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SAA-Regularized Methods for Multiproduct Price Optimization under the Pure Characteristics Demand Model

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Abstract Utility-based choice models are often used to determine a consumer's purchase decision among a list of available products; to provide an estimate of product demands; and, when data on purchase decisions or market shares are available, to infer consumers' preferences over observed product characteristics. These models also serve as a building block in modeling firms' pricing and assortment optimization problems. We consider a firm's multiproduct pricing problem, in which product demands are determined by a pure characteristics model. A sample average approximation (SAA) method is used to approximate the expected market share of products and the firm profit. We propose an SAA-regularized method for the multiproduct price optimization problem. We present convergence analysis and numerical examples to show the efficiency and the effectiveness of the proposed method.

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1 Introduction

In economics, marketing, and operations management literature, utility-based choice models are often used to determine a consumer's purchase decision among a list of available products and to provide an estimate of product demands. Typically in a choice model, a consumer's purchase decision depends on observed product characteristics (such as brand, quality, and price) and the consumer's preference over these observed product characteristics. Once an appropriate choice model is specified, it can be used as a building block for estimating product demands and for modeling and analyzing a given firm's price and assortment decisions.

There has been an increasing interest in studying firms' price and assortment optimization problems under various choice models in the recent operations research and operations management literature; see for example [13, 17, 18, 22, 29] and the reference within. The choice models such as multinomial logit and nested logit, used in these papers typically include an i.i.d. logit error term, an idiosyncratic shock with a type-one extreme value (Gumbel) distribution. The motivations for including such an logit error term are to rationalize consumers' purchase decisions observed in the data and to maintain analytical tractability. However, one undesirable feature with the use of the logit term in a consumer's utility function in a choice model is that every product, regardless of its (low) quality and/or (high) price, admits a positive market share. Aiming to address this concern, Berry and Pakes [3] propose a pure characteristics demand model with two features: first, the logit error term is removed from a consumer's utility function; second, consumers have heterogeneous preferences.

The lack of a logit error term in a consumer's utility function in pure characteristics demand models leads to several interesting and challenging issues in formulating and analyzing the demand estimation problem as well as the product pricing problems. As pointed out in Berry and Pakes [3], solving the demand estimation problem based on pure characteristics demand models is computationally difficult. Pang et al [25] formulate the optimization problem of estimating a pure characteristics demand model as a mathematical program with equilibrium constraints (MPEC). Chen et. al. [8] propose a regularization method to compute a solution of the MPEC model and to prove the convergence of an MPEC solution under SAA to a solution of the original problem with equilibrium constraints.

In this paper, we consider firm F's multiproduct pricing problem in which product demands are determined based on a pure characteristics model. We propose an SAA-regularized method to approximate expected market shares of products and study the convergence of the SAA-regularized solutions when the regularization parameter goes to zero and sample size goes to infinity. Specifically, our main contribution is as follows.

- In [25], the multiproduct pricing problem based on the pure charateristics demand model has been formulated as a stochastic optimization problem with stochastic linear complementarity problem (LCP) constraints. But the problem is still difficult to analyze and solve. The main difficulty in finding a solution of the problem is that the objective function is defined by a special selection of solution functions from the solution set of the stochastic LCP, which does not contain a continuous solution function. Due to the discontinuity, we cannot utilize existing theory and algorithms with convergence analysis for solving the problem.
  In this paper, we propose an SAA-regularized method for the problem. Using the special structure of the problem, we show that the optimal value is given by a sparse solution function with a special order. Moreover, by the closed-form expression of the solution function of the regularized LCP, we prove the convergence of the sequence generated by the SAA-regularized method.

The rest of the paper is organized as follows. In Section 2, we introduce and present the formulation for firm F's price optimization problem under pure characteristics demand models. In Section 3, we introduce an SAA-regularized method to approximate the original price optimization problem and examine the convergence of solutions of the SAA-regularized problems to a solution of the original problem when the regularized parameter goes to zero and the sample size goes to infinity. In Section 4, we present numerical results with several examples to show the efficiency of our SAA-regularized method. We give conclusion remarks in Section 5.

2 The Multiproduct Pricing Problem

Assume that in a market, there are $K+J$ products indexed by $j = 1, \ldots, K+J$. The first K products ($j = 1, \ldots, K$) are produced by firm F and the remaining products ($j = K+1, \ldots, K+J$) are produced by other firms. Each product j is characterized by a vector of observed characteristics $x_j \in \mathbb{R}^\ell$ and price $p_j \in \mathbb{R}$.

There are $\mathcal{M}$ heterogenous consumers in the market. Each consumer is characterized by a (random) vector $\xi \in \mathbb{R}^{\ell+1}$, which represents a consumer's preference or tastes over the observed product characteristics x_j and price p_j.

The vector of random variables $\xi : \Omega \to \Xi \subset \mathbb{R}^{\ell+1}$ is defined in a probability space $(\Omega, \mathcal{F}, \mathscr{P})$.

In the pure characteristics demand model, a consumer with preference ξ and purchasing product j receives the utility $u_j(\xi, p_j)$ defined by the observed characteristics x_j and price p_j. In our study, $x := \{x_1, \ldots, x_{K+J}\}$ is given data. If a consumer decides not to purchase any product (no-purchase outside option), then her utility is 0. To simplify the notation, we define

$$u(\xi, p) = (u_1(\xi, p_1), \ldots, u_K(\xi, p_K), u_{K+1}(\xi, p_{K+1}), \ldots, u_{K+J}(\xi, p_{K+J}))^T, \tag{1}$$

and $p = (p_1, \ldots, p_K) \in \mathbb{R}^K$ for fixed $p_{K+1}, \ldots, p_{K+J}$. The reason that the vector p includes only prices of the K products produced by firm F will be clear later when we define firm F's profit maximization problem.

