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The pulsrodon in $2+1$-dimensional magneto-gasdynamics: Hamiltonian structure and integrability

C. Rogers[1,3] and W. K. Schief[2,3,a)]
[1]*Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong*
[2]*School of Mathematics and Statistics, The University of New South Wales, Sydney, NSW 2052, Australia*
[3]*Australian Research Council Centre of Excellence for Mathematics and Statistics of Complex Systems, School of Mathematics and Statistics, The University of New South Wales, Sydney, NSW 2052, Australia*

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An elliptic vortex-type ansatz introduced into a $2+1$-dimensional system governing rotating homentropic magneto-gasdynamics with a parabolic gas law is shown to lead to a finite-dimensional nonlinear dynamical system which admits exact analytical solution in terms of an elliptic function and integral representation. The dynamical system is demonstrated to be Hamiltonian and equivalent to the stationary reduction of the integrable nonlinear Schrödinger equation coupled with a Steen-Ermakov-Pinney equation. A novel magneto-gasdynamic analogue of the pulsrodon of shallow water f-plane theory is isolated thereby. Confined and time-periodic magneto-gasdynamic flows are constructed explicitly. © *2011 American Institute of Physics*. [doi:10.1063/1.3622595]

I. INTRODUCTION

The analysis of the motion of electrically conducting fluids and plasmas as described by the Lundquist magneto-hydrodynamics (mhd) equations is of considerable importance in astrophysics, geophysics, and engineering applications.[2,21,32] In general, the coupled nonlinear mhd system is analytically intractable. However, under certain physically acceptable approximations, reductions have been made to canonical equations of soliton theory. Indeed, it was in a study of interaction processes in collisionless plasmas that Zabusky and Kruskal[37] made their historic discovery of recurrent solitonic phenomena as described by the Korteweg-de Vries (KdV) equation. The modified KdV equation was likewise originally set down in an analysis of the propagation of nonlinear Alfvén waves in a collisionless plasma.[12] Recent work has established that the uniaxial propagation of magneto-acoustic waves in a cold plasma subject to a purely transverse magnetic field may be modelled by the integrable "resonant" nonlinear Schrödinger equation.[14] In Refs. 26, 29, and 30, a geometric approach has been used to obtain reduction of a steady spatial mhd system to an integrable Regge-Lund model subject to a volume-preserving constraint. Novel Bernoulli-type integrals of motion for certain planar mhd systems have recently been shown to provide a means to construct exact solutions.[27]

In the case of integrable reductions, the well-established methods of modern soliton theory such as the inverse scattering transform[1] and invariance under Bäcklund transformations[25] are available for the analysis of the associated mhd systems. In general, in the absence of approximation, Lie group methods may be applied in a systematic manner to isolate substitution principles and privileged symmetry reductions corresponding to restricted classes of exact solutions of the mhd equations.[8,9,19]

In an interesting series of papers, Neukirch[16] and Neukirch and Priest[18] introduced a novel solution procedure in which the nonlinear acceleration terms in the governing Lundquist momentum

[a)]Author to whom correspondence should be addressed. Electronic mail: schief@maths.unsw.edu.au.

equation either vanish or, more generally, are assumed to be conservative.[17] In this paper, an approach to a $2+1$-dimensional mhd system is adopted which has roots in work of Goldsbrough[10] in classical hydrodynamics wherein a class of exact elliptical vortex solutions was constructed in a study of tidal oscillations in an elliptical basin. This work is, in turn, related to that of Kirchhoff[13] on vortex structures in the classical $2+1$-dimensional Euler system. In Ref. 4, an analytic study of a reduced gravity model descriptive of the time-dependent behaviour of upper-ocean elliptic eddies was undertaken. Subsequently, in Ref. 3, an eight-dimensional dynamical system was derived descriptive of the time evolution of elliptical warm core eddies. This system was solved in general in Ref. 24 and, in particular, a novel rotating, periodically pulsating eddy therein termed a pulsrodon was described analytically. An elegant Lagrangian treatment of an equivalent rotating shallow water system by Holm[11] established that the canonical exact solutions, namely, the rodon, pulson, and pulsrodon, are orbitally Lyapunov stable to perturbations within the class of elliptical vortex solutions.

Here, the procedure presented in Ref. 24 is adapted and extended to analyse a $2+1$-dimensional magneto-gasdynamics system of the type investigated in Ref. 17. An elliptical vortex type ansatz is introduced and reduction obtained thereby to a finite-dimensional nonlinear dynamical system analogous to that of Ref. 24 together with an additional algebraic condition. Time-modulated physical variables are introduced to reduce this system to a form amenable to exact solution. It is demonstrated that the dynamical system is Hamiltonian and that its general solution may be expressed in terms of elliptic functions and integrals. Thereby, remarkably, equivalence with the stationary reduction of the integrable nonlinear Schrödinger equation coupled with a classical Steen-Ermakov-Pinney equation[6,20,35] is proven. The above-mentioned algebraic condition is analysed and shown to hold if either the magnetic field is purely transversal or a compatible constraint is imposed on the dynamical system. In the latter case, a magneto-gasdynamic analogue of the pulsrodon is isolated. A detailed analysis then reveals that the class of solutions so obtained represents confined magneto-gasdynamic flows which are bounded by the surface of vanishing density. The existence of time-periodic flows in the case of the pulsrodon solutions is then investigated.

II. THE MAGNETO-GASDYNAMIC SYSTEM

Here, we consider a $2+1$-dimensional homentropic magneto-gasdynamic system incorporating rotation, namely,

$$\begin{gathered}\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\,\boldsymbol{q}) = 0 \\ \rho\left[\frac{\partial \boldsymbol{q}}{\partial t} + (\boldsymbol{q}\cdot\nabla)\boldsymbol{q}\right] - \mu\,\operatorname{curl}\boldsymbol{H}\times\boldsymbol{H} + \rho f(\boldsymbol{k}\times\boldsymbol{q}) + \nabla p = \boldsymbol{0} \\ \operatorname{div}\boldsymbol{H} = 0 \\ \frac{\partial \boldsymbol{H}}{\partial t} = \operatorname{curl}(\boldsymbol{q}\times\boldsymbol{H}),\end{gathered} \tag{2.1}$$

where

$$\boldsymbol{q} = u\boldsymbol{i} + v\boldsymbol{j}, \quad \boldsymbol{H} = \nabla A \times \boldsymbol{k} + h\boldsymbol{k} \tag{2.2}$$

and the parabolic pressure-density law

$$p = p_0 + \delta\rho + \epsilon\rho^2, \quad \frac{\partial p}{\partial \rho} > 0, \tag{2.3}$$

is adopted. This gas law may be used to approximate real magneto-gas behaviour. In particular, the adiabatic law $p \sim \rho^2$ has previously arisen in astrophysical contexts,[28] while the relation $p \sim \rho$ was adopted in Ref. 17. In the above, the magneto-gas density $\rho(\boldsymbol{x}, t)$, pressure $p(\boldsymbol{x}, t)$, gas velocity $\boldsymbol{q}(\boldsymbol{x}, t)$, magnetic induction $\boldsymbol{H}(\boldsymbol{x}, t)$, and magnetic flux $A(\boldsymbol{x}, t)$ are all assumed to be dependent only on $\boldsymbol{x} = x\boldsymbol{i} + y\boldsymbol{j}$ and t so that $(2.2)_2$ represents the general solution of the magnetic induction equation $(2.1)_3$ with $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ being an adapted orthonormal triad.

