker}(O_s)$, can be proved similarly, thus proving Eq. (18). We can establish the relation

$$\mathrm{Ker}(C_G^\dagger) = \mathrm{Ker}(C_s^\dagger). \qquad (19)$$

similarly. Finally, Eq. (16) follows from Eqs. (17)-(19). ∎

Theorem 3.1 demonstrates that the Kalman decomposition of passive linear quantum systems can contain only co and $\bar{c}\bar{o}$ subsystems. This property is due to the special structure of passive systems, and does not hold for general linear quantum systems; see, e.g., Theorem 4.1. An immediate consequence of this result is that, an uncontrollable mode is necessarily an unobservable mode. As was discussed in Section II, only unitary coordinate transformations preserve the quantum structure of passive linear quantum systems, and are thus allowed to be used to achieve the Kalman decomposition. Although in the case of general linear systems, it will require some effort to construct a Bogoliubov or symplectic transformation for this purpose, the situation is very simple in the passive case. Indeed, in the case of passive linear quantum systems, a decomposition of the space of system variables into a controllable subspace and an uncontrollable subspace will achieve the Kalman decomposition. However, it is a well-known fact that this decomposition can always be performed via a unitary matrix; e.g., see [42] and the references therein.

It is easily seen from Definition 2.1 that $\mathrm{Ker}(O_G)$ is the DFS of system (5)-(6), if it is non-trivial. In [11, Lemma 2], it is shown that for a passive linear quantum system, Hurwitz stability, controllability and observability are all equivalent. From this and Theorem 3.1, we conclude that if the passive linear quantum system (5)-(6) is not Hurwitz stable, it must have a non-trivial DFS. In what follows, we characterize the DFS of a passive linear quantum system.

Theorem 3.2: The DFS of the passive linear quantum system (5)-(6) is spanned by the eigenvectors of the matrix $\mathcal{A}$ whose corresponding eigenvalues are on the imaginary axis.

Proof: It is a well-known fact that $\mathrm{Ker}(O_G)$ is an invariant subspace of $\mathcal{A}$. Hence, it is spanned by its eigenvectors (including generalized ones). First, we show that all eigenvectors of $\mathcal{A}$ with imaginary eigenvalues belong to $\mathrm{Ker}(O_G)$. Let λ be an eigenvalue of $\mathcal{A}$ with $\mathrm{Re}(\lambda) = 0$, and let $\mu \neq 0$ be the corresponding eigenvector. From the proof of Theorem 3.1, it

suffices to show that $\mu \in \mathrm{Ker}(O_s)$. That is, $C_-\Omega_-^k \mu = 0$ for $k = 0, 1, 2, \ldots$ From

$$\mathcal{A}\mu = (-i\Omega_- - \frac{1}{2}C_-^\dagger C_-)\mu = \lambda\mu, \qquad (20)$$

we have $\mu^\dagger(-i\Omega_- - \frac{1}{2}C_-^\dagger C_-)\mu = \lambda\mu^\dagger\mu$ and $\mu^\dagger(i\Omega_- - \frac{1}{2}C_-^\dagger C_-)\mu = \lambda^*\mu^\dagger\mu$. Adding these two equations, we get $-\mu^\dagger C_-^\dagger C_-\mu = 2\mathrm{Re}(\lambda)\mu^\dagger\mu = 0$, which implies $C_-\mu = 0$. Substituting $C_-\mu = 0$ into Eq. (20) yields $\Omega_-\mu = i\lambda\mu$. As a result, $C_-\Omega_-^k\mu = (i\lambda)^k C_-\mu = 0, \quad k = 0, 1, 2, \ldots$. Thus we have $\mu \in \mathrm{Ker}(O_s) = \mathrm{Ker}(O_G)$. Next, we show that if $i\omega, \omega \in \mathbb{R}$, is an eigenvalue of the matrix $\mathcal{A}$ for the passive linear quantum system (5)-(6), then its geometric multiplicity is one. This way, generalized eigenvectors for imaginary eigenvalues are excluded. To see this, suppose that the geometric multiplicity is two. Then, in an appropriate basis, the matrix $\mathcal{A}$ has a Jordan block $\begin{bmatrix} i\omega & 1 \\ 0 & i\omega \end{bmatrix}$. Clearly, the matrix

$$\begin{bmatrix} i\omega & 1 \\ 0 & i\omega \end{bmatrix} + \begin{bmatrix} i\omega & 1 \\ 0 & i\omega \end{bmatrix}^\dagger = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is indefinite. However, from Eq. (8), we have $\mathcal{A} + \mathcal{A}^\dagger = -C_-^\dagger C_-$, which is negative semi-definite, a contradiction. A similar argument excludes cases of higher geometric multiplicity. Finally, to complete the proof, we need to show that $\mathrm{Ker}(O_G)$ is spanned only by eigenvectors of $\mathcal{A}$ with eigenvalues on the imaginary axis. Let $\mu \in \mathrm{Ker}(O_G)$ be an eigenvector of $\mathcal{A} = -i\Omega_- - \frac{1}{2}C_-^\dagger C_-$, with eigenvalue λ. Then, the equations $C_-\mu = 0$, and $\mathcal{A}\mu = \lambda\mu$, imply that $-i\Omega_-\mu = \lambda\mu$. However, Ω_- is a Hermitian matrix, and this implies that λ is imaginary. ∎

We end this section with a simple example.

Example 3.1: Consider a passive linear quantum system with parameters $C_- = [1 \; 1]$ and $\Omega_- = I_2$. The corresponding QSDEs are

$$\dot{\boldsymbol{a}}_1(t) = -(i + \frac{1}{2})\boldsymbol{a}_1(t) - \frac{1}{2}\boldsymbol{a}_2(t) - \boldsymbol{b}(t),$$

$$\dot{\boldsymbol{a}}_2(t) = -\frac{1}{2}\boldsymbol{a}_1(t) - (i + \frac{1}{2})\boldsymbol{a}_2(t) - \boldsymbol{b}(t),$$

$$\boldsymbol{b}_{\mathrm{out}}(t) = \boldsymbol{a}_1(t) + \boldsymbol{a}_2(t) + \boldsymbol{b}(t).$$

If we let $T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, and $\begin{bmatrix} \boldsymbol{a}_{DF} \\ \boldsymbol{a}_D \end{bmatrix} \triangleq T^\dagger\boldsymbol{a}$, the QSDEs for $\boldsymbol{a}_{DF}$ and $\boldsymbol{a}_D$ are the following:

$$\dot{\boldsymbol{a}}_{DF}(t) = -i\boldsymbol{a}_{DF}(t),$$

$$\dot{\boldsymbol{a}}_D(t) = -(1 + i)\,\boldsymbol{a}_D(t) - \sqrt{2}\boldsymbol{b}(t),$$

$$b_{\mathrm{out}}(t) = \sqrt{2}\boldsymbol{a}_D(t) + \boldsymbol{b}(t).$$

Clearly, $\boldsymbol{a}_{DF}$ is a DF mode.

IV. THE KALMAN DECOMPOSITION FOR GENERAL LINEAR QUANTUM SYSTEMS

In this section, we construct the Kalman decomposition for a general linear quantum system and uncover its special structure. In Subsection IV-A, we derive the decomposition in the complex annihilation-creation operator representation, and show that it can be achieved with a unitary and Bogoliubov coordinate transformation. Then, we translate the main results of this subsection, Theorems 4.1 and 4.2, into the real

quadrature operator representation, Theorems 4.3 and 4.4 in Subsection IV-B. Finally, some special cases of the Kalman decomposition are investigated in Subsection IV-C.

A. The Kalman decomposition in the complex annihilation-creation operator representation

To make the presentation easy to follow, we first establish a series of lemmas that are used to prove the main results of this subsection, Theorems 4.1 and 4.2.

Define an auxiliary matrix [11, Eq. (7)]:

$$O_s \triangleq \begin{bmatrix} \mathcal{C} \\ \mathcal{C}(J_n \Omega) \\ \vdots \\ \mathcal{C}(J_n \Omega)^{2n-1} \end{bmatrix}.$$

By [11, Proposition 2], we know that

$$\mathrm{Ker}\,(O_G) = \mathrm{Ker}\,(O_s), \quad \mathrm{Ker}(C_G^\dagger) = \mathrm{Ker}\,(O_s J_n).$$

Instead of working directly with $\mathrm{Ker}\,(O_G)$ and $\mathrm{Ker}(C_G^\dagger)$, it will be easier to work with $\mathrm{Ker}\,(O_s)$ and $\mathrm{Ker}\,(O_s J_n)$.

We start by characterizing the controllable subspace $\mathrm{Im}(C_G)$, the uncontrollable subspace $\mathrm{Ker}(C_G^\dagger)$, the observable subspace $\mathrm{Im}(O_G^\dagger)$, and the unobservable subspace $\mathrm{Ker}\,(O_G)$. We first establish the following result.

Lemma 4.1: The unobservable subspace $\mathrm{Ker}\,(O_s)$ and the uncontrollable subspace $\mathrm{Ker}\,(O_s J_n)$ are related by

$$\mathrm{Ker}\,(O_s) = J_n \mathrm{Ker}\,(O_s J_n). \quad (21)$$

Similarly, the controllable subspace $\mathrm{Im}(C_G)$ and the observable subspace $\mathrm{Im}(O_G^\dagger)$ are related by

$$\mathrm{Im}(C_G) = J_n \mathrm{Im}(O_G^\dagger). \quad (22)$$

Proof: Eq. (21) can be established in a straightforward way. Hence, we concentrate on Eq. (22). Noticing that $\mathrm{Im}(C_G) = \mathrm{Ker}(C_G^\dagger)^\perp = \mathrm{Ker}\,(O_s J_n)^\perp$, and $\mathrm{Im}(O_G^\dagger) = \mathrm{Ker}\,(O_G)^\perp = \mathrm{Ker}\,(O_s)^\perp = (J_n \mathrm{Ker}\,(O_s J_n))^\perp$, where Eq. (21) is used in the last step, it suffices to show that

$$\mathrm{Ker}\,(O_s J_n)^\perp = J_n (J_n \mathrm{Ker}\,(O_s J_n))^\perp. \quad (23)$$

However,

$$\begin{aligned}
&(J_n \mathrm{Ker}\,(O_s J_n))^\perp \\
&= \{x : (J_n y)^\dagger x = 0, \forall y \in \mathrm{Ker}\,(O_s J_n)\} \\
&= J_n \{w : w^\dagger y = 0, \forall y \in \mathrm{Ker}\,(O_s J_n)\} \\
&= J_n (\mathrm{Ker}(O_s J_n))^\perp.
\end{aligned}$$

Therefore, Eq. (23) holds, and so does Eq. (22). ∎

Now, let us define the four subspaces used in the Kalman decomposition:

$$\begin{aligned}
R_{c\bar{o}} &\triangleq \mathrm{Im}(C_G) \cap \mathrm{Ker}(O_G), & (24) \\
R_{co} &\triangleq \mathrm{Im}(C_G) \cap \mathrm{Im}(O_G^\dagger), & (25) \\
R_{\bar{c}\bar{o}} &\triangleq \mathrm{Ker}(C_G^\dagger) \cap \mathrm{Ker}(O_G), & (26) \\
R_{\bar{c}o} &\triangleq \mathrm{Ker}(C_G^\dagger) \cap \mathrm{Im}(O_G^\dagger). & (27)
\end{aligned}$$

That is, $R_{c\bar{o}}$, R_{co}, $R_{\bar{c}\bar{o}}$, and $R_{\bar{c}o}$ are respectively the controllable/unobservable ($c\bar{o}$), controllable/observable (co), uncontrollable/unobservable ($\bar{c}\bar{o}$), and uncontrollable/observable ($\bar{c}o$) subspaces of system (1)-(2).

The following lemma, which is an immediate consequence of Lemma 4.1, reveals relations among the subspaces $R_{c\bar{o}}$, R_{co}, $R_{\bar{c}\bar{o}}$, and $R_{\bar{c}o}$.

Lemma 4.2: The subspaces $R_{c\bar{o}}$, R_{co}, $R_{\bar{c}\bar{o}}$, and $R_{\bar{c}o}$ can be expressed as

$$\begin{aligned}
R_{c\bar{o}} &= \mathrm{Ker}\,(O_s J_n)^\perp \cap \mathrm{Ker}(O_s), \\
R_{co} &= \mathrm{Ker}\,(O_s J_n)^\perp \cap (\mathrm{Ker}\,(O_s)^\perp + \mathrm{Ker}\,(O_s J_n)), \\
R_{\bar{c}\bar{o}} &= \mathrm{Ker}\,(O_s) \cap (\mathrm{Ker}\,(O_s)^\perp + \mathrm{Ker}\,(O_s J_n)), \\
R_{\bar{c}o} &= \mathrm{Ker}\,(O_s J_n) \cap \mathrm{Ker}\,(O_s)^\perp.
\end{aligned}$$

Moreover, they enjoy the following properties: $R_{c\bar{o}} \perp R_{co} \perp R_{\bar{c}\bar{o}} \perp R_{\bar{c}o}$, and

$$R_{c\bar{o}} = J_n R_{\bar{c}o}, \ R_{co} = J_n R_{co}, \ R_{\bar{c}\bar{o}} = J_n R_{\bar{c}\bar{o}}. \quad (28)$$

Furthermore, the vector space $\mathbb{C}^{2n}$ is the direct sum of these orthogonal subspaces. That is, $\mathbb{C}^{2n} = R_{c\bar{o}} \oplus R_{co} \oplus R_{\bar{c}\bar{o}} \oplus R_{\bar{c}o}$.

$$\color{red} R_{co} \oplus R_{\bar{c}\bar{o}} = J_n(R_{co} \oplus R_{\bar{c}\bar{o}}) \ ?????$$

The next lemma shows that we can choose bases with a special structure for the subspaces R_{co} and $R_{\bar{c}\bar{o}}$.

