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Dynamical analysis of quantum linear systems driven by multi-channel multi-photon states ⋆

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Abstract

In this paper, we investigate the dynamics of quantum linear systems where the input signals are multi-channel multi-photon states, namely states determined by a definite number of photons superposed in multiple input channels. In contrast to most existing studies on separable input states in the literature, we allow the existence of quantum correlation (for example quantum entanglement) in these multi-channel multi-photon input states. Due to the prevalence of quantum correlations in the quantum regime, the results presented in this paper are very general. Moreover, the multi-channel multi-photon states studied here are reasonably mathematically tractable. Three types of multi-photon states are considered: 1) m photons superposed among m channels, 2) N photons superposed among m channels where $N \geq m$, and 3) N photons superposed among m channels where N is an arbitrary positive integer. Formulae for intensities and states of output fields are derived. Examples are used to demonstrate the effectiveness of the results.

Key words: quantum linear systems, multi-photon states, intensity, tensor computation.

1 Introduction

Dynamical response analysis is an essential ingredient of control engineering, and is also the basis of controller design. For example, impulse response, step response, and frequency response are standard materials in modern control textbooks, see, e.g., [21], [1]. Fluctuation analysis of a dynamical system driven by white noise underlies the celebrated Kalman filter and linear quadratic Gaussian (LQG) control. Likewise, in the quantum regime, the response of quantum linear systems to quantum Gaussian white noise is the basis of quantum filtering and measurement-based feedback control, see, e.g., [5], [37], [9], [45], [16] and references therein.

In addition to quantum Gaussian noise commonly dealt with in quantum optical laboratories, in recent years, highly nonclassical signals such as single-photon states, multi-photon states, and Schrödinger's cat states have been attracting growing interest due to their promising applications in quantum information technology. Roughly speaking, an ℓ-photon state of a light beam means that the light field contains *exactly* ℓ photons. In this paper, we are concerned with continuous-mode ℓ-photon states, that is, these photons are specified by their frequency profiles centered at the carrier frequency of the light field. Continuous-mode single- and multi-photon states have found important applications in quantum computing, quantum communication, and quantum metrology, see, e.g., [12], [23], [29], [24], [15], [3], [4], [44], [39], and [28].

In the quantum control community, the response of quantum systems to single- and multi-photon states has been studied in the past few years. The phenomenon of cross phase shift on a coherent signal induced by a single photon pulse was investigated in [25]. Gough *et al.* derived quantum filters for Markov quantum systems driven by single-photon states or Schrödinger's cat states [15]. The theory in [15] was applied to the study of phase modulation in [8]. Quantum master equations for an arbitrary quantum system driven by multi-photon states were derived in [3]. Quantum filters for multi-photon states were derived in [35], for both homodyne detection and photodetection measurements. Numerical simulations carried out in [35] for a two-level system driven by a 2-photon state revealed interesting and com-

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plicated nonlinear behavior in the photon-atom interaction. When a two-level atom, initialized in the ground state, is driven by a single photon, the exact form of the output field state was derived in [30]. More discussions can be found in, e.g., [10], [28] and references therein.

In [44], an analytic expression of the output field state of a quantum linear system driven by a single-photon state was derived. The research initialized in [44] was continued and extended in [40], where multi-photon states were considered. Unfortunately, the multi-photon states studied in [40] are either with very limited quantum correlation or mathematically formidable. More specifically, the multi-photon states defined in [40, Eqs. (23) and (41)] are separable states, i.e., there exists no entanglement among the channels. A class of photon-Gaussian states was defined in [40, Eq. (34)]. On the one hand, this class of states appears mathematically complicated. On the other hand, because each pulse shape is indexed by three parameters only, the feature of the multi-channel entanglement is unclear. A class of multi-channel multi-photon states was defined in [40, Eq. (43)], which, due to the presence of an m-fold product, looks rather complicated mathematically.

The purpose of this paper is to provide a direct study of the dynamical response of quantum linear systems to initially entangled multi-channel multi-photon states. Unlike those separable states studied in [44] and [40], the multi-channel multi-photon states proposed in this paper are able to capture the entanglement among channels. Examples presented in this paper demonstrate that these types of multi-channel multi-photon states can be easily processed by quantum linear systems. Furthermore, the proposed multi-channel multi-photon states are very general as they contain many types of multi-channel multi-photon states as special cases, see, e.g., [23, Chapter 6], [33], [7]. Finally, these states are mathematically more tractable than those in [40, Eqs. (34) and (43)]. Therefore, the study carried out in this paper is more relevant to quantum linear feedback networks and control.

Three types of multi-channel multi-photon states are studied in this paper. Case 1): m photons are superposed among m channels. Specifically, the m-channel m-photon states are defined in Subsection 3.1. When the underlying quantum linear system is passive, an analytic expression of the output intensity is presented in Subsection 3.2, see Theorem 1. Moreover, the steady-state output field state is investigated in Subsections 3.3 and 3.4, see Theorems 2 and 3. When the underlying quantum linear system is non-passive, the steady-state output field state is no longer an m-channel m-photon state, an explicit form of the output field state is given in Subsection 3.5, see Theorem 4. Case 2): N photons are superposed among m channels where $N \geq m$. For this case, we assume the underlying quantum linear system is passive. The analytic expressions of the output field state are derived, see Theorems 5 and 6. Case 3): N photons are superposed among m channels, where N is an arbitrary positive integer. Specifically, a class of m-channel N-photon states are first presented in Subsection 5.1, then in Subsection 5.2, the steady-state output field state of a quantum linear passive system driven by an m-channel N-photon input state is derived, see Theorem 7.

Notation. The imaginary unit $\sqrt{-1}$ is denoted by i. Given a column vector of complex numbers or operators $x = [x_1 \ \cdots \ x_k]^T$, define a column vector $x^\# \triangleq [x_1^* \ \cdots \ x_k^*]^T$, where the superscript "$*$" stands for complex conjugation of a complex number or Hilbert space adjoint of an operator. Define a row vector $x^\dagger \triangleq (x^\#)^T = [x_1^* \ \cdots \ x_k^*]$. Define a doubled-up column vector $\breve{x} \triangleq [x^T \ x^\dagger]^T$. Let I_k be an identity matrix and 0_k a zero square matrix, both of dimension k. Denote $J_k = \mathrm{diag}(I_k, -I_k)$. Given a matrix $X \in \mathbb{C}^{2j\times 2k}$, define $X^\flat \triangleq J_k X^\dagger J_j$. Given a matrix A, let A^{jk} denote the entry on the jth row and kth column. Let m be the number of input channels. Let n be the number of degrees of freedom of a given quantum linear system, namely the number of quantum harmonic oscillators. The ket $|\phi\rangle$ denotes the initial state of the system of interest, and $|0\rangle$ stands for the vacuum state of free fields. The convolution of two functions f and g is denoted as $f \circledast g$. Given two matrices $U, V \in \mathbb{C}^{r\times k}$, define a doubled-up matrix $\Delta(U,V) \triangleq [U \ V; V^\# \ U^\#]$. Given two operators $\mathcal{A}$ and $\mathcal{B}$, their commutator is defined to be $[\mathcal{A},\mathcal{B}] \triangleq \mathcal{AB} - \mathcal{BA}$. The Kronecker delta function is denoted by δ_{jk}, whereas the Dirac delta function is denoted by $\delta(t)$. The m-fold integral $\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} dt_1\cdots dt_m$ is sometimes denoted by $\int d\overrightarrow{t}$. Given a function $f(t)$ in the time domain, define its *two-sided* Laplace transform [34, Eq. (13)] to be $F[s] \equiv \mathscr{L}_b\{f(t)\}(s) \triangleq \int_{-\infty}^{\infty} e^{-st} f(t)dt$. The m dimensional Fourier transform of an m-variable function $f(t_1,\ldots,t_m)$ is, [6],

$$f(\mathrm{i}\omega_1,\ldots,\mathrm{i}\omega_m) \triangleq \frac{1}{(2\pi)^{m/2}} \int_{-\infty}^{\infty}\cdots \int_{-\infty}^{\infty} dt_1\cdots dt_m e^{-\mathrm{i}(\omega_1 t_1+\cdots+\omega_m t_m)} f(t_1,\ldots,t_m). \tag{1}$$

We set $\hbar = 1$ throughout this paper.

2 Preliminaries

In this section, quantum linear systems are briefly introduced; more discussions can be found in, e.g., [11], [36], [13], [37], [18], [31], [42], [43], [38], and [26]. Some tensors and their associated operations are also discussed.

2.1 Quantum linear systems

A quantum linear system G is shown schematically in Fig. 1. In this model, the quantum linear system G con-

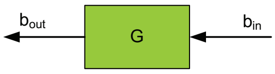

Fig. 1. Quantum linear system G with input $b_{\rm in}$ and output $b_{\rm out}$.

sists of a collection of n (interacting) quantum harmonic oscillators represented by $a = [a_1 \ \cdots \ a_n]^T$. Here, a_j ($j = 1, \ldots, n$), defined on a Hilbert space $\mathfrak{H}$, is the annihilation operator of the jth quantum harmonic oscillator. The adjoint operator of a_j, denoted by a_j^*, is called a creation operator. These operators satisfy the following canonical commutation relations: $[a_j, a_k^*] = \delta_{jk}$, and $[a_j, a_k] = [a_j^*, a_k^*] = 0$, ($j, k = 1, \ldots, n$). The input fields are represented by a vector of annihilation operators $b_{\rm in}(t) = [b_{{\rm in},1}(t) \ \cdots \ b_{{\rm in},m}(t)]^T$; the entry $b_{{\rm in},j}(t)$ ($j = 1, \ldots, m$), defined on a Fock space $\mathfrak{F}$, is the annihilation operator for the jth input channel. The adjoint operator of $b_{{\rm in},j}(t)$, denoted by $b_{{\rm in},j}^*(t)$, is also called a creation operator. However, unlike a_j and a_k^*, the annihilation and creation operators for the input fields satisfy the following singular commutation relations, [11], [14, Eq. (20)],

$$
\begin{aligned}
&[b_{{\rm in},j}(t), b_{{\rm in},k}^*(r)] = \delta_{jk}\delta(t-r), \\
&[b_{{\rm in},j}(t), b_{{\rm in},k}(r)] = [b_{{\rm in},j}^*(t), b_{{\rm in},k}^*(r)] = 0,
\end{aligned} \tag{2}
$$

for $j, k = 1, \ldots, m$ and $\forall t, r \in \mathbb{R}$. Notice the presence of the Dirac delta function $\delta(t-r)$ in Eq. (2). Mathematically, it is often more convenient to work with integrated annihilation and creation operators, which are defined respectively to be $B_{\rm in}(t) \triangleq \int_{t_0}^t b_{\rm in}(\tau)d\tau$ and $B_{\rm in}^{\#}(t) \triangleq \int_{t_0}^t b_{\rm in}^{\#}(\tau)d\tau$, where the lower limit t_0 of the integrals is the initial time, namely the time when the system and the fields start to interact. The input gauge process (also called *number* process) is defined by the following m-by-m matrix of operators, [11, Chapter 11], [14, Section III.A], [44, Eq. (11)],

$$
\Lambda_{\rm in}(t) \triangleq \int_{t_0}^t b_{\rm in}^{\#}(\tau) b_{\rm in}^T(\tau)d\tau. \tag{3}
$$

In this paper, we deal with *canonical* quantum input fields, that is, the only non-zero Itô products for the input fields are: $dB_{{\rm in},j}(t)dB_{{\rm in},k}^*(t) = \delta_{jk}dt$, $d\Lambda_{\rm in}^{jk}(t)dB_{{\rm in},l}^*(t) = \delta_{kl}dB_{{\rm in},j}^*(t)$, $dB_{{\rm in},j}(t)d\Lambda_{\rm in}^{kl}(t) = \delta_{jk}dB_{{\rm in},l}(t)$, and $d\Lambda_{\rm in}^{jk}(t)d\Lambda_{\rm in}^{lr}(t) = \delta_{kl}d\Lambda_{\rm in}^{jr}(t)$, for $j, k, l, r = 1, \ldots, m$, $\forall t \in \mathbb{R}$, where $\Lambda_{\rm in}^{jk}(t)$ is the entry of the matrix $\Lambda_{\rm in}(t)$ on the jth row and kth column, as introduced in the *Notation* part, see e.g., [11, Chapter 11], [13] , [14], [44, Eq. (12)].