Insertion of the representation (2.2) into Faraday's law $(2.1)_4$ produces the convective constraint

$$\frac{\partial A}{\partial t} + (\boldsymbol{q} \cdot \nabla) A = 0 \tag{2.4}$$

together with

$$\frac{\partial h}{\partial t} + \operatorname{div}(h\, \boldsymbol{q}) = 0. \tag{2.5}$$

By virtue of the continuity equation $(2.1)_1$, the latter is identically satisfied if we set

$$h = \lambda \rho, \quad \lambda = \text{const.} \tag{2.6}$$

and insertion of the representation (2.2) together with the constitutive law (2.3) into the momentum equation $(2.1)_2$ then yields

$$\frac{\partial \boldsymbol{q}}{\partial t} + (\boldsymbol{q} \cdot \nabla)\boldsymbol{q} + \frac{\mu}{\rho}(\nabla^2 A)\nabla A + f\boldsymbol{k} \times \boldsymbol{q} + \delta \nabla(\ln \rho) + (2\epsilon + \mu\lambda^2)\nabla \rho = \boldsymbol{0} \tag{2.7}$$

together with

$$\frac{\partial A}{\partial y}\frac{\partial \rho}{\partial x} = \frac{\partial A}{\partial x}\frac{\partial \rho}{\partial y},$$

whence

$$A = A(\rho, t). \tag{2.8}$$

In the sequel, attention is restricted to the separable case

$$A = \Phi(\rho) T(t), \tag{2.9}$$

whence, on substitution into (2.4) and use of the continuity equation $(2.1)_1$, it is seen that

$$\dot{T} = \frac{\rho \Phi'}{\Phi} T \operatorname{div} \boldsymbol{q}. \tag{2.10}$$

Here, we proceed with the simplest case $\Phi = \rho$ and (2.9) becomes

$$A = \rho T(t), \tag{2.11}$$

whence, on insertion into (2.7),

$$\frac{\partial \boldsymbol{q}}{\partial t} + (\boldsymbol{q} \cdot \nabla)\boldsymbol{q} + (\mu T^2 \nabla^2 \rho + \delta)\nabla \ln \rho + (2\epsilon + \mu\lambda^2)\nabla \rho + f\boldsymbol{k} \times \boldsymbol{q} = \boldsymbol{0} \tag{2.12}$$

to be solved in conjunction with the continuity equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\, \boldsymbol{q}) = 0 \tag{2.13}$$

and the time evolution (2.10), that is,

$$\dot{T} = T \operatorname{div} \boldsymbol{q}. \tag{2.14}$$

The inherent nonlinearity of the coupled magneto-gasdynamic system (2.12)–(2.14) remains a major impediment to analytic progress. It is noted that this system is overdetermined since (2.12) is implicitly constrained by the requirement that $\operatorname{div} \boldsymbol{q}$ be a function of t only.

III. A FINITE-DIMENSIONAL DYNAMICAL SYSTEM

An elliptic vortex type ansatz is now introduced with

$$\begin{gathered} \boldsymbol{q} = \boldsymbol{L}(t)\bar{\boldsymbol{x}} + \boldsymbol{m}(t), \quad \bar{\boldsymbol{x}} = \begin{pmatrix} x - \bar{q}(t) \\ y - \bar{p}(t) \end{pmatrix} \\ \rho = \frac{\bar{\boldsymbol{x}}^T \boldsymbol{E}(t)\bar{\boldsymbol{x}} + h_0(t)}{2\epsilon + \mu\lambda^2}, \quad 2\epsilon + \mu\lambda^2 \neq 0, \end{gathered} \tag{3.1}$$

where

$$\boldsymbol{L} = \begin{pmatrix} u_1(t) & u_2(t) \\ v_1(t) & v_2(t) \end{pmatrix}, \quad \boldsymbol{m} = \begin{pmatrix} \dot{\bar{q}}(t) \\ \dot{\bar{p}}(t) \end{pmatrix}$$
$$\boldsymbol{E} = \begin{pmatrix} a(t) & b(t) \\ b(t) & c(t) \end{pmatrix}. \tag{3.2}$$

Here and in the following, three-dimensional purely "horizontal" vectors are identified with their canonical two-dimensional counterparts and vice versa. Insertion into the continuity equation (2.13) yields

$$\begin{pmatrix} \dot{a} \\ \dot{b} \\ \dot{c} \end{pmatrix} + \begin{pmatrix} 3u_1 + v_2 & 2v_1 & 0 \\ u_2 & 2(u_1 + v_2) & v_1 \\ 0 & 2u_2 & u_1 + 3v_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \boldsymbol{0} \tag{3.3}$$

together with

$$\dot{h}_0 + (u_1 + v_2)h_0 = 0. \tag{3.4}$$

It is observed that the momentum equation (2.12) does not contain the function $T(t)$ if it is so chosen that the terms in $\nabla \ln \rho$ are eliminated. Accordingly, it is demanded that

$$\mu T^2 = -\frac{\delta(2\epsilon + \mu\lambda^2)}{2(a + c)}, \tag{3.5}$$

whence (2.12) is reduced to

$$\frac{\partial \boldsymbol{q}}{\partial t} + (\boldsymbol{q} \cdot \nabla)\boldsymbol{q} + f\boldsymbol{k} \times \boldsymbol{q} + (2\epsilon + \mu\lambda^2)\nabla\rho = 0. \tag{3.6}$$

Insertion of (3.1) into (3.6) now gives

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{v}_1 \\ \dot{v}_2 \end{pmatrix} + \begin{pmatrix} \boldsymbol{L}^T & -f\boldsymbol{I} \\ f\boldsymbol{I} & \boldsymbol{L}^T \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix} + 2\begin{pmatrix} a \\ b \\ b \\ c \end{pmatrix} = \boldsymbol{0}, \tag{3.7}$$

augmented by the linear auxiliary equations

$$\ddot{\bar{p}} + f\dot{\bar{q}} = 0, \quad \ddot{\bar{q}} - f\dot{\bar{p}} = 0. \tag{3.8}$$

Thus, the solution of the original magneto-gasdynamic system is encoded in the seven-dimensional nonlinear dynamical system (3.3) and (3.7). Once, the solution of the latter is known, the quantities h_0 and T are obtained via integration of (3.4) and (2.14), that is,

$$\dot{T} = (u_1 + v_2)T. \tag{3.9}$$

However, *a priori*, the ansatz (3.5) imposes a constraint on the dynamical system, the admissibility of which is examined below. It is noted that if $f = 0$ or the divergence $u_1 + v_2 = 0$ then the relations (3.7) show that the class of motions governed by (3.1) constitutes "accelerated" motions in the sense of Ref. 17.

A. Canonical variables

In the sequel, it proves convenient to proceed in terms of new variables

$$G = u_1 + v_2, \quad G_R = \frac{1}{2}(v_1 - u_2),$$
$$G_S = \frac{1}{2}(v_1 + u_2), \quad G_N = \frac{1}{2}(u_1 - v_2), \tag{3.10}$$
$$B = a + c, \quad B_S = b, \quad B_N = \frac{1}{2}(a - c).$$

Here, G and G_R represent, in turn, the divergence and spin of the velocity field, while G_S and G_N represent shear and normal deformation rates. On use of the expressions (3.10), the system (3.3), (3.4), (3.7), and (3.9) adopts the form

$$\begin{aligned}
\dot{B} &= -2[BG + 2(B_N G_N + B_S G_S)],\\
\dot{B}_S &= -2B_S G - G_S B + 2B_N G_R,\\
\dot{B}_N &= -2B_N G - G_N B - 2B_S G_R\\
\dot{G} &= -\frac{1}{2}G^2 - 2(G_N^2 + G_S^2 - G_R^2) + 2fG_R - 2B,\\
\dot{G}_R &= -GG_R - \frac{1}{2}fG,\\
\dot{G}_N &= -GG_N + fG_S - 2B_N,\\
\dot{G}_S &= -GG_S - fG_N - 2B_S,
\end{aligned} \tag{3.11}$$

together with

$$\dot{h}_0 = -Gh_0, \quad \dot{T} = GT. \tag{3.12}$$

The form of $(3.11)_4$ suggests introducing a function Ω via

$$G = \frac{2\dot{\Omega}}{\Omega} \tag{3.13}$$

so that $(3.11)_5$ and (3.12) yield, in turn,

$$G_R + \frac{1}{2}f = \frac{c_0}{\Omega^2} \tag{3.14}$$

and

$$h_0 = \frac{c_1}{\Omega^2}, \quad T = \tilde{c}\Omega^2, \tag{3.15}$$

where c_1 and $\tilde{c}$ are arbitrary constants of integration.