Lemma 4.3: We have:

(i) There exists a unitary and Bogoliubov matrix T_{co} of the form

$$T_{co} = \begin{bmatrix} Z_1 & 0 \\ 0 & Z_1^\# \end{bmatrix}, \quad (29)$$

where $Z_1 \in \mathbb{C}^{n \times n_1}$ ($n_1 \geq 0$), such that its columns form an orthonormal basis for R_{co}.

(ii) Similarly, there exists a unitary and Bogoliubov matrix $T_{\bar{c}\bar{o}}$ of the form

$$T_{\bar{c}\bar{o}} = \begin{bmatrix} Z_2 & 0 \\ 0 & Z_2^\# \end{bmatrix},$$

where $Z_2 \in \mathbb{C}^{n \times n_2}$ ($n_2 \geq 0$), such that its columns form an orthonormal basis for $R_{\bar{c}\bar{o}}$.

Proof: We first establish Item (i). Let $\begin{bmatrix} e_1 \\ f_1 \end{bmatrix}$ be a nonzero vector in the subspace R_{co}. Then, from the second relation in Eq. (28), we have that $\begin{bmatrix} e_1 \\ -f_1 \end{bmatrix} \in R_{co}$. Therefore, the vectors $\begin{bmatrix} e_1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ f_1 \end{bmatrix} \in R_{co}$. Moreover, due to the doubled-up structure of the system matrices, it can be readily shown that $\begin{bmatrix} 0 \\ e_1^\# \end{bmatrix}$, $\begin{bmatrix} f_1^\# \\ 0 \end{bmatrix} \in R_{co}$, as well. Because $\begin{bmatrix} e_1 \\ f_1 \end{bmatrix} \neq 0$, e_1 and f_1 cannot both be zero. If $e_1 \neq 0$, define $z_1 \triangleq \frac{1}{\|e_1\|} e_1$; otherwise, define $z_1 \triangleq \frac{1}{\|f_1^\#\|} f_1^\#$. Then, $\begin{bmatrix} z_1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ z_1^\# \end{bmatrix}$ are nonzero orthonormal vectors in R_{co}. Take another nonzero vector $\begin{bmatrix} e_2 \\ f_2 \end{bmatrix} \in R_{co}$ which is orthogonal to both $\begin{bmatrix} z_1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ z_1^\# \end{bmatrix}$. Then $z_1^\dagger e_2 = 0$ and $z_1^\dagger f_2^\# = 0$. If $e_2 \neq 0$, define $z_2 \triangleq \frac{1}{\|e_2\|} e_2$; otherwise, define $z_2 \triangleq \frac{1}{\|f_2^\#\|} f_2^\#$. Then, $\begin{bmatrix} z_1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ z_1^\# \end{bmatrix}$, $\begin{bmatrix} z_2 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ z_2^\# \end{bmatrix}$ are orthonormal vectors in R_{co}. Repeat this procedure to get the matrix T_{co} in Eq. (29), with $Z_1 = [z_1 \ z_2 \ldots z_{n_1}]$. Clearly, the columns of T_{co} form an orthonormal basis of R_{co}. Moreover, by the

construction given above, $Z_1^\dagger Z_1 = I_{n_1}$ holds. As a result, $T_{co}^\dagger T_{co} = I_{2n_1}$ and $T_{co}^\dagger J_n T_{co} = J_{n_1}$.

Item (ii) can be established in a similar way. ∎

Remark 4.1: The above proof is more rigorous than that of [11, Lemma 1], which fails to discuss the case where $e_j = 0$ or $f_j = 0$.

Remark 4.2: It follows from Lemma 4.3 that the dimensions of the subspaces R_{co} and $R_{\bar{c}\bar{o}}$ are both even ($2n_1$ and $2n_2$, respectively). Let the dimensions of the subspaces $R_{c\bar{o}}$ and $R_{\bar{c}o}$ be n_3 and n_4 respectively. Due to first relation in Eq. (28), and the fact that J_n is invertible, we must have that $n_4 = n_3$. Hence, $2(n_1 + n_2 + n_3) = 2n$.

In order to construct special orthonormal bases for the subspaces $R_{c\bar{o}}$ and $R_{\bar{c}o}$, the following three lemmas are needed.

Lemma 4.4: Let $M, N \in \mathbb{C}^{r \times k}$ and $x_1, x_2 \in \mathbb{C}^k$. If

$$\Delta(M, N) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0, \tag{30}$$

then

$$\Delta(M, N) \begin{bmatrix} x_1 + x_2^\# \\ x_1^\# + x_2 \end{bmatrix} = 0. \tag{31}$$

Proof: Eq. (30) is equivalent to

$$Mx_1 + Nx_2 = 0, \tag{32}$$
$$N^\# x_1 + M^\# x_2 = 0. \tag{33}$$

Conjugating both sides of Eq. (33) yields

$$Mx_2^\# + Nx_1^\# = 0. \tag{34}$$

Adding Eqs. (32) and (34) yields $M(x_1 + x_2^\#) + N(x_1^\# + x_2) = 0$. Conjugating both sides of the above equation gives $N^\#(x_1 + x_2^\#) + M^\#(x_1^\# + x_2) = 0$. In compact form, the above two equations become

$$\Delta(M, N) \begin{bmatrix} x_1 + x_2^\# \\ x_1^\# + x_2 \end{bmatrix} = 0,$$

which is Eq. (31). ∎

Lemma 4.5: If $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \text{Im}(C_G)$, then $\begin{bmatrix} x_1 + x_2^\# \\ x_2 + x_1^\# \end{bmatrix} \in \text{Im}(C_G)$.

Proof: The matrices $\mathcal{A}$ and $\mathcal{B}$ in Eq. (3) are doubled-up. Hence, $\mathcal{A}^k \mathcal{B}$ is also doubled-up, for all $k = 1, \ldots$. That is, each block column of the controllability matrix C_G is doubled-up. As a result, upon a column permutation, C_G is of the form $\Delta(C_{G,+}, C_{G,-})$. Then, given $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \text{Im}(C_G)$, there exist vectors z_+ and z_- such that

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \Delta(C_{G,+}, C_{G,-}) \begin{bmatrix} z_+ \\ z_- \end{bmatrix}.$$

Consequently, it can be easily shown that

$$\begin{bmatrix} x_1 + x_2^\# \\ x_2 + x_1^\# \end{bmatrix} = \Delta(C_{G,+}, C_{G,-}) \begin{bmatrix} z_+ + z_-^\# \\ z_- + z_+^\# \end{bmatrix}.$$

That is, $\begin{bmatrix} x_1 + x_2^\# \\ x_2 + x_1^\# \end{bmatrix} \in \text{Im}(C_G)$. ∎

Lemmas 4.4 and 4.5 can be used to establish the following result.

Lemma 4.6: If $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in R_{c\bar{o}}$, then $\begin{bmatrix} x_1 + x_2^\# \\ x_1^\# + x_2 \end{bmatrix} \in R_{c\bar{o}}$.

Proof: Consider a vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in R_{c\bar{o}}$. From Eq. (24), $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \text{Im}(C_G) \cap \text{Ker}(O_G)$. According to Lemma 4.5,

$$\begin{bmatrix} x_1 + x_2^\# \\ x_1^\# + x_2 \end{bmatrix} \in \text{Im}(C_G). \tag{35}$$

Also, since $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \text{Ker}(O_G)$, by Lemma 4.4,

$$\begin{bmatrix} x_1 + x_2^\# \\ x_1^\# + x_2 \end{bmatrix} \in \text{Ker}(O_G). \tag{36}$$

Eqs. (35) and (36) yield

$$\begin{bmatrix} x_1 + x_2^\# \\ x_1^\# + x_2 \end{bmatrix} \in \text{Im}(C_G) \cap \text{Ker}(O_G) = R_{c\bar{o}}.$$

∎

Remark 4.3: Lemma 4.6 also holds for the subspaces $R_{\bar{c}o}$, R_{co}, and $R_{\bar{c}\bar{o}}$.

We are ready to construct special orthonormal bases for the subspaces $R_{c\bar{o}}$ and $R_{\bar{c}o}$ in the following lemma.

Lemma 4.7: There exists a matrix $T_{c\bar{o}}$ of the form

$$T_{c\bar{o}} \triangleq \frac{1}{\sqrt{2}} \begin{bmatrix} X & Y \\ X^\# & -Y^\# \end{bmatrix},$$

where $X \in \mathbb{C}^{n \times n_a}$ and $Y \in \mathbb{C}^{n \times n_b}$ ($n_a \geq 0$, $n_b \geq 0$, $n_a + n_b = n_3$) satisfy $X^\dagger X = I_{n_a}$, $Y^\dagger Y = I_{n_b}$ and $X^\dagger Y = 0$, such that its columns form an orthonormal basis of $R_{c\bar{o}}$. Also, the columns of $T_{\bar{c}o} \triangleq J_n T_{c\bar{o}}$ form an orthonormal basis of $R_{\bar{c}o}$.

Proof: Let $X = [x_1, \cdots, x_{n_a}]$ and $Y = [y_1, \cdots, y_{n_b}]$, for some non-negative integers $n_a, n_b \geq 0$ such that $n_a + n_b = n_3$. We use the following algorithm to construct the vectors $x_1, \ldots, x_{n_a}$ and $y_1, \ldots, y_{n_b}$ sequentially.

Step 0. Set indices $j = k = 0$.

Step 1. Pick a nonzero vector $\begin{bmatrix} u \\ v \end{bmatrix} \in R_{c\bar{o}}$. By Lemma 4.6, $\begin{bmatrix} u + v^\# \\ u^\# + v \end{bmatrix} \in R_{c\bar{o}}$. There are two possibilities:

Case (I). $u + v^\# \neq 0$. In this case, define $x_1 \triangleq \frac{1}{\|u + v^\#\|}(u + v^\#)$, where $\|\cdot\|$ denotes the vector Euclidean norm. Clearly, $x_1^\dagger x_1 = 1$, and $\begin{bmatrix} x_1 \\ x_1^\# \end{bmatrix} \in R_{c\bar{o}}$. Set $j \to j + 1$.

Case (II). $u + v^\# = 0$. In this case, $v = -u^\#$. Define $y_1 \triangleq \frac{u}{\|u\|}$. We have $y_1^\dagger y_1 = 1$ and $\begin{bmatrix} y_1 \\ -y_1^\# \end{bmatrix} \in R_{c\bar{o}}$. Set $k \to k + 1$.

According to the above, in the first step of the algorithm we generate either x_1 or y_1.

Step $p = j + k$. Up to this step, we have generated $x_1, \ldots, x_j$, and $y_1, \ldots, y_k$. Now, let us take a nonzero vector $\begin{bmatrix} x \\ y \end{bmatrix} \in R_{c\bar{o}}$ which satisfies

$$x_i^\dagger x + x_i^\top y = 0, \quad i = 1, \ldots, j \tag{37}$$

and

$$y_l^\dagger x - y_l^\top y = 0, \quad l = 1, \ldots, k. \tag{38}$$

Complex conjugating both sides of Eqs. (37) and (38) yields

$$x_i^\top x^\# + x_i^\dagger y^\# = 0, \quad i = 1, \ldots, j \tag{39}$$

and

$$-y_l^\top x^\# + y_l^\dagger y^\# = 0, \quad l = 1, \ldots, k, \tag{40}$$

respectively. Adding Eqs. (37) and (39) we get

$$x_i^\dagger (x + y^\#) + x_i^\top (x^\# + y) = 0. \tag{41}$$

That is, $\begin{bmatrix} x+y^{\#} \\ x^{\#}+y \end{bmatrix}$ is orthogonal to $\begin{bmatrix} x_i \\ x_i^{\#} \end{bmatrix}$, $i = 1, \ldots, j$. Similarly, using Eqs. (38) and (40) we get $y_l^{\dagger}(x + y^{\#}) - y_l^{\top}(x^{\#} + y) = 0$, $l = 1, \ldots, k$. That is, $\begin{bmatrix} x+y^{\#} \\ x^{\#}+y \end{bmatrix}$ is orthogonal to $\begin{bmatrix} y_j \\ -y_j^{\#} \end{bmatrix}$, for all $l = 1, \ldots, k$. Again, there are two possibilities:

Case (I). $x + y^{\#} \neq 0$. In this case, define $x_{j+1} \triangleq \frac{1}{\|x+y^{\#}\|}(x + y^{\#})$. Clearly, $\begin{bmatrix} x_{j+1} \\ x_{j+1}^{\#} \end{bmatrix}$ is orthogonal to all of the vectors $\begin{bmatrix} x_i \\ x_i^{\#} \end{bmatrix}$ and $\begin{bmatrix} y_j \\ -y_j^{\#} \end{bmatrix}$ for $i = 1, \ldots, j$ and $l = 1, \ldots, k$. Set $j \to j + 1$.

