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Kerr-like neutrino-gravitational solutions

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Two distinct stationary axisymmetric Kerr-like neutrino-gravitational space–times are presented. They reduce to the Kerr solution when the neutrino field vanishes. In addition, the first solution possesses physical line singularities along the axis of symmetry; whereas the second solution is asymptotically flat and its locally bounded Riemann tensor is discontinuous at the equatorial plane. Both metrics are of Petrov type II and their neutrino fields belong to type $[2N\text{-}2S]_{(2-1)}$ in the Plebański classification.

1. INTRODUCTION

Recently a lot of interest has been generated in connection with investigations into the possibilities of gravitational effects on the structure of elementary particles by solving the Einstein–Dirac equations.[1–3] Neutrino-gravitation interactions are governed by the zero rest-mass Einstein–Dirac equations[4] which can be expressed in the two-component spinor formalism as follows:

$$\begin{aligned} R_{\mu\nu} = & -i[\sigma_{\mu AB'}(\Phi^A\Phi^{B'}{}_{;\nu} - \Phi^A{}_{;\nu}\Phi^{B'}) \\ & + \sigma_{\nu AB'}(\Phi^A\Phi^{B'}{}_{;\mu} - \Phi^A{}_{;\mu}\Phi^{B'})], \\ & \sigma^{\mu}{}_{AB'}\Phi^A{}_{;\mu} = 0. \end{aligned} \tag{1}$$

Here $\{\sigma_{\mu AB'}\}, \mu = 1,...,4$, are the generalized Pauli spin matrices and Φ^A, $A = 1,2$, are components of a one-spinor which describes the neutrino field. Exact solutions of equations (1) are given by a number of authors (see Kuchowicz's review[5] and references cited therein). These known solutions imply that spherically symmetric space–times are not compatible with the presence of neutrino-gravitational fields which possess a shearfree geodetic null congruence.[6–8] Furthermore, Madore[9] proved that there exists no neutrino-gravitational field which is static and axisymmetric. In a recent paper, Herrera and Jiménez[2] using asymptotic series expansions showed that the assumptions of axisymmetry and asymptotic flatness would ultimately lead to physically singular solutions of the Einstein–Dirac neutrino field equations. One should note that all the results mentioned above on neutrino-gravitational space–times rest implicitly on the neutrino field being locally smooth[10] in an appropriate coordinate neighborhood where the corresponding metric is defined.

In this paper we present two distinct twisting neutrino-gravitational solutions both of which are reducible to the Kerr metric when the corresponding neutrino field vanishes. In terms of the Kerr coordinates (u,r,θ,ϕ), the first metric (G 1) is stationary axisymmetric, locally smooth, and possesses physical singularities along the axis of symmetry. These physical singularities are induced by the presence of the neutrino field. Thus the G 1 metric does not contradict the existing known theorems on neutrino-gravitational solutions. In contrast with G 1, the second metric (G 2) represents a stationary, axisymmetric asymptotically flat neutrino-gravitational field and it is of class C^{2-}.[11,12] In this case the Riemann tensor remains locally bounded but it is discontinuous at the equatorial plane. This is due directly to the existence of a continuous locally bounded current 4-vector[13] of the neutrino field which has a discontinuous θ-coordinate derivative. The apparent paradox between the G 2 solution and known results can be explained by the order of differentiability of the metric.

2. REDUCED NEUTRINO-GRAVITATIONAL FIELD EQUATIONS

The solutions in here are obtained by assuming that (i) the principal null congruence of the Weyl tensor coincides with the principal null congruence of the neutrino field[13] (we denote such a principal null vector by $\mathbf{L}$); (ii) $\mathbf{L}$ is geodesic and shearfree with nonvanishing twist; (iii) the neutrino field Φ^A is time independent; and (iv) these solutions reduce to the Kerr solution when the neutrino field Φ^A vanishes. Assumptions (i) and (ii) imply that the Weyl tensor is algebraically special. In addition, if a space–time admits a Killing vector of the type $\mathbf{K} = \partial_{u'}$, then, using a theorem by Kerr and Debney,[14] there exists local coordinates $(u', r, \zeta, \bar{\zeta})$ such that the metric is given by

$$ds^2 = 2\Theta^{(1}\Theta^{2)} - 2\Theta^{(3}\Theta^{4)}. \tag{2}$$

Here the basis 1-forms are

$$\begin{aligned} \Theta^1 &= dr + 2\,\mathrm{Im}(\rho^0{}_{\zeta}d\zeta) - U\Theta^2, \\ \Theta^2 &= du' + bd\zeta + \bar{b}d\bar{\zeta}, \\ \Theta^3 &= -e^{p}(r - i\rho^0)d\bar{\zeta}, \quad \Theta^4 = \bar{\Theta}^3, \end{aligned} \tag{3}$$

where

$$U = R^{(2)} + \frac{mr + \rho^0(M - \Phi^0\bar{\Phi}^0)}{r^2 + \rho^{0^2}}. \tag{4}$$