Among the $K + J$ products, a consumer chooses one product or the no-purchase outside option that gives her the highest utility. Given this decision rule, the purchase decision of a consumer with a taste vector ξ is a solution to the linear program

$$\begin{aligned} &\max_y \; y^T u(\xi, p) \\ &\text{s.t.} \quad y \geq 0, e^T y \leq 1, \end{aligned} \tag{2}$$

where $e \in \mathbb{R}^{K+J}$ is a column vector of all ones, and $u : \Xi \times \mathbb{R}^K \to \mathbb{R}^{K+J}$ is a vector of utilities received from purchasing each product.

Let c_j be the marginal cost of product j. Given the price vector p of firm F and prices of the J products produced by other firms, the total expected profit of firm F is

$$\mathcal{M} \sum_{j=1}^{K} \mathbb{E}[y_j(\xi, p)](p_j - c_j), \tag{3}$$

where $y_j(\xi, p)$ is the j-th element of a solution of the linear program (2), given ξ and p (and p_j, $j = K + 1, \ldots, K + J$). The term $\mathbb{E}[y_j(\xi, p)]$ is the expected market share of product j, where the expectation is taken over the distribution of ξ.

In the profit maximization problem for firm F, one potential issue is that problem (2) may have multiple solutions. Taking the possibility of multiple solutions into account, we denote by $S(\xi, p)$ the solution set mapping of the linear program (2) for given ξ and p. Since the number of consumers in a market, $\mathcal{M}$, is a constant, we can omit it from the profit function in (3) and the remainder of the paper. Let $y_{\mathcal{K}}(\xi, p) = (y_1(\xi, p), \ldots, y_K(\xi, p))^T$ and $c = (c_1, \ldots, c_K)^T$. Then the profit maximization problem for firm F is formulated as

$$\begin{aligned} &\max_{p \in P} \; f(p) := \mathbb{E}[y_{\mathcal{K}}(\xi, p)]^T (p - c) \\ &\text{s.t.} \quad y(\xi, p) \in S(\xi, p), \;\; \xi \in \Xi, \end{aligned} \tag{4}$$

where $P := \{p \in \mathbb{R}^K : \underline{p} \leq p \leq \bar{p}\}$ is a compact set and $\underline{p} < c$. Note that in the real world application, the price of products will not be infinity. We can choose a reasonable upper bound of price $\bar{p}$ based on experts' suggestions.

It is also easy to see from (2) and (4) that there always exists an optimal solution $p^* \geq c$. Therefore, we can assume that the feasible region for p is a compact set.

Note that the first-order necessary and sufficient optimality conditions of the linear problem (2) can be written as the following linear complementarity problem (LCP):

$$0 \leq \begin{pmatrix} y \\ \gamma \end{pmatrix} \perp M \begin{pmatrix} y \\ \gamma \end{pmatrix} + \begin{pmatrix} -u(\xi, p) \\ 1 \end{pmatrix} \geq 0 \tag{5}$$

with

$$M = \begin{pmatrix} 0 & e \\ -e^T & 0 \end{pmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}.$$

Thus, the solution set mapping $S(\xi, p)$ of the linear program (2) is the y part of the solution set mapping of the LCP (5).

To solve the price optimization problem (4), we need to first approximate the expected market share $\mathbb{E}[y_{\mathcal{K}}(\xi, p)]$ in the profit function $f(p)$. The SAA approach is often used for this purpose. The SAA version of the pricing problem (4) is

$$\begin{aligned} \max_{p \leq \bar{p}} \ & \frac{1}{N} \sum_{i=1}^{N} [y_{\mathcal{K}}(\xi^i, p)]^T (p - c) \\ \text{s.t.} \ \ & y(\xi^i, p) \in S(\xi^i, p), \ \ i = 1, \dots, N, \end{aligned} \tag{6}$$

where $\{\xi^1, \dots, \xi^N\}$ are i.i.d. samples of random variables ξ.

A natural question that arises is whether a sequence of solutions of the SAA problem (6) will converge to a solution of the original problem when the sample size increases. Providing such convergence analysis in the context of price optimization under pure characteristics demand models is challenging because the linear program (2) or the LCP (5) can admit multiple solutions. Moreover, we can not find a single-valued function $y(\xi, p) \in S(\xi, p)$ which is continuous with respect to p. Consider a simple example [8]: $u(\xi, p) = (\xi_1 + p, \xi_2) \in \mathbb{R}^2$, where $\xi_1 \in \mathbb{R}$ and $\xi_2 > 0$, the solution set has the form

$$S(\xi, p) = \begin{cases} (1, 0) & \text{if } p > \xi_2 - \xi_1, \\ \{(\alpha, 1-\alpha) \mid \alpha \in [0, 1]\} & \text{if } p = \xi_2 - \xi_1, \\ (0, 1) & \text{otherwise.} \end{cases}$$

As a result, the convergence of the SAA scheme (6) cannot be established directly by using results in existing literatures. In the next section, we introduce a regularized LCP method proposed by Chen et. al. [8] to study the convergence of a sequence of solutions of the SAA profit maximization problem when the sample size increases.

3 An SAA-Regularized Method for the Pricing Problem

In this section, we consider an SAA-regularized method for solving firm F's multiproduct pricing problem (4) and establish convergence analysis.

3.1 An SAA-regularized method

Chen et. al. [8] consider a generalized method of moments (GMM) estimation problem for estimating the parameters of the pure characteristics demand model [3,25]. In such an estimation problem, the consumers' purchased decisions and the observed market share data of products are characterized by the LCP (5) [8,25]. To overcome the difficulty that the LCP (5) can admit multiple solutions, they propose a regularized LCP method to approximate the LCP (5) as follows

$$0 \le \begin{pmatrix} y \\ \gamma \end{pmatrix} \perp M^\epsilon \begin{pmatrix} y \\ \gamma \end{pmatrix} + \begin{pmatrix} -u(\xi, p) \\ 1 \end{pmatrix} \ge 0, \tag{7}$$

where $M^\epsilon = M + \epsilon I$, with $\epsilon > 0$. The benefits of using the regularized LCP (7) are as follows

- (i) the regularized LCP has a unique solution $(y^\epsilon(\xi,p), \gamma^\epsilon(\xi,p))$ with a closed-form;
- (ii) the solution mapping $y^\epsilon(\xi,p)$ is globally Lipschitz continuous in price p when the utility function u is so.