Case (II). $x + y^{\#} = 0$. In this case, define $y_{k+1} \triangleq \frac{1}{\|x\|}x$. Then we have,

$$x_i^{\dagger} y_{k+1} + x_i^{\top}(-y_{k+1}^{\#}) = \frac{1}{\|x\|}(x_i^{\dagger}x + x_i^{\top}y) = 0$$

for all $i = 1, \ldots, j$, and

$$y_l^{\dagger} y_{k+1} - y_l^{\top}(-y_{k+1}^{\#}) = \frac{1}{\|x\|}\left(y_l^{\dagger}x - y_l^{\top}y\right) = 0$$

for all $l = 1, \ldots, k$. That is, $\begin{bmatrix} y_{k+1} \\ -y_{k+1}^{\#} \end{bmatrix}$ is orthogonal to all of the vectors $\begin{bmatrix} x_i \\ x_i^{\#} \end{bmatrix}$ and $\begin{bmatrix} y_j \\ -y_j^{\#} \end{bmatrix}$ for $= 1, \ldots, j$ and $l = 1, \ldots, k$. Set $k \to k + 1$.

Step n_3. The algorithm terminates.

When the above algorithm terminates, we will have constructed the matrices X, and Y in the definition of $T_{c\bar{o}}$. It is clear from the above construction that the columns of $T_{c\bar{o}}$ form an orthonormal basis for the subspace $R_{c\bar{o}}$. From the first relation in Eq. (28), it follows that the columns of $T_{\bar{c}o} \triangleq J_n T_{c\bar{o}}$ form an orthonormal basis of $R_{\bar{c}o}$. Finally, we prove that X and Y satisfy the relations $X^{\dagger}X = I_{n_a}$, $Y^{\dagger}Y = I_{n_b}$, and $X^{\dagger}Y = 0$. Indeed, from the fact that the columns of $T_{c\bar{o}}$ form an orthonormal basis of $R_{c\bar{o}}$, we have that $T_{c\bar{o}}^{\dagger}T_{c\bar{o}} = I_{n_3}$, from which it follows that $X^{\dagger}X + X^{\top}X^{\#} = 2I_{n_a}$. Similarly, since $R_{c\bar{o}} \perp R_{\bar{c}o}$, we have that $T_{\bar{c}o}^{\dagger}T_{c\bar{o}} = T_{c\bar{o}}^{\dagger}J_n T_{c\bar{o}} = 0$, which implies the relation $X^{\dagger}X - X^{\top}X^{\#} = 0_{n_a}$. Adding these equations gives $X^{\dagger}X = I_{n_a}$. The other two relations can be proved similarly. ∎

We are now ready to construct a unitary and Bogoliubov transformation matrix that achieves the Kalman decomposition of the system (1)-(2). From now on, we will use the notation $R_h = R_{c\bar{o}} \oplus R_{\bar{c}o}$.

Theorem 4.1: Let

$$T_{co} = \begin{bmatrix} Z_1 & 0 \\ 0 & Z_1^{\#} \end{bmatrix}, \quad T_{\bar{c}\bar{o}} = \begin{bmatrix} Z_2 & 0 \\ 0 & Z_2^{\#} \end{bmatrix},$$

$$T_{c\bar{o}} = \frac{1}{\sqrt{2}}\begin{bmatrix} X & Y \\ X^{\#} & -Y^{\#} \end{bmatrix}, \quad T_{\bar{c}o} = \frac{1}{\sqrt{2}}\begin{bmatrix} X & Y \\ -X^{\#} & Y^{\#} \end{bmatrix}$$

be constructed as in Lemmas 4.3 and 4.7, and let $Z_3 \triangleq [X \ Y]$. Then the matrix

$$\tilde{T} \triangleq \begin{bmatrix} Z_3 & Z_1 & Z_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & Z_3^{\#} & Z_1^{\#} & Z_2^{\#} \end{bmatrix} \quad (42)$$

is a unitary and Bogoliubov matrix (i.e., it satisfies $\tilde{T}^{\dagger}\tilde{T} = I_{2n}$ and $\tilde{T}^{\dagger}J_n\tilde{T} = J_n$), and decomposes the system variables of the linear quantum system (1)-(2) as follows:

$$\begin{bmatrix} a_h^{\top} & a_{co}^{\top} & a_{\bar{c}\bar{o}}^{\top} & a_h^{\dagger} & a_{co}^{\dagger} & a_{\bar{c}\bar{o}}^{\dagger} \end{bmatrix}^{\top} = \tilde{T}^{\dagger}\breve{a}. \quad (43)$$

Proof: From Lemmas 4.3, and 4.7, we have that $Z_1^{\dagger}Z_1 = I_{n_1}$, $Z_2^{\dagger}Z_2 = I_{n_2}$, and $X^{\dagger}X = I_{n_a}$, $Y^{\dagger}Y = I_{n_b}$, and $X^{\dagger}Y = 0$. From the last three equations, we deduce that $Z_3^{\dagger}Z_3 = I_{n_3}$. Also, from the orthogonality of the subspaces R_{co}, $R_{\bar{c}\bar{o}}$, and $R_h = R_{c\bar{o}} \oplus R_{\bar{c}o}$, we have that $Z_i^{\dagger}Z_j = 0$, $i, j = 1, 2, 3, i \neq j$. Then, the equations $\tilde{T}^{\dagger}\tilde{T} = I_{2n}$ and $\tilde{T}^{\dagger}J_n\tilde{T} = J_n$ follow immediately. From Lemma 4.3, we have that

$$\breve{a}_{co} = \begin{bmatrix} a_{co} \\ a_{co}^{\#} \end{bmatrix} \triangleq T_{co}^{\dagger}a = \begin{bmatrix} Z_1^{\dagger}a \\ (Z_1^{\dagger}a)^{\#} \end{bmatrix} \quad (44)$$

are the co variables. Similarly,

$$\breve{a}_{\bar{c}\bar{o}} = \begin{bmatrix} a_{\bar{c}\bar{o}} \\ a_{\bar{c}\bar{o}}^{\#} \end{bmatrix} \triangleq T_{\bar{c}\bar{o}}^{\dagger}a = \begin{bmatrix} Z_2^{\dagger}a \\ (Z_2^{\dagger}a)^{\#} \end{bmatrix} \quad (45)$$

are the $\bar{c}\bar{o}$ variables. Finally, from Lemma 4.7 we have that the columns of

$$\frac{1}{\sqrt{2}}\begin{bmatrix} X & Y & X & Y \\ X^{\#} & -Y^{\#} & -X^{\#} & Y^{\#} \end{bmatrix}$$

form an orthonormal basis for $R_h = R_{c\bar{o}} \oplus R_{\bar{c}o}$. Using simple manipulations, it is easy to see that the same is true for

$$\begin{bmatrix} X & Y & 0 & 0 \\ 0 & 0 & X^{\#} & Y^{\#} \end{bmatrix} = \begin{bmatrix} Z_3 & 0 \\ 0 & Z_3^{\#} \end{bmatrix} \triangleq T_h. \quad (46)$$

Hence,

$$\breve{a}_h = \begin{bmatrix} a_h \\ a_h^{\#} \end{bmatrix} \triangleq T_h^{\dagger}a = \begin{bmatrix} Z_3^{\dagger}a \\ (Z_3^{\dagger}a)^{\#} \end{bmatrix} \quad (47)$$

are the $h = c\bar{o} \cup \bar{c}o$ variables. Hence, (43) follows. ∎

Although $\tilde{T}$ is useful in decomposing the space of variables of the system (1)-(2) into its R_{co}, $R_{\bar{c}\bar{o}}$, and $R_h = R_{c\bar{o}} \oplus R_{\bar{c}o}$ subspaces, it is not directly useful in putting (1)-(2) into the Kalman canonical form. The reason is that the evolution equation for $\tilde{T}^{\dagger}\breve{a}$ mixes the evolution of variables in different subspaces in a non-obvious way. To put (1)-(2) into a Kalman-like canonical form, we introduce the following transformation:

$$\begin{aligned} T &\triangleq \begin{bmatrix} T_h & | & T_{co} & | & T_{\bar{c}\bar{o}} \end{bmatrix} \\ &= \begin{bmatrix} Z_3 & 0 & | & Z_1 & 0 & | & Z_2 & 0 \\ 0 & Z_3^{\#} & | & 0 & Z_1^{\#} & | & 0 & Z_2^{\#} \end{bmatrix}, \end{aligned} \quad (48)$$

where T_h was defined in Eq. (46). Similarly to T_{co} and $T_{\bar{c}\bar{o}}$, T_h satisfies $T_h^{\dagger}T_h = I_{2n_3}$, and $T_h^{\dagger}J_nT_h = J_{n_3}$. From the identities $Z_i^{\dagger}Z_j = \delta_{ij}I_{n_i}$, $i, j = 1, 2, 3$, established in Lemmas 4.3 and 4.7, and Theorem 4.1, it follows that $T^{\dagger}T = I_{2n}$, that is, T is unitary, and also that

$$T^{\dagger}J_nT = \begin{bmatrix} J_{n_3} & 0 & 0 \\ 0 & J_{n_1} & 0 \\ 0 & 0 & J_{n_2} \end{bmatrix}.$$

That is, T is *blockwise Bogoliubov*. From this, we have the following theorem.

Theorem 4.2: The unitary and blockwise Bogoliubov coordinate transformation

$$\begin{bmatrix} \breve{a}_h \\ \breve{a}_{co} \\ \breve{a}_{\bar{c}\bar{o}} \end{bmatrix} = T^{\dagger}\breve{a} \quad (49)$$

transforms the linear quantum system (1)-(2) into the form

$$
\begin{bmatrix} \dot{\breve{\boldsymbol{a}}}_h(t) \\ \dot{\breve{\boldsymbol{a}}}_{co}(t) \\ \dot{\breve{\boldsymbol{a}}}_{\bar{c}\bar{o}}(t) \end{bmatrix} = \bar{\mathcal{A}} \begin{bmatrix} \breve{\boldsymbol{a}}_h(t) \\ \breve{\boldsymbol{a}}_{co}(t) \\ \breve{\boldsymbol{a}}_{\bar{c}\bar{o}}(t) \end{bmatrix} + \bar{\mathcal{B}}\,\breve{\boldsymbol{b}}(t), \qquad (50)
$$

$$
\breve{\boldsymbol{b}}_{\text{out}}(t) = \bar{\mathcal{C}} \begin{bmatrix} \breve{\boldsymbol{a}}_h(t) \\ \breve{\boldsymbol{a}}_{co}(t) \\ \breve{\boldsymbol{a}}_{\bar{c}\bar{o}}(t) \end{bmatrix} + \breve{\boldsymbol{b}}(t), \qquad (51)
$$

where

$$
\begin{aligned}
\bar{\mathcal{A}} &\triangleq T^\dagger \mathcal{A} T = \begin{bmatrix} \mathcal{A}_h & \mathcal{A}_{12} & \mathcal{A}_{13} \\ \mathcal{A}_{21} & \mathcal{A}_{co} & 0 \\ \mathcal{A}_{31} & 0 & \mathcal{A}_{\bar{c}\bar{o}} \end{bmatrix}, \\
\bar{\mathcal{B}} &\triangleq T^\dagger \mathcal{B} = \begin{bmatrix} \mathcal{B}_h \\ \mathcal{B}_{co} \\ 0 \end{bmatrix}, \\
\bar{\mathcal{C}} &\triangleq \mathcal{C} T = \begin{bmatrix} \mathcal{C}_h & \mathcal{C}_{co} & 0 \end{bmatrix}. \qquad (52)
\end{aligned}
$$

Proof: The proof follows from the following well-known invariance properties of linear systems; e.g., see [19, Chapter 2]:

$$
\mathcal{A} R_{c\bar{o}} \subset R_{c\bar{o}}, \ \mathcal{A} R_{co} \subset R_{c\bar{o}} \oplus R_{co}, \ \mathcal{A} R_{\bar{c}\bar{o}} \subset R_{c\bar{o}} \oplus R_{\bar{c}\bar{o}} \quad (53)
$$

and

$$
\begin{aligned}
\operatorname{Im}(\mathcal{B}) &\subset \operatorname{Im}(C_G) = R_{c\bar{o}} \oplus R_{co}, \\
\operatorname{Ker}(O_G) &= R_{c\bar{o}} \oplus R_{\bar{c}\bar{o}} \subset \operatorname{Ker}(C), \qquad (54)
\end{aligned}
$$

which imply

$$
\begin{aligned}
\mathcal{A} R_{co} \subset R_h \oplus R_{co}, &\quad \mathcal{A} R_{\bar{c}\bar{o}} \subset R_h \oplus R_{\bar{c}\bar{o}}, \\
\operatorname{Im}(\mathcal{B}) \subset R_h \oplus R_{co}, &\quad R_{\bar{c}\bar{o}} \subset \operatorname{Ker}(C).
\end{aligned}
$$