The dynamics of the open quantum linear system G can be described conveniently in the (S_-, L, H) formalism [13], [43]. Here, S_- is a constant unitary matrix of dimension m, which can be used to model static devices such as phase shifters and beamsplitters. The operator L describes how the system is coupled to the fields, and is of the form $L = C_-a + C_+a^{\#}$ with $C_-, C_+ \in \mathbb{C}^{m\times n}$. For example, when a single-mode (namely, $n = 1$) optical cavity is driven by a light field, L can be of the form $L = \sqrt{\kappa}a$, where a is the annihilation operator of the quantum harmonic operator for the cavity (also called the cavity mode) and $\kappa > 0$ is the coupling strength between the cavity and the field. The operator H stands for the initial system Hamiltonian, which can be written as $H = \frac{1}{2}\breve{a}^\dagger \Delta(\Omega_-, \Omega_+)\breve{a}$ with constant matrices $\Omega_-, \Omega_+ \in \mathbb{C}^{n\times n}$ satisfying $\Omega_- = \Omega_-^\dagger$ and $\Omega_+ = \Omega_+^T$. For example, for the optical cavity above mentioned, $H = \frac{1}{2}[a^* \ \ a]\left[\begin{smallmatrix}\omega_d & 0\\ 0 & \omega_d\end{smallmatrix}\right][a \ \ a^*]^T = \omega_d a^* a + \frac{1}{2}\omega_d$, where $\omega_d \in \mathbb{R}$ is the detuning frequency between the cavity mode and the center frequency of the input light field. (The term $\frac{1}{2}\omega_d$ introduces a global phase shift and leads to no consequence.) With these parameters, in Itô form, Schrödinger's equation for the temporal evolution of the open quantum linear system in Fig. 1 is, [17], [13, Eq. (30)], [14, Eq. (22)], [44, Eq. (13)],

$$
\begin{aligned}
dU(t,t_0) = \Big\{ & {\rm Tr}[(S_- - I_m)d\Lambda_{\rm in}(t)^T] + dB_{\rm in}^\dagger(t)L \\
& - L^\dagger S_- dB_{\rm in}(t) - (\frac{1}{2}L^\dagger L + {\rm i}H)dt \Big\} U(t,t_0)
\end{aligned} \tag{4}
$$

for $t \geq t_0$ with $U(t,t_0) = I$ (identity operator) for all $t \leq t_0$.

In the Heisenberg picture, system operators evolve according to $\breve{a}(t) = U(t,t_0)^*\breve{a}(t_0)U(t,t_0)$ (component-wise for the components of $\breve{a}(t_0)$). Moreover, the output field $b_{\rm out}(t)$ carries away information of the system after interaction, and is defined by

$$
\breve{b}_{\rm out}(t) \triangleq U(t,t_0)^* \breve{b}_{\rm in}(t) U(t,t_0) \tag{5}
$$

(component-wise for the components of $\breve{b}_{\rm in}(t)$). Consequently, by Eq. (4) and quantum Itô calculus [17], Heisenberg's equation of motion for the system in Fig. 1 is, [14, Eq. (26)], [44, Eqs. (14)-(15)],

$$
\begin{aligned}
\dot{\breve{a}}(t) &= \mathbf{A}\breve{a}(t) + \mathbf{B}\breve{b}_{\rm in}(t), \\
\breve{b}_{\rm out}(t) &= \mathbf{C}\breve{a}(t) + \mathbf{S}\breve{b}_{\rm in}(t), \quad \breve{a}(t_0) = \breve{a},
\end{aligned} \tag{6}
$$

in which the constant system matrices are

$$
\begin{aligned}
&\mathbf{S} = \Delta(S_-, 0), \ \mathbf{C} = \Delta(C_-, C_+), \ \mathbf{B} = -\mathbf{C}^\flat \mathbf{S}, \\
&\mathbf{A} = -\frac{1}{2}\mathbf{C}^\flat \mathbf{C} - {\rm i}J_n \Delta(\Omega_-, \Omega_+)
\end{aligned} \tag{7}
$$

with the matrix $J_n = {\rm diag}(I_n, -I_n)$ introduced in the *Notation* part. The gauge process $\Lambda_{\rm out}(t)$ of the output fields,

$$\Lambda_{\rm out}(t) \triangleq \int_{t_0}^{t} b_{\rm out}^{\#}(\tau)b_{\rm out}^{T}(\tau)d\tau = U(t,t_0)^*\Lambda_{\rm in}(t)U(t,t_0), \tag{8}$$

satisfies the following quantum stochastic differential equation (QSDE), [13], [43, Eq. (16)],

$$\begin{aligned} d\Lambda_{\rm out}(t) = & S_-^{\#}d\Lambda_{\rm in}(t)S_-^T + S_-^{\#}dB_{\rm in}^{\#}(t)L^T(t) \\ & + L^{\#}(t)dB_{\rm in}^T(t)S_-^T + L^{\#}(t)L^T(t)dt. \end{aligned} \tag{9}$$

In quantum optics, the diagonal elements of $\Lambda_{\rm out}(t)$ are operators for the total number of photons in each of the m output channels, counted from time t_0 to t. The intensity of the output field, namely the rate of change of the number process $\Lambda_{\rm out}(t)$, is given by, [44, Eq. (45)],

$$\bar{n}_{\rm out}(t) \triangleq \langle \phi\Psi_{\rm in}| b_{\rm out}^{\#}(t)b_{\rm out}^T(t)|\phi\Psi_{\rm in}\rangle. \tag{10}$$

In Eq. (10), $|\phi\rangle$ is the initial system state and $|\Psi_{\rm in}\rangle$ is the initial input field state. Therefore, the ket vector $|\phi\rangle\otimes|\Psi_{\rm in}\rangle \equiv |\phi\Psi_{\rm in}\rangle$ is the initial joint system-field state. The bra vector $\langle\phi\Psi_{\rm in}|$ is the Hilbert space conjugate of the ket vector $|\phi\Psi_{\rm in}\rangle$. In this paper, $|\phi\rangle$ is always assumed to be the vacuum state, while the specific form of $|\Psi_{\rm in}\rangle$ will be given in due course.

The quantum linear system G is said to be *asymptotically stable* if the matrix $\mathbf{A}$ in Eq. (7) is Hurwitz stable, [42, Sec. III-A]. In analogy to classical (namely non-quantum) control theory, the impulse response function of the system G is, [44, Eq. (18)],

$$g_G(t) \triangleq \begin{cases} \delta(t)\mathbf{S} - \mathbf{C}e^{\mathbf{A}t}\mathbf{C}^\flat\mathbf{S}, & t \geq 0, \\ 0, & t<0, \end{cases}$$

which enjoys the following doubled-up form

$$g_G(t) = \Delta(g_{G^-}(t), g_{G^+}(t)), \tag{11}$$

with matrix functions

$$\begin{aligned} g_{G^-}(t) &\triangleq \begin{cases} \delta(t)S_- - [\,C_-\ \ C_+\,]e^{\mathbf{A}t}\begin{bmatrix} C_-^\dagger \\ -C_+^\dagger \end{bmatrix} S_-, & t\geq 0, \\ 0, & t<0, \end{cases} \\ g_{G^+}(t) &\triangleq \begin{cases} -[\,C_-\ \ C_+\,]e^{\mathbf{A}t}\begin{bmatrix} -C_+^T \\ C_-^T \end{bmatrix} S_-^{\#}, & t\geq 0, \\ 0, & t<0. \end{cases} \end{aligned} \tag{12}$$

Next, we express the output field in terms of the impulse function $g_G(t)$. In fact, solving Eq. (6) we have

$$\breve{b}_{\rm out}(t) = \mathbf{C}e^{\mathbf{A}(t-t_0)}\breve{a} + \int_{t_0}^{t} g_G(t-r)\breve{b}_{\rm in}(r)dr. \tag{13}$$

Furthermore, if the system is asymptotically stable, then in the limit $t_0\to-\infty$, Eq. (13) reduces to

$$\breve{b}_{\rm out}(t) = \int_{-\infty}^{t} g_G(t-r)\breve{b}_{\rm in}(r)dr = g_G \circledast \breve{b}_{\rm in}(t). \tag{14}$$

Remark 1 *If the interaction starts in the remote past, namely $t_0\to-\infty$, and if the system is asymptotically stable, Eq. (14) indicates that the initial system information has no influence on the output field. This is also true in classical control theory, see, e.g., [21].*

Define a matrix function

$$g_{G^{-1}}(t) \triangleq \Delta(g_{G^-}(-t)^\dagger, -g_{G^+}(-t)^T). \tag{15}$$

It can be verified that the following convolution relations

$$g_G \circledast g_{G^{-1}} \circledast f(t) = g_{G^{-1}} \circledast g_G \circledast f(t) = f(t) \tag{16}$$

hold for any function $f(t)$ of suitable dimension provided that the involved integrals converge. Thus, $g_{G^{-1}}(t)$ is the inverse function of the impulse response function $g_G(t)$. According to Eqs. (14) and (16), in the limit $t_0\to-\infty$ we have

$$\breve{b}_{\rm in}(t) = g_{G^{-1}} \circledast \breve{b}_{\rm out}(t). \tag{17}$$

A class of *passive* quantum linear systems is obtained when $C_+=0$ and $\Omega_+=0$ in Eq. (7). For this type of systems, it is sufficient to work in the annihilation-operator representation. To be specific, it suffices to study

$$\begin{aligned} \dot{a}(t) &= Aa(t) + Bb_{\rm in}(t), \\ b_{\rm out}(t) &= Ca(t) + S_-b_{\rm in}(t), \quad a(t_0)=a, \end{aligned} \tag{18}$$

where

$$A = -\mathrm{i}\Omega_- - \frac{1}{2}C_-^\dagger C_-, \quad B = -C_-^\dagger S_-, \quad C = C_-.$$

In this case, Eq. (12) reduces to

$$g_{G^-}(t) = \begin{cases} \delta(t)S_- - C_-e^{At}C_-^\dagger S_-, & t\geq 0, \\ 0, & t<0, \end{cases} \quad g_{G^+}(t) = 0. \tag{19}$$

Accordingly, Eqs. (11) and (15) reduce to

$$g_G(t) = \Delta\left(g_{G^-}(t),0\right), g_{G^{-1}}(t) \triangleq \Delta\left(g_{G^-}(-t)^\dagger,0\right), \tag{20}$$

respectively.

It is well-known that in linear classical control theory, if an asymptotically stable finite-dimensional linear time-invariant (FDLTI) system is driven by Gaussian white noise, then the steady-state output is again a Gaussian stationary process, see, e.g., [21, Section 11, Chapter 1], and [1]. The following result is the quantum counterpart.

Lemma 1 *[44, Theorem 2] Let the asymptotically stable quantum linear system G be initialized in the vacuum state $|\phi\rangle$ and let the input field $|\Psi_{\rm in}\rangle$ be in the vacuum state $|0\rangle$. Then the steady-state output field state is a zero-mean Gaussian state, whose power spectral density matrix $R_{\rm out}[\mathrm{i}\omega]$ is given by*

$$R_{\rm out}[\mathrm{i}\omega] = G[\mathrm{i}\omega]\begin{bmatrix} I_m & 0 \\ 0 & 0_m \end{bmatrix} G[\mathrm{i}\omega]^\dagger, \tag{21}$$

where $G[\mathrm{i}\omega] = \int_{-\infty}^{\infty} e^{-\mathrm{i}\omega t} g_G(t)dt$ is the two-sided Laplace transform of $g_G(t)$ with $s = \mathrm{i}\omega$, as introduced in the Notation part. In particular, if the system is passive, then the output is in a vacuum state with power spectral density matrix

$$R_{\rm out}[\mathrm{i}\omega] = \begin{bmatrix} I_m & 0 \\ 0 & 0_m \end{bmatrix}.$$

2.2 Tensors

Tensors and their associated operations are essential mathematical machinery for the research carried out in this paper [32], [19], [40]. In this subsection, we discuss several tensors.

