

Analytical solutions to the Navier–Stokes equations

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With the previous results for the analytical blowup solutions of the N -dimensional ($N \geq 2$) Euler–Poisson equations, we extend the same structure to construct an analytical family of solutions for the isothermal Navier–Stokes equations and pressureless Navier–Stokes equations with density-dependent viscosity. © 2008 American Institute of Physics. [DOI: 10.1063/1.3013805]

I. INTRODUCTION

The Navier–Stokes equations can be formulated in the following form:

$$\rho_t + \nabla \cdot (\rho u) = 0, \tag{1}$$

$$(\rho u)_t + \nabla \cdot (\rho u \otimes u) + \delta \nabla P = \text{vis}(\rho, u).$$

As usual, $\rho = \rho(x, t)$ and $u(x, t)$ are the density and the velocity, respectively. $P = P(\rho)$ is the pressure. We use a γ -law on the pressure, i.e.,

$$P(\rho) = K\rho^\gamma, \tag{2}$$

with $K > 0$, which is a universal hypothesis. The constant $\gamma = c_p/c_v \geq 1$, where c_p and c_v are the specific heats per unit mass under constant pressure and constant volume, respectively, is the ratio of the specific heats. γ is the adiabatic exponent in (2). In particular, the fluid is called isothermal if $\gamma = 1$. It can be used for constructing models with nondegenerate isothermal fluid. δ can be the constant 0 or 1. When $\delta = 0$, we call the system pressureless; when $\delta = 1$, we call that it is with pressure. Additionally, $\text{vis}(\rho, u)$ is the viscosity function. When $\text{vis}(\rho, u) = 0$, the system (1) becomes the Euler equations. For the detailed study of the Euler and Navier–Stokes equations, see Refs. 1 and 4. In the first part of this article, we study the solutions of the N -dimensional ($N \geq 1$) isothermal equations in radial symmetry,

$$\rho_t + u\rho_r + \rho u_r + \frac{N-1}{r} \rho u = 0, \tag{3}$$

$$\rho(u_t + uu_r) + \nabla K\rho = \text{vis}(\rho, u).$$

Definition 1: (Blowup) *We say a solution blows up if one of the following conditions is satisfied.*

- (1) *The solution becomes infinitely large at some point x and some finite time T .*
- (2) *The derivative of the solution becomes infinitely large at some point x and some finite time T .*

For the formation of singularity in the three-dimensional case for the Euler equations, please refer to the paper of Sideris.¹⁰ In this article, we extend the results from the study of the (blowup) analytical solutions in the N -dimensional ($N \geq 2$) Euler–Poisson equations, which describes the

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evolution of the gaseous stars in astrophysics^{2,3,7,12,13} to the Navier–Stokes equations. For the same kinds of blowup results in the nonisothermal case of the Euler or Navier–Stokes equations, please refer to Refs. 5 and 12.

Recently, in Yuen’s results,¹³ there exists a family of the blowup solution for the Euler–Poisson equations in the two-dimensional radial symmetry case,

$$\begin{aligned} \rho_t + u\rho_r + \rho u_r + \frac{1}{r}\rho u &= 0, \\ \rho(u_t + uu_r) + K\rho_r &= -\frac{2\pi\rho}{r} \int_0^r \rho(t,s) s ds. \end{aligned} \tag{4}$$

The solutions are

$$\begin{aligned} \rho(t,r) &= \frac{1}{a(t)^2} e^{y(r/a(t))}, \quad u(t,r) = \frac{\dot{a}(t)}{a(t)} r, \\ \ddot{a}(t) &= -\frac{\lambda}{a(t)}, \quad a(0) = a_0 > 0, \quad \dot{a}(0) = a_1, \end{aligned} \tag{5}$$

$$\ddot{y}(x) + \frac{1}{x}\dot{y}(x) + \frac{2\pi}{K} e^{y(x)} = \mu, \quad y(0) = \alpha, \quad \dot{y}(0) = 0,$$

where $K > 0$, $\mu = 2\lambda/K$ with a sufficiently small λ , and α are constants.

- (1) When $\lambda > 0$, the solutions blow up in a finite time T .
- (2) When $\lambda = 0$, if $a_1 < 0$, the solutions blow up at $t = -a_0/a_1$.