In what follows, we apply the SAA-regularized method proposed in Chen et. al. [8] to firm F's multiproduct pricing problem (4). For a given p and every $\xi \in \Xi$, let $y^\epsilon_{\mathcal{K}}(\xi,p)$ be the first K components of the unique solution of the LCP (7).

We approximate the pricing problem (4) in two steps. The first step is to consider the pricing problem with the regularized LCP (7):

$$\max_{p \in P} f^\epsilon(p) := \mathbb{E}[y^\epsilon_{\mathcal{K}}(\xi,p)]^T (p-c). \tag{8}$$

The second step is to consider the SAA of the regularized pricing problem (8):

$$\max_{p \in P} f^\epsilon_N(p) := \frac{1}{N}\sum_{i=1}^{N} (y^\epsilon_{\mathcal{K}}(\xi^i,p))^T (p-c), \tag{9}$$

where N is the number of samples and $\{\xi^1,\ldots,\xi^N\}$ are the i.i.d. samples of ξ. For the three problems (4), (8) and (9), we denote the optimal solution sets by $\mathcal{P}^*, \mathcal{P}_\epsilon, \mathcal{P}_{\epsilon,N} \subset \mathbb{R}^K$ and the optimal values by ν^*, ν_ϵ, and $\nu_{\epsilon,N}$, respectively. We say $p^* \in \mathcal{P}^*$ is an optimal solution of (4) if $p^* \in P$ and there exists

$y(\xi, p^*) \in S(\xi, p^*)$, $\forall \xi \in \Xi$ such that $\mathbb{E}[y_{\mathcal{K}}(\xi, p^*)](p^* - c) = \nu^*$. Note that the feasible sets of these three problems are same.

Although we apply the SAA-regularized method proposed in [8] to the pricing problem (4), since the GMM estimation problem and problem (4) are different, the contributions on convergence analysis of the SAA-regularized methods are very different. We clarify the difference between GMM problem in [8] and the pricing problem (4) in the following remark.

Remark 1 In [8], the authors focus on the GMM estimation problem

$$\begin{aligned} &\min_{x \in X} x^T Q x + c^T x \\ &\text{s.t.} \;\; \mathbb{E}[S(\xi, x)] \ni b, \end{aligned} \tag{10}$$

where X is a convex set, Q is a positive definite matrix, c and b are vectors, and $S(\xi, x)$ is the y part of the solution set of the LCP (5) when the utility function $u(\xi, p)$ is replaced by the consumers' utility function with unknown parameter x and known price p. Problem (10) is used to estimate the parameter x of the pure characteristics demand model when the true market prices and market shares are observed.

In this paper, we consider problem (4) which is used to find optimal prices for maximizing firm F's profit.

It is obvious that the purposes and the structures (include both objective functions and constraints) of the two problems are very different.

One of key assumptions in the convergence analysis established in [8] is that at the optimal solution x^* of problem (10), the least norm solution of the LCP (5), $y(\xi, x^*)$ satisfies $\mathbb{E}[y(\xi, x^*)] = b$. Without this assumption, the convergence between original GMM estimation problem and its regularized problem may not hold.

However, for the optimal solution p^* of the pricing problem (4), the least norm solution $y(\xi, p^*)$ of the LCP (5) may not achieve the optimal value of (4). Hence, in this paper, we introduce a new concept of sparse solutions with some orders (Definitions 1 and 2), which can give the optimal value of (4) (Lemmas 3 and 5).

In the following subsections, we prove the convergence of the solution set $\mathcal{P}_{\epsilon,N}$ of problem (9) to the solution set $\mathcal{P}^*$ of problem (4) as $\epsilon \downarrow 0$ and $N \to \infty$.

To present the convergence analysis clearly, we first give some preliminaries, including assumptions and previous results in other literatures in subsection 3.2. Then we start our convergence analysis for a simple case when the pricing problem with one product ($K = 1$) in subsection 3.3. Using the results in subsection 3.3, we establish the convergence analysis for the multiproduct pricing problem ($K \geq 1$) in subsection 3.4.

In subsections 3.3 and 3.4, we first consider the convergence of the optimal solution set $\mathcal{P}_\epsilon$ of the regularized problem (8) to that of the pricing problem (4) when $\epsilon \downarrow 0$. Then we establish the convergence of the SAA-regularized solution set $\mathcal{P}_{\epsilon,N}$ of (9) to that of problem (8) when $N \to \infty$.

3.2 Convergence analysis of the SAA-regularized methods: preliminaries

In Assumption 1 below, we give conditions on the utility function for the convergence analysis. Note that these conditions are very mild for the utility function u.