New modulated variables are now introduced according to

$$\begin{gathered}
\bar{B} = \Omega^4 B, \quad \bar{B}_S = \Omega^4 B_S, \quad \bar{B}_N = \Omega^4 B_N,\\
\bar{G}_S = \Omega^2 G_S, \quad \bar{G}_N = \Omega^2 G_N,
\end{gathered} \tag{3.16}$$

whence the system (3.11) reduces to

$$\begin{aligned}
\dot{\bar{B}} &= -4\frac{\bar{B}_N\bar{G}_N + \bar{B}_S\bar{G}_S}{\Omega^2},\\
\dot{\bar{B}}_S &= -\frac{\bar{B}\bar{G}_S - 2c_0\bar{B}_N}{\Omega^2} - f\bar{B}_N,\\
\dot{\bar{B}}_N &= -\frac{\bar{B}\bar{G}_N + 2c_0\bar{B}_S}{\Omega^2} + f\bar{B}_S,\\
\dot{\bar{G}}_N &= f\bar{G}_S - 2\frac{\bar{B}_N}{\Omega^2},\\
\dot{\bar{G}}_S &= -f\bar{G}_N - 2\frac{\bar{B}_S}{\Omega^2},
\end{aligned} \tag{3.17}$$

together with the second-order nonlinear differential equation

$$\Omega^3\ddot{\Omega} + \frac{f^2}{4}\Omega^4 - c_0^2 + \bar{G}_N^2 + \bar{G}_S^2 + \bar{B} = 0. \tag{3.18}$$

B. The constraint (3.5)

Comparison of the two expressions for T given by $(3.15)_2$ and (3.5), that is

$$\mu T^2 = -\frac{\delta(2\epsilon + \mu\lambda^2)}{2\bar{B}}\Omega^4, \tag{3.19}$$

reveals that

$$2\mu\tilde{c}^2\bar{B} = -\delta(2\epsilon + \mu\lambda^2) \tag{3.20}$$

and hence

$$\tilde{c} = \delta = 0 \quad \text{or} \quad \bar{B} = \text{const.} \tag{3.21}$$

In the former case, the magnetic flux A vanishes so that the magnetic induction $\boldsymbol{H}$ is purely transversal and the dynamical system (3.17), (3.18) is unconstrained. In the latter case, the evolution equation $(3.17)_1$ implies that the dynamical system (3.17), (3.18) is constrained by

$$\bar{B}_N\bar{G}_N + \bar{B}_S\bar{G}_S = 0. \tag{3.22}$$

In order to satisfy this condition, we introduce the parametrisation

$$\bar{B}_N = \alpha\bar{G}_S, \quad \bar{B}_S = -\alpha\bar{G}_N. \tag{3.23}$$

The system (3.17) is then readily shown to reduce to the pair

$$\dot{\bar{G}}_N = \left(f - 2\frac{\alpha}{\Omega^2}\right)\bar{G}_S, \quad \dot{\bar{G}}_S = -\left(f - 2\frac{\alpha}{\Omega^2}\right)\bar{G}_N, \tag{3.24}$$

with

$$\bar{B} = 2\alpha(c_0 - \alpha), \quad \dot{\alpha} = 0. \tag{3.25}$$

Accordingly, the constraint (3.5) is admissible. The above linear system is equivalent to a harmonic oscillator equation and gives rise to pulsrodon solutions of the magneto-gasdynamic system as discussed in Sec. V.

In conclusion, it is noted that, in general, the dynamical system (3.17), (3.18) admits the three integrals of motion

$$\begin{gathered}\bar{B}_S^2 + \bar{B}_N^2 - \frac{\bar{B}^2}{4} = c_2, \quad \bar{G}_S^2 + \bar{G}_N^2 - \bar{B} = c_3,\\ 2(\bar{B}_N\bar{G}_S - \bar{B}_S\bar{G}_N) - c_0\bar{B} = c_4,\end{gathered} \tag{3.26}$$

the relevance of which is analysed in Sec. IV. In particular, if $\bar{B} = \text{const.}$ then the second-order equation (3.18) reduces to the classical Steen-Ermakov-Pinney equation[6,20,35]

$$\ddot{\Omega} + \frac{f^2}{4}\Omega = \frac{(c_0 - 2\alpha)^2 - c_3}{\Omega^3}, \tag{3.27}$$

with constant "frequency" $f/2$. In fact, it is shown in Sec. IV that the Steen-Ermakov-Pinney equation with variable frequency arises in the general case. A connection with a standard two-component generalisation of the Steen-Ermakov-Pinney equation leading to another "hidden" integral of motion is revealed in Sec. VII.

IV. HAMILTONIAN STRUCTURE AND INTEGRABILITY

It is now demonstrated that the seven-dimensional dynamical system (3.17), (3.18) is Hamiltonian and, in fact, equivalent to the stationary integrable nonlinear Schrödinger equation coupled with the Steen-Ermakov-Pinney equation.

A. Hamiltonian formulation

We first eliminate the "rotation coefficient" f by considering the change of variables

$$\begin{aligned} \sigma &= \bar{B}_S \cos ft + \bar{B}_N \sin ft, \quad \varphi = \bar{G}_S \cos ft + \bar{G}_N \sin ft, \\ \tau &= \bar{B}_N \cos ft - \bar{B}_S \sin ft, \quad \psi = \bar{G}_N \cos ft - \bar{G}_S \sin ft. \end{aligned} \tag{4.1}$$

In terms of the new variables, the dynamical system adopts the form

$$\begin{aligned} \dot{\sigma} &= -\frac{\bar{B}\varphi - 2c_0\tau}{\Omega^2}, \quad \dot{\varphi} = -2\frac{\sigma}{\Omega^2} \\ \dot{\tau} &= -\frac{\bar{B}\psi + 2c_0\sigma}{\Omega^2}, \quad \dot{\psi} = -2\frac{\tau}{\Omega^2} \end{aligned} \tag{4.2}$$

and

$$\dot{\bar{B}} = -4\frac{\sigma\varphi + \tau\psi}{\Omega^2} \tag{4.3}$$

together with

$$\Omega^3\ddot{\Omega} + \frac{f^2}{4}\Omega^4 - c_0^2 + \varphi^2 + \psi^2 + \bar{B} = 0. \tag{4.4}$$

If we now introduce a new time measure s according to

$$s = \int \frac{dt}{\Omega^2(t)} \tag{4.5}$$

then the above system "decouples" into the five-dimensional dynamical system

$$\begin{aligned} \sigma' &= -\bar{B}\varphi + 2c_0\tau, \quad \varphi' = -2\sigma, \\ \tau' &= -\bar{B}\psi - 2c_0\sigma, \quad \psi' = -2\tau, \\ \bar{B}' &= -4(\sigma\varphi + \tau\psi), \end{aligned} \tag{4.6}$$

and the Steen-Ermakov-Pinney equation

$$\Sigma'' + (c_0^2 - \varphi^2 - \psi^2 - \bar{B})\Sigma = \frac{f^2}{4\Sigma^3}, \qquad \Sigma = \frac{1}{\Omega}, \tag{4.7}$$

which may be solved once σ, τ, φ, ψ, and $\bar{B}$ are known. The associated integrals of motion become

$$\begin{aligned} \sigma^2 + \tau^2 - \frac{\bar{B}^2}{4} &= c_2, \quad \varphi^2 + \psi^2 - \bar{B} = c_3, \\ 2(\tau\varphi - \sigma\psi) - c_0\bar{B} &= c_4. \end{aligned} \tag{4.8}$$