In this paper, we extend the above result to the isothermal Navier–Stokes equations in radial symmetry with the usual viscous function

$$\text{vis}(\rho, u) = v\Delta u,$$

where v is a positive constant,

$$\rho_t + u\rho_r + \rho u_r + \frac{N-1}{r}\rho u = 0, \tag{6a}$$

$$\rho(u_t + uu_r) + K\rho_r = v \left(u_{rr} + \frac{N-1}{r} u_r - \frac{N-1}{r^2} u \right). \tag{6b}$$

Theorem 2: For the N -dimensional isothermal Navier–Stokes equations in radial symmetry (6a) and (6b), there exists a family of solutions; those are

$$\begin{aligned} \rho(t,r) &= \frac{1}{a(t)^N} e^{y(r/a(t))}, \quad u(t,r) = \frac{\dot{a}(t)}{a(t)} r, \\ \ddot{a}(t) &= \frac{-\lambda}{a(t)}, \quad a(0) = a_0 > 0, \quad \dot{a}(0) = a_1, \end{aligned} \tag{7}$$

$$y(x) = \frac{\lambda}{2K} x^2 + \alpha,$$

where α and λ are arbitrary constants.

In particular, for $\lambda > 0$, the solutions blow up in finite time T .

In the last part, the corresponding solutions to the pressureless Navier–Stokes equations with density-dependent viscosity are also studied.

II. THE ISOTHERMAL ($\gamma=1$) CASES

Before we present the proof of Theorem 2, Lemma 6 of Ref. 13 could be needed to further extend to the N -dimensional space.

Lemma 3: (The Extension of Lemma 6 of Ref. 13) *For the equation of conservation of mass in radial symmetry,*

$$\rho_t + u\rho_r + \rho u_r + \frac{N-1}{r}\rho u = 0, \quad (8)$$

there exist solutions

$$\rho(t,r) = \frac{f(r/a(t))}{a(t)^N}, \quad u(t,r) = \frac{\dot{a}(t)}{a(t)}r, \quad (9)$$

with the form $f \geq 0 \in C^1$ and $a(t) > 0 \in C^1$.

Proof: We just plug (9) into (8). Then

$$\begin{aligned} \rho_t + u\rho_r + \rho u_r + \frac{N-1}{r}\rho u &= \frac{-N\dot{a}(t)f(r/a(t))}{a(t)^{N+1}} - \frac{\dot{a}(t)r\dot{f}(r/a(t))}{a(t)^{N+2}} + \frac{\dot{a}(t)r\dot{f}(r/a(t))}{a(t)} \frac{f(r/a(t))}{a(t)^{N+1}} + \frac{f(r/a(t))}{a(t)^N} \frac{\dot{a}(t)}{a(t)} \\ &+ \frac{N-1}{r} \frac{f(r/a(t))}{a(t)^N} \frac{\dot{a}(t)}{a(t)} r = 0. \end{aligned}$$

The proof is completed. ■

Besides, Lemma 7 of Ref. 13 is also useful. For the better understanding of the lemma, the proof is given here.

Lemma 4: (Lemma 7 of Ref. 13) *For the Emden equation,*

$$\ddot{a}(t) = -\frac{\lambda}{a(t)}, \quad (10)$$

$$a(0) = a_0 > 0, \quad \dot{a}(0) = a_1,$$

we have that, if $\lambda > 0$, there exists a finite time $T_- < +\infty$ such that $a(T_-) = 0$.

Proof: By integrating (10), we have

$$0 \leq \frac{1}{2}\dot{a}(t)^2 = -\lambda \ln a(t) + \theta, \quad (11)$$

where $\theta = \lambda \ln a_0 + \frac{1}{2}a_1^2$.

From (11), we get

$$a(t) \leq e^{\theta/\lambda}.$$

If the statement is not true, we have

$$0 < a(t) \leq e^{\theta/\lambda} \quad \text{for all } t \geq 0.$$

However, since

$$\ddot{a}(t) = -\frac{\lambda}{a(t)} \leq \frac{-\lambda}{e^{\theta/\lambda}},$$

we integrate this twice to deduce

$$a(t) \leq \int_0^t \int_0^\tau \frac{-\lambda}{e^{\theta/\lambda}} ds d\tau + C_1 t + C_0 = \frac{-\lambda t^2}{2e^{\theta/\lambda}} + C_1 t + C_0.$$

By taking t large enough, we get

$$a(t) < 0.$$

As a contradiction is met, the statement of the lemma is true. ■

By extending the structure of the solutions (5) to the two-dimensional isothermal Euler–Poisson equations (4) in Ref. 13, it is a natural result to get the proof of Theorem 2.