Assumption 1 *The utility function $u : \Xi \times P \to \mathbb{R}^{K+J}$ has the following properties:*

1. *$u(\xi, p)$ is continuous w.r.t. $(\xi, p) \in \Xi \times P$. Moreover, for any fixed $\xi \in \Xi$, $u(\xi, \cdot)$ is Lipschitz continuous with a Lipschitz modulus $\kappa(\xi)$ and there exists an integrable function $L(\xi)$ such that $|\kappa(\xi)| \le L(\xi)$ for almost all $\xi \in \Xi$;*
2. *$u(\xi, p) := (u_1(\xi, p_1), \ldots, u_K(\xi, p_K), u_{K+1}(\xi), \ldots, u_{K+J}(\xi))$, where $u_j(\xi, p_j)$ is the utility for purchasing product j at price p_j. The function $u_j(\xi, p_j)$ is a strictly decreasing function w.r.t. p_j, for $j = 1, \ldots, K$ and almost all $\xi \in \Xi$.*
3. *$\|u(\xi, p)\|_1$ is integrable for every $p \in P$, that is $\mathbb{E}[\|u(\xi, p)\|] < \infty$ for every $p \in P$.*
4. *$u(\xi, \cdot)$ is continuously differentiable w.r.t. p. For any compact subset $\bar{\Xi} \subseteq \Xi$, there exists a constant $c > 0$ such that $\nabla_p u(\xi, \cdot) \le -ce$ for all $\xi \in \bar{\Xi}$, where $e \in \mathbb{R}^K$ with all components 1.*

By [8, Lemma 2.2], we have a closed-form formula for $y^\epsilon(\xi, p)$, the first $K + J$ components of the unique solution of the LCP (7). We state the results below for convenience.

Lemma 2.2 in [8]. *For any fixed (ξ, p), we have $u_{j_1}(\xi, p) \ge u_{j_2}(\xi, p) \ge \ldots \ge u_{j_{J+K}}(\xi, p)$, where $\{j_1, \cdots, j_{J+K}\}$ is a reordering of $\{1, \cdots, J + K\}$. Denote for $k = 1, \ldots, K + J$,*

$$\alpha_{j_k}(\xi, p, \epsilon) = \sum_{i=1}^{k} u_{j_i}(\xi, p) - (k + \epsilon^2) u_{j_k}(\xi, p) - \epsilon,$$

with $\mathcal{J}_\xi = \{k \,|\, \alpha_{j_k}(\xi, p, \epsilon) \le 0, k = 1, \ldots, J + K\}$, $\quad J_\xi = |\mathcal{J}_\xi|$, $\quad \sigma = -\sum_{i=1}^{J_\xi} u_{j_i}(\xi, p)$.

If $\|(u(\xi, p))_+\|_1 \ge \epsilon$, we have

$$y^\epsilon_{j_k}(\xi, p) = \begin{cases} \frac{\sigma + (J_\xi + \epsilon^2) u_{j_k}(\xi, p) + \epsilon}{J_\xi \epsilon + \epsilon^3} & \text{if } j \in \mathcal{J}_\xi, \\ 0 & \text{otherwise}, \end{cases} \tag{11}$$

otherwise,

$$y^\epsilon_{j_k}(\xi,p) = \begin{cases} u_{j_k}(\xi,p)/\epsilon & \text{if } u_{j_k}(\xi,p) > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

Moreover, if Assumption 1 holds, by [8, Remark 2.5], $y^\epsilon(\xi,p)$ is globally Lipschitz continuous w.r.t. p.

Lemma 1 *Suppose Assumption 1 holds. Then for any fixed $p \in P$ and $\epsilon \in (0,1]$, we have*

$$\|y^\epsilon(\xi,p)\|_\infty \le (n+1)(\|u(\xi,p)\|_1 + 1). \tag{13}$$

Moreover, for any sequence $\{(p^\epsilon,\epsilon)\} \to (p,0)$, $\epsilon_{k_0} \in (0,1]$ and $\epsilon \le \epsilon_{k_0}$ with any fixed $p \in P$, we have

$$\sup_{k\ge k_0} \|y^{\epsilon_k}(\xi,p^{\epsilon_k})\|_\infty \le (n+1)(\|u(\xi,p)\|_1 + \|p^{\epsilon_{k_0}} - p\|\kappa(\xi) + 1). \tag{14}$$

Proof For any fixed $p \in P$, $\epsilon_{k_0} > 0$ and $\epsilon \le \epsilon_{k_0}$, we consider three cases: 1. $\|(u(\xi,p))_+\|_1 = 0$; 2. $\|(u(\xi,p))_+\|_1 \le \epsilon$; 3. $\|(u(\xi,p))_+\|_1 \ge \epsilon$. By [8, Lemma 2.2], it is easy to see $y^\epsilon(\xi,p) = 0$ in case 1 and $0 \le y^\epsilon(\xi,p) \le e$ in case 2. Then we consider case 3. Since (11) and

$$\alpha_{j_{J_\xi}}(\xi,p,\epsilon) = \sum_{i=1}^{J_\xi} u_{j_i}(\xi,p) - (J_\xi + \epsilon^2)u_{j_{J_\xi}}(\xi,p) - \epsilon \le 0,$$

we have

$$\begin{aligned} \|y^\epsilon(\xi,p)\|_\infty &\le \left\| \frac{(J_\xi+\epsilon^2)u_{j_1}(\xi,p) - \sum_{i=1}^{J_\xi} u_{j_i}(\xi,p) + \epsilon}{J_\xi\epsilon + \epsilon^3} \right\| \\ &\le \left\| \frac{J_\xi(u_{j_1}(\xi,p) - u_{j_{J_\xi}}(\xi,p)) + \epsilon^2 u_{j_1}(\xi,p) + \epsilon}{J_\xi\epsilon + \epsilon^3} \right\| \end{aligned}$$

and

$$(u_{j_1}(\xi,p) - u_{j_{J_\xi}}(\xi,p)) \le \sum_{i=1}^{J_\xi} u_{j_i}(\xi,p) - J_\xi u_{j_{J_\xi}}(\xi,p) \le \epsilon^2 u_{j_{J_\xi}}(\xi,p) + \epsilon,$$

which imply

$$\begin{aligned} \|y^\epsilon(\xi,p)\|_\infty &\le \left\| \frac{J_\xi(\epsilon^2 u_{j_{J_\xi}}(\xi,p) + \epsilon) + \epsilon^2 u_{j_1}(\xi,p) + \epsilon}{J_\xi\epsilon + \epsilon^3} \right\| \\ &\le \|(n+1)(u_{j_1}(\xi,p)\epsilon + 1)\|. \end{aligned}$$

Then (13) holds in all three cases.

