Analytical solutions to the Navier–Stokes equations

Yuen Manwai

Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong

(Received 1 August 2008; accepted 14 October 2008; published online 12 November 2008)

With the previous results for the analytical blowup solutions of the \(N\)-dimensional \(N\approx2\) Euler–Poisson equations, we extend the same structure to construct an analytical family of solutions for the isothermal Navier–Stokes equations and pressureless Navier–Stokes equations with density-dependent viscosity. © 2008 American Institute of Physics. [DOI: 10.1063/1.3013805]

I. INTRODUCTION

The Navier–Stokes equations can be formulated in the following form:

\[
\rho_t + \nabla \cdot (\rho u) = 0, \\
(\rho u)_t + \nabla \cdot (\rho u \otimes u) + \delta \nabla P = \text{vis}(\rho, u).
\]

As usual, \(\rho=\rho(x,t)\) and \(u(x,t)\) are the density and the velocity, respectively. \(P=P(\rho)\) is the pressure. We use a \(\gamma\)-law on the pressure, i.e.,

\[
P(\rho) = K\rho^\gamma,
\]

with \(K>0\), which is a universal hypothesis. The constant \(\gamma=c_p/c_v\geq1\), where \(c_p\) and \(c_v\) are the specific heats per unit mass under constant pressure and constant volume, respectively, is the ratio of the specific heats. \(\gamma\) is the adiabatic exponent in (2). In particular, the fluid is called isothermal if \(\gamma=1\). It can be used for constructing models with nondegenerate isothermal fluid. \(\delta\) can be the constant 0 or 1. When \(\delta=0\), we call the system pressureless; when \(\delta=1\), we call that it is with pressure. Additionally, \(\text{vis}(\rho, u)\) is the viscosity function. When \(\text{vis}(\rho, u)=0\), the system (1) becomes the Euler equations. For the detailed study of the Euler and Navier–Stokes equations, see Refs. 1 and 4. In the first part of this article, we study the solutions of the \(N\)-dimensional \((N\approx1)\) isothermal equations in radial symmetry,

\[
\rho_t + uu_r + \rho u_r + \frac{N-1}{r} \rho u = 0, \\
\rho(u_t + uu_r) + \nabla P = \text{vis}(\rho, u).
\]

Definition 1: (Blowup) We say a solution blows up if one of the following conditions is satisfied.

1. The solution becomes infinitely large at some point \(x\) and some finite time \(T\).
2. The derivative of the solution becomes infinitely large at some point \(x\) and some finite time \(T\).

For the formation of singularity in the three-dimensional case for the Euler equations, please refer to the paper of Sideris. In this article, we extend the results from the study of the (blowup) analytical solutions in the \(N\)-dimensional \((N\approx2)\) Euler–Poisson equations, which describes the

---

Electronic mail: nevetsyuen@hotmail.com.
evolution of the gaseous stars in astrophysics\textsuperscript{2,3,7,12,13} to the Navier–Stokes equations. For the same kinds of blowup results in the nonisothermal case of the Euler or Navier–Stokes equations, please refer to Refs. 5 and 12.

Recently, in Yuen’s results,\textsuperscript{13} there exists a family of the blowup solution for the Euler–Poisson equations in the two-dimensional radial symmetry case,

\[
\rho_t + u\rho_r + \rho u_r + \frac{1}{r}\rho u = 0, \\
\rho(u_t + uu_r) + K\rho_r = -\frac{2\pi p}{r}\int_0^r \rho(t,s)sds.
\]

The solutions are

\[
\rho(t,r) = \frac{1}{a(t)^2}e^{\lambda r/a(t)}, \quad u(t,r) = \frac{\dot{a}(t)}{a(t)}r,
\]

\[
\dot{a}(t) = -\frac{\lambda}{a(t)}, \quad a(0) = a_0 > 0, \quad \dot{a}(0) = a_1,
\]

\[
y(x) + \frac{1}{x}y(x) + \frac{2\pi}{K}e^{\rho(x)} = \mu, \quad y(0) = \alpha, \quad \dot{y}(0) = 0,
\]

where \(K > 0, \mu = 2\lambda/K\) with a sufficiently small \(\lambda\), and \(\alpha\) are constants.