∎

Remark 4.4: From a physics perspective, one expects that the co subsystem

$$
\begin{aligned}
\dot{\breve{\boldsymbol{a}}}_{co}(t) &= \mathcal{A}_{co} \breve{\boldsymbol{a}}_{co}(t) + \mathcal{B}_{co} \breve{\boldsymbol{b}}(t), \\
\breve{\boldsymbol{b}}_{\text{out}}(t) &= \mathcal{C}_{co} \breve{\boldsymbol{a}}_{co}(t) + \breve{\boldsymbol{b}}(t),
\end{aligned}
$$

and the $\bar{c}\bar{o}$ subsystem

$$
\dot{\breve{\boldsymbol{a}}}_{\bar{c}\bar{o}}(t) = \mathcal{A}_{\bar{c}\bar{o}} \breve{\boldsymbol{a}}_{\bar{c}\bar{o}}(t),
$$

are respectively linear quantum systems in their own right. The proof is as follows: From the second of Eqs. (52), we have that

$$
\begin{aligned}
\begin{bmatrix} \mathcal{B}_h \\ \mathcal{B}_{co} \\ 0 \end{bmatrix} &= \bar{\mathcal{B}} = T^\dagger \mathcal{B} = -T^\dagger \mathcal{C}^\flat = -T^\dagger J_n C^\dagger J_m \\
&= -(T^\dagger J_n T)(CT)^\dagger J_m = - \begin{bmatrix} J_{n_3} & 0 & 0 \\ 0 & J_{n_1} & 0 \\ 0 & 0 & J_{n_2} \end{bmatrix} \bar{\mathcal{C}}^\dagger J_m \\
&= - \begin{bmatrix} J_{n_3} \mathcal{C}_h^\dagger J_m \\ J_{n_1} \mathcal{C}_{co}^\dagger J_m \\ 0 \end{bmatrix} = - \begin{bmatrix} \mathcal{C}_h^\flat \\ \mathcal{C}_{co}^\flat \\ 0 \end{bmatrix},
\end{aligned}
$$

from which follows that

$$
\mathcal{B}_{co} = -\mathcal{C}_{co}^\flat. \qquad (55)
$$

From this, we also conclude that

$$
\mathcal{B}^\flat T = -\mathcal{C}T = -\bar{\mathcal{C}} = \begin{bmatrix} \mathcal{B}_h^\flat & \mathcal{B}_{co}^\flat & 0 \end{bmatrix},
$$

and, hence,

$$
\begin{aligned}
T^\dagger \mathcal{B}\mathcal{B}^\flat T &= \begin{bmatrix} \mathcal{B}_h \\ \mathcal{B}_{co} \\ 0 \end{bmatrix} \begin{bmatrix} \mathcal{B}_h^\flat & \mathcal{B}_{co}^\flat & 0 \end{bmatrix} \\
&= \begin{bmatrix} \mathcal{B}_h \mathcal{B}_h^\flat & \mathcal{B}_h \mathcal{B}_{co}^\flat & 0 \\ \mathcal{B}_{co} \mathcal{B}_h^\flat & \mathcal{B}_{co} \mathcal{B}_{co}^\flat & 0 \\ 0 & 0 & 0 \end{bmatrix}. \qquad (56)
\end{aligned}
$$

Also,

$$
\begin{aligned}
T^\dagger \mathcal{A}^\flat T &= T^\dagger J_n \mathcal{A}^\dagger J_n T \\
&= (T^\dagger J_n T)(T^\dagger \mathcal{A} T)^\dagger (T^\dagger J_n T) \\
&= \begin{bmatrix} J_{n_3} & 0 & 0 \\ 0 & J_{n_1} & 0 \\ 0 & 0 & J_{n_2} \end{bmatrix} \bar{\mathcal{A}}^\dagger \begin{bmatrix} J_{n_3} & 0 & 0 \\ 0 & J_{n_1} & 0 \\ 0 & 0 & J_{n_2} \end{bmatrix} \\
&= \begin{bmatrix} \mathcal{A}_h^\flat & \mathcal{A}_{21}^\flat & \mathcal{A}_{31}^\flat \\ \mathcal{A}_{12}^\flat & \mathcal{A}_{co}^\flat & 0 \\ \mathcal{A}_{13}^\flat & 0 & \mathcal{A}_{\bar{c}\bar{o}}^\flat \end{bmatrix}. \qquad (57)
\end{aligned}
$$

Now, we multiply both sides of the first of the Eqs. (4) by $T^\dagger$ from the left, and T from the right:

$$
T^\dagger \mathcal{A} T + T^\dagger \mathcal{A}^\flat T + T^\dagger \mathcal{B}\mathcal{B}^\flat T = 0.
$$

Using Eqs. (56) and (57), the $(2,2)$ and $(3,3)$ blocks of the resulting block-matrix equation are, respectively,

$$
\begin{aligned}
\mathcal{A}_{co} + \mathcal{A}_{co}^\flat + \mathcal{B}_{co}\mathcal{B}_{co}^\flat &= 0, \qquad (58) \\
\mathcal{A}_{\bar{c}\bar{o}} + \mathcal{A}_{\bar{c}\bar{o}}^\flat &= 0. \qquad (59)
\end{aligned}
$$

Eqs. (55) and (58) are the physical realizability conditions for the co subsystem, while (59) is the physical realizability condition for the $\bar{c}\bar{o}$ subsystem (which has no inputs/outputs).

Remark 4.5: We emphasize the fact that the transformation matrices $\tilde{T}$ in Eq. (43) and T in Eq. (49) are unitary, in addition to being Bogoliubov or blockwise Bogoliubov, respectively. This property is due to the special structure of linear quantum systems and does not hold in general for linear systems. A consequence of this is that these transformations can be applied in a numerically stable way. Also, similarly to the classical case, they are not unique.

Remark 4.6: The sub-matrices of the matrix $\bar{\mathcal{A}}$ defined in Eq. (52) satisfy the following identity:

$$
\begin{bmatrix} \mathcal{A}_{21} \\ \mathcal{A}_{31} \end{bmatrix} (sI - \mathcal{A}_h)^{-1} \begin{bmatrix} \mathcal{A}_{12} & \mathcal{A}_{13} \end{bmatrix} = 0. \qquad (60)
$$

This result is established in Remark 4.7, in the next subsection. It follows from (60) that

$$
\sigma(\bar{\mathcal{A}}) = \sigma(\mathcal{A}_{co}) \cup \sigma(\mathcal{A}_{\bar{c}\bar{o}}) \cup \sigma(\mathcal{A}_h). \qquad (61)
$$

We end this subsection with an illustrative example.

Example 4.1: Consider the linear quantum system (1)-(2) with parameters

$$
C_- = [1 \ 0], \ C_+ = [0 \ 0], \ \Omega_- = \Omega_+ = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
$$

Then,

$$\boldsymbol{H} = (\boldsymbol{a}_1 + \boldsymbol{a}_1^*)(\boldsymbol{a}_2 + \boldsymbol{a}_2^*), \qquad (62)$$

$$\boldsymbol{L} = \boldsymbol{a}_1. \qquad (63)$$

The transformation matrix T in Eq. (48) is computed to be

$$T = \left[\ T_h\ |\ T_{co}\ \right] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Hence, using Eq. (49) we have

$$\breve{\boldsymbol{a}}_h = T_h^\dagger\ \breve{\boldsymbol{a}} = \begin{bmatrix} a_2 \\ a_2^* \end{bmatrix}, \quad \breve{\boldsymbol{a}}_{co} = T_{co}^\dagger\ \breve{\boldsymbol{a}} = \begin{bmatrix} a_1 \\ a_1^* \end{bmatrix}.$$

The corresponding Kalman canonical form is

$$\dot{\breve{\boldsymbol{a}}}_h(t) = \imath \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \breve{\boldsymbol{a}}_{co}(t),$$

$$\dot{\breve{\boldsymbol{a}}}_{co}(t) = \imath \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \breve{\boldsymbol{a}}_h(t) - \frac{1}{2}\breve{\boldsymbol{a}}_{co}(t) - \breve{\boldsymbol{b}}(t),$$

$$\breve{\boldsymbol{b}}_{\text{out}}(t) = \breve{\boldsymbol{a}}_{co}(t) + \breve{\boldsymbol{b}}(t).$$

It can be easily seen that the transfer function

$$\mathcal{A}_{21}\left(sI - \mathcal{A}_h\right)^{-1}\mathcal{A}_{12} = -\frac{1}{s}\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} = 0,$$

as required by (60).

B. The Kalman decomposition in the real quadrature operator representation

In this subsection, we present the Kalman decomposition for a linear quantum system in the real quadrature operator representation, namely a system of the form (10)-(11).

First, let us introduce the following system variables for R_{co}, $R_{\bar{c}\bar{o}}$, and R_h, in the real quadrature operator representation:

$$\boldsymbol{x}_{co} \equiv \begin{bmatrix} \boldsymbol{q}_{co} \\ \boldsymbol{p}_{co} \end{bmatrix} \triangleq V_{n_1}\breve{\boldsymbol{a}}_{co}, \ \boldsymbol{x}_{\bar{c}\bar{o}} \equiv \begin{bmatrix} \boldsymbol{q}_{\bar{c}\bar{o}} \\ \boldsymbol{p}_{\bar{c}\bar{o}} \end{bmatrix} \triangleq V_{n_2}\breve{\boldsymbol{a}}_{\bar{c}\bar{o}},$$

$$\tilde{\boldsymbol{x}}_h \equiv \begin{bmatrix} \tilde{\boldsymbol{q}}_h \\ \tilde{\boldsymbol{p}}_h \end{bmatrix} \triangleq V_{n_3}\breve{\boldsymbol{a}}_h. \qquad (64)$$

Then, the following result is a direct consequence of Theorem 4.1, which gives the Kalman decomposition for the linear quantum system (10)-(11):

Theorem 4.3: Let $\tilde{S} \triangleq V_n\tilde{T}V_n^\dagger$, where $\tilde{T}$ is given by Eq. (42). $\tilde{S}$ is a real orthogonal and symplectic coordinate transformation that decomposes the space of variables of the linear quantum system (10)-(11), as follows:

$$\left[\ \tilde{\boldsymbol{q}}_h^\top\ \ \boldsymbol{q}_{co}^\top\ \ \boldsymbol{q}_{\bar{c}\bar{o}}^\top\ \ \tilde{\boldsymbol{p}}_h^\top\ \ \boldsymbol{p}_{co}^\top\ \ \boldsymbol{p}_{\bar{c}\bar{o}}^\top\ \right]^\top = \tilde{S}^\top \boldsymbol{x}. \qquad (65)$$

Proof: Firstly, since $\tilde{T}$ is Bogoliubov and V_n is unitary, $\tilde{S} = V_n\tilde{T}V_n^\dagger$ in Eq. (65) is real symplectic. Secondly, $\tilde{S}$ is unitary because it is a product of three unitary matrices. A real unitary matrix is orthogonal. Thus, $\tilde{S}$ is a real orthogonal and symplectic coordinate transformation.

$$\tilde{T} = T\Lambda, \qquad (66)$$

where

$$\Lambda \triangleq \begin{bmatrix} I_{n_3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n_3} & 0 & 0 \\ 0 & I_{n_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{n_1} & 0 \\ 0 & 0 & I_{n_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{n_2} \end{bmatrix} \qquad (67)$$

Finally, by Eqs. (9), (43), (44), (45), and (47), we get

$$\tilde{S}^\top \boldsymbol{x} = \tilde{S}^\dagger \boldsymbol{x} = V_n\tilde{T}^\dagger V_n^\dagger \boldsymbol{x} = V_n\tilde{T}^\dagger \breve{\boldsymbol{a}}$$

$$= V_n \begin{bmatrix} \boldsymbol{a}_h \\ \boldsymbol{a}_{co} \\ \boldsymbol{a}_{\bar{c}\bar{o}} \\ \boldsymbol{a}_h^\# \\ \boldsymbol{a}_{co}^\# \\ \boldsymbol{a}_{\bar{c}\bar{o}}^\# \end{bmatrix} = \begin{bmatrix} \tilde{\boldsymbol{q}}_h \\ \boldsymbol{q}_{co} \\ \boldsymbol{q}_{\bar{c}\bar{o}} \\ \tilde{\boldsymbol{p}}_h \\ \boldsymbol{p}_{co} \\ \boldsymbol{p}_{\bar{c}\bar{o}} \end{bmatrix},$$

which is Eq. (65). ∎

Now, we proceed to prove the analog of Theorem 4.2, namely Theorem 4.4. However, before we do this, we introduce a new set of variables for the $h = c\bar{o} \cup \bar{c}o$ subspace, in the real quadrature operator representation. The reason for this is that using these new variables, we can reveal more structure in the real quadrature operator representation of Kalman canonical form system matrices (70) than in the creation-annihilation operator representation (52).