Moreover, consider any sequence $\{(p^\epsilon,\epsilon)\} \to (p,0)$, $\epsilon_{k_0} \in (0,1]$ and $\epsilon \le \epsilon_{k_0}$ with any fixed $p \in P$. By Assumption 1, $u(\xi,\cdot)$ is Lipschitz continuous with a Lipschitz modulus $\kappa(\xi)$ and there exists an integrable function $L(\xi)$ such that $|\kappa(\xi)| \le L(\xi)$ for almost all $\xi \in \Xi$ and $\|u(\xi,p)\|_1$ is integrable for any fixed $p \in P$. Then by (13), for any $k_0 > 0$ and $\epsilon_{k_0} < 1$, (14) holds. □

We establish the convergence of the optimal solution set $\mathcal{P}_\epsilon$ and the objective value ν_ϵ of the regularized problem (8) to those of the pricing problem (4) when $\epsilon \downarrow 0$.

The following lemma is based on the graphical convergence result proposed in [8, Theorem 3.2]. By this lemma, we can see that, only by graphical convergence, we may not get the deserved convergence results.

Lemma 2 *Suppose Assumption 1 holds. Then*

$$\nu^* \geq \limsup_{\epsilon\downarrow 0} \nu_\epsilon, \tag{15}$$

where ν^* and ν_ϵ are the optimal values of (4) and (8), respectively.

Proof By [8, Theorem 3.2], we have

$$y^\epsilon(\xi,\cdot) \xrightarrow{g} S(\xi,\cdot), \tag{16}$$

which implies

$$\limsup_{\epsilon\downarrow 0, p^\epsilon\to p} y^\epsilon(\xi,p^\epsilon) = S(\xi,p),$$

where the "$\xrightarrow{g}$" denotes graphical convergence, see [26, Definition 5.32]. Then for any sequence $\{(\epsilon_k, p^{\epsilon_k})\}$ such that $\epsilon_k \downarrow 0$ and $p^{\epsilon_k} \to p$ as $k\to\infty$,

$$\limsup_{k\to\infty} \mathbb{E}[y^{\epsilon_k}(\xi,p^{\epsilon_k})] \subseteq \mathbb{E}[\limsup_{k\to\infty} y^{\epsilon_k}(\xi,p^{\epsilon_k})] \subseteq \mathbb{E}[S(\xi,p)], \tag{17}$$

where the integral of the stochastic set-valued mapping $S(\xi,p)$ is defined in [2] and $\{p^{\epsilon_k}\}$ can be thought as a sequence of feasible solutions from regularized pricing problems. Note that the first $\subseteq$ in (17) is from (14) in Lemma 1 and [1, Theorem 8.6.7], and the second $\subseteq$ is from (16). That suggests

$$\text{h}-\liminf_{\epsilon_k\downarrow 0}(p^{\epsilon_k}-c)^T\mathbb{E}[y^{\epsilon_k}(\xi,p^{\epsilon_k})] \leq \max_{y\in\mathbb{E}[S(\xi,p)]}(p-c)^T y,$$

where h-lim inf denotes lower hypo-limits; see [26, Chapter 7]. Since the two problems (4) and (8) have the same feasible sets for p, we have

$$\limsup_{\epsilon_k\downarrow 0, p\in P} \mathbb{E}[y^{\epsilon_k}_{\mathcal{K}}(\xi,p)]^T(p-c) \leq \sup_{p\in P}\{s_{\mathcal{K}}^T(p-c) | s\in\mathbb{E}[S(\xi,p)]\},$$

where $s_{\mathcal{K}} = (s_1,\dots,s_K)^T$, and consequently, (15) holds. □

In general, since $\mathbb{E}[y^{\epsilon_k}(\xi,p^{\epsilon_k})]$ may not converge to $\mathbb{E}[S(\xi,p)]$ in a graphical sense, it is difficult to prove the convergence of the solution set $\mathcal{P}_\epsilon$ of problem (8) to that of problem (4) when $\epsilon\downarrow 0$ through the way above.

However, special structures of these two problems allow us to construct a specific sequence $\{p^{\epsilon_k}\}$ such that $\{\mathbb{E}[y^{\epsilon_k}(\xi,p^{\epsilon_k})\}$ converges to a specific solution in the set $\mathbb{E}[S(\xi,p)]$. In what follows, we will show how to use the special structure to construct such specific convergence sequence.

3.3 Convergence analysis of the SAA-regularized methods: single product pricing problem

In this subsection, we consider the pricing problem (4), regularized problem (8) and SAA-regularized problem (9) for firm F with one product case ($K = 1$).

We first show the special structure of the problem through a simple example.

Example 1 Consider problem (4), for each $p \in P$ such that $p > c$, we find a $y(\xi, p) \in S(\xi, p)$ such that

$$\begin{aligned} \mathbb{E}[y_1(\xi, p)](p - c) = \max_{y(\xi)} \; & f(p) := \mathbb{E}[y_1(\xi)](p - c) \\ \text{s.t.} \; & y(\xi) \in S(\xi, p), \;\; \forall \xi \in \Xi. \end{aligned} \tag{18}$$

Let $\Xi = \{\bar{\xi}\}$, $J = 2$. For a fixed $p \in P$, suppose that $u_1(p, \bar{\xi}) = u_2(\bar{\xi}) = u_3(\bar{\xi}) > 0$. Then by simple calculation,

$$S(\bar{\xi}, p) := \{(y_1, y_2, y_3) : y_1 + y_2 + y_3 = 1, y_i \geq 0, i = 1, 2, 3\}.$$

Since $p - c > 0$ only affects the first component of $y(\bar{\xi}, p)$, the sparse solution[1] *$(1, 0, 0) \in S(\bar{\xi}, p)$ gives the optimal value, but other sparse solutions do not.*

If $0 \leq u_1(p, \bar{\xi}) < u_2(\bar{\xi}) = u_3(\bar{\xi})$, then

$$S(\bar{\xi}, p) := \{(0, y_2, y_3) : y_2 + y_3 = 1, y_2, y_3 \geq 0\}.$$

Thus every $y \in S(\bar{\xi}, p)$ gives the optimal value.