1. When \(\lambda > 0\), the solutions blow up in a finite time \(T\).
2. When \(\lambda = 0\), if \(a_1 < 0\), the solutions blow up at \(t = -a_0/a_1\).

In this paper, we extend the above result to the isothermal Navier–Stokes equations in radial symmetry with the usual viscous function

\[
\text{vis}(\rho,u) = v\Delta u,
\]

where \(v\) is a positive constant,

\[
\rho_t + u\rho_r + \rho u_r + \frac{N-1}{r}\rho u = 0, \\
\rho(u_t + uu_r) + K\rho_r = v\left(u_{rr} + \frac{N-1}{r}u_r - \frac{N-1}{r^2}u\right).
\]

**Theorem 2:** For the \(N\)-dimensional isothermal Navier–Stokes equations in radial symmetry \((6a)\) and \((6b)\), there exists a family of solutions; those are

\[
\rho(t,r) = \frac{1}{a(t)^2}e^{\lambda r/a(t)}, \quad u(t,r) = \frac{\dot{a}(t)}{a(t)}r,
\]

\[
\dot{a}(t) = -\frac{\lambda}{a(t)}, \quad a(0) = a_0 > 0, \quad \dot{a}(0) = a_1,
\]

\[
y(x) = \frac{\lambda}{2K}\sqrt{x^2 + \alpha},
\]
where $\alpha$ and $\lambda$ are arbitrary constants.

In particular, for $\lambda>0$, the solutions blow up in finite time $T$.

In the last part, the corresponding solutions to the pressureless Navier–Stokes equations with density-dependent viscosity are also studied.

II. THE ISOTHERMAL ($\gamma=1$) CASES

Before we present the proof of Theorem 2, Lemma 6 of Ref. 13 could be needed to further extend to the $N$-dimensional space.

**Lemma 3:** (The Extension of Lemma 6 of Ref. 13) For the equation of conservation of mass in radial symmetry,

$$\rho_t + u \rho_r + \rho u_r + \frac{N-1}{r} \rho u = 0, \quad (8)$$

there exist solutions

$$\rho(t,r) = \frac{f(r/a(t))}{a(t)^N}, \quad u(t,r) = \frac{\dot{a}(t)}{a(t)} r, \quad (9)$$

with the form $f \equiv 0 \in C^1$ and $a(t) > 0 \in C^1$.

**Proof:** We just plug (9) into (8). Then

$$\rho_t + u \rho_r + \rho u_r + \frac{N-1}{r} \rho u = -\frac{Na(t)f(r/a(t))}{a(t)^{N+1}} - \frac{\dot{a}(t)r f(r/a(t))}{a(t)^N} + \frac{\dot{a}(t)r f(r/a(t))}{a(t)^{N+1}} + \frac{f(r/a(t)) \dot{a}(t)}{a(t)^N a(t)}$$

$$+ \frac{N-1 f(r/a(t)) \dot{a}(t)}{a(t)^N} - \frac{f(r/a(t)) \dot{a}(t)}{a(t)^N} f = 0.$$

The proof is completed.

Besides, Lemma 7 of Ref. 13 is also useful. For the better understanding of the lemma, the proof is given here.

**Lemma 4:** (Lemma 7 of Ref. 13) For the Emden equation,

$$\ddot{a}(t) = -\frac{\lambda}{a(t)}, \quad (10)$$

$$a(0) = a_0 > 0, \quad \dot{a}(0) = a_1,$$

we have that, if $\lambda>0$, there exists a finite time $T_- < +\infty$ such that $a(T_-) = 0$.

**Proof:** By integrating (10), we have

$$0 \leq \frac{1}{2} \dot{a}(t)^2 = -\lambda \ln a(t) + \theta,$$

where $\theta = \lambda \ln a_0 + \frac{1}{2} a_1^2$.