The tetrad variables ρ^0, p, m, and M are real-valued functions of ζ and $\bar{\zeta}$. The complex function $b(\zeta,\bar{\zeta})$ is defined by

$$b_{\bar{\zeta}} - \bar{b}_{\zeta} = -2ie^{2p}\rho^0. \tag{5}$$

$R^{(2)}$ is the Gaussian curvature of the 2-surface with metric $dl^2 = e^{2p}\,d\zeta\,d\bar{\zeta}$, i.e.,

$$R^{(2)} = e^{-2p}p_{\zeta\bar{\zeta}}. \tag{6}$$

[a)]Part of the research was carried out in the Department of Mathematics, Monash University, Clayton, Victoria 3168, Australia.

TABLE I. Functions Φ^0, $\Phi^0\overline{\Phi^0}$, N_1, N_2 and N_3 in $(u', r, \zeta, \bar\zeta)$ coordinates.

	$G\,1$ Solution	$G\,2$ solution [a]
Φ^0	$\eta e^{in}[\zeta^{-1}(1+\frac14\zeta\bar\zeta)]^{1/2}$	$\eta e^{in}[\frac12(1+\frac14\zeta\bar\zeta)]^{1/2}, \quad 0\leqslant\lvert\zeta\rvert\leqslant 2$
		$\eta e^{in}\zeta^{-1}[2(1+\frac14\zeta\bar\zeta)]^{1/2}, \quad 2<\lvert\zeta\rvert<\infty$
$\Phi^0\overline{\Phi^0}$ [b]	$\dfrac{\eta^2(1+\frac14\zeta\bar\zeta)}{(\zeta\bar\zeta)^{1/2}}$	$2^{-1}\eta^2(1+\frac14\zeta\bar\zeta), \quad 0\leqslant\lvert\zeta\rvert\leqslant 2$
		$2^{-1}\eta^2\left(1+4\dfrac{1}{\zeta\bar\zeta}\right), \quad 2<\lvert\zeta\rvert<\infty$
N_1	$\dfrac{2\eta^2(\zeta\bar\zeta)^{1/2}}{(1+\frac14\zeta\bar\zeta)}$	$\eta^2(1+\frac14\zeta\bar\zeta)^{-1}\left\{\zeta\bar\zeta-2(1+\frac14\zeta\bar\zeta)\ln(1+\frac14\zeta\bar\zeta)+(1-\frac14\zeta\bar\zeta)\left[\mathrm{diln}(\frac14\zeta\bar\zeta)+\dfrac{\pi^2}{12}\right]\right\}, \quad 0\leqslant\lvert\zeta\rvert\leqslant 2$
		$\eta^2(1+\frac14\zeta\bar\zeta)^{-1}\left\{4-2(1+\frac14\zeta\bar\zeta)\ln\left(1+4\dfrac{1}{\zeta\bar\zeta}\right)-(1-\frac14\zeta\bar\zeta)\left[\mathrm{diln}\left(4\dfrac{1}{\zeta\bar\zeta}\right)+\dfrac{\pi^2}{12}\right]\right\}, \quad 2<\lvert\zeta\rvert<\infty$
N_2	$\dfrac{\eta^2(\bar\zeta)^{1/2}(1-\frac14\zeta\bar\zeta)}{(\zeta)^{1/2}(1+\frac14\zeta\zeta)^2}$	$\eta^2\bar\zeta[2(1+\frac14\zeta\bar\zeta)^2]^{-1}\left\{(1-\frac14\zeta\bar\zeta)-\dfrac{2}{\zeta\bar\zeta}\left(1-\dfrac{\zeta^2\bar\zeta^2}{16}\right)\ln(1+\frac14\zeta\bar\zeta)-\left[\mathrm{diln}(\frac14\zeta\bar\zeta)+\dfrac{\pi^2}{12}\right]\right\}, \quad 0\leqslant\lvert\zeta\rvert\leqslant 2$
		$\eta^2\bar\zeta[2(1+\frac14\zeta\bar\zeta)^2]^{-1}\left\{-\left(1-\dfrac{4}{\zeta\bar\zeta}\right)-\dfrac{2}{\zeta\bar\zeta}\left(1-\dfrac{\zeta^2\bar\zeta^2}{16}\right)\ln\left(1+4\dfrac{1}{\zeta\bar\zeta}\right)+\left[\mathrm{diln}\left(4\dfrac{1}{\zeta\bar\zeta}\right)+\dfrac{\pi^4}{12}\right]\right\}, \quad 2<\lvert\zeta\rvert<\infty$
N_3	$-2i\eta^2\zeta^{-1}\left[\tan^{-1}\dfrac{(\zeta\bar\zeta)^{1/2}}{2}\right.$	$i\eta^2\bar\zeta[2(1+\frac14\zeta\bar\zeta)^2]^{-1}\left\{(1+\frac14\zeta\bar\zeta)-\dfrac{2(1+\frac14\zeta\bar\zeta)^2}{\zeta\bar\zeta}\left[\dfrac{2\ln(1+\frac14\zeta\bar\zeta)}{(1+\frac14\zeta\bar\zeta)}-\ln 2\right]-\left[\mathrm{diln}(\frac14\zeta\bar\zeta)+\dfrac{\pi^2}{12}\right]\right\}, \quad 0\leqslant\lvert\zeta\rvert\leqslant 2$
	$\left.-\dfrac{(\zeta\bar\zeta)^{1/2}(1-\frac14\zeta\bar\zeta)}{2(1+\frac14\zeta\bar\zeta)^2}\right]$	$i\eta^2\bar\zeta[2(1+\frac14\zeta\bar\zeta)^2]^{-1}\left\{-\left(1-4\dfrac{1}{\zeta\bar\zeta}\right)+\dfrac{2(1+\frac14\zeta\bar\zeta)^2}{\zeta\bar\zeta}\left[\dfrac{2\ln[1+(4/\zeta\bar\zeta)]}{[1+(4/\zeta\bar\zeta)]}-\ln 2\right]+\left[\mathrm{diln}\left(4\dfrac{1}{\zeta\bar\zeta}\right)+\dfrac{\pi^2}{12}\right]\right\}, \quad 2<\lvert\zeta\rvert<\infty$