To this end, let us define the matrix $\Pi \in \mathbb{C}^{2n_3 \times 2n_3}$ by

$$\Pi \triangleq \begin{bmatrix} I_{n_a} & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_{n_b} \\ 0 & 0 & I_{n_a} & 0 \\ 0 & I_{n_b} & 0 & 0 \end{bmatrix}.$$

It is easy to verify that $\Pi\Pi^\top = \Pi^\top\Pi = I_{2n_3}$, and $\Pi\mathbb{J}_{n_3}\Pi^\top = \Pi^\top\mathbb{J}_{n_3}\Pi = \mathbb{J}_{n_3}$, that is Π is orthogonal and symplectic. Now, let

$$\tilde{V}_{n_3} \triangleq \Pi V_{n_3}, \qquad (68)$$

and define a new set of system variables for R_h by

$$\boldsymbol{x}_h \equiv \begin{bmatrix} \boldsymbol{q}_h \\ \boldsymbol{p}_h \end{bmatrix} \triangleq \tilde{V}_{n_3}\breve{\boldsymbol{a}}_h = \Pi \begin{bmatrix} \tilde{\boldsymbol{q}}_h \\ \tilde{\boldsymbol{p}}_h \end{bmatrix}, \qquad (69)$$

using Eqs. (64) and (68). Since Π is real symplectic, it follows that $\boldsymbol{q}_h$ and $\boldsymbol{p}_h$ are self-adjoint operators, and that $[\boldsymbol{q}_h, \boldsymbol{q}_h^\top] = 0$, $[\boldsymbol{p}_h, \boldsymbol{p}_h^\top] = 0$, and $[\boldsymbol{q}_h, \boldsymbol{p}_h^\top] = \imath I_{n_3}$. Hence, $\boldsymbol{q}_{h,i}$ and $\boldsymbol{p}_{h,i}$ are conjugate observables for $i = 1, \ldots, n_3$. We find it preferable to work with $\boldsymbol{q}_h$ and $\boldsymbol{p}_h$, rather than $\tilde{\boldsymbol{q}}_h$ and $\tilde{\boldsymbol{p}}_h$, because they allow us to transform the linear quantum system (10)-(11) to the standard Kalman canonical form, as to be given in Theorem 4.4.

To prove the analog of Theorem 4.2, namely Theorem 4.4, we need two lemmas. Lemma 4.8 transforms the structure of the system matrices in Eqs. (50)-(51) to the real quadrature representation with variables $(\boldsymbol{q}_h, \boldsymbol{p}_h, \boldsymbol{x}_{co}, \boldsymbol{x}_{\bar{c}\bar{o}})$, and Lemma 4.9 establishes properties of the matrix that transforms the system to this representation.

Lemma 4.8: Let

$$\tilde{V}_n \triangleq \begin{bmatrix} \tilde{V}_{n_3} & & \mathbf{0} \\ & V_{n_1} & \\ \mathbf{0} & & V_{n_2} \end{bmatrix}.$$

Then,

$$\bar{A} \triangleq \tilde{V}_n \bar{\mathcal{A}} \tilde{V}_n^\dagger = \left[\begin{array}{cc|c|c} A_h^{11} & A_h^{12} & A_{12} & A_{13} \\ 0 & A_h^{22} & 0 & 0 \\ \hline 0 & A_{21} & A_{co} & 0 \\ \hline 0 & A_{31} & 0 & A_{\bar{c}\bar{o}} \end{array}\right],$$

$$\bar{B} \triangleq \tilde{V}_n \bar{\mathcal{B}} V_m^\dagger = \left[\begin{array}{c} B_h \\ 0 \\ \hline B_{co} \\ \hline 0 \end{array}\right],$$

$$\bar{C} \triangleq V_m \bar{\mathcal{C}} \tilde{V}_n^\dagger = \left[\begin{array}{cc|c|c} 0 & C_h & C_{co} & 0 \end{array}\right]. \quad (70)$$

Proof: It follows from the definitions of $\bar{A}$, $\bar{B}$, and $\bar{C}$ in Eq. (70), along with Eq. (52), that $\bar{A} = \hat{T}^\dagger \mathcal{A} \hat{T}$, $\bar{B} = \hat{T}^\dagger \mathcal{B} V_m^\dagger$, and $\bar{C} = V_m \mathcal{C} \hat{T}$, where

$$\hat{T} \triangleq T \tilde{V}_n^\dagger = \left[\begin{array}{ccc} T_h \tilde{V}_{n_3}^\dagger & T_{co} V_{n_1}^\dagger & T_{\bar{c}\bar{o}} V_{n_2}^\dagger \end{array}\right]$$
$$\equiv \left[\begin{array}{ccc} \hat{T}_h & \hat{T}_{co} & \hat{T}_{\bar{c}\bar{o}} \end{array}\right].$$

Since the columns of T_h, T_{co} and $T_{\bar{c}\bar{o}}$ are orthonormal bases of R_h, R_{co} and $R_{\bar{c}\bar{o}}$, respectively, and $\tilde{V}_{n_3}$, V_{n_1}, and V_{n_2} are unitary, the same is true for $\hat{T}_h$, $\hat{T}_{co}$ and $\hat{T}_{\bar{c}\bar{o}}$. Using the definitions of T_h and $\tilde{V}_{n_3}$, we can show that

$$\hat{T}_h = T_h \tilde{V}_{n_3}^\dagger = \frac{1}{\sqrt{2}} \left[\begin{array}{cc|cc} X & -\imath Y & \imath X & Y \\ X\# & \imath Y\# & -\imath X\# & Y\# \end{array}\right]$$
$$=: \left[\begin{array}{c|c} \hat{T}_{c\bar{o}} & \hat{T}_{\bar{c}o} \end{array}\right].$$

That is, the columns of $\hat{T}_h$ are the union of a basis for $R_{c\bar{o}}$, namely the columns of $\hat{T}_{c\bar{o}} = \left[\begin{smallmatrix} X & -\imath Y \\ X\# & \imath Y\# \end{smallmatrix}\right]$, and a basis for $R_{c\bar{o}}$, namely the columns of $\hat{T}_{\bar{c}o} = \left[\begin{smallmatrix} \imath X & Y \\ -\imath X\# & Y\# \end{smallmatrix}\right]$; see also Lemma 4.7. The structure of $\bar{A}$, $\bar{B}$, and $\bar{C}$ in Eq. (70) then follows from the invariance properties Eqs. (53)-(54). For example, $\mathcal{A} R_{c\bar{o}} \subset R_{c\bar{o}}$ implies $\hat{T}_{\bar{c}o}^\dagger \mathcal{A} \hat{T}_{c\bar{o}} = 0$. Hence, the $(2,1)$ block of $\bar{A}$ is zero. Similarly, $\mathcal{A} R_{co} \subset R_{c\bar{o}} + R_{co}$ implies $\hat{T}_{\bar{c}o}^\dagger \mathcal{A} \hat{T}_{co} = 0$. Hence, the $(2,3)$ block of $\bar{A}$ is zero. The rest of the block-zero entries of $\bar{\mathcal{A}}$, $\bar{\mathcal{B}}$, and $\bar{\mathcal{C}}$ can be obtained similarly. ∎

Remark 4.7: The structure of the matrix $\bar{A}$ given in (70) implies

$$\left[\begin{array}{cc} 0 & A_{21} \\ 0 & A_{31} \end{array}\right] \left(sI - \left[\begin{array}{cc} A_h^{11} & A_h^{12} \\ 0 & A_h^{22} \end{array}\right]\right)^{-1} \left[\begin{array}{cc} A_{12} & A_{13} \\ 0 & 0 \end{array}\right]$$
$$= 0. \quad (71)$$

Also, it follows from Eqs. (52) and (70) that

$$\left[\begin{array}{cc} A_h^{11} & A_h^{12} \\ 0 & A_h^{22} \end{array}\right] = \tilde{V}_{n_3} \mathcal{A}_h \tilde{V}_{n_3}^\dagger;$$
$$\left[\begin{array}{cc} A_{12} & A_{13} \\ 0 & 0 \end{array}\right] = \tilde{V}_{n_3} \left[\begin{array}{cc} \mathcal{A}_{12} & \mathcal{A}_{13} \end{array}\right] \left[\begin{array}{cc} V_{n_1}^\dagger & 0 \\ 0 & V_{n_2}^\dagger \end{array}\right];$$
$$\left[\begin{array}{cc} 0 & A_{21} \\ 0 & A_{31} \end{array}\right] = \left[\begin{array}{cc} V_{n_1} & 0 \\ 0 & V_{n_2} \end{array}\right] \left[\begin{array}{c} \mathcal{A}_{21} \\ \mathcal{A}_{31} \end{array}\right] \tilde{V}_{n_3}^\dagger.$$

Then, since the matrices $\tilde{V}_{n_3}$, V_{n_1} and V_{n_2} are unitary, Eq. (71) implies that the condition (60) is satisfied.

Lemma 4.9: Define $S \triangleq V_n T \tilde{V}_n^\dagger$, where T is defined in Eq. (48). Then, S is real, orthogonal and *blockwise symplectic*; i.e., it satisfies

$$S^\top \mathbb{J}_n S = \left[\begin{array}{ccc} \mathbb{J}_{n_3} & & 0 \\ & \mathbb{J}_{n_1} & \\ 0 & & \mathbb{J}_{n_2} \end{array}\right]. \quad (72)$$

Proof: First, notice that,

$$S^\dagger \left[\begin{array}{c} \boldsymbol{q} \\ \boldsymbol{p} \end{array}\right] = \tilde{V}_n T^\dagger \breve{\boldsymbol{a}} = \tilde{V}_n \left[\begin{array}{c} \breve{\boldsymbol{a}}_h \\ \breve{\boldsymbol{a}}_{co} \\ \breve{\boldsymbol{a}}_{\bar{c}\bar{o}} \end{array}\right] = \left[\begin{array}{c} \boldsymbol{q}_h \\ \boldsymbol{p}_h \\ \hline \boldsymbol{x}_{co} \\ \hline \boldsymbol{x}_{\bar{c}\bar{o}} \end{array}\right]. \quad (73)$$

Since $\boldsymbol{q}$, $\boldsymbol{p}$, $\boldsymbol{q}_h$, $\boldsymbol{p}_h$, $\boldsymbol{x}_{co}$, and $\boldsymbol{x}_{\bar{c}\bar{o}}$ are all self-adjoint, S is real. S is also unitary, as a product of unitary matrices, and hence it is orthogonal. Finally, using the equations $V_n^\dagger \mathbb{J}_n V_n = -\imath J_n \Leftrightarrow V_n J_n V_n^\dagger = \imath \mathbb{J}_n$, and $T^\dagger J_n T = \mathrm{diag}(J_{n_3}, J_{n_1}, J_{n_3})$, we have that

$$S^\top \mathbb{J}_n S = -\imath \left[\begin{array}{ccc} \tilde{V}_{n_3} J_{n_3} \tilde{V}_{n_3}^\dagger & & 0 \\ & V_{n_1} J_{n_1} V_{n_1}^\dagger & \\ 0 & & V_{n_2} J_{n_2} V_{n_1}^\dagger \end{array}\right]$$
$$= \left[\begin{array}{ccc} \Pi \mathbb{J}_{n_3} \Pi^\top & & 0 \\ & \mathbb{J}_{n_1} & \\ 0 & & \mathbb{J}_{n_2} \end{array}\right],$$

from which Eq. (72) follows, because $\Pi \mathbb{J}_{n_3} \Pi^\top = \mathbb{J}_{n_3}$. ∎

Now we can state the analog of Theorem 4.2 in the real quadrature operator representation.

Theorem 4.4: The real orthogonal and blockwise symplectic coordinate transformation

$$\left[\begin{array}{c} \boldsymbol{q}_h \\ \boldsymbol{p}_h \\ \hline \boldsymbol{x}_{co} \\ \hline \boldsymbol{x}_{\bar{c}\bar{o}} \end{array}\right] = S^\top \boldsymbol{x} \quad (74)$$

transforms the linear quantum system (10)-(11) into the form

$$\left[\begin{array}{c} \dot{\boldsymbol{q}}_h(t) \\ \dot{\boldsymbol{p}}_h(t) \\ \hline \dot{\boldsymbol{x}}_{co}(t) \\ \hline \dot{\boldsymbol{x}}_{\bar{c}\bar{o}}(t) \end{array}\right] = \bar{A} \left[\begin{array}{c} \boldsymbol{q}_h(t) \\ \boldsymbol{p}_h(t) \\ \hline \boldsymbol{x}_{co}(t) \\ \hline \boldsymbol{x}_{\bar{c}\bar{o}}(t) \end{array}\right] + \bar{B} \boldsymbol{u}(t), \quad (75)$$

$$\boldsymbol{y}(t) = \bar{C} \left[\begin{array}{c} \boldsymbol{q}_h(t) \\ \boldsymbol{p}_h(t) \\ \hline \boldsymbol{x}_{co}(t) \\ \hline \boldsymbol{x}_{\bar{c}\bar{o}}(t) \end{array}\right] + \boldsymbol{u}(t), \quad (76)$$

where matrices $\bar{A}, \bar{B}, \bar{C}$ were given in Eq. (70). After a re-arrangement, the system (75)-(76) becomes

$$\left[\begin{array}{c} \dot{\boldsymbol{q}}_h(t) \\ \dot{\boldsymbol{x}}_{co}(t) \\ \dot{\boldsymbol{x}}_{\bar{c}\bar{o}}(t) \\ \dot{\boldsymbol{p}}_h(t) \end{array}\right] = \left[\begin{array}{cccc} A_h^{11} & A_{12} & A_{13} & A_h^{12} \\ 0 & A_{co} & 0 & A_{21} \\ 0 & 0 & A_{\bar{c}\bar{o}} & A_{31} \\ 0 & 0 & 0 & A_h^{22} \end{array}\right] \left[\begin{array}{c} \boldsymbol{q}_h(t) \\ \boldsymbol{x}_{co}(t) \\ \boldsymbol{x}_{\bar{c}\bar{o}}(t) \\ \boldsymbol{p}_h(t) \end{array}\right]$$
$$+ \left[\begin{array}{c} B_h \\ B_{co} \\ 0 \\ 0 \end{array}\right] \boldsymbol{u}(t), \quad (77)$$

$$\boldsymbol{y}(t) = \left[\begin{array}{cccc} 0 & C_{co} & 0 & C_h \end{array}\right] \left[\begin{array}{c} \boldsymbol{q}_h(t) \\ \boldsymbol{x}_{co}(t) \\ \boldsymbol{x}_{\bar{c}\bar{o}}(t) \\ \boldsymbol{p}_h(t) \end{array}\right] + \boldsymbol{u}(t). \quad (78)$$

A block diagram for the system (75)-(76) is given in Fig. 2.