From (11), we get

$$a(t) \leq e^{\theta \lambda}.$$

If the statement is not true, we have

$$0 < a(t) \leq e^{\theta \lambda} \quad \text{for all } t \geq 0.$$
\[ \ddot{a}(t) = -\frac{\lambda}{a(t)} \leq -\frac{\lambda}{e^{\theta x}}, \]

we integrate this twice to deduce

\[ a(t) \leq \int_0^t \int_0^r \frac{-\lambda}{e^{\theta x}} ds \, dr + C_1 t + C_0 = \frac{-\lambda t^2}{2e^{\theta x}} + C_1 t + C_0. \]

By taking \( t \) large enough, we get

\[ a(t) < 0. \]

As a contradiction is met, the statement of the lemma is true. \( \square \)

By extending the structure of the solutions (5) to the two-dimensional isothermal Euler–Poisson equations (4) in Ref. 13, it is a natural result to get the proof of Theorem 2.

**Proof of Theorem 2:** By using Lemma 3, we can get that (7) satisfy (6a). For the momentum equation, we have

\[
\rho(u_t + u \cdot u_r) + K \rho_r - v \left( u_{rr} + \frac{N-1}{r} u_r - \frac{N-1}{r^2} u \right) = \rho \frac{a(t)}{a(t)} \frac{a(t)}{a(t)} - \frac{\lambda r}{a(t)} + K \frac{\dot{r}}{a(t)}. \]

By choosing

\[ \gamma(x) = \frac{\lambda}{2K} x^2 + \alpha, \]

we have verified that (7) satisfies the above (6b). If \( \lambda > 0 \), by Lemma 4, there exists a finite time \( T \) for such that \( a(T) = 0 \). Thus, there exist blowup solutions in finite time \( T \). The proof is completed. \( \square \)

With the assistance of the blowup rate results of the Euler–Poisson equations, i.e., Theorem 3 in Ref. 13, it is trivial to have the following theorem.

**Theorem 5:** With \( \lambda > 0 \), the blowup rate of the solutions (7) is

\[ \lim_{t \to T_0} \rho(t,0)(T_0 - t)^\alpha = O(1), \]

where the blowup time \( T_0 \) and \( \alpha < N \) are constants.

**Remark 6:** If we are interested in the mass of the solutions, the mass of the solutions can be calculated by

\[ M(t) = \int_{\mathbb{R}^N} \rho(t,s) ds = \alpha(N) \int_0^{\infty} \rho(t,s) s^{N-1} ds, \]

where \( \alpha(N) \) denotes some constant related to the unit ball in \( \mathbb{R}^N \): \( \alpha(1) = 1; \alpha(2) = 2\pi; \) for \( N \geq 3, \)

\[ \alpha(N) = N(N-2)V(N) = N(N-2) \frac{\pi^{N/2}}{\Gamma(N/2 + 1)}, \]

where \( V(N) \) is the volume of the unit ball in \( \mathbb{R}^N \) and \( \Gamma \) is the gamma function. We observe the following for the mass of the initial time 0:

(1) For \( \lambda \geq 0, \)
The mass is infinitive. The very large density comes from the ends outside the origin $O$.

For $\lambda < 0$,

$$M(0) = \frac{\alpha(N)}{a_0^N} \int_0^{+\infty} e^{(\lambda/2K)r^2 + N-1} ds.$$ 

The mass of the solution can be arbitrarily small but without compact support if $\alpha$ is taken to be a very small negative number.