Given an m-variable function $\psi(t_1,\ldots,t_m)$ and an m-dimensional column vector $x(t) = [x_1(t) \ \cdots \ x_m(t)]^T$, denote

$$\psi \circ^m x \equiv \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} dt_1\cdots dt_m\, \psi(t_1,\ldots,t_m)\prod_{j=1}^m x_j(t_j). \tag{22}$$

Given an m-way m-dimensional tensor function $\varphi = (\varphi_{j_1\ldots j_m}(t_1,\ldots,t_m)), (j_1,\ldots,j_m = 1,\ldots,m)$, and an m-dimensional column vector $x(t) = [x_1(t) \ \cdots \ x_m(t)]^T$, denote

$$\begin{aligned}\varphi \circledcirc^m x \equiv & \sum_{j_1=1}^m\cdots\sum_{j_m=1}^m \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} dt_1\cdots dt_m \\ & \varphi_{j_1\ldots j_m}(t_1,\ldots,t_m)x_{j_1}(t_1)\cdots x_{j_m}(t_m).\end{aligned} \tag{23}$$

We may update an m-variable function $\psi(t_1,\ldots,t_m)$ to an m-way m-dimensional tensor function $\psi^\uparrow = (\psi^\uparrow_{j_1\ldots j_m}(t_1,\ldots,t_m))$ with entries

$$\begin{aligned}&\psi^\uparrow_{j_1\ldots j_m}(t_1,\ldots,t_m) \\ \triangleq &\begin{cases}\psi(t_1,\ldots,t_m), & \text{if } j_1=1, j_2=2,\ldots,j_m=m,\\ 0, & \text{otherwise.}\end{cases}\end{aligned} \tag{24}$$

Then Eq. (22) can be re-written as Eq. (23), specifically,

$$\psi \circ^m x = \psi^\uparrow \circledcirc^m x. \tag{25}$$

Let $\psi(t_1^1,\ldots,t_{k_1}^1,\ldots,t_1^m,\ldots,t_{k_m}^m)$ be an N-variable function, where the positive integers $k_1,\ldots,k_m$ satisfy $\sum_{i=1}^m k_i = N$. Let $x(t) = [x_1(t) \ \cdots \ x_m(t)]^T$ be an m-dimensional column vector. Denote

$$\begin{aligned}\psi \ \cdot^N_{k_1\cdots k_m} \ x \equiv & \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} dt_1^1\cdots dt_{k_1}^1\cdots \\ & dt_1^m\cdots dt_{k_m}^m\, \psi(t_1^1,\ldots,t_{k_1}^1,\ldots,t_1^m,\ldots,t_{k_m}^m)x_1(t_1^1)\cdots \\ & x_1(t_{k_1}^1)\cdots x_m(t_1^m)\cdots x_m(t_{k_m}^m).\end{aligned} \tag{26}$$

Given an N-way m-dimensional tensor function $\varphi = \left(\varphi_{j_1^1\ldots j_{k_1}^1\ldots j_1^m\ldots j_{k_m}^m}(t_1^1,\ldots,t_{k_1}^1,\ldots,t_1^m,\ldots,t_{k_m}^m)\right)$ and an m-dimensional vector $x(t) = [x_1(t) \ \cdots \ x_m(t)]^T$, denote

$$\begin{aligned}&\varphi \odot^N_{k_1\cdots k_m} x \\ \equiv & \sum_{j_1^1=1}^m\cdots\sum_{j_{k_1}^1=1}^m\cdots\sum_{j_1^m=1}^m\cdots\sum_{j_{k_m}^m=1}^m\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} dt_1^1\cdots dt_{k_1}^1\cdots \\ & dt_1^m\cdots dt_{k_m}^m\ \varphi_{j_1^1\ldots j_{k_1}^1\ldots j_1^m\ldots j_{k_m}^m}(t_1^1,\ldots,t_{k_1}^1,\ldots,t_1^m,\ldots,t_{k_m}^m) \\ & x_{j_1^1}(t_1^1)\cdots x_{j_{k_1}^1}(t_{k_1}^1)\cdots x_{j_1^m}(t_1^m)\cdots x_{j_{k_m}^m}(t_{k_m}^m).\end{aligned} \tag{27}$$

Update the N-variable function $\psi(t_1^1,\ldots,t_{k_1}^1,\ldots,t_1^m,\ldots,t_{k_m}^m)$ in Eq. (26) to an N-way m-dimensional tensor function $\psi^\uparrow(t_1^1,\ldots,t_{k_1}^1,\ldots,t_1^m,\ldots,t_{k_m}^m)$, whose none-zero elements are defined as

$$\begin{aligned}&\psi^\uparrow_{j_1^1\ldots j_{k_1}^1\ldots j_1^m\ldots j_{k_m}^m}(t_1^1,\ldots,t_{k_1}^1,\ldots,t_1^m,\ldots,t_{k_m}^m) \\ \triangleq\ & \psi(t_1^1,\ldots,t_{k_1}^1,\ldots,t_1^m,\ldots,t_{k_m}^m)\end{aligned} \tag{28}$$

when $j_1^1 = 1,\cdots,j_{k_1}^1 = k_1,\cdots,j_1^m = \sum_{j=1}^{m-1}k_j + 1,\cdots,j_{k_m}^m = N$. Then

$$\psi \ \cdot^N_{k_1\cdots k_m} \ x = \psi^\uparrow \odot^N_{k_1\cdots k_m} x. \tag{29}$$

In the above, we have defined several operations between tensors and vectors. In the following, we look at operations between tensors and matrices.

Given an $m\times m$ matrix function $\mathcal{A}(t)$ and an m-way m-dimensional tensor function $\varphi = (\varphi_{j_1\ldots j_m}(t_1,\ldots,t_m))$, $(j_1,\ldots,j_m = 1,\ldots,m)$, define another m-way m-dimensional tensor function $\tilde{\varphi} = (\tilde{\varphi}_{i_1\ldots i_m}(r_1,\ldots,r_m))$ in such a way that, for all $i_1,\ldots,i_m = 1,\ldots,m$,

$$\begin{aligned}\tilde{\varphi}_{i_1\ldots i_m}(r_1,\ldots,r_m) = & \sum_{j_1,\ldots,j_m=1}^m\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} dt_1\cdots dt_m \\ & \mathcal{A}^{i_1j_1}(r_1-t_1)\cdots\mathcal{A}^{i_mj_m}(r_m-t_m)\varphi_{j_1\ldots j_m}(t_1,\ldots,t_m).\end{aligned} \tag{30}$$

Eq. (30) may be re-written in a more compact form

$$\tilde{\varphi} = \varphi \circledast_t^m \mathcal{A}, \tag{31}$$

where the subscript "t" indicates the time domain, while the superscript "m" implies the m-fold convolution. Applying the m-dimensional Fourier transform (1) to Eq. (30), we get

$$\begin{aligned}&\tilde{\varphi}_{i_1 \dots i_m}(\mathrm{i}\omega_1, \dots, \mathrm{i}\omega_m) \\ &= \sum_{j_1,\dots,j_m=1}^{m} \mathcal{A}^{i_1 j_1}[\mathrm{i}\omega_1] \cdots \mathcal{A}^{i_m j_m}[\mathrm{i}\omega_m] \varphi_{j_1 \dots j_m}(\mathrm{i}\omega_1, \dots, \mathrm{i}\omega_m),\end{aligned} \tag{32}$$

where

$$\mathcal{A}^{i_k j_k}[\mathrm{i}\omega_k] = \int_{-\infty}^{\infty} e^{-\mathrm{i}\omega_k t} \mathcal{A}^{i_k j_k}(t) dt$$

is the two-sided Laplace transform of the matrix function $\mathcal{A}^{i_k j_k}(t)$. In analogy to Eq. (31), we may also write Eq. (32) in the following compact form $\tilde{\varphi} = \varphi \circledast_\omega^m \mathcal{A}$, where the subscript "$\omega$" indicates the frequency domain.

Given an N-way m-dimensional tensor function $\varphi = \left(\varphi_{j_1^1 \dots j_{k_1}^1 \dots j_1^m \dots j_{k_m}^m}(t_1^1, \dots, t_{k_1}^1, \dots, t_1^m, \dots, t_{k_m}^m)\right)$, and an $m \times m$ matrix function $\mathcal{A}(t)$, define a new N-way m-dimensional tensor function $\tilde{\varphi}$ by

$$\begin{aligned}&\tilde{\varphi}_{l_1^1 \dots l_{k_1}^1 \dots l_1^m \dots l_{k_m}^m}(r_1^1, \dots, r_{k_1}^1, \dots, r_1^m, \dots, r_{k_m}^m) \\ &\triangleq \sum_{i_1^1=1}^{m} \cdots \sum_{i_{k_1}^1=1}^{m} \cdots \sum_{i_1^m=1}^{m} \cdots \sum_{i_{k_m}^m=1}^{m} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dt_1^1 \cdots dt_{k_1}^1 \\ &\quad \cdots dt_1^m \cdots dt_{k_m}^m \ \mathcal{A}^{l_1^1 i_1^1}(r_1^1 - t_1^1) \cdots \mathcal{A}^{l_{k_1}^1 i_{k_1}^1}(r_{k_1}^1 - t_{k_1}^1) \cdots \\ &\quad \mathcal{A}^{l_1^m i_1^m}(r_1^m - t_1^m) \cdots \mathcal{A}^{l_{k_m}^m i_{k_m}^m}(r_{k_m}^m - t_{k_m}^m) \\ &\qquad \varphi_{i_1^1 \dots i_{k_1}^1 \dots i_1^m \dots i_{k_m}^m}(t_1^1, \dots, t_{k_1}^1, \dots, t_1^m, \dots, t_{k_m}^m),\end{aligned}$$

which may be re-written in a more compact form

$$\tilde{\varphi} = \varphi \circledast_{t, k_1 \cdots k_m}^N \mathcal{A}. \tag{33}$$

Given an m-way m-dimensional tensor function $\varphi = \varphi_{j_1 \dots j_m}(\mathrm{i}\omega_1, \dots, \mathrm{i}\omega_m)$ in the frequency domain, denote

$$\|\varphi(\mathrm{i}\omega_1, \dots, \mathrm{i}\omega_m)\| \equiv \sqrt{\sum_{j_1,\dots,j_m=1}^{m} |\varphi_{j_1 \dots j_m}(\mathrm{i}\omega_1, \dots, \mathrm{i}\omega_m)|^2}$$

for all $\omega_1, \dots, \omega_m \in \mathbb{R}$. We end this subsection by citing the following result.

Lemma 2 *[32, Theorem 3.3] Let two tensors φ and $\tilde{\varphi}$ be related by Eq. (32), or equivalently Eq. (30). If $\mathcal{A}[\mathrm{i}\omega]$ is unitary for all $\omega \in \mathbb{R}$, then*

$$\|\tilde{\varphi}(\mathrm{i}\omega_1, \dots, \mathrm{i}\omega_m)\| = \|\varphi(\mathrm{i}\omega_1, \dots, \mathrm{i}\omega_m)\|, \forall \omega_1, \dots, \omega_m \in \mathbb{R}.$$

3 m photons superposed among m input channels

In this section, we investigate how a quantum linear system responds to a class of m-photon input states. We first define m-photon input states in Subsection 3.1, then derive the output intensity in Subsection 3.2, after that, we present an analytic form of the output field state when the underlying quantum linear system is passive in Subsections 3.3 and 3.4, finally we turn to the non-passive case in Subsection 3.5.

3.1 m-photon input states

In this subsection, we introduce a class of m-photon input states. For ease of presentation, we start with the *single-channel single-photon* state case. In this case, $m = 1$. A single-channel single-photon input state can be defined by

$$|\Psi_{\mathrm{in}}\rangle \triangleq \int_{-\infty}^{\infty} dt \ \psi_{\mathrm{in}}(t) b_{\mathrm{in}}^*(t) |0\rangle .$$

Here, the function ψ_{in} is square integrable, more specifically, $\psi_{\mathrm{in}} \in L_2(\mathbb{R}, \mathbb{C})$. The Euclidean norm of ψ_{in}, $\|\psi_{\mathrm{in}}\| \triangleq \sqrt{\int_{-\infty}^{\infty} |\psi_{\mathrm{in}}(t)|^2 dt}$, is equal to 1. Consequently, the inner product $\langle \Psi_{\mathrm{in}} | \Psi_{\mathrm{in}} \rangle = 1$. That is, $|\Psi_{\mathrm{in}}\rangle$ is a normalized state. Moreover, it can be easily shown that

$$\lim_{t_0 \to -\infty, t \to \infty} \langle \Psi_{\mathrm{in}} | \Lambda_{\mathrm{in}}(t) | \Psi_{\mathrm{in}} \rangle = 1, \tag{34}$$

where $\Lambda_{\mathrm{in}}(t)$ is the input gauge process defined in Eq. (3). Eq. (34) indicates that there is one photon in the field. On the other hand, it can be readily verified that

$$\langle \Psi_{\mathrm{in}} | b_{\mathrm{in}}(t) | \Psi_{\mathrm{in}} \rangle = \langle \Psi_{\mathrm{in}} | b_{\mathrm{in}}^*(t) | \Psi_{\mathrm{in}} \rangle = 0, \ \forall t \in \mathbb{R}. \tag{35}$$

That is, the average field amplitude is zero. Finally, it is worth noting that $|\Psi_{\mathrm{in}}\rangle$ is not a single-photon *coherent* state which can be defined to be

$$\begin{aligned}&|\alpha_{\psi_{\mathrm{in}}}\rangle \\ &\triangleq \exp\left(\int_{-\infty}^{\infty} dt \alpha \psi_{\mathrm{in}}(t) b_{\mathrm{in}}^*(t) - \int_{-\infty}^{\infty} dt (\alpha \psi_{\mathrm{in}}(t))^* b_{\mathrm{in}}(t)\right) |0\rangle,\end{aligned}$$

where $\alpha = e^{i\theta}$ is a complex number. In fact, for the single-photon coherent state $|\alpha_{\psi_{\mathrm{in}}}\rangle$, Eq. (34) still holds, but Eq. (35) does not.

Next, we look at *single-channel two-photon* states, which can be defined as

$$|\Psi_{\mathrm{in}}\rangle \triangleq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt_1 dt_2 \ \psi_{\mathrm{in}}(t_1, t_2) b_{\mathrm{in}}^*(t_1) b_{\mathrm{in}}^*(t_2) |0\rangle. \tag{36}$$

Swapping t_1 and t_2 in Eq. (36) yields

$$|\Psi_{\rm in}\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dt_1 dt_2\ \psi_{\rm in}(t_2,t_1) b_{\rm in}^*(t_1) b_{\rm in}^*(t_2)|0\rangle. \tag{37}$$

Comparing Eqs. (36) and (37) we see that $\psi_{\rm in}(t_1,t_2) = \psi_{\rm in}(t_2,t_1)$. Moreover, it can be verified that

$$\lim_{t_0\to-\infty, t\to\infty} \langle \Psi_{\rm in}|\Lambda_{\rm in}(t)|\Psi_{\rm in}\rangle = 2,$$

i.e., there are two photons in the field.