Proof of Theorem 2: By using Lemma 3, we can get that (7) satisfy (6a). For the momentum equation, we have

$$\begin{aligned} \rho(u_t + u \cdot u_r) + K\rho_r - v\left(u_{rr} + \frac{N-1}{r}u_r - \frac{N-1}{r^2}u\right) &= \rho \frac{\ddot{a}(t)}{a(t)}r + \frac{K}{a(t)}\rho y\left(\frac{r}{a(t)}\right) \\ &= \frac{\rho}{a(t)}\left[-\frac{\lambda r}{a(t)} + Ky\left(\frac{r}{a(t)}\right)\right]. \end{aligned}$$

By choosing

$$y(x) = \frac{\lambda}{2K}x^2 + \alpha,$$

we have verified that (7) satisfies the above (6b). If $\lambda > 0$, by Lemma 4, there exists a finite time T for such that $a(T_-) = 0$. Thus, there exist blowup solutions in finite time T . The proof is completed. ■

With the assistance of the blowup rate results of the Euler–Poisson equations, i.e., Theorem 3 in Ref. 13, it is trivial to have the following theorem.

Theorem 5: With $\lambda > 0$, the blowup rate of the solutions (7) is

$$\lim_{t \rightarrow T_*} \rho(t, 0)(T_* - t)^\alpha \geq O(1),$$

where the blowup time T_* and $\alpha < N$ are constants.

Remark 6: If we are interested in the mass of the solutions, the mass of the solutions can be calculated by

$$M(t) = \int_{R^N} \rho(t, s) ds = \alpha(N) \int_0^{+\infty} \rho(t, s) s^{N-1} ds,$$

where $\alpha(N)$ denotes some constant related to the unit ball in R^N : $\alpha(1) = 1$; $\alpha(2) = 2\pi$; for $N \geq 3$,

$$\alpha(N) = N(N-2)V(N) = N(N-2) \frac{\pi^{N/2}}{\Gamma(N/2 + 1)},$$

where $V(N)$ is the volume of the unit ball in R^N and Γ is the gamma function. We observe the following for the mass of the initial time 0:

- (1) For $\lambda \geq 0$,

$$M(0) = \frac{\alpha(N)}{a_0^N} \int_0^{+\infty} e^{(\lambda/2K)s^2 + \alpha s^{N-1}} ds.$$

The mass is infinite. The very large density comes from the ends outside the origin O .

(2) For $\lambda < 0$,

$$M(0) = \frac{\alpha(N)}{a_0^N} \int_0^{+\infty} e^{(\lambda/2K)s^2 + \alpha s^{N-1}} ds = \frac{\alpha(N)e^\alpha}{a_0^N} \int_0^{+\infty} e^{(\lambda/2K)s^2} s^{N-1} ds.$$

The mass of the solution can be arbitrarily small but without compact support if α is taken to be a very small negative number.

Remark 7: Our results can be easily extended to the isothermal Euler/Navier–Stokes equations with frictional damping term with the assistance of Lemma 7 in Ref. 12,

$$\rho_t + u\rho_r + \rho u_r + \frac{N-1}{r}\rho u = 0,$$

$$\rho(u_t + u \cdot u_r) + K\rho_r + \beta\rho u = v \left(u_{rr} + \frac{N-1}{r}u_r - \frac{N-1}{r^2}u \right),$$

where $\beta \geq 0$ and $v \geq 0$. The solutions are

$$\rho(t, r) = \frac{e^{y(r/a(t))}}{a(t)^N}, \quad u(t, r) = \frac{\dot{a}(t)}{a(t)}r,$$

$$\ddot{a}(t) + \beta\dot{a}(t) = \frac{-\lambda}{a(t)}, \quad a(0) = a_0 > 0, \quad \dot{a}(0) = a_1,$$

$$y(x) = \frac{\lambda}{2K}x^2 + \alpha.$$

Remark 8: Our results can be easily extended to the isothermal Euler/Navier–Stokes equations with frictional damping term with the assistance of Lemma 7 in Ref. 12,

$$\rho_t + u\rho_r + \rho u_r + \frac{N-1}{r}\rho u = 0,$$

$$\rho(u_t + u \cdot u_r) + K\rho_r + \beta\rho u = v \left(u_{rr} + \frac{N-1}{r}u_r - \frac{N-1}{r^2}u \right),$$

where $\beta \geq 0$ and $v \geq 0$. The solutions are

$$\rho(t, r) = \frac{e^{y(r/a(t))}}{a(t)^N}, \quad u(t, r) = \frac{\dot{a}(t)}{a(t)}r,$$

$$\ddot{a}(t) + \beta\dot{a}(t) = \frac{-\lambda}{a(t)}, \quad a(0) = a_0 > 0, \quad \dot{a}(0) = a_1,$$

$$y(x) = \frac{\lambda}{2K}x^2 + \alpha.$$

Remark 9: The solutions (5) to the Euler–Poisson equations only work for the two-dimensional case. However, the solutions (7) to the Navier–Stokes equations work for the N -dimensional ($N \geq 1$) case.