**Remark 7:** Our results can be easily extended to the isothermal Euler/Navier–Stokes equations with frictional damping term with the assistance of Lemma 7 in Ref. 12,

$$\rho_t + u\rho_r + \rho u_r + \frac{N-1}{r} \rho u = 0,$$

$$\rho(u_t + u \cdot u_r) + K \rho_r + \beta \rho u = v \left( u_{rr} + \frac{N-1}{r} u_r - \frac{N-1}{r^2} u \right),$$

where $\beta \geq 0$ and $v \geq 0$. The solutions are

$$\rho(t,r) = \frac{\rho(x(t))}{a(t)^N}, \quad u(t,r) = \frac{u(x(t))}{a(t)} r,$$

$$\ddot{a}(t) + \beta \dot{a}(t) = -\frac{\lambda}{a(t)^2}, \quad a(0) = a_0 > 0, \quad \dot{a}(0) = a_1,$$

$$y(x) = \frac{\lambda}{2K} x^2 + \alpha.$$ 

**Remark 8:** Our results can be easily extended to the isothermal Euler/Navier–Stokes equations with frictional damping term with the assistance of Lemma 7 in Ref. 12,

$$\rho_t + u\rho_r + \rho u_r + \frac{N-1}{r} \rho u = 0,$$

$$\rho(u_t + u \cdot u_r) + K \rho_r + \beta \rho u = v \left( u_{rr} + \frac{N-1}{r} u_r - \frac{N-1}{r^2} u \right),$$

where $\beta \geq 0$ and $v \geq 0$. The solutions are

$$\rho(t,r) = \frac{\rho(x(t))}{a(t)^N}, \quad u(t,r) = \frac{u(x(t))}{a(t)} r,$$

$$\ddot{a}(t) + \beta \dot{a}(t) = -\frac{\lambda}{a(t)^2}, \quad a(0) = a_0 > 0, \quad \dot{a}(0) = a_1,$$

$$y(x) = \frac{\lambda}{2K} x^2 + \alpha.$$
Remark 9: The solutions (5) to the Euler–Poisson equations only work for the two-dimensional case. However, the solutions (7) to the Navier–Stokes equations work for the N-dimensional case.

Remark 10: We may extend the solutions to the two-dimensional Euler/Navier–Stokes equations with a solid core,

\[ \rho_r + u p_r + \frac{1}{r} \rho u = 0, \]

\[ \rho(u_t + uu_r) + K p_t + \beta \rho u = \frac{M_0}{r} + v \left( u_{rr} + \frac{1}{r} u_r - \frac{1}{r^2} u \right), \]

where \( M_0 > 0 \); there is a unit stationary solid core located \([0, r_0]\), where \( r_0 \) is a positive constant, surrounded by the distribution density. The corresponding solutions are

\[ \rho(t, r) = \frac{e^{\nu(t)}}{a(t)}, \quad u(t, r) = \frac{\dot{a}(t)}{a(t)} r \quad \text{for} \quad r > r_0, \]

\[ \ddot{a}(t) + \beta \dot{a}(t) = -\frac{\lambda}{a(t)}, \quad a(0) = a_0 > 0, \quad \dot{a}(0) = a_1, \]

\[ y(x) = \frac{\lambda}{2K} x^2 + M_0 \ln x + \alpha, \]

where \( \alpha > -\lambda/2K \) is a constant.

III. PRESSURELESS NAVIER–STOKES EQUATIONS WITH DENSITY-DEPENDENT VISCOSITY

Now we consider the pressureless Navier–Stokes equations with density-dependent viscosity,

\[ \text{vis} (\rho, u) = \nabla (\mu(\rho) \nabla \cdot u), \]

in radial symmetry,

\[ \rho_r + u p_r + \frac{1}{r} \rho u = 0, \]

\[ \rho(u_t + uu_r) = (\mu(\rho)) u_r + \mu(\rho) \left( u_{rr} + \frac{N-1}{r} u_r - \frac{N-1}{r^2} u \right), \]

where \( \mu(\rho) \) is a density-dependent viscosity function, which is usually written as \( \mu(\rho) = \kappa \rho^\theta \) with the constants \( \kappa, \theta > 0 \). For the study of this kind of the above system, the readers may refer to Refs. 8, 9, and 11.