Moreover, if $0 = u_1(p, \bar{\xi}) > u_2(\bar{\xi}) = u_3(\bar{\xi})$, then

$$S(\bar{\xi}, p) := \{(y_1, 0, 0) : y_1 \in [0, 1]\}.$$

Note that only $y = (1, 0, 0)$ gives the optimal value and it is not a sparse solution over the solution set $S(\bar{\xi}, p)$.

From Example 1, the selection of $y(\xi, p) \in S(\xi, p)$ depends on the $u_1(\xi, p)$.

Definition 1 For any fixed (ξ, p), we call $y(\xi, p)$ a sparse solution with respect to the order 1 (w.r.t. 1) in $S(\xi, p)$ if when $u_1(\xi, p) = \|(u(\xi, p))_+\|_\infty$, $y_1(\xi, p) = 1$, otherwise, $y_1(\xi, p) = 0$.

As we explained in Example 1, a sparse solution w.r.t. 1 in $S(\xi, p)$ may not be a sparse solution in $S(\xi, p)$.

[1] y^* is called a sparse solution in $S(\xi, p)$ if $\|y^*\|_0 = \min_{y \in S(\xi, p)} \|y\|_0$.

Lemma 3 *For any optimal solution $p^* \in \mathcal{P}^*$ of problem (4), let $y(\xi, p^*)$ be any sparse solution w.r.t.* 1 *in $S(\xi, p^*)$ for almost all $\xi \in \Xi$. Then*

$$v^* = \mathbb{E}[y_1(\xi, p^*)](p^* - c).$$

Proof Let v^* be the optimal value of (4). Since p^* is the optimal solution of (4), then there exists $y(\xi, p^*) \in S(\xi, p^*)$ such that

$$v^* = \mathbb{E}[y_1(\xi, p^*)](p^* - c).$$

Assume for a contradiction that $y(\xi, p^*)$ is not a sparse solution w.r.t. 1 in $S(\xi, p^*)$ for some ξ, then there exists a set $\bar{\Xi} \subset \Xi$ and a sparse solution w.r.t. 1, $\bar{y}(\xi, p^*) \in S(\xi, p^*)$ such that

$$y_1(\xi, p^*) < \bar{y}_1(\xi, p^*),\ \forall \xi \in \bar{\Xi},$$

$$y_1(\xi, p^*) \le \bar{y}_1(\xi, p^*),\ \forall \xi \notin \bar{\Xi}$$

and $\mu(\bar{\Xi}) > 0$. That means $v^* < \mathbb{E}[\bar{y}_1(\xi, p^*)](p^* - c)$, a contradiction. Then $y(\xi, p^*)$ is a sparse solution w.r.t. 1 in $S(\xi, p^*)$ for almost all $\xi \in \Xi$. Moreover, for any other sparse solution w.r.t. 1 in $S(\xi, p^*)$ for almost all $\xi \in \Xi$, $y'(\xi, p^*)$, it is easy to observe that if $y_1(\xi, p^*) = y'_1(\xi, p^*)$, then

$$v^* = \mathbb{E}[y_1(\xi, p^*)](p^* - c) = \mathbb{E}[y'_1(\xi, p^*)](p^* - c).$$

□

Lemma 3 shows that for an optimal solution $p^* \in \mathcal{P}^*$, any sparse solution w.r.t. 1 in $S(\xi, p^*)$, $y(\xi, p^*)$ has the property that $\mathbb{E}[y_1(\xi, p^*)](p^* - c) = v^*$. The following lemma shows that, for any p, there exists a special sequence $\{(\epsilon_k, p^{\epsilon_k})\} \to (0, p)$ such that $y^{\epsilon_k}(\xi, p^{\epsilon_k}) \to \bar{y}(\xi, p)$ for almost all $\xi \in \Xi$, where $\bar{y}(\xi, p)$ is a sparse solution w.r.t. 1 in $S(\xi, p)$.

Lemma 4 *Suppose Assumption 1 holds, then for any p^0 and sequence $\{\epsilon_k\} \downarrow 0$, there exists a sequence $\{p^{\epsilon_k}\} \uparrow p^0$ such that $\mathbb{E}[y(\xi, p^0)] = \lim_{k \to \infty} \mathbb{E}[y^{\epsilon_k}(\xi, p^{\epsilon_k})]$ and $y(\xi, p^0)$ is the sparse solution w.r.t.* 1 *in $S(\xi, p^0)$ for almost every $\xi \in \Xi$.*

Proof Since the random variable ξ is finitely dimensional, by [4, Theorem 1.4], for any $\tau \in (0, 1]$, we can find a compact set Ξ^τ such that $\mu(\Xi^\tau) \ge 1 - \tau$, where $\mu(\Xi^\tau) := \text{Prob}\{\xi \in \Xi^\tau\}$. Now, we consider the case when $\xi \in \Xi^\tau$ for a fixed $\tau \in (0, 1]$. Let

$$\Xi_1 := \{\xi \in \Xi^\tau, u_1(\xi, p^0) = \|(u(\xi, p^0))_+\|_\infty\}.$$

The proof is given by considering two cases: Case A1 is for $\xi \in \Xi_1$; Case A2 is for $\xi \notin \Xi_1$.