Next, let us look at *two-channel two-photon* states, which can be defined to be

$$|\Psi_{\rm in}\rangle \triangleq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dt_1 dt_2\, \psi_{\rm in}(t_1,t_2) b_{{\rm in},1}^*(t_1) b_{{\rm in},2}^*(t_2)|0_1 0_2\rangle. \tag{38}$$

Again, the function $\psi_{\rm in}(t_1,t_2)$ is required to normalize the state. This is guaranteed by

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dt_1 dt_2\ |\psi_{\rm in}(t_1,t_2)|^2 = 1. \tag{39}$$

(Notice that in this case, the condition $\psi_{\rm in}(t_1,t_2) = \psi_{\rm in}(t_2,t_1)$ is not necessary.) It can be easily shown that

$$\lim_{t_0\to-\infty, t\to\infty} \langle \Psi_{\rm in}|\Lambda_{\rm in}(t)|\Psi_{\rm in}\rangle = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{40}$$

Eq. (40) implies that each channel contains one photon. However, these two photons can form an entangled state. Moreover, if we use the single-photon state $\int_{-\infty}^{\infty} \gamma(\tau) b_{{\rm in},2}^*(\tau) d\tau\ |0_2\rangle$ to measure the second channel, the resulting state for the first channel is given by

$$\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} \gamma^*(\tau)\psi_{\rm in}(t,\tau) d\tau\right] b_{{\rm in},1}^*(t) dt\ |0_1\rangle.$$

In general, Eq. (38) defines a state for which the two photons are entangled. However, for the special case that $\psi_{\rm in}(t_1,t_2) = \xi_1(t_1)\xi_2(t_2)$, we end up with a product state

$$\begin{aligned} &|\Psi_{\rm in}\rangle \\ &= \int_{-\infty}^{\infty} \xi_1(t_1) b_{{\rm in},1}^*(t_1) dt_1 |0_1\rangle \otimes \int_{-\infty}^{\infty} \xi_2(t_2) b_{{\rm in},2}^*(t_2) dt_2 |0_2\rangle. \end{aligned} \tag{41}$$

For the state defined in Eq. (41), there exists no entanglement between these two photons.

We are ready to introduce a class of m-channel m-photon input states. Such states can be of the form

$$\begin{aligned} |\Psi_{\rm in}\rangle \triangleq \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} & dt_1 \cdots dt_m\ \psi_{\rm in}(t_1,\dots,t_m) \\ & b_{{\rm in},1}^*(t_1)\cdots b_{{\rm in},m}^*(t_m)|0_1\rangle \otimes \cdots \otimes |0_m\rangle. \end{aligned} \tag{42}$$

For convenience, in the sequel we use the shorthand notation $|0^{\otimes m}\rangle$ for the tensor product of the vacuum input fields $|0_1\rangle \otimes \cdots \otimes |0_m\rangle$. In the notation introduced in Subsection 2.2, Eq. (42) may be re-written as

$$|\Psi_{\rm in}\rangle = \psi_{\rm in} \circ^m b_{\rm in}^{\#}\ |0^{\otimes m}\rangle.$$

For the m-channel m-photon state $|\Psi_{\rm in}\rangle$, it is clear that

$$\langle \Psi_{\rm in}|\breve{b}_{\rm in}(t)|\Psi_{\rm in}\rangle = 0,\ \forall t\in\mathbb{R}. \tag{43}$$

That is, the average field amplitude of the input light field is 0. Next, we look at two-time correlations $\langle \Psi_{\rm in}|\breve{b}_{\rm in}(t)\breve{b}_{\rm in}^\dagger(r)|\Psi_{\rm in}\rangle$ with $t,r\in\mathbb{R}$. For each $k=1,\dots,m$, introduce the notation

$$\zeta_k(\tau,r) \equiv \psi_{\rm in}(\tau_1,\dots,\tau_{k-1},r,\tau_{k+1},\dots,\tau_m). \tag{44}$$

Specifically,

$$\begin{aligned} \zeta_1(\tau,r) &= \psi_{\rm in}(r,\tau_2,\dots,\tau_m), \\ \zeta_2(\tau,r) &= \psi_{\rm in}(\tau_1,r,\tau_3,\dots,\tau_m), \\ &\vdots \\ \zeta_m(\tau,r) &= \psi_{\rm in}(\tau_1,\dots,\tau_{m-1},r). \end{aligned}$$

Also, define a diagonal matrix function

$$\begin{aligned} &\Lambda(t,r) \\ &\triangleq {\rm diag}\Big(\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} d\tau_2\cdots d\tau_m\ \zeta_1(\tau,t)^*\zeta_1(\tau,r), \\ &\quad \dots, \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} d\tau_1\cdots d\tau_{m-1}\ \zeta_m(\tau,t)^*\zeta_m(\tau,r)\Big) \end{aligned} \tag{45}$$

for all $t,r\in\mathbb{R}$. Clearly, $\Lambda(t,r)^\dagger = \Lambda(r,t)$, and $\int_{-\infty}^{\infty} dt\ \Lambda(t,t) = I_m$. Furthermore, it can be shown that the two-time correlation $\langle \Psi_{\rm in}|\breve{b}_{\rm in}(t)\breve{b}_{\rm in}^\dagger(r)|\Psi_{\rm in}\rangle$ has the form

$$\begin{aligned} &\left\langle \Psi_{\rm in}|\breve{b}_{\rm in}(t)\breve{b}_{\rm in}^\dagger(r)|\Psi_{\rm in}\right\rangle \\ &= \delta(t-r)\begin{bmatrix} I_m & 0 \\ 0 & 0_m \end{bmatrix} + \begin{bmatrix} \Lambda(r,t) & 0 \\ 0 & \Lambda(t,r) \end{bmatrix}. \end{aligned} \tag{46}$$

Remark 2 *If all the input fields are in the vacuum state, i.e., $|\Psi_{\rm in}\rangle = |0^{\otimes m}\rangle$, it is well-known that*

$$\left\langle 0^{\otimes m}|\breve{b}_{\rm in}(t)\breve{b}_{\rm in}^\dagger(r)|0^{\otimes m}\right\rangle = \delta(t-r)\begin{bmatrix} I_m & 0 \\ 0 & 0_m \end{bmatrix}. \tag{47}$$

In this case, the field is Markovian. The second term on the right-hand side of Eq. (46) reveals the non-Markovian *nature of the m-channel m-photon input fields. Moreover, due to the presence of the pulse shape $\psi_{\rm in}$ in all the diagonal entries of $\Lambda(t,r)$, the inputs can be regarded as correlated non-Markovian noise inputs.*

3.2 The passive case: output intensity

In this subsection, for the *passive* quantum linear system (18) driven by an m-photon input state $|\Psi_{\rm in}\rangle$ defined in Eq. (42), we derive a formula for the output intensity $\bar{n}_{\rm out}(t)$ defined in Eq. (10).

Recall that in the passive case the matrix $C_+=0$. Substitution of $L(t)=C_-a(t)$ into Eq. (9) yields

$$\begin{aligned} d\Lambda_{\rm out}(t) = {} & S_-^{\#}d\Lambda_{\rm in}(t)S_-^T + S_-^{\#}dB_{\rm in}^{\#}(t)a^T(t)C_-^T \\ & + C_-^{\#}a^{\#}(t)dB_{\rm in}^T(t)S_-^T + C_-^{\#}a^{\#}(t)a^T(t)C_-^T dt. \end{aligned} \tag{48}$$

Inspired by the second term on the right-hand side of Eq. (48), we define an n-by-m matrix function $f(t)$ as

$$f(t) \triangleq \langle \phi\Psi_{\rm in}|b_{\rm in}^{\#}(t)a^T(t)|\phi\Psi_{\rm in}\rangle^T. \tag{49}$$

Moreover, define an $n\times n$ matrix function $\Sigma(t)$ to be

$$\Sigma(t) \triangleq \langle \phi\Psi_{\rm in}|a(t)a^\dagger(t)|\phi\Psi_{\rm in}\rangle, \quad t\geq t_0. \tag{50}$$

Clearly, $\Sigma(t)=\Sigma(t)^\dagger$.

The following theorem is the main result of this subsection, which gives an explicit procedure for computing the output intensity $\bar{n}_{\rm out}(t)$.

Theorem 1 *For the passive quantum linear system (18) initialized in the vacuum state $|\phi\rangle$ and driven by the m-channel m-photon input state $|\Psi_{\rm in}\rangle$ defined in Eq. (42), the matrix function $f(t)$ defined in Eq. (49) has the following form*

$$f(t) = -\int_{t_0}^t e^{A(t-r)}C_-^\dagger S_-\Lambda(t,r)\,dr, \tag{51}$$

where the matrix function $\Lambda(t,r)$ is given in Eq. (45). The output intensity $\bar{n}_{\rm out}(t)$ is given by

$$\begin{aligned} \bar{n}_{\rm out}(t) = {} & S_-^{\#}\Lambda(t,t)S_-^T + S_-^{\#}f(t)^TC_-^T + C_-^{\#}f(t)^{\#}S_-^T \\ & - C_-^{\#}C_-^T + C_-^{\#}\Sigma(t)^TC_-^T, \end{aligned} \tag{52}$$

in which the covariance function $\Sigma(t)$ solves the following matrix equation

$$\dot{\Sigma}(t) = A\Sigma(t)+\Sigma(t)A^\dagger+C_-^\dagger C_- - C_-^\dagger S_- f(t)^\dagger - f(t)S_-^\dagger C_- \tag{53}$$

with the initial condition $\Sigma(t_0)=I_n$.

Proof. We prove this theorem in three steps.

Step 1. We establish Eq. (51). Firstly, it can be readily shown that

$$b_{\rm in}(t)|\Psi_{\rm in}\rangle = \begin{bmatrix} |\zeta_1(t)\rangle \\ \vdots \\ |\zeta_m(t)\rangle \end{bmatrix}, \tag{54}$$

where the following *notation*

$$\begin{aligned} & |\zeta_j(t)\rangle \\ \equiv & \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} d\tau_1\cdots d\tau_{j-1}d\tau_{j+1}\cdots d\tau_m \\ & \psi_{\rm in}(\tau_1,\ldots,\tau_{j-1},t,\tau_{j+1},\ldots,\tau_m)\prod_{k=1,k\neq j}^m b_{{\rm in},k}^*(\tau_k)|0^{\otimes m}\rangle \end{aligned} \tag{55}$$

has been used, $(j=1,\ldots,m)$. As a result,

$$\begin{aligned} & \langle \phi\Psi_{\rm in}|b_{\rm in}^{\#}(t)b_{\rm in}^T(r)|\phi\Psi_{\rm in}\rangle = \langle \Psi_{\rm in}|b_{\rm in}^{\#}(t)b_{\rm in}^T(r)|\Psi_{\rm in}\rangle \\ = & \begin{bmatrix} \langle\zeta_1(t)|\zeta_1(r)\rangle & & \\ & \ddots & \\ & & \langle\zeta_m(t)|\zeta_m(r)\rangle \end{bmatrix} = \Lambda(t,r), \end{aligned} \tag{56}$$

where Eq. (45) has been used in the last step. Moreover,

$$\begin{aligned} & \langle \phi\Psi_{\rm in}|b_{\rm in}^{\#}(t)a^T(t)|\phi\Psi_{\rm in}\rangle = \begin{bmatrix} \langle \phi\Psi_{\rm in}|b_{{\rm in},m}^*(t)a^T(t)|\phi\Psi_{\rm in}\rangle \\ \vdots \\ \langle \phi\Psi_{\rm in}|b_{{\rm in},m}^*(t)a^T(t)|\phi\Psi_{\rm in}\rangle \end{bmatrix} \\ = & \begin{bmatrix} \langle \phi\zeta_1(t)|a^T(t)|\phi\Psi_{\rm in}\rangle \\ \vdots \\ \langle \phi\zeta_m(t)|a^T(t)|\phi\Psi_{\rm in}\rangle \end{bmatrix}. \end{aligned} \tag{57}$$

Substituting Eq. (57) into Eq. (49) yields

$$f(t) = \left[\, \langle \phi\zeta_1(t)|a(t)|\phi\Psi_{\rm in}\rangle \ \cdots \ \langle \phi\zeta_m(t)|a(t)|\phi\Psi_{\rm in}\rangle \,\right]. \tag{58}$$

Secondly, solving Eq. (18) we get

$$a(t) = e^{A(t-t_0)}a - \int_{t_0}^t e^{A(t-r)}C_-^\dagger S_- b_{\rm in}(r)dr, \quad t\geq t_0. \tag{59}$$

Partition the n-by-m matrix function $e^{At}C_-^\dagger S_-$ into m columns, specifically,

$$e^{At}C_-^\dagger S_- = [\, c_1(t) \ \cdots \ c_m(t)\,]. \tag{60}$$

By Eqs. (59), (60), (54), and (56), we may derive

$$\begin{aligned}&\left[\ \langle\phi\zeta_1(t)|a(t)|\phi\Psi_{\rm in}\rangle\ \cdots\ \langle\phi\zeta_m(t)|a(t)|\phi\Psi_{\rm in}\rangle\ \right]\\ &=-\int_{t_0}^{t} e^{A(t-r)}C_-^\dagger S_-\Lambda\left(t,r\right)dr.\end{aligned} \tag{61}$$

Substituting Eq. (61) into Eq. (58) gives Eq. (51).