Remark 10: We may extend the solutions to the two-dimensional Euler/Navier–Stokes equations with a solid core,⁶

$$\rho_t + u\rho_r + \rho u_r + \frac{1}{r}\rho u = 0,$$

$$\rho(u_t + uu_r) + K\rho_r + \beta\rho u = \frac{M_0}{r} + v\left(u_{rr} + \frac{1}{r}u_r - \frac{1}{r^2}u\right),$$

where $M_0 > 0$; there is a unit stationary solid core locating $[0, r_0]$, where r_0 is a positive constant, surrounded by the distribution density. The corresponding solutions are

$$\rho(t, r) = \frac{e^{y(r/a(t))}}{a(t)^2}, \quad u(t, r) = \frac{\dot{a}(t)}{a(t)}r \quad \text{for } r > r_0,$$

$$\ddot{a}(t) + \beta\dot{a}(t) = \frac{-\lambda}{a(t)}, \quad a(0) = a_0 > 0, \quad \dot{a}(0) = a_1,$$

$$y(x) = \frac{\lambda}{2K}x^2 + M_0 \ln x + \alpha,$$

where $\alpha > -\lambda/2K$ is a constant.

III. PRESSURELESS NAVIER–STOKES EQUATIONS WITH DENSITY-DEPENDENT VISCOSITY

Now we consider the pressureless Navier–Stokes equations with density-dependent viscosity,

$$\text{vis}(\rho, u) \doteq \nabla(\mu(\rho) \nabla \cdot u),$$

in radial symmetry,

$$\rho_t + u\rho_r + \rho u_r + \frac{N-1}{r}\rho u = 0, \tag{12}$$

$$\rho(u_t + uu_r) = (\mu(\rho))_r u_r + \mu(\rho)\left(u_{rr} + \frac{N-1}{r}u_r - \frac{N-1}{r^2}u\right),$$

where $\mu(\rho)$ is a density-dependent viscosity function, which is usually written as $\mu(\rho) \doteq \kappa\rho^\theta$ with the constants $\kappa, \theta > 0$. For the study of this kind of the above system, the readers may refer to Refs. 8, 9, and 11.

We can obtain the same estimate about Lemma 4 to the following ordinary differential equation (ODE):

$$\ddot{a}(t) = \frac{\lambda\dot{a}(t)}{a(t)^2}, \tag{13}$$

$$a(0) = a_0 > 0, \quad \dot{a}(0) = a_1 \leq \frac{\lambda}{a_0}.$$

Lemma 11: For the ODE (13), with $\lambda > 0$, there exists a finite time $T_- < +\infty$ such that $a(T_-) = 0$.

Proof: If $a(t) > 0$ and $\dot{a}(0) = a_1 \leq \lambda/a_0$ for all time t , by integrating (13), we have

$$\dot{a}(t) = -\frac{\lambda}{a(t)} - \frac{\lambda}{a_0} + a_1 \leq -\frac{\lambda}{a(t)}. \quad (14)$$

Take the integration for (14),

$$\int_0^t a(s)\dot{a}(s)ds \leq -\int_0^t \lambda ds,$$

$$\frac{1}{2}[a(t)]^2 \leq -\lambda t + \frac{1}{2}a_0^2.$$

When t is very large, we have

$$\frac{1}{2}[a(t)]^2 \leq -1.$$

A contradiction is met. The proof is completed. ■

Here we present another lemma before proceeding to the next theorem.

Lemma 12: For the ODE,

$$\dot{y}(x)y(x)^n - \xi x = 0, \quad (15)$$

$$y(0) = \alpha > 0, \quad n \neq -1,$$

where ξ and n are constants, we have the solution

$$y(x) = \sqrt{\frac{n+1}{2}(\xi x^2 + \alpha^{n+1})}.$$

Proof: The above ODE (15) may be solved by the separation method,

$$\dot{y}(x)y(x)^n - \xi x = 0,$$

$$\dot{y}(x)y(x)^n = \xi x.$$

By taking the integration with respect to x ,

$$\int_0^x \dot{y}(x)y(x)^n dx = \int_0^x \xi x dx,$$

we have

$$\int_0^x y(x)^n d[y(x)] = \frac{1}{2}\xi x^2 + C_1, \quad (16)$$

where C_1 is a constant.