Thus, the dynamical system (4.6) may be formulated as

$$\begin{aligned} \varphi'' - 2(\varphi^2 + \psi^2 - c_3)\varphi - 2c_0\psi' &= 0, \\ \psi'' - 2(\varphi^2 + \psi^2 - c_3)\psi + 2c_0\varphi' &= 0, \end{aligned} \tag{4.9}$$

with

$$\sigma = -\frac{1}{2}\varphi', \quad \tau = -\frac{1}{2}\psi', \quad \bar{B} = \varphi^2 + \psi^2 - c_3, \tag{4.10}$$

and the Steen-Ermakov-Pinney equation becomes

$$\Sigma'' + [c_0^2 + c_3 - 2(\varphi^2 + \psi^2)]\Sigma = \frac{f^2}{4\Sigma^3}. \tag{4.11}$$

Finally, the remaining integrals of motion are given by

$$\begin{aligned} \varphi'^2 + \psi'^2 - (\varphi^2 + \psi^2 - c_3)^2 &= \text{const.}, \\ \varphi'\psi - \psi'\varphi - c_0(\varphi^2 + \psi^2) &= \text{const.} \end{aligned} \tag{4.12}$$

It turns out that the integral of motion $(4.12)_1$ essentially constitutes a Hamiltonian associated with the dynamical system (4.9). Indeed, if we set

$$\mathcal{H}_1 = \frac{1}{2}(p_\varphi + c_0\psi)^2 + \frac{1}{2}(p_\psi - c_0\varphi)^2 - \frac{1}{2}(\varphi^2 + \psi^2 - c_3)^2 \tag{4.13}$$

then the Hamilton equations

$$\varphi' = \frac{\partial \mathcal{H}_1}{\partial p_\varphi}, \quad \psi' = \frac{\partial \mathcal{H}_1}{\partial p_\psi}, \tag{4.14}$$

deliver the generalised momenta

$$p_\varphi = \varphi' - c_0\psi, \quad p_\psi = \psi' + c_0\varphi, \tag{4.15}$$

while the remaining Hamilton equations

$$p'_\varphi = -\frac{\partial \mathcal{H}_1}{\partial \varphi}, \quad p'_\psi = -\frac{\partial \mathcal{H}_1}{\partial \psi} \tag{4.16}$$

are equivalent to the "equations of motion" (4.9).

B. The stationary integrable nonlinear Schrödinger equation

The integrability of the dynamical system (4.9) may also be seen by relating it to the stationary reduction of the integrable nonlinear Schrödinger equation. Thus, if we set

$$\gamma = \varphi + i\psi \tag{4.17}$$

then the two second-order differential equations (4.9) may be combined to obtain

$$ic_0\gamma' + c_3\gamma = -\frac{1}{2}\gamma'' + |\gamma|^2\gamma. \tag{4.18}$$

The latter is nothing but the stationary reduction

$$\gamma(\eta, \xi) = \gamma(\eta + c_0\xi)e^{-ic_3\xi} \tag{4.19}$$

of the integrable (defocusing) nonlinear Schrödinger equation[1]

$$i\gamma_\xi = -\frac{1}{2}\gamma_{\eta\eta} + |\gamma|^2\gamma. \tag{4.20}$$

Moreover, the Steen-Ermakov-Pinney equation (4.11) adopts the compact form

$$\Sigma'' + (c_0^2 + c_3 - 2|\gamma|^2)\Sigma = \frac{f^2}{4\Sigma^3}. \tag{4.21}$$

1. The solution of the stationary Schrödinger equation

It is well known that the stationary nonlinear Schrödinger equation may be solved in terms of elliptic functions and integrals (see, e.g., Ref. 30, and references therein). Here, in view of the Steen-Ermakov-Pinney equation, it is convenient to express the solution in terms of the Weierstrass $\wp$ function.[34] Thus, if we set

$$\varphi = \kappa \cos\chi, \quad \psi = \kappa \sin\chi \tag{4.22}$$

then the dynamical system (4.9) represented by the integrals of motion (4.12) assumes the form

$$\kappa'^2 + \kappa^2\chi'^2 - (\kappa^2 - c_3)^2 = c_5, \quad \kappa^2\chi' + c_0\kappa^2 = c_6. \tag{4.23}$$

Thus, elimination of χ'^2 leads to the differential equation

$$\kappa'^2 + \left(\frac{c_6}{\kappa^2} - c_0\right)^2\kappa^2 - (\kappa^2 - c_3)^2 = c_5, \tag{4.24}$$

for κ only, while χ is obtained via integration of

$$\chi' = \frac{c_6}{\kappa^2} - c_0. \tag{4.25}$$

If we now set

$$\kappa^2 = P + \frac{1}{3}c_0^2 + \frac{2}{3}c_3 \tag{4.26}$$

then (4.24) reduces to the differential equation

$$P'^2 = 4P^3 - g_2 P - g_3, \tag{4.27}$$

with

$$\begin{aligned} g_2 &= \frac{4}{3}c_0^4 + \frac{16}{3}c_0^2 c_3 + \frac{4}{3}c_3^2 - 4c_5 - 8c_0 c_6, \\ g_3 &= \frac{8}{27}c_0^6 + \frac{16}{9}c_0^4 c_3 + \frac{20}{9}c_0^2 c_3^2 - \frac{4}{3}c_0^2 c_5, \\ &\quad -\frac{8}{3}c_0^3 c_6 - \frac{16}{3}c_0 c_3 c_6 - \frac{8}{27}c_3^3 - \frac{8}{3}c_3 c_5 + 4c_6^2. \end{aligned} \tag{4.28}$$

Accordingly, the general solution of the above differential equation is given by

$$P(s) = \wp(s + s_0), \tag{4.29}$$

where s_0 is an arbitrary constant of integration, and the Steen-Ermakov-Pinney equation (4.21) becomes

$$\Sigma'' - (2P + C)\Sigma = \frac{f^2}{4\Sigma^3} \tag{4.30}$$

with

$$C = \frac{1}{3}(c_3 - c_0^2). \tag{4.31}$$

2. The solution of the Steen-Ermakov-Pinney equation

The general Steen-Ermakov-Pinney equation is given by

$$\Sigma'' + \omega^2(s)\Sigma = \frac{\delta_0}{\Sigma^3}, \tag{4.32}$$

where ω is an arbitrary but given function of the independent variable and δ_0 is a constant. It originated in a paper by Steen[35] and arises in a wide range of areas of physical importance, most notably in quantum mechanics, optics, and nonlinear elasticity (see, e.g., Refs. 7, 15, and 31). It is characterised by its admittance of a nonlinear superposition principle. Thus, the general solution of (4.32) is given by

$$\Sigma^2 = \delta_1\Sigma_1^2 + 2\delta_2\Sigma_1\Sigma_2 + \delta_3\Sigma_2^2, \tag{4.33}$$

where Σ_1 and Σ_2 are linearly independent solutions of

$$\Sigma_L'' + \omega^2(s)\Sigma_L = 0 \tag{4.34}$$

with unit Wronskian, that is,

$$W(\Sigma_1, \Sigma_2) = \Sigma_1\Sigma_2' - \Sigma_2\Sigma_1' = 1 \tag{4.35}$$

and the constants δ_i are constrained by the relation

$$\delta_1\delta_3 - \delta_2^2 = \delta_0. \tag{4.36}$$

In the current context,

$$\omega^2 = -(2P + C), \quad \delta_0 = \frac{f^2}{4} \tag{4.37}$$

so that, remarkably, (4.34) turns out to be the classical Lamé equation[5]

$$\Sigma_L'' - l(l+1)P\Sigma_L = C\Sigma_L, \quad l = 1, \tag{4.38}$$

that is, the Schrödinger equation with a constant multiple of the Weierstrass $\wp$ function as its potential. If l is an integer, the Lamé equation may be solved explicitly in terms of Lamé polynomials[5,36] and hence the integration of the dynamical system (3.17), (3.18) has been achieved.