Step 2. We establish Eq. (53). By Itô calculus and Eq. (49), we have

$$\begin{aligned}d\Sigma(t) &= A\Sigma(t)dt+\Sigma(t)A^\dagger dt+C_-^\dagger C_- dt\\ &\quad -C_-^\dagger S_- f(t)^\dagger dt - f(t)S_-^\dagger C_- dt.\end{aligned} \tag{62}$$

In Eq. (62), the commutation relations $[a_j(t), dB_{{\rm in},k}(t)] = [a_j^*(t), dB_{{\rm in},k}(t)] = [a_j(t), dB^*_{{\rm in},k}(t)] = [a_j^*(t), dB^*_{{\rm in},k}(t)] = 0$ ($j=1\ldots,n$, $k=1,\ldots,m$) have been used to get the 4th step, and Eq. (58) has been used to get the 6th step (which is the last step). Dividing both sides of Eq. (62) by dt yields Eq. (53).

Step 3. We establish Eq. (52). By the canonical commutation relation $[a_j, a_k^*] = \delta_{jk}$ ($j,k=1,\ldots,n$), we have

$$\begin{aligned}\Sigma(t) &= \langle\phi\Psi_{\rm in}|a(t)a^\dagger(t)|\phi\Psi_{\rm in}\rangle\\ &= I+\langle\phi\Psi_{\rm in}|a^\#(t)a^T(t)|\phi\Psi_{\rm in}\rangle^T.\end{aligned} \tag{63}$$

This, together with Eqs. (48) and (58), yields

$$\begin{aligned}&\langle\phi\Psi_{\rm in}|d\Lambda_{\rm out}(t)|\phi\Psi_{\rm in}\rangle\\ &= S_-^\#\langle\Psi_{\rm in}|d\Lambda_{\rm in}(t)|\Psi_{\rm in}\rangle S_-^T + S_-^\# f(t)^T C_-^T dt\\ &\quad + C_-^\# f(t)^\# S_-^T dt - C_-^\# C_-^T dt + C_-^\#\Sigma(t)^T C_-^T dt.\end{aligned} \tag{64}$$

By Eq. (56),

$$\langle\phi\Psi_{\rm in}|d\Lambda_{\rm in}(t)|\phi\Psi_{\rm in}\rangle = \Lambda\left(t,t\right)dt. \tag{65}$$

Substituting Eq. (65) into Eq. (64) and dividing both sides of the resulting equation by dt yield Eq. (52). ■

3.3 The passive case: state transfer

In this subsection, we derive an analytical form of the output field state of the *passive* quantum linear system (18) driven by the m-photon input state $|\Psi_{\rm in}\rangle$ defined in Eq. (42).

The following is the main result of this subsection.

Theorem 2 *If the asymptotically stable passive quantum linear system (18) is initialized in the vacuum state and is driven by the m-channel m-photon input state $|\Psi_{\rm in}\rangle$ defined in Eq. (42), then the steady-state output field state is an m-channel m-photon state of the form*

$$|\Psi_{\rm out}\rangle = \psi_{\rm out}\circledcirc^m b_{\rm in}^\# |0^{\otimes m}\rangle, \tag{66}$$

where the operation $\circledcirc^m$ has been defined in Eq. (23), and the output pulse $\psi_{\rm out}$ is given by the m-fold convolution

$$\begin{aligned}&\psi_{{\rm out},j_1\ldots j_m}(r_1,\ldots,r_m) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}dt_1\cdots\\ &dt_m\, g_{G^-}^{j_1 1}(r_1-t_1)\cdots g_{G^-}^{j_m m}(r_m-t_m)\psi_{\rm in}(t_1,\ldots,t_m)\end{aligned} \tag{67}$$

($j_1,\ldots,j_m=1,\ldots,m$) with the impulse response function $g_{G^-}(t)$ given in Eq. (19). If we update the m-variable function $\psi_{\rm in}$ in Eq. (42) to a tensor $\psi_{\rm in}^\uparrow$ with entries

$$\begin{aligned}&\psi^\uparrow_{{\rm in},j_1\ldots j_m}(r_1,\ldots,r_m)\\ &\triangleq\begin{cases}\psi_{\rm in}(r_1,\ldots,r_m), & \text{if } j_1=1, j_2=2,\ldots,j_m=m,\\ 0, & \text{otherwise},\end{cases}\end{aligned} \tag{68}$$

as has been done in Eq. (24), then the output pulse $\psi_{\rm out}$ can be written in a compact form

$$\psi_{\rm out} = \psi_{\rm in}^\uparrow \circledast_t^m g_{G^-}, \tag{69}$$

where the operation $\circledast_t^m$ has been defined in Eq. (31).

Proof. To prove this result, we use both the Schrödinger picture and Heisenberg picture. We first work in the Heisenberg picture. By Eqs. (18), (59), and (19),

$$b_{\rm out}(t) = Ce^{A(t-t_0)}a + \int_{t_0}^{t} g_{G^-}(t-r)b_{\rm in}(r)dr,\ t\geq t_0,$$

whose adjoint operator $b_{\rm out}^\#(t)$ satisfies

$$b_{\rm out}^\#(t) = C^\# e^{A^\#(t-t_0)}a^\# + \int_{t_0}^{t} g_{G^-}(t-r)^\# b_{\rm in}^\#(r)dr,\ t\geq t_0. \tag{70}$$

On the other hand, notice that in the Heisenberg picture, Eq. (5) gives

$$b_{\rm out}^\#(t) = U(t,t_0)^* b_{\rm in}^\#(t) U(t,t_0),\quad t\geq t_0 \tag{71}$$

(component-wise for the components of $b_{\rm in}^\#(t)$). Eqs. (70)-(71) yield

$$\begin{aligned}b_{\rm in}^\#(t) &= C^\# e^{A^\#(t-t_0)}U(t,t_0)a^\# U(t,t_0)^*\\ &\quad + \int_{t_0}^{t} g_{G^-}(t-r)^\# U(r,t_0)b_{\rm in}^\#(r)U(r,t_0)^* dr.\end{aligned} \tag{72}$$

Since the system is asymptotically stable, sending $t_0 \to -\infty$, Eq. (72) becomes

$$
\begin{aligned}
b_{\mathrm{in}}^{\#}(t) &= \int_{-\infty}^{t} g_{G^-}(t-r)^{\#} U(r,-\infty) b_{\mathrm{in}}^{\#}(r) U(r,-\infty)^{*} dr \\
&= \int_{-\infty}^{\infty} g_{G^-}(t-r)^{\#} U(r,-\infty) b_{\mathrm{in}}^{\#}(r) U(r,-\infty)^{*} dr. \quad (73)
\end{aligned}
$$

This, together with Eq. (20), yields

$$
U(t,-\infty) b_{\mathrm{in}}^{\#}(t) U(t,-\infty)^{*} = \int_{-\infty}^{\infty} g_{G^-}(r-t)^{T} b_{\mathrm{in}}^{\#}(r) dr. \quad (74)
$$

Next, we switch to the Schrödinger picture. In the Schrödinger picture, the joint system-field state at time $t \geq t_0$ is $U(t,t_0)|\phi\Psi_{\mathrm{in}}\rangle$. Thus, the steady-state output field state can be obtained by tracing out the system. That is,

$$
|\Psi_{\mathrm{out}}\rangle = \langle\phi| \lim_{t_0\to-\infty, t\to\infty} U(t,t_0)|\phi\Psi_{\mathrm{in}}\rangle. \quad (75)
$$

We have

$$
\begin{aligned}
& |\Psi_{\mathrm{out}}\rangle \\
=& \lim_{t_0\to-\infty, t\to\infty} \langle\phi| U(t,t_0)|\phi\Psi_{\mathrm{in}}\rangle \\
=& \lim_{t_0\to-\infty, t\to\infty} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} dt_1\cdots dt_m \ \psi_{\mathrm{in}}(t_1,\ldots,t_m) \\
& \quad \langle\phi| U(t,t_0) b_{\mathrm{in},1}^{*}(t_1)\cdots b_{\mathrm{in},m}^{*}(t_m)|\phi 0^{\otimes m}\rangle \\
=& \lim_{t_0\to-\infty, t\to\infty} \int_{t_0}^{t}\cdots\int_{t_0}^{t} dt_1\cdots dt_m \ \psi_{\mathrm{in}}(t_1,\ldots,t_m) \\
& \quad \langle\phi| U(t_1,t_0) b_{\mathrm{in},1}^{*}(t_1) U(t_1,t_0)^{*} \\
& \qquad \cdots U(t_m,t_0) b_{\mathrm{in},m}^{*}(t_m) U(t_m,t_0)^{*} U(t_m,t_0)|\phi 0^{\otimes m}\rangle \\
=& \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} dt_1\cdots dt_m \ \psi_{\mathrm{in}}(t_1,\ldots,t_m) \\
& \langle\phi| \int_{-\infty}^{\infty}\sum_{j_1=1}^{m} g_{G^-}^{j_1 1}(r_1-t_1) b_{\mathrm{in},j_1}^{*}(r_1) dr_1 \\
& \quad \cdots \int_{-\infty}^{\infty}\sum_{j_m=1}^{m} g_{G^-}^{j_m m}(r_m-t_m) b_{\mathrm{in},j_m}^{*}(r_m) dr_m |\phi 0^{\otimes m}\rangle \\
& \qquad (76) \\
=& \sum_{j_1=1}^{m}\cdots\sum_{j_m=1}^{m}\int d\overrightarrow{r}\ b_{\mathrm{in},j_1}^{*}(r_1)\cdots b_{\mathrm{in},j_m}^{*}(r_m)\int d\overrightarrow{t} \\
& g_{G^-}^{j_1 1}(r_1-t_1)\cdots g_{G^-}^{j_m m}(r_m-t_m)\psi_{\mathrm{in}}(t_1,\ldots,t_m)|0^{\otimes m}\rangle \\
=& \sum_{j_1,\ldots,j_m=1}^{m}\int d\overrightarrow{r}\ b_{\mathrm{in},j_1}^{*}(r_1)\cdots b_{\mathrm{in},j_m}^{*}(r_m) \\
& \psi_{\mathrm{out},j_1\ldots j_m}(r_1,\ldots,r_m)\ |0^{\otimes m}\rangle \quad (77) \\
=& \psi_{\mathrm{out}} \circledcirc b_{\mathrm{in}}^{\#}\ |0^{\otimes m}\rangle,
\end{aligned}
$$

which is exactly Eq. (66). Notice that the following fact,

$$
U(t,t_0)|\phi 0^{\otimes m}\rangle = |\phi 0^{\otimes m}\rangle, \quad t \geq t_0 \quad (78)
$$

upon a global phase, has been used in the above derivation. In fact, Eq. (78) holds for general (not necessarily linear) passive systems, see, e.g., [30, Lemma 3]. Eq. (74) has been used to derive Eq. (76). Finally, it is clear from the second and third last steps that

$$
\begin{aligned}
& \psi_{\mathrm{out},j_1\ldots j_m}(r_1,\ldots,r_m) \\
=& \int d\overrightarrow{t}\ g_{G^-}^{j_1 1}(r_1-t_1)\cdots g_{G^-}^{j_m m}(r_m-t_m)\psi_{\mathrm{in}}(t_1,\ldots,t_m),
\end{aligned}
$$

which is exactly Eq. (67). ■

Remark 3 *In quantum mechanics, the Schrödinger picture describes how quantum states evolve; on the other hand, the Heisenberg picture describes how operators evolve. Eq. (66) tells us how the input state $|\Psi_{\mathrm{in}}\rangle$ evolves and becomes the output state $|\Psi_{\mathrm{out}}\rangle$. That is, it is in the Schrödinger picture. In the Schrödinger picture, operators do not evolve. This is the reason why the input operator $b_{\mathrm{in}}^{\#}$ appears in Eq. (66).*

Remark 4 *When the input pulse is of a product form*

$$
\psi_{\mathrm{in}}(t_1,\ldots,t_m) = \xi_1(t_1)\cdots\xi_m(t_m), \quad (79)
$$

the input state $|\Psi_{\mathrm{in}}\rangle$ in Eq. (42) becomes a separable state

$$
|\Psi_{\mathrm{in}}\rangle = \prod_{k=1}^{m} \mathbf{B}_{\mathrm{in},k}^{*}(\xi_k)|0_k\rangle, \quad (80)
$$

where the notation

$$
\mathbf{B}_{\mathrm{in},k}^{*}(\xi) \equiv \int_{-\infty}^{\infty} \xi(t) b_{\mathrm{in},k}^{*}(t) dt, \quad k = 1,\ldots,m \quad (81)
$$

has been used. In this case, by Eq. (67) we have

$$
\psi_{\mathrm{out},j_1\ldots j_m}(r_1,\ldots,r_m) = \prod_{k=1}^{m}\int_{-\infty}^{\infty} g_{G^-}^{j_k k}(r_k-t_k)\xi_k(t_k)dt_k \quad (82)
$$

for $j_1,\ldots,j_m = 1,\ldots,m$. Define

$$
\xi_{\mathrm{out},jk}(r) \triangleq \int_{-\infty}^{\infty} g_{G^-}^{jk}(r-t)\xi_k(t)dt, \quad j,k = 1,\ldots,m. \quad (83)
$$