[a] The function diln(z) in the $G\,2$ solution is defined by $\mathrm{diln}(z)=-\int_0^z[\ln(1+t)/t]\,dt=\Sigma_{n=1}^{\infty}[(-1)^n/n^2]z^n$, $|z|\leqslant 1$. It is known as the Dilogarithm function (see Ref. 19).

[b] The neutrino flux $j^\mu=\rho\bar\rho\Phi^0\overline{\Phi}{}^0L^\mu$.

TABLE II. Functions Φ^0, $\Phi^0\overline{\Phi^0}$, N'_1, N'_2 and N'_3 in the Kerr coordinates.

	$G\,1$ Solution	$G\,2$ Solution [a]
Φ^0	$\dfrac{\eta e^{\frac12 i(\phi-2n)}}{(\sin\theta)^{1/2}}$	$\dfrac{\eta e^{in}}{(1+\cos\theta)^{1/2}}, \quad 0\leqslant\theta\leqslant\dfrac{\pi}{2}$
		$\dfrac{\eta e^{-i(\phi-n)}}{(1-\cos\theta)^{1/2}}, \quad \dfrac{\pi}{2}<\theta\leqslant\pi$
$\Phi^0\overline{\Phi^0}$	$\dfrac{\eta^2}{\sin\theta}$	$\dfrac{\eta^2}{1+\lvert\cos\theta\rvert}$
N'_1	$2\eta^2\sin\theta$	$2\eta^2\left\{(1-\lvert\cos\theta\rvert)+[\ln(1+\lvert\cos\theta\rvert)-\ln 2]+\dfrac{\lvert\cos\theta\rvert}{2}\left[\mathrm{diln}\left(\dfrac{1-\lvert\cos\theta\rvert}{1+\lvert\cos\theta\rvert}\right)+\dfrac{\pi^2}{12}\right]\right\}$
N'_2	$2\eta^2\sin\theta\cos\theta$	$2\eta^2\cos\theta\left\{(1-\lvert\cos\theta\rvert)+[\ln(1+\lvert\cos\theta\rvert)-\ln 2]-\dfrac{\sin^2\theta}{2\lvert\cos\theta\rvert}\left[\mathrm{diln}\left(\dfrac{1-\lvert\cos\theta\rvert}{1+\lvert\cos\rvert}\right)+\dfrac{\pi^2}{12}\right]\right\}$
N'_3	$2\eta^2(\sin\theta\cos\theta-\theta)$	$2\eta^2\cos\theta\left\{(1-\lvert\cos\theta\rvert)+\dfrac{(1+\lvert\cos\theta\rvert)}{\lvert\cos\theta\rvert}\left[\ln(1+\lvert\cos\theta\rvert)-\dfrac{(\ln 2)\lvert\cos\theta\rvert}{1+\lvert\cos\theta\rvert}\right]-\dfrac{\sin^2\theta}{2\lvert\cos\theta\rvert}\left[\mathrm{diln}\left(\dfrac{1-\lvert\cos\theta\rvert}{1+\lvert\cos\theta\rvert}\right)+\dfrac{\pi^2}{12}\right]\right\}$