Proof: By Lemma 4.9, S is real. Therefore, Eq. (74) is a re-statement of Eq. (73). As a result, Theorem 4.4 follows from

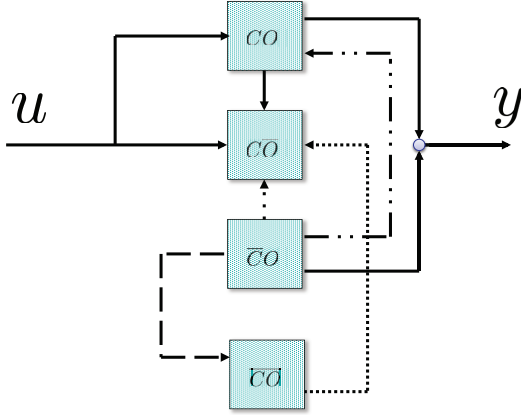the transformation (9), the transformation (73), Theorem 4.2, and Lemma 4.8. ∎

Fig. 2. Kalman decomposition of a linear quantum system. The solid lines indicate that the blocks can be either controlled by the input $\boldsymbol{u}$ or observed via the output $\boldsymbol{y}$.

Remark 4.8: From Eq. (77), we conclude that $\sigma(A) = \sigma(\bar{A}) = \sigma(A_{co}) \cup \sigma(A_{\bar{c}\bar{o}}) \cup \sigma(A_h^{11}) \cup \sigma(A_h^{22})$. The definitions of A_{co}, $A_{\bar{c}\bar{o}}$, A_h^{11}, and A_h^{22} in Eq. (70), also imply that (61) is satisfied.

Remark 4.9: It can be seen from (75)-(76) or (77)-(78) that, $\boldsymbol{q}_{h,i}$, $i = 1, \ldots, n_3$, are controllable but unobservable, while $\boldsymbol{p}_{h,i}$, $i = 1, \ldots, n_3$, are observable but uncontrollable. We see that every $c\bar{o}$ variable must have an associated $\bar{c}o$ variable. That is, they appear in conjugate pairs. Notice that the variables $\boldsymbol{p}_{h,i}$ commute with each other at equal times. Also, as seen from Eq. (77), they evolve without any influence from the inputs or other system variables. As shown in [39], the set of $\boldsymbol{p}_{h,i}$, $i = 1, ..., n_3$, is a QMFS satisfying Eq. (15), see Definition 2.3 and References [39] and [47]. This implies that each $\boldsymbol{p}_{h,i}$ satisfies Eq. (14), hence, each $\boldsymbol{p}_{h,i}$ is a QND variable, see Definition 2.2 and References [44] and [49]. Moreover, $\boldsymbol{x}_{\bar{c}\bar{o},i}$, $i = 1, \ldots, n_2$, are DF modes, see Definition 2.1 and References [6], [48], [49], and [11]. Finally, we emphasize the fact that, not all linear quantum systems contain QND variables and DF modes. Indeed, as in the classical case, for a specific system, some of the subsystems may not be present; see Examples 4.1, 4.2 and Section V for more details.

Remark 4.10: In (77)-(78), recall that $\boldsymbol{u} = \left[\begin{smallmatrix} \boldsymbol{q}_{\text{in}} \\ \boldsymbol{p}_{\text{in}} \end{smallmatrix}\right]$ and $\boldsymbol{y} = \left[\begin{smallmatrix} \boldsymbol{q}_{\text{out}} \\ \boldsymbol{p}_{\text{out}} \end{smallmatrix}\right]$, as defined in Eq. (9). Partition the matrices B_{co} and C_{co} accordingly as

$$B_{co} = \left[\begin{array}{cc} B_{co,q} & B_{co,p} \end{array}\right], \ C_{co} = \left[\begin{array}{c} C_{co,q} \\ C_{co,p} \end{array}\right].$$

If the transfer function $\Xi_{\boldsymbol{p}_{\text{in}} \to \boldsymbol{q}_{\text{out}}}(s) = C_{co,q}(sI - A_{co})^{-1} B_{co,p} = 0$, then the input noise quadrature $\boldsymbol{p}_{\text{in}}$ has no influence on the output quadrature $\boldsymbol{q}_{\text{out}}$. In this case, the system (77)-(78) realizes the BAE measurement of the output $\boldsymbol{q}_{\text{out}}$ with respect to the input $\boldsymbol{p}_{\text{in}}$. Similarly, if the transfer function $\Xi_{\boldsymbol{q}_{\text{in}} \to \boldsymbol{p}_{\text{out}}}(s) = C_{co,p}(sI - A_{co})^{-1} B_{co,q} = 0$, then the system (77)-(78) realizes the BAE measurement of the output $\boldsymbol{p}_{\text{out}}$ with respect to the input $\boldsymbol{q}_{\text{in}}$. These properties will be demonstrated

in Example 4.2 and Example 5.2. In the special case when there is no mode $\boldsymbol{x}_{co}$ in the system, we have

$$\left[\begin{array}{c} \boldsymbol{q}_{\text{out}} \\ \boldsymbol{p}_{\text{out}} \end{array}\right] = C_h \boldsymbol{p}_h + \left[\begin{array}{c} \boldsymbol{q}_{\text{in}} \\ \boldsymbol{p}_{\text{in}} \end{array}\right].$$

Since $\boldsymbol{p}_h$ is uncontrollable, it is clear that in this case, BAE measurements are naturally achieved.

Finally, we have the following result as a corollary of Theorems 4.4 and 3.2.

Corollary 4.1: The Kalman canonical form of a passive linear quantum system in the real quadrature operator representation can be achieved by a real orthogonal transformation, and is as follows:

$$\left[\begin{array}{c} \dot{\boldsymbol{x}}_{co}(t) \\ \dot{\boldsymbol{x}}_{\bar{c}\bar{o}}(t) \end{array}\right] = \left[\begin{array}{cc} A_{co} & 0 \\ 0 & A_{\bar{c}\bar{o}} \end{array}\right] \left[\begin{array}{c} \boldsymbol{x}_{co}(t) \\ \boldsymbol{x}_{\bar{c}\bar{o}}(t) \end{array}\right]$$
$$+ \left[\begin{array}{c} B_{co} \\ 0 \end{array}\right] \boldsymbol{u}(t), \tag{79}$$

$$\boldsymbol{y}(t) = \left[\begin{array}{cc} C_{co} & 0 \end{array}\right] \left[\begin{array}{c} \boldsymbol{x}_{co}(t) \\ \boldsymbol{x}_{\bar{c}\bar{o}}(t) \end{array}\right] + \boldsymbol{u}(t). \tag{80}$$

Here, all eigenvalues of the matrix $A_{\bar{c}\bar{o}}$ are located on the imaginary axis, and have geometric multiplicity one. Also, the real parts of the eigenvalues of the matrix $\tilde{A}_{co}$ are strictly negative.

We end this subsection with an illustrative example.

Example 4.2: For the system in Example 4.1, the Hamiltonian $\boldsymbol{H}$ and the coupling operator $\boldsymbol{L}$ in Eqs. (62)-(63) are given in the real quadrature representation of the system by $\boldsymbol{H} = 2\boldsymbol{q}_1\boldsymbol{q}_2$ and $\boldsymbol{L} = \frac{1}{\sqrt{2}}(\boldsymbol{q}_1 + \imath\boldsymbol{p}_1)$, respectively. By applying Theorem 4.4, we find that the system variables in the real quadrature representation form of the Kalman decomposition are given by $\boldsymbol{q}_h = -\boldsymbol{p}_2$, $\boldsymbol{p}_h = \boldsymbol{q}_2$, $\boldsymbol{q}_{co} = \boldsymbol{q}_1$, $\boldsymbol{p}_{co} = \boldsymbol{p}_1$. Also, the corresponding QSDEs are as follows:

$$\begin{array}{rcl} \dot{\boldsymbol{p}}_2(t) & = & -2\boldsymbol{q}_1(t), \\ \dot{\boldsymbol{q}}_1(t) & = & -0.5\boldsymbol{q}_1(t) - \boldsymbol{q}_{\text{in}}(t), \\ \dot{\boldsymbol{p}}_1(t) & = & -0.5\boldsymbol{p}_1(t) - 2\boldsymbol{q}_2(t) - \boldsymbol{p}_{\text{in}}(t), \\ \dot{\boldsymbol{q}}_2(t) & = & 0, \\ \boldsymbol{q}_{\text{out}}(t) & = & \boldsymbol{q}_1(t) + \boldsymbol{q}_{\text{in}}(t), \\ \boldsymbol{p}_{\text{out}}(t) & = & \boldsymbol{p}_1(t) + \boldsymbol{p}_{\text{in}}(t). \end{array}$$

It can be readily shown that

(i) $\boldsymbol{p}_2$ is controllable but unobservable, while $\boldsymbol{q}_2$ is observable but uncontrollable. So, $\boldsymbol{q}_2$ is a QND variable.

(ii) Because the transfer function $\Xi_{\boldsymbol{q}_{\text{in}} \to \boldsymbol{p}_{\text{out}}}(s) = 0$, the system realizes a BAE measurement of $\boldsymbol{p}_{\text{out}}$ with respect to $\boldsymbol{q}_{\text{in}}$.

(iii) Similarly, the system realizes a BAE measurement of $\boldsymbol{q}_{\text{out}}$ with respect to $\boldsymbol{p}_{\text{in}}$.

C. Some special cases of the Kalman decomposition

In this subsection, we study two special cases of the Kalman decomposition.

Proposition 4.1: If $\text{Ker}(O_s)$ is an invariant space of Ω, then $\mathcal{A}_{13} = 0$ and $\mathcal{A}_{31} = 0$ in Eq. (52).

Proof: Suppose $x \in R_{\bar{c}\bar{o}}$. Then $O_s x = 0$. As a result, $O_s \mathcal{A} x = -\imath O_s J_n \Omega x - \frac{1}{2} O_s \mathcal{C}^\flat \mathcal{C} x = 0$. That is, $\mathcal{A} x \in \mathrm{Ker}(O_s)$. On the other hand, if $\mathrm{Ker}(O_s)$ is an invariant space of Ω, then $\Omega x \in \mathrm{Ker}(O_s)$ for all $x \in \mathrm{Ker}(O_s)$. As a result, $O_s J_n \mathcal{A} x = -\imath O_s J_n J_n \Omega x - \frac{1}{2} O_s J_n \mathcal{C}^\flat \mathcal{C} x = -\imath O_s \Omega x = 0$. That is, $\mathcal{A} x \in \mathrm{Ker}(O_s J_n)$. Consequently, $\mathcal{A} R_{\bar{c}\bar{o}} \subset R_{\bar{c}\bar{o}}$. Hence, $\mathcal{A}_{13} = 0$. Next we show that $\mathcal{A}_{31} = 0$. For $x \in R_{\bar{c}\bar{o}}$, by Eq. (28) we have $J_n x \in J_n R_{\bar{c}\bar{o}} = R_{\bar{c}\bar{o}}$. Consequently, $\mathcal{A}^\dagger x = \imath \Omega J_n x - \frac{1}{2} \mathcal{C}^\dagger J_m \mathcal{C} J_n x = \imath \Omega J_n x \in R_{\bar{c}\bar{o}}$. So we have

$$\mathcal{A}^\dagger R_{\bar{c}\bar{o}} \subset R_{\bar{c}\bar{o}}. \tag{81}$$

Given $x \in R_{\bar{c}o} + R_{c\bar{o}} + R_{co}$, let $\mathcal{A} x = y_1 + y_2$ where $y_1 \in R_{\bar{c}\bar{o}}$ while $y_2 \in (R_{\bar{c}\bar{o}})^\perp = R_{\bar{c}o} + R_{c\bar{o}} + R_{co}$. Then $y_1^\dagger \mathcal{A} x = y_1^\dagger y_1 + y_1^\dagger y_2 = y_1^\dagger y_1$. However, by Eq. (81), we have $\mathcal{A}^\dagger y_1 \in R_{\bar{c}\bar{o}}$, and hence $y_1^\dagger \mathcal{A} x = (\mathcal{A}^\dagger y_1)^\dagger x = 0$. As a result, $y_1^\dagger y_1 = 0$, i.e., $y_1 = 0$ and $\mathcal{A} x \in R_{\bar{c}o} + R_{c\bar{o}} + R_{co}$. Thus, we have

$$\mathcal{A}(R_{\bar{c}o} + R_{c\bar{o}}) \subset \mathcal{A}(R_{\bar{c}o} + R_{c\bar{o}} + R_{co}) \subset (R_{\bar{c}o} + R_{c\bar{o}} + R_{co}).$$

This implies $\mathcal{A}_{31} = 0$. ∎

Remark 4.11: In some sense, Proposition 4.1 slightly relaxes the condition that $\mathrm{Ker}(O_s) \cap \mathrm{Ker}(O_s J_n)$ is an invariant subspace of Ω in [11, Lemma 1].