Then, by Theorem 2 and Eq. (82),

$$
|\Psi_{\mathrm{out}}\rangle = \prod_{k=1}^{m}\sum_{j=1}^{m} \mathbf{B}_{\mathrm{in},j}^{*}(\xi_{\mathrm{out},jk})|0^{\otimes m}\rangle. \quad (84)
$$

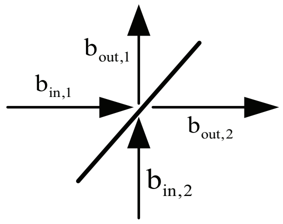

Fig. 2. Schematic representation of a beamsplitter

Interestingly, $|\Psi_{\rm out}\rangle$ *in Eq. (84) can also be derived by means of [44, Theorem 5]. Therefore, Theorem 2 generalizes one of the main results in [44].*

Example 1 *(beamsplitter.) A beamsplitter is a static device widely used in optical laboratories, [22], [2], [27], see Fig. 2. In the* (S_-, L, H) *formalism, a beamsplitter may be modeled by* $L = 0$*,* $H = 0$*, and*

$$S_- = \begin{bmatrix} R & T \\ T & R \end{bmatrix}, \quad R, T \in \mathbb{C}, \ |R|^2 + |T|^2 = 1. \tag{85}$$

Let the 2-channel 2-photon input state be

$$|\Psi_{\rm in}\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dt_1 dt_2 \ \psi_{\rm in}(t_1,t_2) b_{{\rm in},1}^*(t_1) b_{{\rm in},2}^*(t_2)|0_1 0_2\rangle. \tag{86}$$

By Theorem 2, the steady-state output field state is

$$\begin{aligned}
&|\Psi_{\rm out}\rangle \\
= &RT \int dr_1 dr_2 \ b_{{\rm in},1}^*(r_1) b_{{\rm in},1}^*(r_2)\psi_{\rm in}(r_1,r_2)|0_1\rangle \otimes |0_2\rangle \\
&+R^2 \int dr_1 dr_2 \ b_{{\rm in},1}^*(r_1) b_{{\rm in},2}^*(r_2)\psi_{\rm in}(r_1,r_2)|0_1\rangle \otimes |0_2\rangle \\
&+T^2 \int dr_1 dr_2 \ b_{{\rm in},1}^*(r_1) b_{{\rm in},2}^*(r_2)\psi_{\rm in}(r_2,r_1)|0_1\rangle \otimes |0_2\rangle \\
&+|0_1\rangle \otimes RT \int dr_1 dr_2 \ b_{{\rm in},2}^*(r_1) b_{{\rm in},2}^*(r_2)\psi_{\rm in}(r_1,r_2)|0_2\rangle,
\end{aligned}$$

which is exactly [23, Eq. (6.8.7)].

Example 2 *(optical cavity.) An optical cavity is a system composed of reflecting and/or transmitting mirrors [2, Chapter 5.3], [36, Chapter 7], [13], [27]. A widely used type of optical cavities is the so-called Fabry-Perot cavity. In the* (S_-, L, H) *formalism, a single-mode Fabry-Perot cavity with two input channels, as shown in Fig. 3, can be modeled with parameters*

$$\left(S_- = I_2, \ L = \begin{bmatrix} \sqrt{\kappa_1} a \\ \sqrt{\kappa_2} a \end{bmatrix}, \ H = \omega_d a^* a \right). \tag{87}$$

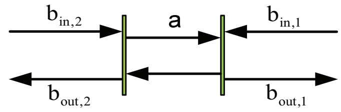

Fig. 3. Schematic representation of a single-mode Fabry-Perot cavity with two inputs

Here, κ_1 *and* κ_2 *are coupling strengths between the cavity and the external fields, and* ω_d *is the detuning frequency between the resonant frequency of the cavity and the external fields. (Here we assume that the two input light fields have the same carrier frequency.) By Eq. (18) we have the following QSDEs*

$$\begin{aligned}
\dot{a}(t) &= -(\frac{\kappa_1+\kappa_2}{2} + {\rm i}\omega_d)a(t) \\
&\quad -\sqrt{\kappa_1} b_{{\rm in},1}(t) - \sqrt{\kappa_2} b_{{\rm in},2}(t), \\
b_{{\rm out},1}(t) &= \sqrt{\kappa_1} a(t) + b_{{\rm in},1}(t), \\
b_{{\rm out},2}(t) &= \sqrt{\kappa_2} a(t) + b_{{\rm in},2}(t).
\end{aligned} \tag{88}$$

Let the input state be that given in Eq. (86). In what follows, we calculate the steady-state output field state. Define the following two-variable functions

$$\Phi_1(r,t_2) \triangleq \int_{-\infty}^{r} dt_1 \ e^{-\left({\rm i}\omega_d + \frac{\kappa_1+\kappa_2}{2}\right)(r-t_1)} \psi_{\rm in}(t_1,t_2), \tag{89}$$

$$\Phi_2(t_1,r) \triangleq \int_{-\infty}^{r} dt_2 \ e^{-\left({\rm i}\omega_d + \frac{\kappa_1+\kappa_2}{2}\right)(r-t_2)} \psi_{\rm in}(t_1,t_2), \tag{90}$$

and

$$\Phi(r,\tau) \triangleq \int_{-\infty}^{r}\int_{-\infty}^{\tau} dt_1 dt_2 \ e^{-\left({\rm i}\omega_d + \frac{\kappa_1+\kappa_2}{2}\right)(r+\tau-t_1-t_2)} \psi_{\rm in}(t_1,t_2). \tag{91}$$

By Theorem 2, the steady-state output field state is

$$\begin{aligned}
&|\Psi_{\rm out}\rangle \\
= &\sqrt{\kappa_1\kappa_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dr_1 dr_2 \ b_{{\rm in},1}^*(r_1) b_{{\rm in},1}^*(r_2) \\
&\quad [\kappa_1 \Phi(r_1,r_2) - \Phi_2(r_1,r_2)]|0_1\rangle \otimes |0_2\rangle \\
&+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dr_1 dr_2 \ b_{{\rm in},1}^*(r_1) b_{{\rm in},2}^*(r_2) \\
&\quad [\psi_{\rm in}(r_1,r_2) - \kappa_1\Phi_1(r_1,r_2) - \kappa_2\Phi_2(r_1,r_2) \\
&\quad\quad +\kappa_1\kappa_2(\Phi(r_1,r_2) + \Phi(r_2,r_1))] \ |0_1\rangle \otimes |0_2\rangle \\
&+|0_1\rangle \otimes \sqrt{\kappa_1\kappa_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dr_1 dr_2 \ b_{{\rm in},2}^*(r_1) b_{{\rm in},2}^*(r_2) \\
&\quad [\kappa_2\Phi(r_1,r_2) - \Phi_1(r_1,r_2)] \ |0_2\rangle.
\end{aligned} \tag{92}$$

In what follows we discuss two cases.

Case 1) In the limit $\kappa_1 \to 0$, the state in Eq. (92) becomes

$$|\Psi_{\rm out}\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dr_1 dr_2\ b_{{\rm in},1}^*(r_1) b_{{\rm in},2}^*(r_2) \\ [\psi_{\rm in}(r_1,r_2) - \kappa_2 \Phi_2(r_1,r_2)]|0_1\rangle \otimes |0_2\rangle. \quad (93)$$

If the input field state is an entangled state, the output field state in Eq. (93) is also an entangled state. Therefore, even though the system does not affect the first channel directly because of $\kappa_1 = 0$, it does influence the first channel via its influence on the second channel.

Case 2) The input state is a product state. Assume

$$\psi_{\rm in}(t_1,t_2) = \xi_1(t_1)\xi_2(t_2), \quad (94)$$

that is, the input is a tensor product state of two single-photon states, one for each channel. In this case, there exists no entanglement between the two input channels. For this product state, Eqs. (89)-(91) reduce to

$$\Phi_1(r_1,r_2) = \xi_2(r_2)\eta_1(r_1), \quad (95)$$

$$\Phi_2(r_1,r_2) = \xi_1(r_1)\eta_2(r_2), \quad (96)$$

and

$$\Phi(r_1,r_2) = \eta_1(r_1)\eta_2(r_2), \quad (97)$$

where

$$\eta_i(t) \triangleq \int_{-\infty}^{t} e^{-\left({\rm i}\omega_d + \frac{\kappa_1+\kappa_2}{2}\right)(t-r)} \xi_i(r) dr,\ i = 1,2. \quad (98)$$

As a result, Eq. (92) becomes

$$|\Psi_{\rm out}\rangle = \left(\mathbf{B}_{{\rm in},1}^*(\xi_1 - \kappa_1\eta_1) - \sqrt{\kappa_1\kappa_2}\mathbf{B}_{{\rm in},2}^*(\eta_2)\right) \quad (99) \\ \left(\mathbf{B}_{{\rm in},2}^*(\xi_2 - \kappa_2\eta_2) - \sqrt{\kappa_1\kappa_2}\mathbf{B}_{{\rm in},1}^*(\eta_1)\right)|0_1 0_2\rangle.$$

The state in Eq. (99) is an entangled state. Therefore, the system entangled the initially separable input state. Sending $\kappa_1 \to 0$ in Eq. (99) yields

$$|\Psi_{\rm out}\rangle = \mathbf{B}_{{\rm in},1}^*(\xi_1)|0_1\rangle \otimes \mathbf{B}_{{\rm in},2}^*(\xi_2 - \kappa_2\eta_2)|0_2\rangle, \quad (100)$$

which is a product state. That is, if the coupling between the system and the first channel is extremely weak, then the output fields are almost in a product state. This is reasonable: when the coupling strength $\kappa_1 = 0$, the first channel has no interaction with the system, so the state of the first channel does not change if it is not initially entangled with the second channel. On the other hand, the pulse shape of the second channel has been transformed by the system from ξ_2 to $\xi_2 - \kappa_2\eta$.

3.4 The passive case: the invariant set

Define a class of m-channel m-photon states of the form

$$\mathcal{F}_1 \triangleq \{|\Psi\rangle = \psi \circledcirc^m b_{\rm in}^{\#}|0^{\otimes m}\rangle\ |\text{tensor } \psi \text{ normalizes } |\Psi\rangle\}. \quad (101)$$

By means of Eq. (68), it is clear that the m-channel m-photon input field state defined in Eq. (42) can be rewritten as $|\Psi_{\rm in}\rangle = \psi_{\rm in}^{\uparrow} \circledcirc^m b_{\rm in}^{\#}|0^{\otimes m}\rangle$. Therefore, $|\Psi_{\rm in}\rangle \in \mathcal{F}_1$. On the other hand, by Theorem 2, the steady-state output field state $|\psi_{\rm out}\rangle \in \mathcal{F}_1$ too. This motivates us to study more general pulse shape transfer than that in Theorem 2.

The following is the main result of this subsection.

Theorem 3 *Let the input state for the asymptotically stable passive quantum linear system (18) (initialized in the vacuum state) be an element $|\Psi_{\rm in}\rangle \in \mathcal{F}_1$ with pulse shape parametrized by an m-way m-dimensional tensor function $\psi_{\rm in}$. Then, the steady-state output field state*

$$|\Psi_{\rm out}\rangle = \psi_{\rm out} \circledcirc^m b_{\rm in}^{\#}|0^{\otimes m}\rangle \quad (102)$$

is also an element in $\mathcal{F}_1$, where the pulse shape is given by

$$\psi_{\rm out} = \psi_{\rm in} \circledast_t^m g_{G^-}. \quad (103)$$

Alternatively, in the frequency domain,

$$\psi_{\rm out} = \psi_{\rm in} \circledast_\omega^m g_{G^-}. \quad (104)$$

Moreover, we have

$$\|\psi_{\rm out}({\rm i}\omega_1,\ldots,{\rm i}\omega_m)\|^2 = \|\psi_{\rm in}({\rm i}\omega_1,\ldots,{\rm i}\omega_m)\|^2 \quad (105)$$

for all $\omega_1,\ldots,\omega_m \in \mathbb{R}$.

Due to page limitation, the proof of Theorem 3 is omitted.

3.5 The non-passive case

A quantum linear system is said to be *non-passive* if $C_+ \neq 0$ and/or $\Omega_+ \neq 0$ in Eq. (7). Non-passive elements, such as optical parametric oscillators (OPOs), are key ingredients of quantum optical systems, [22], [2], [27]. In this subsection, we study the output field state of a non-passive quantum linear system driven by the m-channel m-photon input field state $|\Psi_{\rm in}\rangle$ defined in Eq. (42).