We can obtain the same estimate about Lemma 4 to the following ordinary differential equation (ODE):

\[ \ddot{a}(t) = \frac{\lambda \dot{a}(t)}{a(t)^2}, \]

\[ a(0) = a_0 > 0, \quad \dot{a}(0) = a_1 = \frac{\lambda}{a_0}. \]
**Lemma 11:** For the ODE (13), with \( \lambda > 0 \), there exists a finite time \( T_+ < +\infty \) such that \( a(T_+) = 0 \).

**Proof:** If \( a(t) > 0 \) and \( \dot{a}(0) = a_1 \leq \lambda/a_0 \) for all time \( t \), by integrating (13), we have

\[
\dot{a}(t) = -\frac{\lambda}{a(t)} - \frac{\lambda}{a_0} + a_1 \leq -\frac{\lambda}{a(t)}.
\] (14)

Take the integration for (14),

\[
\int_0^t a(s)\dot{a}(s)ds \leq -\int_0^t \lambda ds,
\]

\[
\frac{1}{2}[a(t)]^2 \leq -\lambda t + \frac{1}{2}a_0^2.
\]

When \( t \) is very large, we have

\[
\frac{1}{2}[a(t)]^2 \leq -1.
\]

A contradiction is met. The proof is completed. \( \blacksquare \)

Here we present another lemma before proceeding to the next theorem.

**Lemma 12:** For the ODE,

\[
y'(x)y'' - \xi x = 0,
\]

\[
y(0) = \alpha > 0, \quad n \neq -1,
\] (15)

where \( \xi \) and \( n \) are constants, we have the solution

\[
y(x) = \sqrt[2]{(n+1)}\xi x^{2/n} + C_1 x^{n+1}.
\]

**Proof:** The above ODE (15) may be solved by the separation method,

\[
y'(x)y'' - \xi x = 0,
\]

\[
y'(x)y'(x) = \xi x.
\]

By taking the integration with respect to \( x \),

\[
\int_0^x y'(x)y''dx = \int_0^x \xi dx,
\]

we have

\[
\int_0^x y(x)^n dx[y(x)] = \frac{1}{2}\xi x^2 + C_1,
\] (16)

where \( C_1 \) is a constant.

By integration by part, then the identity becomes

\[
y(x)^{n+1} - n\int_0^x y(x)^{n-1}y(x)y(x)dx = \frac{1}{2}\xi x^2 + C_1,
\]
\[ y(x)^{n+1} - n \int_0^x y(x)y(x)^n \, dx = \frac{1}{2} \xi x^2 + C_1. \]

From Eq. (16), we can have the simple expression for \( y(x) \),
\[ y(x)^{n+1} - n \left( \frac{1}{2} \xi x^2 + C_1 \right) = \frac{1}{2} \xi x^2 + C_1, \]
\[ y(x)^{n+1} = \frac{1}{2} (n+1) \xi x^2 + C_2, \]
where \( C_2 = (n+1) C_1. \)

By plugging into the initial condition for \( y(0) \), we have
\[ y(0)^{n+1} = \frac{1}{2} \xi_0 + C_1. \]
Thus, the solution is
\[ y(x) = \sqrt{\frac{1}{2} (n+1) \xi x^2 + \alpha^{n+1}}. \]
The proof is completed.

The family of the solution to the pressureless Navier–Stokes equations with density-dependent viscosity,

\[ \rho_t + u \rho_r + \rho u_r + \frac{N-1}{r} \rho u = 0, \tag{17a} \]
\[ \rho(u_t + uu_r) = (\kappa \rho^\theta), u_r + \kappa \rho \left( u_{rr} + \frac{N-1}{r} u_r - \frac{N-1}{r^2} u \right), \tag{17b} \]
is presented as the following.

**Theorem 13:** For the pressureless Navier–Stokes equations with density-dependent viscosity \((17a)\) and \((17b)\) in radial symmetry, there exists a family of solutions.