[a] The function diln(z) is the Dilogarithm function defined in footnote a in Table I (see Ref. 19).

The complex function $\Phi^0(\zeta,\bar{\zeta})$ gives the neutrino field via the equations

$$\Phi = -\Phi^0\rho, \quad \Phi^A = \Phi\xi^A, \quad L^\mu = \sigma^\mu{}_{AB'}\xi^A\xi^{B'}. \tag{7}$$

Here (ξ^A,η^A) denote the basis of two-component spinors.[15] The reduced neutrino-gravitational field equations corresponding to assumption (iii) and Eqs. (2) and (3) are

$$(m+iM)_{\bar{\zeta}} = ie^{p/2}\Phi^0(e^{-p/2}\bar{\Phi}^0)_{\bar{\zeta}}, \tag{8}$$

$$(e^{-2p}p_{\zeta\bar{\zeta}})_{\zeta\bar{\zeta}} = 0, \tag{9}$$

$$e^{-2p}\rho^0{}_{\zeta\bar{\zeta}} - 2R^{(2)}\rho^0 = M, \tag{10}$$

$$(e^{p/2}\Phi^0)_{\bar{\zeta}} = 0. \tag{11}$$

These equations can also be derived from the paper by Trim and Wainwright[6,16] by making appropriate assumptions. General solutions of the reduced field equations (8)–(11) are not available. However, assumption (iv) implies that the 2-surfaces $dl^2 = 2e^{2p}\,d\zeta d\bar{\zeta}$ has constant negative Gaussian curvature. In general one can then choose $R^{(2)} = -\frac{1}{2}$.

3. KERR-LIKE NEUTRINO-GRAVITATIONAL SOLUTIONS

Integration of the above field equations with $R^{(2)} = -\frac{1}{2}$ leads to Kerr-NUT[17,18] type neutrino-gravitational space–times characterized by four essential parameters: m^0, M^0, a and η. Here m^0 corresponds to mass, M^0 is the NUT parameter, a is the Kerr parameter while η determines the neutrino field up to a constant phase change $\Phi^A \to e^{in}\Phi^A$, n = constant. Both the $G\,1$ and $G\,2$ metrics are given via Eqs. (2) and (3), where the unknown functions m, M, p, ρ^0, $\rho^0{}_\zeta$, b, and U are written as follows:

$$\begin{aligned}
&m = m^0, \quad M = M^0 + \Phi^0\bar{\Phi}^0, \quad e^{-2p} = 2(1+\tfrac{1}{4}\zeta\bar{\zeta})^2,\\
&\rho^0 = -a\left(\frac{1-\frac{1}{4}\zeta\bar{\zeta}}{1+\frac{1}{4}\zeta\bar{\zeta}}\right) + M^0 + N_1,\\
&\rho^0{}_\zeta = \frac{a\bar{\zeta}}{2(1+\frac{1}{4}\zeta\bar{\zeta})^2} + N_2,\\
&b = \frac{ia\bar{\zeta}}{2(1+\frac{1}{4}\zeta\bar{\zeta})^2} + \frac{aiM^0}{\zeta(1+\frac{1}{4}\zeta\bar{\zeta})} + N_3,\\
&U = -\frac{1}{2} + \frac{m^0 r + M^0\rho^0}{r^2+\rho^{0^2}}.
\end{aligned} \tag{12}$$

The real-valued function N_1 and the complex functions Φ^0, N_2, N_3 are directly attributable to the presence of neutrino fields. Together with $\Phi^0\bar{\Phi}^0$, these functions are listed for the $G\,1$ and $G\,2$ metrics in Table I.