Firstly, we introduce some notation. Define operators

$$b_j^d(t) \triangleq \begin{cases} b_{{\rm in},j}^*(t), & d = -1, \\ b_{{\rm in},j}(t), & d = 1, \end{cases} \quad j = 1,\ldots,m. \quad (106)$$

Then, define an $\underbrace{m \times \cdots \times m}_{m} \times \underbrace{2 \times \cdots \times 2}_{m}$ tensor operator $\mathfrak{b} = \mathfrak{b}^{d_1 \ldots d_m}_{j_1 \cdots j_m}(t_1, \ldots, t_m)$, whose entries are

$$\mathfrak{b}^{d_1 \ldots d_m}_{j_1 \cdots j_m}(t_1, \ldots, t_m) \triangleq b^{d_1}_{j_1}(t_1) \cdots b^{d_m}_{j_m}(t_m) \tag{107}$$

for $j_1, \ldots, j_m = 1, \ldots, m,\ d_1, \ldots, d_m = \pm 1$. Denote

$$g^{kj}_{G^d}(t) \triangleq \begin{cases} g^{kj}_{G^-}(t), & d = -1, \\ -g^{kj}_{G^+}(t)^*, & d = 1, \end{cases}, \quad j, k = 1, \ldots, m. \tag{108}$$

Then, define an $\underbrace{m \times \cdots \times m}_{m} \times \underbrace{2 \times \cdots \times 2}_{m}$ tensor function ψ with entries

$$\begin{aligned} \psi^{d_1 \ldots d_m}_{j_1 \cdots j_m}(r_1, \ldots, r_m) \triangleq & \int_{-\infty}^{\infty} dt_1 \cdots dt_m\ g^{j_1 1}_{G^{d_1}}(r_1 - t_1) \\ & \cdots g^{j_m m}_{G^{d_m}}(r_m - t_m) \psi_{\mathrm{in}}(t_1, \ldots, t_m) \end{aligned} \tag{109}$$

$(j_1, \ldots, j_m = 1, \ldots, m,\ d_1, \ldots, d_m = \pm 1)$. Finally, define the following operation between tensors $\mathfrak{b}$ and ψ

$$\begin{aligned} \langle \mathfrak{b}, \psi \rangle \triangleq & \sum_{j_1, \ldots, j_m = 1}^{m} \sum_{d_1, \ldots, d_m = \pm 1} \int dr_1 \cdots dr_m \\ & \psi^{d_1 \ldots d_m}_{j_1 \cdots j_m}(r_1, \ldots, r_m) b^{d_1}_{j_1}(r_1) \cdots b^{d_m}_{j_m}(r_m). \end{aligned} \tag{110}$$

The following result shows how a non-passive quantum linear system processes m-channel m-photon input states.

Theorem 4 *Let G be an asymptotically stable non-passive quantum linear system which is initialized in the vacuum state $|\phi\rangle$ and is driven by the m-channel m-photon input state $|\Psi_{\mathrm{in}}\rangle$ defined in Eq. (42). The steady-state output field state is*

$$\rho_{\mathrm{out}} = \langle \mathfrak{b}, \psi \rangle \langle \phi | \rho_\infty | \phi \rangle \langle \mathfrak{b}, \psi \rangle^*, \tag{111}$$

where $\langle \mathfrak{b}, \psi \rangle$ is given in Eq. (110), and

$$\rho_\infty \triangleq \lim_{t_0 \to -\infty, t \to \infty} U(t, t_0) |\phi 0^{\otimes m}\rangle \langle \phi 0^{\otimes m}| U(t, t_0)^* \tag{112}$$

is a zero-mean Gaussian state for the joint system whose power spectral density matrix is given by Eq. (21) in Lemma 1.

Due to page limitation, the proof of Theorem 4 is omitted.

4 N ($N \geq m$) photons superposed over m channels

In this section, we study how a *passive* quantum linear system processes N photons that are superposed over m input channels, thus generalizing the results in Section 3.

4.1 State transfer

Let the input field be in a state where N photons are superposed over m input channels. Specifically, the input state considered in this subsection is defined to be

$$\begin{aligned} |\Psi_{\mathrm{in}}\rangle \triangleq & \int d\overrightarrow{t}\ \psi_{\mathrm{in}}(t^1_1, \ldots, t^1_{k_1}, \ldots, t^m_1, \ldots, t^m_{k_m}) b^*_{\mathrm{in},1}(t^1_1) \\ & \cdots b^*_{\mathrm{in},1}(t^1_{k_1}) \cdots b^*_{\mathrm{in},m}(t^m_1) \cdots b^*_{\mathrm{in},m}(t^m_{k_m}) |0^{\otimes m}\rangle. \end{aligned} \tag{113}$$

In Eq. (113), the positive integers k_i satisfy $\sum_{i=1}^m k_i = N$. It is also implicitly assumed in Eq. (113) that the N-variable function ψ_{in} normalizes the state $|\Psi_{\mathrm{in}}\rangle$.

Remark 5 *For the ith input channel ($i = 1, \ldots, m$), the creation operator $b^*_{\mathrm{in},i}$ appears k_i times in Eq. (113), thus there are k_i photons in the ith input channel.*

With the notation introduced in Eq. (26), Eq. (113) can be re-written as

$$|\Psi_{\mathrm{in}}\rangle = \psi_{\mathrm{in}} \cdot^N_{k_1 \cdots k_m} b^\#_{\mathrm{in}} |0^{\otimes m}\rangle. \tag{114}$$

Moreover, inspired by Eq. (28), update the N-variable function $\psi_{\mathrm{in}}(t^1_1, \ldots, t^1_{k_1}, \ldots, t^m_1, \ldots, t^m_{k_m})$ to an N-way m-dimensional tensor $\psi^\uparrow_{\mathrm{in}}(t^1_1, \ldots, t^1_{k_1}, \ldots, t^m_1, \ldots, t^m_{k_m})$, whose non-zero elements are defined as

$$\begin{aligned} & \psi^\uparrow_{\mathrm{in}, i^1_1 \cdots i^1_{k_1} \cdots i^m_1 \cdots i^m_{k_m}}(t^1_1, \ldots, t^1_{k_1}, \ldots, t^m_1, \ldots, t^m_{k_m}) \\ & \triangleq \psi_{\mathrm{in}}(t^1_1, \ldots, t^1_{k_1}, \ldots, t^m_1, \ldots, t^m_{k_m}), \end{aligned} \tag{115}$$

where $i^1_1 = 1, \cdots, i^1_{k_1} = k_1, \cdots, i^m_1 = \sum_{j=1}^{m-1} k_j + 1, \cdots, i^m_{k_m} = N$. Then, Eq. (114) can be re-written as

$$|\Psi_{\mathrm{in}}\rangle = \psi^\uparrow_{\mathrm{in}} \odot^N_{k_1 \cdots k_m} b^\#_{\mathrm{in}} |0^{\otimes m}\rangle, \tag{116}$$

where the operation $\odot^N_{k_1 \cdots k_m}$ has been introduced in Eq. (27).

The following result gives an explicit form of the steady-state output field state.

Theorem 5 *If the asymptotically stable passive quantum linear system (18) is initialized in the vacuum state* *and is driven by the m-channel N-photon input state $|\Psi_{\rm in}\rangle$ defined in Eq. (113), then the steady-state output field state is*

$$|\Psi_{\rm out}\rangle = \psi_{\rm out} \odot_{k_1\cdots k_m}^N b_{\rm in}^{\#}|0^{\otimes m}\rangle \tag{117}$$

with the pulse shape

$$\psi_{\rm out} = \psi_{\rm in}^{\uparrow} \circledast_{t,k_1\cdots k_m}^N g_{G^-}. \tag{118}$$

In Eq. (118), the tensor $\psi_{\rm in}^{\uparrow}$ has been given in Eq. (115), and the operation $\circledast_{t,k_1\cdots k_m}^N$ has been defined in Eq. (33).

Proof. The proof is similar to that of Theorem 2, so is omitted.

Example 3 *Consider a beamsplitter*

$$S_- = \begin{bmatrix} R & -T \\ T & R \end{bmatrix}, \quad R, T \in \mathbb{C}, \ |R|^2 + |T|^2 = 1, \tag{119}$$

and a 3-photon input state of the form

$$\begin{aligned}|\Psi_{\rm in}\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} & dt_1dt_2dt_3 \ \psi_{\rm in}(t_1,t_2,t_3) \\ & b_{{\rm in},1}^*(t_1)b_{{\rm in},2}^*(t_2)b_{{\rm in},2}^*(t_3)|0_1\rangle\otimes|0_2\rangle. \end{aligned}\tag{120}$$

In this case, there are two input channels ($m = 2$). The total number of photons is $N = 3$. In fact, as explained in Remark 5 above, there is one photon in the first channel ($k_1 = 1$) and two photons in the second channel ($k_2 = 2$). According to Theorem 5, the steady-state output field state is

$$\begin{aligned}
&|\Psi_{\rm out}\rangle \\
=&RT^2\int d\overrightarrow{t}\ b_{{\rm in},1}^*(t_1)b_{{\rm in},1}^*(t_2)b_{{\rm in},1}^*(t_3)\psi_{\rm in}(t_1,t_2,t_3)|0_10_2\rangle \\
&-T\int d\overrightarrow{t}\ b_{{\rm in},1}^*(t_1)b_{{\rm in},1}^*(t_2)b_{{\rm in},2}^*(t_3)[R^2\psi_{\rm in}(t_1,t_2,t_3) \\
&\quad+R^2\psi_{\rm in}(t_1,t_3,t_2)-T^2\psi_{\rm in}(t_3,t_2,t_1)]|0_10_2\rangle \\
&-R\int d\overrightarrow{t}\ b_{{\rm in},1}^*(t_1)b_{{\rm in},2}^*(t_2)b_{{\rm in},2}^*(t_3)[T^2\psi_{\rm in}(t_2,t_1,t_3) \\
&\quad+T^2\psi_{\rm in}(t_3,t_2,t_1)-R^2\psi_{\rm in}(t_1,t_2,t_3)]|0_10_2\rangle \\
&+R^2T\int d\overrightarrow{t}\, b_{{\rm in},2}^*(t_1)b_{{\rm in},2}^*(t_2)b_{{\rm in},2}^*(t_3)\psi_{\rm in}(t_1,t_2,t_3)|0_10_2\rangle.
\end{aligned}\tag{121}$$

In what follows, we assume 1) $R = T = \frac{1}{\sqrt{2}}$, 2) $\psi_{\rm in}(t_1,t_2,t_3)$ is permutation-invariant, and 3) $\|\psi_{\rm in}(t_1,t_2,t_3)\| = \frac{1}{\sqrt{2}}$. Define states

$$\begin{aligned}
&|\Pi_{30}\rangle \\
&\triangleq \frac{1}{\sqrt{3}}\int d\overrightarrow{t}\ b_{{\rm in},1}^*(t_1)b_{{\rm in},1}^*(t_2)b_{{\rm in},1}^*(t_3)\psi_{\rm in}(t_1,t_2,t_3)|0_1\rangle, \\
&|\Pi_{21}\rangle \\
&\triangleq \int d\overrightarrow{t}\, b_{{\rm in},1}^*(t_1)b_{{\rm in},1}^*(t_2)b_{{\rm in},2}^*(t_3)\psi_{\rm in}(t_1,t_2,t_3)|0_10_2\rangle, \\
&|\Pi_{12}\rangle \\
&\triangleq \int d\overrightarrow{t}\ b_{{\rm in},1}^*(t_1)b_{{\rm in},2}^*(t_2)b_{{\rm in},2}^*(t_3)\psi_{\rm in}(t_1,t_2,t_3)|0_10_2\rangle, \\
&|\Pi_{03}\rangle \\
&\triangleq \frac{1}{\sqrt{3}}\int d\overrightarrow{t}\ b_{{\rm in},2}^*(t_1)b_{{\rm in},2}^*(t_2)b_{{\rm in},2}^*(t_3)\psi_{\rm in}(t_1,t_2,t_3)|0_2\rangle.
\end{aligned}\tag{122}$$

It is easy to show that all the states in Eq. (122) are normalized. Moreover, $|\Pi_{30}\rangle$ is a 3-photon state for the first channel, $|\Pi_{03}\rangle$ is a 3-photon state for the second channel, and $|\Pi_{21}\rangle$ and $|\Pi_{12}\rangle$ are states where two channels share three photons. With these new notations, Eq. (121) becomes

$$\begin{aligned}|\Psi_{\rm out}\rangle =& \frac{\sqrt{6}}{4}|\Pi_{30}\rangle\otimes|0_2\rangle - \frac{\sqrt{2}}{4}|\Pi_{21}\rangle \\ &-\frac{\sqrt{2}}{4}|\Pi_{12}\rangle + |0_1\rangle\otimes\frac{\sqrt{6}}{4}|\Pi_{03}\rangle. \end{aligned}\tag{123}$$

Finally, if $\psi_{\rm in}(t_1,t_2,t_3) = \xi_1(t_1)\xi_2(t_2)\xi_3(t_3)$, then Eq. (120) becomes

$$|\Psi_{\rm in}\rangle = \mathbf{B}_{{\rm in},1}^*(\xi_1)|0_1\rangle\otimes\mathbf{B}_{{\rm in},2}^*(\xi_2)\mathbf{B}_{{\rm in},2}^*(\xi_3)|0_3\rangle.$$