By integration by part, then the identity becomes

$$y(x)^{n+1} - n \int_0^x y(x)^{n-1} \dot{y}(x)y(x) dx = \frac{1}{2}\xi x^2 + C_1,$$

$$y(x)^{n+1} - n \int_0^x \dot{y}(x)y(x)^n dx = \frac{1}{2}\xi x^2 + C_1.$$

From Eq. (16), we can have the simple expression for $y(x)$,

$$y(x)^{n+1} - n\left(\frac{1}{2}\xi x^2 + C_1\right) = \frac{1}{2}\xi x^2 + C_1,$$

$$y(x)^{n+1} = \frac{1}{2}(n+1)\xi x^2 + C_2,$$

where $C_2=(n+1)C_1$.

By plugging into the initial condition for $y(0)$, we have

$$y(0)^{n+1} = \alpha^{n+1} = C_2.$$

Thus, the solution is

$$y(x) = \sqrt[n+1]{\frac{1}{2}(n+1)\xi x^2 + \alpha^{n+1}}.$$

The proof is completed. ■

The family of the solution to the pressureless Navier–Stokes equations with density-dependent viscosity,

$$\rho_t + u\rho_r + \rho u_r + \frac{N-1}{r}\rho u = 0, \tag{17a}$$

$$\rho(u_t + uu_r) = (\kappa\rho^\theta)_r u_r + \kappa\rho^\theta\left(u_{rr} + \frac{N-1}{r}u_r - \frac{N-1}{r^2}u\right), \tag{17b}$$

is presented as the following.

Theorem 13: *For the pressureless Navier–Stokes equations with density-dependent viscosity (17a) and (17b) in radial symmetry, there exists a family of solutions.*

For $\theta=1$,

$$\rho(t,r) = \frac{e^{y(r/a(t))}}{a(t)^N}, \quad u(t,r) = \frac{\dot{a}(t)}{a(t)}r,$$

$$\ddot{a}(t) = \frac{\lambda\dot{a}(t)}{a(t)^2}, \quad a(0) = a_0 > 0, \quad \dot{a}(0) = a_1,$$

$$y(x) = \frac{\lambda}{2\kappa}x^2 + \alpha,$$

where α and λ are arbitrary constants. In particular, for $\lambda > 0$ and $a_1 \leq \lambda/a_0$, the solutions blow up in finite time. For $\theta \neq 1$,

$$\rho(t,r) = \begin{cases} \frac{y(r/a(t))}{a(t)^N} & \text{for } y\left(\frac{r}{a(t)}\right) \geq 0; \\ 0 & \text{for } y\left(\frac{r}{a(t)}\right) < 0 \end{cases}, \quad u(t,r) = \frac{\dot{a}(t)}{a(t)}r,$$

$$\ddot{a}(t) = \frac{-\lambda\dot{a}(t)}{a(t)^{N\theta-2N+2}}, \quad a(0) = a_0 > 0, \quad \dot{a}(0) = a_1, \tag{18}$$

$$y(x) = \sqrt[\theta-1]{\frac{1}{2}(\theta-1)\frac{-\lambda}{\kappa\theta}x^2 + \alpha^{\theta-1}},$$

where $\alpha > 0$.

Proof of Theorem 13: To (17a), we may use Lemma 3 to check it. For $\theta=1$, (17b) becomes

$$\begin{aligned} & \rho(u_t + u \cdot u_r) - (\kappa\rho)_r u_r - \kappa\rho_r \left(u_{rr} + \frac{N-1}{r} u_r - \frac{N-1}{r^2} u \right) \\ &= \rho(u_t + u \cdot u_r) - (\kappa\rho)_r u_r = \rho \frac{\ddot{a}(t)}{a(t)} r - \left(\frac{\kappa e^{y(r/a(t))}}{a(t)^N} \right) \frac{\dot{a}(t)}{r a(t)} = \rho \left(\frac{\lambda \dot{a}(t) r}{a(t)^3} \right) - \frac{\kappa e^{y(r/a(t))} \dot{y} \left(\frac{r}{a(t)} \right) \dot{a}(t)}{a(t)^{N+1}} \\ &= \frac{\rho \dot{a}(t)}{a(t)^2} \left(\frac{\lambda r}{a(t)} - \kappa \dot{y} \left(\frac{r}{a(t)} \right) \right), \end{aligned} \quad (19)$$

where we use

$$\ddot{a}(t) = \frac{\lambda \dot{a}(t)}{a(t)^2}.$$

By choosing

$$y \left(\frac{r}{a(t)} \right) \doteq y(x) = \frac{\lambda}{2\kappa} x^2 + \alpha,$$

(19) is equal to zero.