Case A1. In this case $S_1(\xi, p^0) = [0, 1]$. We prove that there is a sequence $\{(\epsilon_k, p^{\epsilon_k})\}$ such that $y_1^{\epsilon_k}(\xi, p^{\epsilon_k}) \to 1$ as $\{(\epsilon_k, p^{\epsilon_k})\} \to (0, p^0)$ for $\xi \in \Xi_1$. Let

$$\alpha_{j_2}(\xi, p, \epsilon) := u_1(\xi, p) - (1 + \epsilon^2) u_{j_2}(\xi) - \epsilon, \tag{19}$$

where $u_{j_2}(\xi, p^\epsilon) = \max_{i \geq 2}\{u_i(\xi)\}$.

By the compactness of Ξ_1, we can find a sufficiently small ϵ_0 such that $\max_{\xi \in \Xi_1} \|u(\xi, p^0)\| \leq \frac{1}{\epsilon_0}$. Moreover, by the continuity of $u(\cdot, \cdot)$ and Assumption 1, we can find the maximal $p^{\epsilon_0} \leq p^0$ such that

$$v_1(p^{\epsilon_0}) = \min_{\xi \in \Xi_1} u_1(\xi, p^{\epsilon_0}) \geq \epsilon_0$$

and

$$\phi(\epsilon_0, p^{\epsilon_0}) = \min_{\xi \in \Xi_1} \alpha_{j_2}(\xi, p^{\epsilon_0}, \epsilon_0) \geq 0.$$

Note that by [6, Proposition 4.4], Assumption 1 and the definition of α_{j_2}, v_1 is a continuous and strictly monotonically descending function w.r.t. p and ϕ is a continuous and monotonically descending function w.r.t. (ϵ, p).

Note that $\|(u(\xi, p^{\epsilon_0}))_+\|_1 \geq \epsilon_0$, by (11), we have

$$y^{\epsilon_0}(\xi, p^{\epsilon_0}) = \begin{cases} \frac{1 + \epsilon_0 u_{j_1}(\xi, p^{\epsilon_0})}{1 + \epsilon_0^2} & \text{if} \quad j = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{20}$$

Moreover $y^{\epsilon_0}(\xi, p)$ has the same formulation as (20) for all $p \leq p^{\epsilon_0}$.

Similarly, for any $0 < \epsilon'' < \epsilon' < \epsilon_0$, we can find maximal p'' and p' such that

$$v_1(p'') \geq \epsilon'' \quad \text{and} \quad \phi(\epsilon'', p'') \geq 0,$$
$$v_1(p') \geq \epsilon' \quad \text{and} \quad \phi(\epsilon', p') \geq 0,$$

and $p^0 \geq p'' \geq p' \geq p^{\epsilon_0}$. Moreover, we have $\phi(\epsilon'', p^0) \leq \phi(\epsilon'', p'')$.

Hence for any sequence $\{\epsilon_k\} \downarrow 0$, we can find a corresponding monotonically increasing sequence $\{p^{\epsilon_k}\} \uparrow p^0$ such that for $\xi \in \Xi_1$,

$$y_j^{\epsilon_k}(\xi, p^{\epsilon_k}) = \begin{cases} \frac{1 + \epsilon_k u_j(\xi, p^{\epsilon_k})}{1 + \epsilon_k^2} & \text{if } j = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

Note that $u_{j_1}(\xi, p^{\epsilon_k})$ decreases to $u_{j_1}(\xi, p^0)$ and $\epsilon_k \downarrow 0$ with $k \to \infty$,

$$y_1^{\epsilon_k}(\xi, p^{\epsilon_k}) \to y_1(\xi, p^0) = 1$$

and

$$y_j^{\epsilon_k}(\xi, p^{\epsilon_k}) \to 0, \quad j = 2, \ldots, J + 1.$$

Case A2. We consider Ξ^τ / Ξ_1. By [8, Theorem 3.2], $y_1^\epsilon(\xi, \cdot) \xrightarrow{g} S_1(\xi, \cdot)$ as $\epsilon \to 0$, where $S_1(\xi, \cdot)$ is the solution set mapping of the first component of the solutions of the LCP (5). Moreover, for $\xi \in \Xi^\tau / \Xi_1$, $S_1(\xi, p^0) = \{0\}$, which means $y_1^\epsilon(\xi,p^\epsilon)\to y_1(\xi,p^0)=0$ for any sequence $\{(\epsilon,p^\epsilon)\}\to(0,p^0)$ and all $\xi\in\Xi^\tau/\Xi_1$. Then for the sequence $\{(\epsilon_k,p^{\epsilon_k})\}$ defined in Case A1 and all $\xi\in\Xi^\tau/\Xi_1$, with $k\to\infty$,

$$y_1^{\epsilon_k}(\xi,p^{\epsilon_k})\to y_1(\xi,p^0)=0.$$

Moreover, since τ can be arbitrarily small, for almost all $\xi\in\Xi$, $y_1^{\epsilon_k}(\xi,p^{\epsilon_k})\to y_1(\xi,p^0)$ as $\epsilon_k\to 0$. □

Now we are ready to prove the convergence of the SAA-regularized problem (9). In the rest of the paper, for two sets $A,B\subseteq \mathbb{R}^K$, the deviation from A to B is denoted as

$$\mathbb{D}(A,B):=\sup_{a\in A}\inf_{b\in B}\|a-b\|_2. \tag{22}$$

Theorem 2 *Suppose the conditions of Lemma 4 hold, then* $\mathbb{D}(\mathcal{P}_\epsilon,\mathcal{P}^*)\to 0$ *as* $\epsilon\downarrow 0$.