That is, the input is a product state, with one photon in channel 1 and two photons in channel 2. Moreover, if $\xi_1 = \xi_2 = \xi_3 \equiv \xi$, then the normalization condition requires that $\|\xi\| = 1/\sqrt[6]{2}$. Eq. (122) reduces to

$$\begin{aligned}
|\Pi_{30}\rangle &= \frac{1}{\sqrt{3}}(\mathbf{B}_{{\rm in},1}^*(\xi))^3|0_1\rangle, \\
|\Pi_{21}\rangle &= (\mathbf{B}_{{\rm in},1}^*(\xi))^2|0_1\rangle\otimes\mathbf{B}_{{\rm in},2}^*(\xi)|0_2\rangle, \\
|\Pi_{12}\rangle &= \mathbf{B}_{{\rm in},1}^*(\xi)|0_1\rangle\otimes(\mathbf{B}_{{\rm in},2}^*(\xi))^2|0_2\rangle, \\
|\Pi_{03}\rangle &= \frac{1}{\sqrt{3}}(\mathbf{B}_{{\rm in},2}^*(\xi))^3|0_2\rangle.
\end{aligned}\tag{124}$$

That is, all the states become product states. If we ignore pulse shapes and only count the number of photons in each channel, we may identify $|\Pi_{30}\rangle$ with $|3_1\rangle$, $|\Pi_{03}\rangle$ with $|3_2\rangle$, $|\Pi_{21}\rangle$ with $|2_1\rangle\otimes|1_2\rangle$, and $|\Pi_{12}\rangle$ with $|1_1\rangle\otimes|2_2\rangle$. Accordingly, the state in Eq. (123) reduces to

$$\begin{aligned}|\Psi_{\rm out}\rangle =& \frac{\sqrt{6}}{4}|3_1\rangle\otimes|0_2\rangle - \frac{\sqrt{2}}{4}|2_1\rangle\otimes|1_2\rangle \\ &-\frac{\sqrt{2}}{4}|1_1\rangle\otimes|2_2\rangle + \frac{\sqrt{6}}{4}|0_1\rangle\otimes|3_2\rangle. \end{aligned}$$

4.2 The invariant set

In this subsection, we define a class of m-channel N-photon states and show that this class of states is invariant under the steady-state action of a quantum linear passive system. The discussions here generalize those in Subsection 3.4.

Motivated by Eqs. (116) and (117), define a class of m-channel N-photon states:

$$\mathcal{F}_2 \triangleq \left\{|\Psi\rangle = \psi \odot_{k_1\cdots k_m}^N b_{\rm in}^{\#}|0^{\otimes m}\rangle \ |\text{tensor } \psi \text{ normalizes } |\Psi\rangle.\right\} \tag{125}$$

The following result shows that the set $\mathcal{F}_2$ is invariant under the steady-state action of a passive quantum linear system.

Theorem 6 *The steady-state output field state of the asymptotically stable passive quantum linear system (18), initialized in the vacuum state $|\phi\rangle$ and driven by an m-channel N-photon input state $|\Psi_{\rm in}\rangle \in \mathcal{F}_2$ with pulse information encoded by an N-way m-dimensional tensor function $\psi_{\rm in}$, is another element $|\Psi_{\rm out}\rangle \in \mathcal{F}_2$, whose pulse information is encoded by an N-way m-dimensional tensor function $\psi_{\rm out}$ given by*

$$\psi_{\rm out} = \psi_{\rm in} \circledast_{t,k_1\cdots k_m}^N g_{G^-}. \tag{126}$$

This result can be established in a similar way as Theorem 3. So the proof is omitted.

5 An arbitrary number of photons superposed over m input channels

In all the previous discussions, we have implicitly assumed that the total number of photons is no less than the number of input channels. In this section, we remove this constraint. More specifically, we study a class of m-channel N-photon states where N can be an arbitrary positive integer.

5.1 A class of m-channel N-photon input states

In this subsection, we present a class of m-channel N-photon input states. Two illustrative examples are also given.

Let a normalized m-channel N-photon input state be

$$|\Psi_{\rm in}\rangle = \prod_{j=1}^N \sum_{k=1}^m \int_{-\infty}^{\infty} dt\ \psi_{{\rm in},jk}(t) b_{{\rm in},k}^*(t)|0^{\otimes m}\rangle, \tag{127}$$

where N is an arbitrary positive integer. The input state $|\Psi_{\rm in}\rangle$ is parametrized by the pulse shapes $\psi_{{\rm in},jk}(t)$ ($j=1,\dots,N$ and $k=1,\dots m$). Clearly, different combinations of $\psi_{{\rm in},jk}(t)$ give rise to different m-channel N-photon states. By the notation in Eq. (81), the m-channel N-photon input state in Eq. (127) can be re-written as

$$|\Psi_{\rm in}\rangle = \prod_{j=1}^N \sum_{k=1}^m \mathbf{B}_{{\rm in},k}^*(\psi_{{\rm in},jk})|0^{\otimes m}\rangle. \tag{128}$$

Remark 6 *A class of photon-Gaussian states has been defined in [44, Eq. (95)]. If the density matrix ρ_R there used is of the form $\rho_R = |\phi 0^{\otimes m}\rangle\langle \phi 0^{\otimes m}|$, and moreover, $\xi_{jk}^+ \equiv 0$, then the resulting states are m-channel m-photon states. Actually, they form a special subclass of the m-channel N-photon states defined in Eq. (128) (with $N=m$).*

Remark 7 *Although the positive integer N in Eq. (127) is allowed to be arbitrary, the multi-photon input states defined in Eq. (127) may not be able to include those multi-photon states studied in Sections 3 and 4 as subclasses. This can be easily seen by comparing the forms of multi-photon states in Eqs. (42), (113), and (127).*

Remark 8 *Eq. (128) provides flexibility for specifying multi-channel multi-photon states.*

(i) *if for some j_0 ($1 \le j_0 \le N$) and k_0 ($1 \le k_0 \le m$), $\psi_{{\rm in},j_0k_0} \equiv 0$, then the term $\mathbf{B}_{{\rm in},k_0}^*(\psi_{{\rm in},j_0k_0})$ does not appear on the right-hand side of Eq. (128).*

(ii) *As a special case of item (i) above, if for some j_0 ($1 \le j_0 \le N$), $\psi_{{\rm in},j_0k} \equiv 0$ for all $k=1,\dots m$, then Eq. (128) reduces to*

$$|\Psi_{\rm in}\rangle = \prod_{j=1, j\neq j_0}^N \sum_{k=1}^m \mathbf{B}_{{\rm in},k}^*(\psi_{{\rm in},jk})|0^{\otimes m}\rangle.$$

In this case, there are $\mathbf{N-1}$ photons among m channels. Thus, the term "N-photon" is a bit confusing. Nevertheless, the exact number of photons can be determined easily from the context.

We illustrate Remark 8 with the following two Examples.

Example 4 *When $N=1$ and $m=2$, by Eq. (128), the input state is*

$$|\Psi_{\rm in}\rangle = \mathbf{B}_{{\rm in},1}^*(\psi_{{\rm in},11})|0_1\rangle \otimes |0_2\rangle + |0_1\rangle \otimes \mathbf{B}_{{\rm in},2}^*(\psi_{{\rm in},12})|0_2\rangle. \tag{129}$$

In what follows, we discuss two cases.

Case 1): $\psi_{{\rm in},11} \equiv 0$. In this case, as commented by item (i) in Remark 8, Eq. (129) becomes

$$|\Psi_{\rm in}\rangle = |0_1\rangle \otimes \mathbf{B}_{{\rm in},2}^*(\psi_{{\rm in},12})|0_2\rangle. \tag{130}$$

In this case, the first channel is in the vacuum state and the second channel is in a single-photon state.

Case 2): $\psi_{\text{in},11} = \psi_{\text{in},12} \equiv \xi$. Eq. (129) becomes

$$|\Psi_{\text{in}}\rangle = \boldsymbol{B}^*_{\text{in},1}(\xi)\,|0_1\rangle \otimes |0_2\rangle + |0_1\rangle \otimes \boldsymbol{B}^*_{\text{in},2}(\xi)\,|0_2\rangle\,. \quad (131)$$

The normalization condition requires that $\|\xi\| = \frac{1}{\sqrt{2}}$. Moreover, it can be readily shown that

$$\lim_{t_0\to-\infty, t\to\infty} \langle\Psi_{\text{in}}|\Lambda_{\text{in}}(t)|\Psi_{\text{in}}\rangle = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (132)$$

That is, the photon is not localized in either of the two channels; instead, it is shared by two channels. This reveals the wave property of photons.

Example 5 *Let $N = 2$ and $m = 3$. According to Eq. (128), the input state is*

$$|\Psi_{\text{in}}\rangle = \sum_{k=1}^{3} \mathbf{B}^*_{\text{in},k}(\psi_{\text{in},1k}) \sum_{k=1}^{3} \mathbf{B}^*_{\text{in},k}(\psi_{\text{in},2k})|0^{\otimes 3}\rangle. \quad (133)$$

If $\psi_{\text{in},11} = \psi_{\text{in},22} \equiv 0$, then, as commented by item (i) in Remark 8, the input state in Eq. (133) becomes

$$\begin{aligned} & |\Psi_{\text{in}}\rangle \\ = & \mathbf{B}^*_{\text{in},1}(\psi_{\text{in},21})|0_1\rangle \otimes \mathbf{B}^*_{\text{in},2}(\psi_{\text{in},12})|0_2\rangle \otimes |0_3\rangle \\ & + \mathbf{B}^*_{\text{in},1}(\psi_{\text{in},21})|0_1\rangle \otimes |0_2\rangle \otimes \mathbf{B}^*_{\text{in},3}(\psi_{\text{in},13})|0_3\rangle \\ & + |0_1\rangle \otimes \mathbf{B}^*_{\text{in},2}(\psi_{\text{in},12})|0_2\rangle \otimes \mathbf{B}^*_{\text{in},3}(\psi_{\text{in},23})|0_3\rangle \\ & + |0_1\rangle \otimes |0_2\rangle \otimes \mathbf{B}^*_{\text{in},3}(\psi_{\text{in},13})\mathbf{B}^*_{\text{in},3}(\psi_{\text{in},23})|0_3\rangle. \end{aligned} \quad (134)$$

That is, two photons are shared by three channels. If further $\psi_{\text{in},21} \equiv \psi_{\text{in},13} \equiv 0$, then, as commented by item (i) in Remark 8, Eq. (134) reduces to

$$|\Psi_{\text{in}}\rangle = |0_1\rangle \otimes \mathbf{B}^*_{\text{in},2}(\psi_{\text{in},12})|0_2\rangle \otimes \mathbf{B}^*_{\text{in},3}(\psi_{\text{in},23})|0_3\rangle. \quad (135)$$

In this case, the first channel is in the vacuum state, and there is exactly one photon in each of the second and third channels, respectively. Finally, if further $\psi_{\text{in},23}(t) \equiv 0$, the only existing pulse shape in Eq. (133) is $\psi_{\text{in},12}$, and therefore, as commented by item (ii) in Remark 8, we end up with a single photon state. Indeed, Eq. (128) reduces to $|\Psi_{\text{in}}\rangle = |0_1\rangle \otimes \mathbf{B}^*_{\text{in},2}(\psi_{\text{in},12})|0_2\rangle \otimes |0_3\rangle$. That is, the second channel has one photon while both the first and third channels are in the vacuum state.

5.2 State transfer

In this subsection, we derive an analytic form of the steady-state output field state of a passive quantum linear system driven by an m-channel N-photon input state defined in Eq. (127).

The following is the main result of this section.

Theorem 7 *Let the asymptotically stable passive quantum linear system (18) be initialized in the vacuum state and driven by the m-channel N-photon input $|\Psi_{\text{in}}\rangle$ defined in Eq. (127). The steady-state output field state is another m-channel N-photon state of the form*

$$|\Psi_{\text{out}}\rangle = \prod_{j=1}^{N} \sum_{l=1}^{m} \int_{-\infty}^{\infty} dt\ \psi_{\text{out},jl}(t) b^*_{\text{in},l}(t)|0^{\otimes m}\rangle, \quad (136)$$

where the output pulses are given by

$$\psi_{\text{out},jl}(t) \triangleq \sum_{k=1}^{m} \int_{-\infty}^{\infty} g^{lk}_{G^-}(t-r)\psi_{\text{in},jk}(r)dr \quad (137)$$

for $j = 1, \ldots, N,\ l = 1, \ldots, m$.

The proof is similar to that for Theorem 2; thus is omitted.

6 Conclusion

In this paper, we have studied the dynamics of quantum linear systems in response to multi-channel multi-photon states. We have derived the intensity of the output field which can be used to investigate the influence of quantum linear systems on quantum correlations of multi-photon light fields. We have also presented the explicit formula of the steady-state output field states when a quantum linear system is driven by three classes of multi-channel multi-photon input states. The results presented here are very general and hold promising applications in photon-based quantum coherent feedback networks. One of the future research directions is to study controller synthesis problem on the basis of the system analysis carried out in this paper, for example, via the Lyapunov method [20].

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