V. THE PULSRODON

In general, the solution of the magneto-gasdynamic system obtained in the preceding is only valid for purely transversal magnetic fields, that is $\delta = 0$. However, as discussed in Sec. III if one imposes the admissible constraint $\bar{B} = \text{const.}$ then δ may be arbitrary. In this case, the solutions of the stationary nonlinear Schrödinger equation (4.18) are of the simple form

$$\gamma = \kappa_0 e^{i\chi_0 t} \tag{5.1}$$

and the Lamé equation (4.38) degenerates to the harmonic oscillator equation. Alternatively, it has been shown in Sec. III that this particular case is governed by the linear system

$$\dot{\bar{G}}_N = \left(f - 2\frac{\alpha}{\Omega^2}\right)\bar{G}_S, \quad \dot{\bar{G}}_S = -\left(f - 2\frac{\alpha}{\Omega^2}\right)\bar{G}_N, \tag{5.2}$$

and the Steen-Ermakov-Pinney equation,

$$\ddot{\Omega} + \frac{f^2}{4}\Omega = \frac{\delta_0}{\Omega^3}, \quad \delta_0 = (c_0 - \alpha)^2 + \alpha^2 - G_0^2, \tag{5.3}$$

where

$$\bar{G}_S^2 + \bar{G}_N^2 = G_0^2 = \text{const.} \tag{5.4}$$

The general solution of the linear system (5.2) is given by

$$\begin{gathered}\bar{G}_N = G_0 \sin\eta, \quad \bar{G}_S = G_0\cos\eta, \\ \eta = ft - 2\alpha\int\frac{dt}{\Omega^2(t),}\end{gathered} \tag{5.5}$$

and application of the procedure outlined in Sec. IV with

$$\Omega_1 = \cos\frac{f}{2}, \quad \Omega_2 = \frac{2}{f}\sin\frac{f}{2} \tag{5.6}$$

produces the general solution

$$\Omega^2 = \delta_4\cos(ft + \omega_0) + \delta_5 \tag{5.7}$$

of the Steen-Ermakov-Pinney equation (5.3), where the constants δ_4 and δ_5 are related by

$$f^2(\delta_4^2 - \delta_5^2) + 4\delta_0 = 0. \tag{5.8}$$

It is noted that the reality constraints associated with the relations $(5.3)_2$, (5.7), and (5.8) require that

$$\delta_5 > \delta_4 \geq 0, \quad (c_0 - \alpha)^2 + \alpha^2 > G_0^2 \tag{5.9}$$

without loss of generality.

The quantities $B, B_S, B_N, G, G_R, G_N, G_S$, and h_0, T are now determined via the relations (3.13)–(3.16) and (3.20), (3.23), (3.25). We obtain

$$\begin{gathered} G = 2\frac{\dot{\Omega}}{\Omega}, \quad G_R = \frac{c_0}{\Omega^2} - \frac{1}{2}f, \quad B = \frac{2\alpha(c_0-\alpha)}{\Omega^4}, \\ B_N = \frac{\alpha G_0}{\Omega^4}\cos\eta, \quad B_S = -\frac{\alpha G_0}{\Omega^4}\sin\eta, \\ G_N = \frac{G_0}{\Omega^2}\sin\eta, \quad G_S = \frac{G_0}{\Omega^2}\cos\eta, \end{gathered} \tag{5.10}$$

together with

$$h_0 = \frac{c_1}{\Omega^2}, \quad T^2 = -\frac{\delta(2\epsilon + \mu\lambda^2)}{4\mu\alpha(c_0-\alpha)}\Omega^4, \tag{5.11}$$

subject to $\delta\alpha(c_0-\alpha) < 0$. The velocity and density distributions are given by the relations (3.1) and (3.2), wherein

$$\begin{aligned} u_1 &= \frac{\dot{\Omega}}{\Omega} + \frac{G_0}{\Omega^2}\sin\eta, \\ v_1 &= \frac{G_0}{\Omega^2}\cos\eta + \frac{c_0}{\Omega^2} - \frac{1}{2}f, \\ u_2 &= \frac{G_0}{\Omega^2}\cos\eta - \frac{c_0}{\Omega^2} + \frac{1}{2}f, \\ v_2 &= \frac{\dot{\Omega}}{\Omega} - \frac{G_0}{\Omega^2}\sin\eta, \end{aligned} \tag{5.12}$$

and

$$\begin{aligned} a &= \frac{\alpha}{\Omega^4}(c_0 - \alpha + G_0\cos\eta), \\ b &= -\frac{\alpha G_0}{\Omega^4}\sin\eta, \\ c &= \frac{\alpha}{\Omega^4}(c_0 - \alpha - G_0\cos\eta). \end{aligned} \tag{5.13}$$

The remaining functions $\bar{p}$ and $\bar{q}$ are readily determined from the coupled system of linear differential equations (3.8). Finally, the magnetic flux A is determined by (2.11), while the pressure is obtained from the constitutive law (2.3).

It is noted, parenthetically, that the class of exact multi-parameter $2+1$-dimensional magneto-gasdynamics solutions set down explicitly above may be boosted to produce a class of $3+1$-dimensional solutions via the simple superposition $\boldsymbol{q} \to \boldsymbol{q} + \psi(A)\boldsymbol{k}$. The magneto-gasdynamics solutions presented here are analogous to the pulsrodons constructed in Ref. 24 in the context of a rotating elliptic-warm core hydrodynamic system. These pulsrodon solutions and their duals were later derived via an elegant Lagrangian formulation in Ref. 11.

VI. CONFINED MAGNETO-GASDYNAMIC FLOWS

Here, we analyse in more detail the class of solutions presented in the preceding sections. The following discussion applies to both flows for which the magnetic field is purely transversal and flows of pulsrodon type. Whenever applicable, properties which are specific to either of these flows are discussed.

A. The geometry of the surfaces of constant density

The analysis is based on the surfaces of constant density

$$S_\varrho(t) = \{\boldsymbol{r} = \boldsymbol{x} + z\boldsymbol{k} \in \mathbb{R}^3 : \rho(\boldsymbol{x}, t) = \varrho\}, \tag{6.1}$$

defined for any fixed time t and constant ϱ. The ansatz $(3.1)_3$ shows that the surfaces S_ϱ given by

$$\bar{\boldsymbol{x}}^T \boldsymbol{E}(t)\bar{\boldsymbol{x}} + r^2(t) = 0, \quad r^2(t) = h_0(t) - (2\epsilon + \mu\lambda^2)\varrho \tag{6.2}$$

constitute cylinders with conic cross sections. If the real eigenvalues of the symmetric matrix $\boldsymbol{E}$ are denoted by $\lambda_\pm$ then the cross sections are concentric ellipses if and only if the signs of $r^2, -\lambda_\pm$ coincide. Since $\rho \geq 0$, the region in which the solutions are physically meaningful is bounded by the cylinder S_0. The latter surface of vanishing density is a material surface since it is convected with the flow. Indeed, the continuity equation $(2.1)_1$ implies that

$$\frac{d}{dt}\rho(\boldsymbol{x}(t), t) = -\rho(\boldsymbol{x}(t), t)\,\mathrm{div}\,\boldsymbol{q}(\boldsymbol{x}(t), t) \tag{6.3}$$

for any particle line $\boldsymbol{x}(t)$. Thus, magneto-gasdynamic flows which are confined to the interior of an elliptic cylinder S_0 are obtained by demanding that

$$\lambda_\pm < 0, \quad h_0 > 0. \tag{6.4}$$

Here, we assume that $2\epsilon + \mu\lambda^2 > 0$. A similar analysis may be conducted in the case $2\epsilon + \mu\lambda^2 < 0$. It is noted that exact magneto-hydrostatic solutions for elliptic plasma cylinders bounded by a vacuum have been presented in Refs. 33 and 2.