Proposition 4.2: If

$$\mathrm{Ker}(\mathcal{C})^\perp \perp \mathrm{Ker}(O_s J_n), \tag{82}$$

then $\mathcal{B}_h = 0$ and $\mathcal{C}_h = 0$ in Eq. (52).

Proof: Eq. (82) can be restated as $\mathrm{Ker}(\mathcal{C})^\perp \perp R_{\bar{c}o} \oplus R_{\bar{c}\bar{o}}$, which implies $\mathrm{Ker}(\mathcal{C})^\perp \perp R_{\bar{c}o}$. Since $\mathrm{Im}(\mathcal{B}) = \mathrm{Im}(-J_n \mathcal{C}^\dagger J_m) = J_n \mathrm{Im}(\mathcal{C}^\dagger) = J_n \mathrm{Ker}(\mathcal{C})^\perp$, we have that $\mathrm{Im}(\mathcal{B}) \perp J_n R_{\bar{c}o} = R_{c\bar{o}}$. Hence, $\mathcal{B}_h = 0$. Also, Eq. (82) implies $\mathrm{Ker}(\mathcal{C})^\perp \subseteq (R_{\bar{c}o} \oplus R_{\bar{c}\bar{o}})^\perp$, and equivalently $\mathrm{Ker}(\mathcal{C}) \supseteq R_{\bar{c}o} \oplus R_{\bar{c}\bar{o}}$. Then, $\mathrm{Ker}(\mathcal{C}) \supseteq R_{\bar{c}o}$, which implies $\mathcal{C}_h = 0$. ∎

V. APPLICATIONS

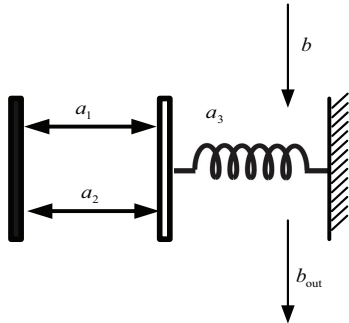In this section, we apply the Kalman decomposition theory developed to two physical systems.

Fig. 3. An opto-mechanical system

Example 5.1: In this example, we investigate an opto-mechanical system, as shown in Fig. 3. The optical cavity has two optical modes, $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$. The cavity is coupled to a mechanical oscillator with mode $\boldsymbol{a}_3$, whose resonant frequency is ω_m. We ignore the optical damping, but keep the mechanical damping as represented by $\boldsymbol{b}$ in Fig. 3. (Although the external mode $\boldsymbol{b}$ is thermal noise [6], we treat it here as a general quantum input, because our purpose is only to demonstrate our results.) The coupling operator of the system is $\boldsymbol{L} = \sqrt{\kappa} \boldsymbol{a}_3$, where $\kappa > 0$ is a coupling constant. Denote the optical detunings for $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ as Δ_1 and Δ_2, respectively. The Hamiltonian of the system is given by

$$\begin{aligned} \boldsymbol{H} = \ & \lambda_1 \frac{\boldsymbol{a}_1 + \boldsymbol{a}_1^*}{\sqrt{2}} \frac{\boldsymbol{a}_3 + \boldsymbol{a}_3^*}{\sqrt{2}} + \lambda_2 \frac{\boldsymbol{a}_2 + \boldsymbol{a}_2^*}{\sqrt{2}} \frac{\boldsymbol{a}_3 + \boldsymbol{a}_3^*}{\sqrt{2}} \\ & - \Delta_1 \boldsymbol{a}_1^* \boldsymbol{a}_1 - \Delta_2 \boldsymbol{a}_2^* \boldsymbol{a}_2 + \omega_m \boldsymbol{a}_3^* \boldsymbol{a}_3, \end{aligned} \tag{83}$$

where $\lambda_1, \lambda_2 > 0$ are the opto-mechanical couplings. In the following, we discuss three cases of opto-mechanical couplings, [2, Sec. III]. Also, we let

$$\lambda = \sqrt{\lambda_1^2 + \lambda_2^2}, \quad \rho_1 = \lambda_1/\lambda, \quad \rho_2 = \lambda_2/\lambda. \tag{84}$$

Case 1: Red-detuned regime (laser detuning for the two cavity modes below the resonant frequency of the mechanical oscillator). In this case, $\Delta_1 = \Delta_2 = -\omega_m$. In this regime, the existence of an opto-mechanical dark mode has been experimentally demonstrated in [6]. The opto-mechanical dark mode is a coherent superposition of the two optical modes $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, and is decoupled from the mechanical mode $\boldsymbol{a}_3$. Therefore, it is immune to thermal noise, the major source of decoherence in this type of opto-mechanical systems. In what follows, we apply the theory proposed in this paper to derive the opto-mechanical dark mode in [6]. In the rotating frame $\boldsymbol{a}_1(t) \to \boldsymbol{a}_1(t) e^{\imath \omega_m t}$, $\boldsymbol{a}_2(t) \to \boldsymbol{a}_2(t) e^{\imath \omega_m t}$, and $\boldsymbol{a}_3(t) \to \boldsymbol{a}_3(t) e^{\imath \omega_m t}$ (see, e.g., [2, Eq. (31)]), the Hamiltonian (83) can be approximated by

$$\begin{aligned} \boldsymbol{H}_R = \ & \omega_m(\boldsymbol{a}_1^* \boldsymbol{a}_1 + \boldsymbol{a}_2^* \boldsymbol{a}_2 + \boldsymbol{a}_3^* \boldsymbol{a}_3) \\ & + \frac{\lambda_1}{2}(\boldsymbol{a}_1 \boldsymbol{a}_3^* + \boldsymbol{a}_1^* \boldsymbol{a}_3) + \frac{\lambda_2}{2}(\boldsymbol{a}_2 \boldsymbol{a}_3^* + \boldsymbol{a}_2^* \boldsymbol{a}_3). \end{aligned}$$

In this case, the system is passive. The coordinate transformation

$$\begin{bmatrix} \boldsymbol{a}_{DF} \\ \boldsymbol{a}_D \end{bmatrix} = T^\dagger \breve{\boldsymbol{a}} = \begin{bmatrix} \rho_2 \boldsymbol{a}_1 - \rho_1 \boldsymbol{a}_2 \\ \rho_1 \boldsymbol{a}_1 + \rho_2 \boldsymbol{a}_2 \\ \boldsymbol{a}_3 \end{bmatrix}$$

yields the following Kalman decomposition:

$$\begin{aligned} \dot{\boldsymbol{a}}_{DF}(t) &= -\imath \omega_m \boldsymbol{a}_{DF}(t), \\ \dot{\boldsymbol{a}}_D(t) &= - \begin{bmatrix} \imath \omega_m & \imath \frac{\lambda}{2} \\ \imath \frac{\lambda}{2} & \frac{\kappa}{2} + \imath \omega_m \end{bmatrix} \boldsymbol{a}_D(t) - \begin{bmatrix} 0 \\ \sqrt{\kappa} \end{bmatrix} \boldsymbol{b}(t) \\ \boldsymbol{b}_{\mathrm{out}}(t) &= \begin{bmatrix} 0 & \sqrt{\kappa} \end{bmatrix} \boldsymbol{a}_D(t) + \boldsymbol{b}(t). \end{aligned}$$

Clearly, $\boldsymbol{a}_{DF}$ is a DF mode (which is denoted $\hat{\boldsymbol{a}}_D$ in [6]). It is a linear combination of the two cavity modes and is decoupled from the mechanical mode, thus being immune from the mechanical damping. This phenomenon has been observed in [40], where the mode has been called "mechanically dark". Finally, in the real quadrature operator representation, the DF mode is

$$V_1 \begin{bmatrix} \boldsymbol{a}_{DF} \\ \boldsymbol{a}_{DF}^* \end{bmatrix} = \begin{bmatrix} \rho_2 \boldsymbol{q}_1 - \rho_1 \boldsymbol{q}_2 \\ \rho_2 \boldsymbol{p}_1 - \rho_1 \boldsymbol{p}_2 \end{bmatrix}. \tag{85}$$

Case 2: Blue-detuned regime (laser detuning for the two cavity modes above the resonant frequency of the mechanical

oscillator). In this case, $\Delta_1 = \Delta_2 = \omega_m$. Under the rotating frame approximation $\boldsymbol{a}_1(t) \to \boldsymbol{a}_1(t)e^{-\imath\omega_m t}$, $\boldsymbol{a}_2(t) \to \boldsymbol{a}_2(t)e^{-\imath\omega_m t}$, and $\boldsymbol{a}_3(t) \to \boldsymbol{a}_3(t)e^{\imath\omega_m t}$ (see, e.g., [2, Eq. (32)]), the Hamiltonian (83) can be approximated by

$$\boldsymbol{H}_B = \lambda_1 \frac{\boldsymbol{a}_1\boldsymbol{a}_3 + \boldsymbol{a}_1^*\boldsymbol{a}_3^*}{2} + \lambda_2 \frac{\boldsymbol{a}_2\boldsymbol{a}_3 + \boldsymbol{a}_2^*\boldsymbol{a}_3^*}{2}$$
$$-\omega_m\boldsymbol{a}_1^*\boldsymbol{a}_1 - \omega_m\boldsymbol{a}_2^*\boldsymbol{a}_2 + \omega_m\boldsymbol{a}_3^*\boldsymbol{a}_3.$$

In this case, we find that there are no $\bar{c}o$ or $c\bar{o}$ subsystems. The unitary and blockwise Bogoliubov transformation matrix T in Eq. (48) is given by

$$T = [T_{co} \,|\, T_{\bar{c}\bar{o}}] = \begin{bmatrix} 0 & \rho_1 & 0 & 0 & \rho_2 & 0 \\ 0 & \rho_2 & 0 & 0 & -\rho_1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_1 & 0 & \rho_2 \\ 0 & 0 & 0 & \rho_2 & 0 & -\rho_1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

where ρ_1 and ρ_2 are defined as in (84). Thus using Eq. (74), we obtain By Theorem 4.4,

$$\boldsymbol{x}_{co} = \begin{bmatrix} \boldsymbol{q}_3 \\ \rho_1\boldsymbol{q}_1 + \rho_2\boldsymbol{q}_2 \\ \boldsymbol{p}_3 \\ \rho_1\boldsymbol{p}_1 + \rho_2\boldsymbol{p}_2 \end{bmatrix}, \quad \boldsymbol{x}_{\bar{c}\bar{o}} = \begin{bmatrix} \rho_2\boldsymbol{q}_1 - \rho_1\boldsymbol{q}_2 \\ \rho_2\boldsymbol{p}_1 - \rho_1\boldsymbol{p}_2 \end{bmatrix}.$$

Also, Eqs. (77)-(78) take the form

$$\dot{\boldsymbol{x}}_{co}(t) = A_{co}\boldsymbol{x}_{co}(t) - \begin{bmatrix} \sqrt{\kappa} & 0 \\ 0 & 0 \\ 0 & \sqrt{\kappa} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{q}_{\rm in}(t) \\ \boldsymbol{p}_{\rm in}(t) \end{bmatrix},$$

$$\dot{\boldsymbol{x}}_{\bar{c}\bar{o}}(t) = \begin{bmatrix} 0 & -\omega_m \\ \omega_m & 0 \end{bmatrix}\boldsymbol{x}_{\bar{c}\bar{o}}(t),$$

$$\begin{bmatrix} \boldsymbol{q}_{\rm out}(t) \\ \boldsymbol{p}_{\rm out}(t) \end{bmatrix} = \begin{bmatrix} \sqrt{\kappa} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\kappa} & 0 \end{bmatrix}\boldsymbol{x}_{co}(t) + \begin{bmatrix} \boldsymbol{q}_{\rm in}(t) \\ \boldsymbol{p}_{\rm in}(t) \end{bmatrix},$$

where

$$A_{co} = \begin{bmatrix} -\kappa & 0 & \omega_m & -\frac{\lambda}{2} \\ 0 & 0 & -\frac{\lambda}{2} & -\omega_m \\ -\omega_m & -\frac{\lambda}{2} & -\kappa & 0 \\ -\frac{\lambda}{2} & \omega_m & 0 & 0 \end{bmatrix}.$$

Clearly, $\boldsymbol{x}_{\bar{c}\bar{o}}$ is a DF mode. Indeed, it is exactly the same as that in Eq. (85) for the red-detuned regime case.