For \( \theta = 1 \),
\[ \rho(t,r) = e^{y(r/a(t))}, \quad u(t,r) = \frac{\dot{a}(t)}{a(t)} r, \]
\[ \dot{a}(t) = \frac{\lambda \dot{a}(t)}{a(t)^2}, \quad a(0) = a_0 > 0, \quad \dot{a}(0) = a_1, \]
\[ y(x) = \frac{\lambda}{2 \kappa} x^2 + \alpha, \]
where \( \alpha \) and \( \lambda \) are arbitrary constants. In particular, for \( \lambda > 0 \) and \( a_1 \leq \lambda/a_0 \), the solutions blow up in finite time. For \( \theta \neq 1 \),
\[ \rho(t,r) = \begin{cases} 
\frac{y(r/a(t))}{a(t)^N} & \text{for } y \left( \frac{r}{a(t)} \right) \geq 0; \\
0 & \text{for } y \left( \frac{r}{a(t)} \right) < 0
\end{cases}, \quad u(t,r) = \frac{\dot{a}(t)}{a(t)} r, \]
\[ \ddot{a}(t) = -\frac{\lambda \dot{a}(t)}{a(t)^{N-2N+2}}, \quad a(0) = a_0 > 0, \quad \dot{a}(0) = a_1, \tag{18} \]
where \( \alpha > 0 \).

**Proof of Theorem 13:** To (17a), we may use Lemma 3 to check it. For \( \theta = 1 \), (17b) becomes

\[
\rho (u_t + u \cdot u_r) - (\kappa \rho \theta)_r = \kappa \rho \theta (u_{rr} + \frac{N - 1}{r} u_r - \frac{N - 1}{r^2} u) = \rho \frac{\ddot{a}(t)}{a(t)} - \kappa \left( \frac{\lambda a(t)}{a(t)^{N+1}} \right) \dot{a}(t) = \rho \left( \frac{\lambda a(t)}{a(t)^{N+1}} \right) \dot{a}(t)
\]

where we use

\[
\dot{a}(t) = \frac{\lambda a(t)}{a(t)^2}.
\]

By choosing

\[
y \left( \frac{r}{a(t)} \right) = y(x) = \frac{\lambda}{2\kappa} x^2 + \alpha,
\]

(19) is equal to zero.

For the case of \( \theta \neq 1 \), (17b) can be calculated,

\[
\rho (u_t + u \cdot u_r) - (\kappa \rho \theta)_r = \kappa \rho \theta (u_{rr} + \frac{N - 1}{r} u_r - \frac{N - 1}{r^2} u) = \rho \frac{\ddot{a}(t)}{a(t)} - \kappa \left( \frac{\lambda a(t)}{a(t)^{N+1}} \right) \dot{a}(t) = \rho \left( \frac{\lambda a(t)}{a(t)^{N+1}} \right) \dot{a}(t)
\]

Define \( x = r/a(t) \), \( n = \theta - 2 \); it follows

\[
- \rho \frac{\dot{a}(t)}{a(t)^{N+2}} \left[ - \frac{\lambda}{a(t)} + \kappa \theta \left( \frac{r}{a(t)} \right)^{\theta-1} \dot{a}(t) \right] = \rho \left( 1 - \frac{\lambda a(t)}{a(t)^{N+1}} \right) \dot{a}(t)
\]

and \( \xi = \lambda / \kappa \theta \) in Lemma 12, and choose
Moreover this is easy to check that

\[ \dot{y}(0) = 0. \]

Equation (22) is equal to zero. The proof is completed.

**Remark 14:** By controlling the initial conditions in some solutions (18), we may get the blowup solutions. Additionally the modified solutions can be extended to the system in radial symmetry with frictional damping,

\[
\rho_r + u \rho_r + \rho u_r + \frac{N-1}{r} \rho u = 0,
\]

\[
\rho(u_r + uu_r) + \beta pu = (\mu(\rho))u_r + \mu(\rho) \left( u_{rr} + \frac{N-1}{r} u_r - \frac{N-1}{r^2} u \right),
\]

where \( \beta > 0 \). With the assistance of the ODE,

\[
\ddot{a}(t) + \beta \dot{a}(t) = -\frac{\lambda a(t)}{a(t)^5},
\]

where \( S \) is a constant.