Since the magnetic induction $\boldsymbol{H}$ is of the form

$$\boldsymbol{H} = T(t)\nabla\rho \times \boldsymbol{k} + \lambda\rho\,\boldsymbol{k}, \tag{6.5}$$

we conclude that $\boldsymbol{H} \cdot \nabla\rho = 0$ and hence the magnetic field lines lie on the constant density surfaces S_ϱ. Thus, in the case of a purely transversal magnetic field, the field lines are the straight generators of the elliptic cylinders S_ϱ and the magnetic field vanishes on the boundary S_0. In the presence of pulsrodons (with non-vanishing T), the magnetic field lines are horizontal ellipses on the boundary S_0 and "elliptic" helices on the interior cylinders S_ϱ with constant vertical component $\lambda\varrho$ (cf., Figure 2). At any fixed time, the density ρ assumes its maximum value

$$\varrho_{\max} = \frac{h_0}{2\epsilon + \mu\lambda^2}, \tag{6.6}$$

on the (central) magnetic axis $S_{\varrho_{\max}}$. Specifically, for any fixed time t, the parametrised magnetic field line $\boldsymbol{r}(s)$ passing through the point $\boldsymbol{r}(0) = \boldsymbol{r}_0$ is the solution of the initial value problem

$$\frac{d}{ds}\boldsymbol{r}(s) = \boldsymbol{H}(\boldsymbol{r}(s)), \quad \boldsymbol{r}(0) = \boldsymbol{r}_0 \tag{6.7}$$

so that

$$\frac{d}{ds}(\bar{\boldsymbol{x}}(s) + z(s)\,\boldsymbol{k}) = \frac{2T}{2\epsilon + \mu\lambda^2}\boldsymbol{E}\bar{\boldsymbol{x}}(s) \times \boldsymbol{k} + \lambda\rho(\bar{\boldsymbol{x}}(s))\,\boldsymbol{k}. \tag{6.8}$$

Since ρ is constant along the magnetic field lines, that is,

$$\rho(\bar{\boldsymbol{x}}(s)) = \rho_0 = \frac{\bar{\boldsymbol{x}}_0^T \boldsymbol{E}\bar{\boldsymbol{x}}_0 + h_0}{2\epsilon + \mu\lambda^2}, \tag{6.9}$$

the solution of the above initial value problem is given by

$$\begin{aligned} \bar{\boldsymbol{x}} &= \bar{\boldsymbol{x}}_0 \cos\nu s + \boldsymbol{E}\bar{\boldsymbol{x}}_0 \times \boldsymbol{k}\frac{\sin\nu s}{\sqrt{ac - b^2}} \\ z &= \lambda\rho_0 s + z_0 \end{aligned} \tag{6.10}$$

with

$$\nu = \frac{2T}{2\epsilon + \mu\lambda^2}\sqrt{ac - b^2}. \tag{6.11}$$

B. The motion of the boundary

The motion of the boundary surface S_0 is determined by the temporal behaviour of the elliptic cross section

$$\bar{\boldsymbol{x}}^T \boldsymbol{E} \bar{\boldsymbol{x}} + h_0 = 0, \quad h_0 = \frac{c_1}{\Omega^2}, \tag{6.12}$$

with $c_1 > 0$ by virtue of the constraint (6.4)$_2$. In general, the motion of this ellipse may be decomposed into four components. Thus, the ellipse rotates about its centre $\bar{\boldsymbol{x}} = \boldsymbol{0}$ and a fixed point which may be taken to be the origin of the $\boldsymbol{x}$-plane and it changes its eccentricity and size.

The definition (3.1)$_2$ of the reduced coordinate $\bar{\boldsymbol{x}}$ shows that the centre of the ellipse is located at

$$\boldsymbol{x} = \begin{pmatrix} \bar{q} \\ \bar{p} \end{pmatrix}, \tag{6.13}$$

where $\bar{p}$ and $\bar{q}$ are a solution of the linear system (3.8) and hence

$$\bar{q} = r_0 \sin ft, \quad \bar{p} = r_0 \cos ft \tag{6.14}$$

without loss of generality. Thus, the ellipse rotates about the origin at an angular velocity of $-f$ (measured anti-clockwise).

The eigenvalues of the matrix $\boldsymbol{E}$ are given by

$$\begin{aligned} \lambda_\pm &= \frac{a + c \pm \sqrt{(a-c)^2 + 4b^2}}{2} \\ &= \frac{\bar{B} \pm 2\sqrt{\bar{B}_N^2 + \bar{B}_S^2}}{2\Omega^4} \\ &= \frac{\bar{B} \pm \sqrt{\bar{B}^2 + 4c_2}}{2\Omega^4}, \end{aligned} \tag{6.15}$$

where the relations

$$a + c = \frac{\bar{B}}{\Omega^4}, \quad a - c = 2\frac{\bar{B}_N}{\Omega^4}, \quad b = \frac{\bar{B}_S}{\Omega^4}, \tag{6.16}$$

and the first integral (3.26)$_1$ have been used. Hence, the condition (6.4)$_1$ of negative eigenvalues is satisfied if and only if

$$c_2 < 0, \quad \bar{B} < 0. \tag{6.17}$$

In the case of pulsrodons, these inequalities reduce to a restriction of the space of parameters since $\bar{B}$ is constant. The ratio of the eigenvalues is a measure of the eccentricity of the ellipse and reads

$$\frac{\lambda_+}{\lambda_-} = \frac{\bar{B} + \sqrt{\bar{B}^2 + 4c_2}}{\bar{B} - \sqrt{\bar{B}^2 + 4c_2}}. \tag{6.18}$$

Accordingly, the eccentricity is a function of $\bar{B}$ only and is therefore constant for pulsrodons. It is noted in passing that the relation,

$$|\gamma|^2 = \bar{B} + c_3, \tag{6.19}$$

reveals that the eccentricity of the boundary surface is directly encoded in the modulus of the solution γ of the stationary nonlinear Schrödinger equation (4.18) or, equivalently, the associated Weierstrass $\wp$ function.

The semi-axes of the ellipse are parallel to the eigenvectors

$$\boldsymbol{v}_\pm = \begin{pmatrix} \lambda_\pm - c \\ b \end{pmatrix} = \begin{pmatrix} \dfrac{a - c \pm \sqrt{(a-c)^2 + 4b^2}}{2} \\ b \end{pmatrix} = \frac{1}{\Omega^4} \begin{pmatrix} \bar{B}_N \pm \sqrt{\bar{B}_S^2 + \bar{B}_N^2} \\ \bar{B}_S \end{pmatrix} \tag{6.20}$$

of $\boldsymbol{E}$. Thus, if we apply the orthogonal transformation

$$\bar{\boldsymbol{x}} = (\hat{\boldsymbol{v}}_+ \ \hat{\boldsymbol{v}}_-)\boldsymbol{X}, \tag{6.21}$$

where $\hat{\boldsymbol{v}}_\pm$ are the normalised eigenvectors, then the representation (6.12) of the elliptic cross section becomes

$$\boldsymbol{X}^T \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} \boldsymbol{X} + h_0 = 0. \tag{6.22}$$

Accordingly, the lengths of the semi-axes are given by

$$r_\pm = \sqrt{-\frac{c_1}{\Omega^2 \lambda_\pm}} \tag{6.23}$$

so that Ω is a measure of the size of the pulsrodon since

$$r_\pm \sim \Omega \quad \text{if} \quad \bar{B} = \text{const.} \tag{6.24}$$

In view of conservation of mass, this is in agreement with the behaviour

$$\varrho_{\max} \sim \Omega^{-2} \tag{6.25}$$

of the density on the magnetic axis. It is observed that the rotation about the origin and the pulsation of the pulsrodon occur at the same frequency $f/2\pi$. In Sec. VII it is demonstrated that, remarkably, the lengths of the semi-axes $r_\pm$ obey a particular Hamiltonian two-component generalisation of the Steen-Ermakov-Pinney equation known as Ermakov-Ray-Reid system.[22,23]

Finally, if $\omega_\pm$ denote the angles between the semi-axes and the x axis then

$$\tan \omega_\pm = \frac{\bar{B}_S}{\bar{B}_N \pm \sqrt{\bar{B}_N^2 + \bar{B}_S^2}}. \tag{6.26}$$

Hence, $\omega_\pm$ depends on the ratio $\bar{B}_N/\bar{B}_S$ only and $\dot{\omega}_+ = \dot{\omega}_-$ represents the angular velocity at which the ellipse rotates about its centre.