For the case of $\theta \neq 1$, (17b) can be calculated,

$$\begin{aligned} & \rho(u_t + u \cdot u_r) - (\kappa\rho^\theta)_r u_r - \kappa\rho^\theta \left(u_{rr} + \frac{N-1}{r} u_r - \frac{N-1}{r^2} u \right) \\ &= \rho \frac{\ddot{a}(t)}{a(t)} r - \left[\left(\frac{\kappa y \left(\frac{r}{a(t)} \right)}{a(t)^N} \right)^\theta \right] \frac{\dot{a}(t)}{r a(t)} \\ &= \rho \left(- \frac{\lambda \dot{a}(t) r}{a(t)^{N\theta-2N+2} a(t)} \right) - \frac{\kappa \theta y \left(\frac{r}{a(t)} \right)^{\theta-1} \dot{y} \left(\frac{r}{a(t)} \right) \dot{a}(t)}{a(t)^{N(\theta-1)} a(t)} \\ &= \rho \left(- \frac{\lambda \dot{a}(t) r}{a(t)^{N\theta-2N+2} a(t)} \right) - \frac{\kappa \theta y \left(\frac{r}{a(t)} \right) y \left(\frac{r}{a(t)} \right)^{\theta-2} \dot{y} \left(\frac{r}{a(t)} \right) \dot{a}(t)}{a(t)^N a(t)^{N\theta-2N+2}} \\ &= \rho \left(- \frac{\lambda \dot{a}(t) r}{a(t)^{N\theta-2N+2} a(t)} \right) - \frac{\kappa \theta \rho y \left(\frac{r}{a(t)} \right)^{\theta-2} \dot{y} \left(\frac{r}{a(t)} \right) \dot{a}(t)}{a(t)^{N\theta-2N+2}} \end{aligned} \quad (20)$$

$$= \frac{-\rho \dot{a}(t)}{a(t)^{N\theta-2N+2}} \left(- \frac{\lambda r}{a(t)} + \kappa \theta y \left(\frac{r}{a(t)} \right)^{\theta-2} \dot{y} \left(\frac{r}{a(t)} \right) \right). \quad (21)$$

Define $x \doteq r/a(t)$, $n \doteq \theta-2$; it follows

$$= \frac{-\rho \dot{a}(t)}{a(t)^{N\theta-2N+2}} (\lambda x + \kappa \theta y(x)^n \dot{y}(x)) \quad (22)$$

$$= \frac{-\lambda \rho \dot{a}(t)}{a(t)^{N\theta-2N+2}} \left(x + \frac{\kappa \theta}{\lambda} y(x)^n \dot{y}(x) \right), \quad (23)$$

and $\xi \doteq \lambda/\kappa\theta$ in Lemma 12, and choose

$$y\left(\frac{r}{a(t)}\right) \doteq y(x) = \sqrt{\frac{1}{2}(\theta-1)\frac{-\lambda}{\kappa\theta}x^2 + \alpha^{\theta-1}}.$$

Moreover this is easy to check that

$$\dot{y}(0) = 0.$$

Equation (22) is equal to zero. The proof is completed. ■

Remark 14: By controlling the initial conditions in some solutions (18), we may get the blowup solutions. Additionally the modified solutions can be extended to the system in radial symmetry with frictional damping,

$$\rho_t + u\rho_r + \rho u_r + \frac{N-1}{r}\rho u = 0,$$

$$\rho(u_t + uu_r) + \beta\rho u = (\mu(\rho))_r u_r + \mu(\rho)\left(u_{rr} + \frac{N-1}{r}u_r - \frac{N-1}{r^2}u\right),$$

where $\beta > 0$. With the assistance of the ODE,

$$\ddot{a}(t) + \beta\dot{a}(t) = \frac{-\lambda\dot{a}(t)}{a(t)^S},$$

$$a(0) = a_0 > 0, \quad \dot{a}(0) = a_1,$$

where S is a constant.

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