Introducing angular coordinates θ, ϕ by $\zeta = 2e^{i\phi}\times\tan(\theta/2)$ and a new u coordinate by $u = u' - 2M^0\phi$, one transforms $(u',r,\zeta,\bar{\zeta})$ to the Kerr coordinates (u,r,θ,ϕ). The line element defined by Eqs. (2) and (3) then assumes the form

$$\begin{aligned}
ds^2 = {}&(r^2+\rho^{0^2})^{-1}\{(\Delta - a^2\sin^2\theta)du^2 + 2[aR\sin^2\theta - H\Delta\\
&+ N_2'(r^2+\rho^{0^2})]dud\phi - (R^2\sin^2\theta + 2HN_2'\\
&- H^2\Delta)d\phi^2\} + 2dudr - 2Hdrd\phi - (r^2+\rho^{0^2})d\theta^2,
\end{aligned} \tag{13}$$

with

$$\begin{aligned}
&\rho^0 = -a\cos\theta + M^0 + N_1',\\
&H = a\sin^2\theta + 2M^0\cos\theta + N_3',\\
&\Delta = r^2 - 2m^0 r + a^2 - M^{0^2} + N_1'(N_1' - 2a\cos\theta),\\
&R = r^2 + a^2 + M^{0^2} + N_1'(N_1' - 2a\cos\theta + 2M^0) + aN_3'.
\end{aligned} \tag{14}$$

In Table II, the functions Φ^0, $\Phi^0\bar{\Phi}^0$, N_1', N_2', and N_3' for the $G\,1$ and $G\,2$ solutions are listed. Note that N_1', N_2' and N_3' vanish when the neutrino field is set equal to zero. The resulting vacuum Kerr solution is in the form given by Kinnersley.[18]

4. CONCLUDING REMARKS

From Table II it is obvious that both $G\,1$ and $G\,2$ admit a pair of commuting Killing vectors, $\mathbf{K}_1 = \partial_u$ and $\mathbf{K}_2 = \partial_\phi$. Using the Newman–Penrose formalism[15] one can show that these solutions possess the properties mentioned in the opening paragraphs (see Appendix). Their Weyl tensors are of Petrov type II and their respective neutrino fields belong to type $[2N\text{-}2S]_{(2\text{-}1)}$ in the Plebański classification.[20,21] These solutions are stationary axisymmetric.

From Eqs. (A3), the asymptotic behavior[22] of the nonzero tetrad components of the Ricci tensor is

$$\begin{aligned}
&\Phi_{11} = \Phi^0_{11}r^{-4},\\
&\Phi_{21} = \Phi^0_{21}r^{-3} + O(r^{-4}) = \bar{\Phi}_{12},\\
&\Phi_{22} = -e^{-2p}[(e^p\Phi^0_{12})_\zeta + (e^p\Phi^0_{21})_{\bar{\zeta}}]r^{-3} + O(r^{-4}),
\end{aligned}$$

where Φ^0_{11} and Φ^0_{21} are given by Eqs. (A2). Consequently, at large r

$$\Phi_{11}\Phi_{22} - \Phi_{12}\Phi_{21} < 0. \tag{15}$$

For a neutrino field subject to assumptions (i) and (ii), the energy momentum tensor assumes the form

$$\begin{aligned}
T_{\mu\nu} = {}&2\Phi_{22}L_\mu L_\nu + 2\Phi_{11}[4L_{(\mu}n_{\nu)} - g_{\mu\nu}]\\
&+ 4\Phi_{21}L_{(\mu}m_{\nu)} + 4\Phi_{12}L_{(\mu}\bar{m}_{\nu)}.
\end{aligned}$$

Here the vector fields L^μ, n^μ, m^μ and $\bar{m}^\mu$ are pseudo-orthonormal vectors dual to the basis 1-form (3). Using a theorem by Wainwright,[23] Eq. (15) implies that both $G\,1$ and $G\,2$ violate the energy conditions E_1, E_2 and E_3 of Wainwright.[5,23] Moreover, one can show that the condition

$$Q_\mu(u)Q^\mu(u) \geqslant 0 \tag{16}$$

may not be valid with reference to these solutions, where

$$Q_\mu(u) = T_{\mu\nu}u^\nu$$

is the energy flow vector of the neutrino field with respect to an observer moving along a future-pointing unit velocity u^μ.[24]

The author would like to thank the referee's suggestions and for pointing out the work of Wainwright[24] on the nonexistence of physical solutions to the Einstein–Weyl equations in static space–times.