C. The pulsrodon solution

We conclude by examining the pulsrodon solution as given by (5.5) and (5.7)–(5.13) in more detail. Since

$$\bar{B} = 2\alpha(c_0 - \alpha) < 0, \tag{6.27}$$

it is required that

$$\alpha(c_0 - \alpha) < 0 \tag{6.28}$$

so that $\delta > 0$ by virtue of $(5.11)_2$. Evaluation of the first integral $(3.26)_1$ produces

$$c_2 = \alpha^2[G_0^2 - (c_0 - \alpha)^2] \tag{6.29}$$

so that the requirement $c_2 < 0$ leads to the constraint,

$$(c_0 - \alpha)^2 > G_0^2. \tag{6.30}$$

The condition (5.9)$_2$ is then automatically satisfied and

$$\delta_5 = \sqrt{\delta_4^2 + 4\frac{(c_0-\alpha)^2 + \alpha^2 - G_0^2}{f^2}}, \quad \delta_4 \geq 0. \tag{6.31}$$

The eigenvalues simplify to

$$\lambda_{\pm} = \frac{\alpha(c_0-\alpha) \pm |\alpha G_0|}{\Omega^4} \tag{6.32}$$

and hence the cross section of the boundary S_0 constitutes a circle if $G_0 = 0$.

In order to determine the angular velocity of the elliptic cross section about its centre, we note that

$$\tan\omega_{\pm} = -\frac{\sin\eta}{\cos\eta \pm \operatorname{sgn}(\alpha G_0)} = -\tan\left(\frac{\eta}{2} + \hat{\varepsilon}\right), \tag{6.33}$$

where $\hat{\varepsilon}$ is either 0 or $\pi/2$ and hence

$$\dot{\omega}_{\pm} = -\frac{\dot{\eta}}{2} = -\frac{f}{2} + \frac{\alpha}{\Omega^2}. \tag{6.34}$$

Thus, if $f\alpha < 0$ then the angular velocities $\dot{\omega}_{\pm}$ and $-f$ have the same sign. If $f\alpha > 0$ then, depending on the choice of parameters, the sign of $\dot{\omega}_{\pm}$ may change periodically. However, on average, the two rotational motions undergone by the elliptic boundary of the magneto-gasdynamic flow have the same orientation. Indeed, evaluation of

$$\Delta\omega = \int_{-\pi/|f|}^{\pi/|f|} \dot{\omega}_{\pm}(t)\, dt \tag{6.35}$$

yields

$$\sigma\Delta\omega = -\pi + \frac{2\alpha\pi}{f\sqrt{\delta_5^2 - \delta_4^2}} < 0, \qquad \sigma = \operatorname{sgn}(f) \tag{6.36}$$

since the constraint (6.30) applied to (6.31) delivers

$$\delta_5^2 - \delta_4^2 > 4\frac{\alpha^2}{f^2}. \tag{6.37}$$

Accordingly, we conclude that $-f\Delta\omega > 0$.

Periodic motions are obtained if and only if

$$\frac{\Delta\omega}{\pi} \in \mathbb{Q}, \tag{6.38}$$

that is

$$\frac{\alpha}{\sqrt{(c_0-\alpha)^2 + \alpha^2 - G_0^2}} \in \mathbb{Q}, \tag{6.39}$$

since

$$\sigma\Delta\omega = \pi\left(\frac{\alpha\sigma}{\sqrt{(c_0-\alpha)^2 + \alpha^2 - G_0^2}} - 1\right). \tag{6.40}$$

For instance, if

$$G_0 = \alpha = -c_0 \tag{6.41}$$

then

$$\sigma\Delta\omega = \begin{cases} -\dfrac{3\pi}{2} & \text{for} \quad \alpha f < 0 \\ -\dfrac{\pi}{2} & \text{for} \quad \alpha f > 0 \end{cases} \tag{6.42}$$

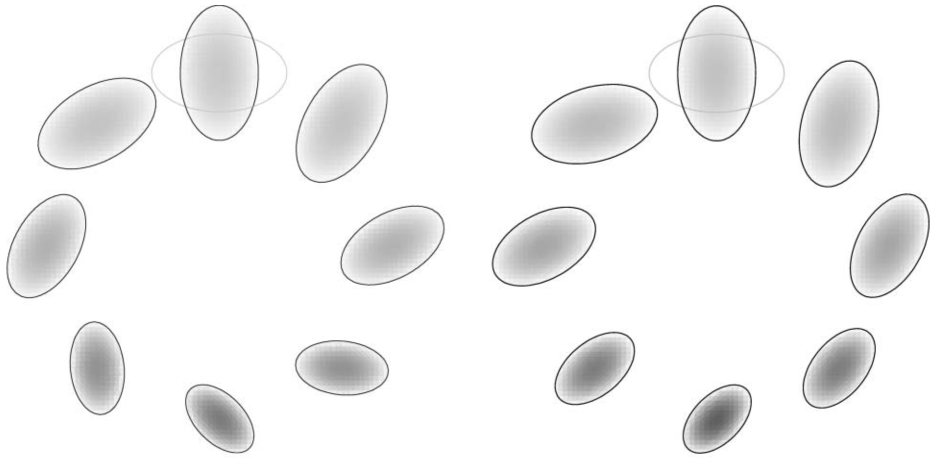

FIG. 1. A density plot of a pulsrodon solution at different times for $\Delta\omega = -3\pi/2$ (a) and $\Delta\omega = -\pi/2$ (b). The black ellipses represent the boundary of the flow (vanishing density). In the interior, darker grey corresponds to higher density ρ. The grey ellipse indicates the position of the pulsrodon after one revolution about the origin.

and, in both cases, the flow returns to its initial state after four revolutions about the origin. This is indicated in Figure 1. A corresponding snapshot of the flow displaying the magnetic field lines on the surfaces of constant density is displayed in Figure 2.

VII. A ERMAKOV-RAY-REID CONNECTION

Here, it is demonstrated that, remarkably, the dynamical system considered in this paper may also be reformulated in terms of a particular Ermakov-Ray-Reid system which turns out to be Hamiltonian, leading to an additional "hidden" first integral. Thus, the Ermakov-Ray-Reid system constitutes a two-component generalisation of the Steen-Ermakov-Pinney equation which adopts the form

$$\begin{aligned} \ddot{\mathcal{A}} + \omega^2(t)\mathcal{A} &= \frac{1}{\mathcal{A}^2\mathcal{B}}F(\mathcal{B}/\mathcal{A}),\\ \ddot{\mathcal{B}} + \omega^2(t)\mathcal{B} &= \frac{1}{\mathcal{B}^2\mathcal{A}}G(\mathcal{A}/\mathcal{B}), \end{aligned} \tag{7.1}$$

where F, G, and ω are given functions of their respective argument. Such systems have their origin in work of Ermakov[6] and were introduced by Ray[22] and Reid and Ray[23] some 100 years later. The main theoretical interest in this system resides in its admittance of a distinctive integral of motion, namely, the Ray-Reid invariant

$$I = \frac{1}{2}(\mathcal{A}\dot{\mathcal{B}} - \mathcal{B}\dot{\mathcal{A}})^2 + H(\mathcal{A}/\mathcal{B}), \tag{7.2}$$

with

$$H(w) = \int F(w^{-1})\,dw^{-1} + \int G(w)\,dw. \tag{7.3}$$

The Ermakov-Ray-Reid system may be solved ("linearised") in various equivalent ways. In the present context, ω is assumed to be constant and one may employ the change of variables

$$\mathcal{A} = \Omega\sqrt{w}, \quad \mathcal{B} = \frac{\Omega}{\sqrt{w}}. \tag{7.4}$$