Case 3: Phase-shift regime (the two cavity modes resonant with their respective driving lasers). In this case, $\Delta_1 = \Delta_2 = 0$ (see, e.g., [2, Eq. (33)]). The unitary and blockwise Bogoliubov transformation matrix T in Eq. (48) is given by

$$T = [T_h \,|\, T_{co} \,|\, T_{\bar{c}\bar{o}}]$$
$$= \begin{bmatrix} \rho_1 & 0 & 0 & 0 & \rho_2 & 0 \\ \rho_2 & 0 & 0 & 0 & -\rho_1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \rho_1 & 0 & 0 & 0 & \rho_2 \\ 0 & \rho_2 & 0 & 0 & 0 & -\rho_1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

where ρ_1 and ρ_2 are defined as in (84). Thus by Eq. (74), By Theorem 4.4,

$$\begin{bmatrix} \boldsymbol{q}_h(t) \\ \boldsymbol{p}_h(t) \\ \hline \boldsymbol{x}_{co}(t) \\ \hline \boldsymbol{x}_{\bar{c}\bar{o}}(t) \end{bmatrix} = \begin{bmatrix} -\rho_1\boldsymbol{p}_1 - \rho_2\boldsymbol{p}_2 \\ \rho_1\boldsymbol{q}_1 + \rho_2\boldsymbol{q}_2 \\ \hline \boldsymbol{q}_3 \\ \boldsymbol{p}_3 \\ \hline \rho_2\boldsymbol{q}_1 - \rho_1\boldsymbol{q}_2 \\ \rho_2\boldsymbol{p}_1 - \rho_1\boldsymbol{p}_2 \end{bmatrix}.$$

Then, Eqs. (77)-(78) take the form

$$\begin{bmatrix} \dot{\boldsymbol{q}}_h(t) \\ \dot{\boldsymbol{p}}_h(t) \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix}\boldsymbol{x}_{co}(t),$$

$$\dot{\boldsymbol{x}}_{co}(t) = \begin{bmatrix} -\kappa/2 & \omega_m \\ -\omega_m & -\kappa/2 \end{bmatrix}\boldsymbol{x}_{co}(t)$$
$$-\lambda\begin{bmatrix} 0 \\ \boldsymbol{p}_h(t) \end{bmatrix} - \sqrt{\kappa}\begin{bmatrix} \boldsymbol{q}_{\rm in}(t) \\ \boldsymbol{p}_{\rm in}(t) \end{bmatrix},$$

$$\dot{\boldsymbol{x}}_{\bar{c}\bar{o}}(t) = 0,$$

$$\begin{bmatrix} \boldsymbol{q}_{\rm out}(t) \\ \boldsymbol{p}_{\rm out}(t) \end{bmatrix} = \sqrt{\kappa}\boldsymbol{x}_{co}(t) + \begin{bmatrix} \boldsymbol{q}_{\rm in}(t) \\ \boldsymbol{p}_{\rm in}(t) \end{bmatrix}.$$

Clearly, $\boldsymbol{x}_{\bar{c}\bar{o}}(t)$ is a DF mode (which is the same as Cases 1 and 2 above). On the other hand, $\boldsymbol{p}_h(t)$ is a constant for all $t \geq 0$, thus is a QND variable. Actually, $\boldsymbol{p}_h$ could be measured continuously with no quantum limit on the predictability of these measurements as the measurement back-action only drives its conjugate operator $\boldsymbol{q}_h$.
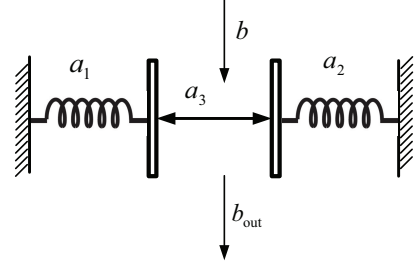

Fig. 4. Schematic diagram of an opto-mechanical system studied in [29] and [47].

Example 5.2: The opto-mechanical system, as shown in Fig. 4, has been studied theoretically in [47], and implemented experimentally in [29]. Back-action evading measurements of collective quadratures of the two mechanical oscillators were demonstrated in this system. Here, the two mechanical oscillators with modes $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, are not directly coupled. Instead, they are coupled to a microwave cavity, with mode $\boldsymbol{a}_3$. In this system, the mechanical damping is much smaller than the optical damping. (In the experimental paper [29], the mechanical damping is around 10^{-6} times that of the optical damping.) Therefore, in what follows we neglect the mechanical damping. The system Hamiltonian ([29, Eq. (1)], [47, Eq. (A6)]) is the following:

$$\boldsymbol{H} = \Omega(\boldsymbol{a}_1^*\boldsymbol{a}_1 - \boldsymbol{a}_2^*\boldsymbol{a}_2) + g(\boldsymbol{a}_1 + \boldsymbol{a}_1^*)(\boldsymbol{a}_3 + \boldsymbol{a}_3^*)$$
$$+g(\boldsymbol{a}_2 + \boldsymbol{a}_2^*)(\boldsymbol{a}_3 + \boldsymbol{a}_3^*).$$

The g used here is G in [29, Eq. (1)] and equals $g_a \bar{c} = g_b \bar{c}$ in [47, Eq. (A6)]). The optical coupling is $\boldsymbol{L} = \sqrt{\kappa} \boldsymbol{a}_3$. The unitary and blockwise Bogoliubov transformation matrix of (48) is given by

$$
\begin{aligned}
T &= \begin{bmatrix} T_h & | & T_{co} \end{bmatrix} \\
&= \begin{bmatrix}
-1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 & 0 & 0 \\
1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\
0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},
\end{aligned}
$$

and thus by Eq. (74), By Theorem 4.4,

$$
\begin{bmatrix} \boldsymbol{q}_h \\ \hline \boldsymbol{p}_h \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \boldsymbol{q}_2 - \boldsymbol{q}_1 \\ -\boldsymbol{p}_1 - \boldsymbol{p}_2 \\ \hline \boldsymbol{p}_2 - \boldsymbol{p}_1 \\ \boldsymbol{q}_1 + \boldsymbol{q}_2 \end{bmatrix}, \quad \boldsymbol{x}_{co} = \begin{bmatrix} \boldsymbol{q}_3 \\ \boldsymbol{p}_3 \end{bmatrix}.
$$

Then, Eqs. (77)-(78) take the form

$$
\begin{aligned}
\dot{\boldsymbol{q}}_h(t) &= \begin{bmatrix} 0 & \Omega \\ -\Omega & 0 \end{bmatrix} \boldsymbol{q}_h(t) + \begin{bmatrix} 0 & 0 \\ 2\sqrt{2}g & 0 \end{bmatrix} \boldsymbol{x}_{co}(t), \\
\dot{\boldsymbol{x}}_{co}(t) &= -\frac{\kappa}{2} \boldsymbol{x}_{co}(t) - \begin{bmatrix} 0 & 0 \\ 0 & 2\sqrt{2}g \end{bmatrix} \boldsymbol{p}_h(t) \\
&\quad - \sqrt{\kappa} \begin{bmatrix} \boldsymbol{q}_{\text{in}}(t) \\ \boldsymbol{p}_{\text{in}}(t) \end{bmatrix}, \qquad (86) \\
\dot{\boldsymbol{p}}_h(t) &= \begin{bmatrix} 0 & \Omega \\ -\Omega & 0 \end{bmatrix} \boldsymbol{p}_h(t), \\
\begin{bmatrix} \boldsymbol{q}_{\text{out}}(t) \\ \boldsymbol{p}_{\text{out}}(t) \end{bmatrix} &= \sqrt{\kappa} \boldsymbol{x}_{co}(t) + \begin{bmatrix} \boldsymbol{q}_{\text{in}}(t) \\ \boldsymbol{p}_{\text{in}}(t) \end{bmatrix}.
\end{aligned}
$$

The components of $\boldsymbol{p}_h$ are linear combinations of variables of the two mechanical oscillators, are immune from optical damping, and form a QMFS. Moreover, the second entry of $\boldsymbol{p}_h$, can be measured via a measurement on the optical cavity, and the back-action will only affect the dynamics of the mechanical quadratures in $\boldsymbol{q}_h$, which are conjugate to those in $\boldsymbol{p}_h$. It can be readily shown that the system realizes a BAE measurement of $\boldsymbol{q}_{\text{out}}$ with respect to $\boldsymbol{p}_{\text{in}}$, and a BAE measurement of $\boldsymbol{p}_{\text{out}}$ with respect to $\boldsymbol{q}_{\text{in}}$. Finally, notice that $\frac{\boldsymbol{q}_1 + \boldsymbol{q}_2}{\sqrt{2}}$, the second entry of $\boldsymbol{p}_h$, is exactly X_+ in [29] and [47], which couples to the microwave cavity dynamics $\boldsymbol{x}_{co}$, as can be seen in Eq. (86).

VI. CONCLUSION

In this paper, we have studied the Kalman decomposition for linear quantum systems. We have shown that it can always be performed with a unitary Bogoliubov coordinate transformation in the complex annihilation-creation operator representation. Alternatively, it can be performed with an orthogonal symplectic coordinate transformation in the real quadrature representation. These are the only coordinate transformations allowed by quantum mechanics to preserve the physical realizability conditions for linear quantum systems. Because the coordinate transformations are unitary, they can be performed in a numerically stable way. Furthermore, the decomposition is performed in a constructive way, as in the classical case. We have shown that a system in the Kalman canonical form has an interesting structure. For passive linear quantum systems, only co and $\bar{c}o$ subsystems may exist, because the uncontrollable and the unobservable subspaces are identical; a characterization of these subspaces has also been given. In the general case, $c\bar{o}$ and $\bar{c}o$ subsystems may be present, but their respective system variables must be conjugates of each other. The Kalman canonical decomposition naturally exposes the system's decoherence-free modes, quantum-nondemolition variables, quantum-mechanics-free-subspaces, and back-action evasion measurement, which are important resources in quantum information science. The methodology proposed in this paper should be helpful in the analysis and synthesis of linear quantum control systems.

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APPENDIX

In this Appendix we show how to construct matrices T_{co} and $T_{\bar{c}\bar{o}}$ used in Eq. (48) by means of the singular value decomposition (SVD).

First, we review SVD. Performing SVD to a matrix W gives

$$W = USV^{\dagger}, \tag{87}$$

where S is a diagonal matrix and U and V are both unitary. Without loss of generality, assume

$$S = \left[\begin{array}{cc} S_1 & 0 \\ 0 & 0 \end{array} \right], \tag{88}$$

where the diagonal entries of the matrix S_1 are all nonzero. Accordingly, partition

$$U = [U_1 \ U_2], \quad V = [V_1 \ V_2]. \tag{89}$$

Then Eq. (87) is equivalent to

$$W = U_1 S_1 V_1^{\dagger}. \tag{90}$$

It is well known that

$$U_1 = \text{Im}(W), \quad U_2 = \text{Im}(W)^{\perp}, \tag{91}$$

$$V_1 = \text{Ker}(W)^{\perp}, \quad V_2 = \text{Ker}(W). \tag{92}$$

Next, we construct T_{co} and $T_{\bar{c}\bar{o}}$ in four steps outlined below.

Step 1. Use Eq. (13) to compute the controllability matrix $\mathcal{C}_G$ and the observability matrix $\mathcal{O}_G$.

Step 2. Apply the SVD to $\mathcal{C}_G$ and $\mathcal{O}_G^{\dagger}$ to get $\text{Im}(\mathcal{C}_G)$ and $\text{Im}(\mathcal{O}_G^{\dagger})$. More specifically, assume

$$\mathcal{C}_G = [U_{co,1} \ U_{co,1}] \left[\begin{array}{cc} S_{co} & 0 \\ 0 & 0 \end{array} \right] [V_{co,1} \ V_{co,2}]^{\dagger} \tag{93}$$

$$\mathcal{O}_G^{\dagger} = [U_{\bar{c}\bar{o},1} \ U_{\bar{c}\bar{o},2}] \left[\begin{array}{cc} S_{\bar{c}\bar{o}} & 0 \\ 0 & 0 \end{array} \right] [V_{\bar{c}\bar{o},1} \ V_{\bar{c}\bar{o},2}]^{\dagger}, \tag{94}$$

where both S_{co} and $S_{\bar{c}\bar{o}}$ are nonsingular matrices. Then

$$U_{co,1} = \text{Im}(\mathcal{C}_G), \tag{95}$$

$$U_{\bar{c}\bar{o},1} = \text{Im}(\mathcal{O}_G^{\dagger}) \tag{96}$$

Step 3. By Eq. (25), $R_{co} = \text{Im}(U_{co,1}) \cap \text{Im}(U_{\bar{c}\bar{o},1})$. Pick n_3 columns from R_{co} constructed above to get T_{co}.

Step 4. Construct $T_{\bar{c}\bar{o}}$ in a similar way.

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