APPENDIX

In the Newman–Penrose formalism,[15] the nonvanishing spin coefficients with respect to the basis 1-form (3) and $R^{(2)}$ = constant are

$$\rho = -(r + i\rho^0)^{-1}, \quad \beta = \tfrac{1}{2} e^{-P} p_{\xi}\bar{\rho},$$
$$\bar{\alpha} + \beta = 0, \quad \gamma = -\tfrac{1}{2}\Psi^0_2\rho^2,$$
$$\mu = -R^{(2}\bar{\rho} - \tfrac{1}{2}\Psi^0_2(\rho^2 + \rho\bar{\rho}) + \Phi^0_{11}\rho^2\bar{\rho},$$
$$\nu = -\tfrac{1}{2}e^{-P}(\Psi^0_2)_{\xi}\rho^2 - ie^{-P}\rho^0{}_{\xi}\Psi^0_2\rho^3 - \Phi^0_{21}\rho\bar{\rho} + e^{-P}(\Phi^0_{11})_{\xi}\rho^2\bar{\rho} + 2ie^{-P}\rho^0{}_{\xi}\Phi^0_{11}\rho^3\bar{\rho}. \tag{A1}$$

Here

$$\Psi^0_2 = -(m + iM), \quad \Phi^0_{11} = \rho^0\Phi^0\bar{\Phi}^0,$$
$$\Phi^0_{21} = \frac{i}{2}e^{-P}(\Phi^0\bar{\Phi}^0)_{\zeta}. \tag{A2}$$

The nonvanishing tetrad components of the Riemann tensor are

$$\Psi_2 = -\Psi^0_2\rho^3 + 2\Phi^0_{11}\rho^3\bar{\rho},$$
$$\Psi_3 = -e^{-P}(\Psi^0_2)_{\xi}\rho^3 - 3ie^{-P}\rho^0{}_{\zeta}\Psi^0_2\rho^4 - \Phi^0_{21}\rho^2\bar{\rho} + 2e^{-P}(\Phi^0_{11})_{\xi}\rho^3\bar{\rho} + 6ie^{-P}\rho^0{}_{\zeta}\Phi^0_{11}\rho^4\bar{\rho},$$
$$\Psi_4 = -\tfrac{1}{2}[e^{-2P}(\Psi^0_2)_{\zeta}]_{\xi}\rho^3 - i[\Psi^0_2(e^{-2P}\rho^0{}_{\zeta})_{\zeta} + 3e^{-2P}\rho^0{}_{\zeta}(\Psi^0_2)_{\zeta}]\rho^4 + 6e^{-2P}(\rho^0{}_{\zeta})^2\Psi^0_2\rho^5 - (e^{-P}\Phi^0_{21})_{\zeta}\rho^2\bar{\rho} + [(e^{-2P}(\Phi^0_{11})_{\zeta})_{\zeta} + 2ie^{-P}\rho^0{}_{\zeta}\Phi^0_{21}]\rho^3\bar{\rho} + 2i[\Phi^0_{11}(e^{-2P}\rho^0{}_{\zeta})_{\zeta} + 3e^{-2P}\rho^0{}_{\zeta}(\Phi^0_{11})_{\zeta}]\rho^4\bar{\rho} - 12e^{-2P}(\rho^0{}_{\zeta})^2\Phi^0_{11}\rho^5\bar{\rho},$$
$$\Phi_{11} = \Phi^0_{11}\rho^2\bar{\rho}^2,$$
$$\Phi_{21} = -\Phi^0_{21}\rho\bar{\rho}^2 + e^{-P}(\Phi^0_{11})_{\zeta}\rho^2\bar{\rho}^2 + 2ie^{-P}\rho^0{}_{\zeta}\Phi^0_{11}\rho^3\bar{\rho}^2,$$
$$\Phi_{22} = [e^{-2P}(e^{P}\Phi^0_{12})_{\zeta}\rho^2\bar{\rho} - 2ie^{-P}\rho^0{}_{\zeta}\Phi^0_{12}\rho^3\bar{\rho} + \tfrac{1}{2}e^{-2P}(\Phi^0_{11})_{\zeta\bar{\zeta}}\rho^2\bar{\rho}^2 + 2ie^{-2P}\rho^0{}_{\zeta}(\Phi^0_{11})_{\bar{\zeta}}\rho^3\bar{\rho}^2 + 2e^{-2P}\rho^0{}_{\zeta}\rho^0{}_{\bar{\zeta}}\Phi^0_{11}\rho^3\bar{\rho}^3] + \text{complex conjugate}. \tag{A3}$$

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