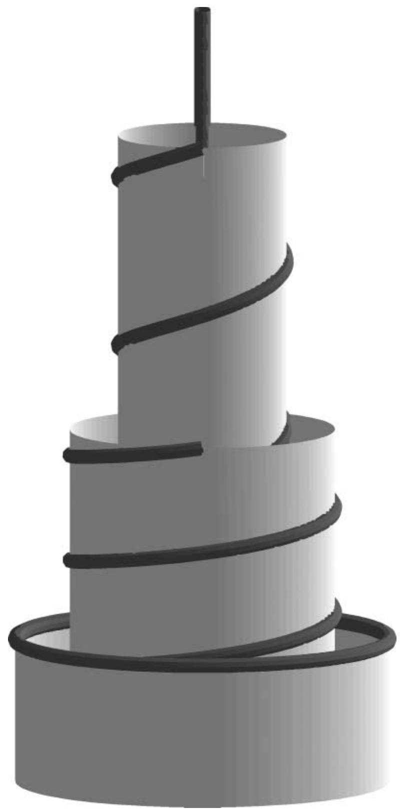

FIG. 2. A typical magnetic field line distribution on the elliptic cylinders of constant density S_ϱ.

The Ray-Reid invariant may then be formulated as

$$\dot{w}^2 = 2\frac{w^2}{\Omega^4}[I - H(w)], \tag{7.5}$$

while the Ermakov-Ray-Reid system reduces to

$$\ddot{\Omega} + \omega^2\Omega = \frac{1}{\Omega^3}K(w) \tag{7.6}$$

with

$$K(w) = \frac{1}{2}[w^{-1}F(w^{-1}) + wG(w) + H(w) - I]. \tag{7.7}$$

Finally, introduction of the new independent variable

$$s = \int \frac{dt}{\Omega^2(t)}, \tag{7.8}$$

leads to the Steen-Ermakov-Pinney equation

$$\Sigma'' + K(w)\Sigma = \frac{\omega^2}{\Sigma^3} \tag{7.9}$$

in which the "frequency" $\sqrt{K(w(s))}$ is determined via a quadrature associated with the first-order differential equation

$$w'^2 = 2w^2[I - H(w)]. \tag{7.10}$$

Hamiltonian Ermakov-Ray-Reid systems are obtained by postulating the existence of a Hamiltonian of the form

$$\mathcal{H}_2 = \frac{1}{2}(\dot{\mathcal{A}}^2 + \dot{\mathcal{B}}^2) + \frac{1}{2}\omega^2(\mathcal{A}^2 + \mathcal{B}^2) + \frac{1}{\mathcal{A}\mathcal{B}} J(\mathcal{A}/\mathcal{B}). \tag{7.11}$$

Comparison of the Hamilton equations

$$\ddot{\mathcal{A}} = -\frac{\partial \mathcal{H}_2}{\partial \mathcal{A}}, \quad \ddot{\mathcal{B}} = -\frac{\partial \mathcal{H}_2}{\partial \mathcal{B}}, \tag{7.12}$$

with the Ermakov-Ray-Reid system (7.1) then reveals that

$$\begin{aligned} F(\mathcal{B}/\mathcal{A}) &= J(\mathcal{A}/\mathcal{B}) - \frac{\mathcal{A}}{\mathcal{B}} J'(\mathcal{A}/\mathcal{B}), \\ G(\mathcal{A}/\mathcal{B}) &= J(\mathcal{A}/\mathcal{B}) + \frac{\mathcal{A}}{\mathcal{B}} J'(\mathcal{A}/\mathcal{B}), \end{aligned} \tag{7.13}$$

so that Hamiltonian Ermakov-Ray-Reid systems adopt the symmetric form

$$\begin{aligned} \ddot{\mathcal{A}} + \omega^2 \mathcal{A} &= \frac{1}{\mathcal{A}^2 \mathcal{B}} \frac{d}{d(\mathcal{B}/\mathcal{A})} \left[\frac{\mathcal{B}}{\mathcal{A}} J(\mathcal{A}/\mathcal{B}) \right], \\ \ddot{\mathcal{B}} + \omega^2 \mathcal{B} &= \frac{1}{\mathcal{B}^2 \mathcal{A}} \frac{d}{d(\mathcal{A}/\mathcal{B})} \left[\frac{\mathcal{A}}{\mathcal{B}} J(\mathcal{A}/\mathcal{B}) \right]. \end{aligned} \tag{7.14}$$

The connection with the dynamical system under consideration is obtained by noting that combination of (4.3) and the first integrals (4.8) leads to

$$\dot{\bar{B}}^2 = \frac{4}{\Omega^4}[(\bar{B}^2 + 4c_2)(\bar{B} + c_3) - (c_0 \bar{B} + c_4)^2] \tag{7.15}$$

and the Steen-Ermakov-Pinney equation (4.4) reads

$$\ddot{\Omega} + \frac{f^2}{4}\Omega = \frac{1}{\Omega^3}(c_0^2 - c_3 - 2\bar{B}). \tag{7.16}$$

Comparison with the pair (7.5), (7.6) reveals that the pair (7.15), (7.16) may be reformulated as a Ermakov-Ray-Reid system with appropriately chosen functions F and G. In fact, since (7.15) encodes the stationary nonlinear Schrödinger equation (4.18), the latter encapsulates the associated Ray-Reid invariant (7.2). Furthermore, the Ermakov-Ray-Reid variables $\mathcal{A}$ and $\mathcal{B}$ turn out to be intimately related to the geometry of the magneto-gasdynamics flow. Thus, as discussed in Sec. VI the lengths of the semi-axes of the elliptic boundary S_0 are given by

$$r_\pm = |\Omega| \sqrt{\frac{c_1}{2c_2}(\bar{B} \mp \sqrt{\bar{B}^2 + 4c_2})}. \tag{7.17}$$

Hence, up to an irrelevant scaling of Ω, the definition

$$\mathcal{A} = r_+, \quad \mathcal{B} = r_- \tag{7.18}$$

may be identified with the change of variables (7.4). On taking into account that

$$\bar{B} = -\sqrt{-c_2}\left(\frac{\mathcal{A}}{\mathcal{B}} + \frac{\mathcal{B}}{\mathcal{A}}\right), \quad \sqrt{\bar{B}^2 + 4c_2} = \sqrt{-c_2}\left(\frac{\mathcal{A}}{\mathcal{B}} - \frac{\mathcal{B}}{\mathcal{A}}\right), \tag{7.19}$$

one may then verify that the associated Ermakov-Ray-Reid system may be brought into the Hamiltonian form (7.14) with

$$J(w) = \frac{2c_1^2}{\sqrt{-c_2}} + \frac{(c_4 + 2c_0\sqrt{-c_2})^2 c_1^2}{4c_2^2} \frac{w}{(w+1)^2} + \frac{(c_4 - 2c_0\sqrt{-c_2})^2 c_1^2}{4c_2^2} \frac{w}{(w-1)^2}. \tag{7.20}$$

Accordingly, the dynamical system considered in this paper admits the Hamiltonian (7.11) as an additional first integral which does not appear to be readily available in its original formulation. In conclusion, it is noted that J may be obtained from the Ray-Reid-type relation

$$(\mathcal{A}\dot{\mathcal{B}} - \mathcal{B}\dot{\mathcal{A}})^2 = \frac{c_1^2}{c_2^2}(c_4^2 - 4c_2c_3) - 2\left(\frac{\mathcal{A}}{\mathcal{B}} + \frac{\mathcal{B}}{\mathcal{A}}\right) J(\mathcal{A}/\mathcal{B}). \tag{7.21}$$

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