Proof We consider problems (4) and (8). Let $\{\epsilon_k\}\to 0$ be any sequence, $p^{\epsilon_k}\in\mathcal{P}_{\epsilon_k}$ and $\{(\epsilon_k,p^{\epsilon_k},\mathbb{E}[y^{\epsilon_k}(\xi,p^{\epsilon_k})])\}\to(0,p^*,\mathbb{E}[y^*(\xi,p^*)])$ such that

$$f^{\epsilon_k}(p^{\epsilon_k})\to\mathbb{E}[y_1^*(\xi,p^*)](p^*-c).$$

It is sufficient to prove that $p^*\in\mathcal{P}^*$.

Assume for a contradiction that p^* is not an optimal solution of problem (4). By Lemma 3, there exist an optimal solution $\hat p$ of problem (4) with a corresponding sparse solution w.r.t. 1, $\hat y(\xi,\hat p)$ and $\delta>0$ such that $\hat p\geq c$ and

$$\mathbb{E}[\hat y_1(\xi,\hat p)](\hat p-c)-\mathbb{E}[y_1^*(\xi,p^*)](p^*-c)>\delta. \tag{23}$$

Then by Lemma 4, there exists $\{(\epsilon_k,\hat p_{\epsilon_k},\mathbb{E}[y^{\epsilon_k}(\xi,\hat p_{\epsilon_k})])\}\to(0,\hat p,\mathbb{E}[\hat y(\xi,\hat p)])$ such that $\hat p_{\epsilon_k}\in P$ and

$$f^{\epsilon_k}(\hat p^{\epsilon_k})\to\mathbb{E}[\hat y_1(\xi,\hat p)](\hat p-c). \tag{24}$$

Then we have

$$\begin{aligned}&\mathbb{E}[\hat y_1(\xi,\hat p)](\hat p-c)-\mathbb{E}[y_1^*(\xi,p^*)](p^*-c)\\ &=\mathbb{E}[\hat y_1(\xi,\hat p)](\hat p-c)-f^{\epsilon_k}(\hat p^{\epsilon_k})+f^{\epsilon_k}(\hat p^{\epsilon_k})-f^{\epsilon_k}(p^{\epsilon_k})\\ &+f^{\epsilon_k}(p^{\epsilon_k})-\mathbb{E}[y_1^*(\xi,p^*)](p^*-c).\end{aligned} \tag{25}$$

Moreover, it is obvious that

$$f^{\epsilon_k}(\hat p^{\epsilon_k})-f^{\epsilon_k}(\hat p^{\epsilon_k})\leq 0$$

and

$$f^{\epsilon_k}(p^{\epsilon_k})-\mathbb{E}[y_1^*(\xi,p^*)](p^*-c)\to 0.$$

Combining this with (24), we have

$$\mathbb{E}[\hat y_1(\xi,\hat p)](\hat p-c)-\mathbb{E}[y_1^*(\xi,p^*)](p^*-c)\leq 0,$$

which contradicts (23). □

Now we consider problems (8) and (9).

Theorem 3 *Suppose the random sample $\{\xi^1, \ldots, \xi^N\}$ of ξ is i.i.d. and Assumption 1 holds. Then for any $\epsilon > 0$, $\nu_{\epsilon,N} \to \nu_\epsilon$ and $\mathbb{D}(\mathcal{P}_{\epsilon,N}, \mathcal{P}_\epsilon) \to 0$ w.p.1 as $N \to \infty$.*

Proof Note that the feasible set of problems (8) and (9) are same. Moreover, by the compactness of P, Assumption 1, Lemma 1 and the closed-form expression of $y^\epsilon(\xi, p)$, we have

1. the function $y^\epsilon(\xi, p)$ is Lipschitz continuous with a finite Lipschitz modulus $\kappa_\epsilon(\xi)$ and there exists an integrable function $L(\xi)$ such that $\kappa_\epsilon(\xi) \le L(\xi)$ for almost $\xi \in \Xi$;
2. for all $p \in P$, there exists an integrable function $L_p(\xi)$ such that $\|y^\epsilon(\xi, p)\| \le L_p(\xi)$ for almost $\xi \in \Xi$ and $\mathbb{E}[L_p(\xi)] < \infty$.

Then by [30, Chapter 6, Proposition 7], $f_N^\epsilon \to f^\epsilon$ w.p.1 uniformly on P. Moreover, since $\mathcal{P}$ is compact and f_N^ϵ and f^ϵ are Lipschitz continuous with finite Lipschitz moduli, f_N^ϵ and f^ϵ are finite on feasible set P, and solution sets $\mathcal{P}_\epsilon, \mathcal{P}_{\epsilon,N}$ are nonempty. Then by [30, Chapter 6, Proposition 6], we have

$$\nu_{\epsilon,N} \to \nu_\epsilon \ \text{ and } \ \mathbb{D}(\mathcal{P}_{\epsilon,N}, \mathcal{P}_\epsilon) \to 0 \ \ w.p.1 \ \ as\ N \to \infty.$$

□

Now we are ready to prove the convergence of the solution set $\mathcal{P}_{\epsilon,N}$ of problem (9) as $N \to \infty$ and $\epsilon \downarrow 0$.

Theorem 4 *Suppose the conditions of Theorem 2 and Theorem 3 hold. Then*

$$\lim_{\epsilon \downarrow 0} \lim_{N \to \infty} \mathbb{D}(\mathcal{P}_{\epsilon,N}, \mathcal{P}^*) = 0 \quad \text{w.p.1.} \tag{26}$$

Proof The assertion follows directly from Theorem 2 and Theorem 3. □

3.4 Convergence analysis of the SAA-regularized methods: multiproduct pricing problem

Now we consider the general case when $K \ge 1$.

Definition 2 For any fixed $\xi \in \Xi$ and $p \in P$, $y(\xi, p)$ is called a sparse solution with respect to the order $\{l_1, \ldots, l_t\}$ (w.r.t. $\{l_1, \ldots, l_t\}$) in $S(\xi, p)$, $t \le J + K$, if $y_{l_i}(\xi, p) = 1$, with i being the smallest number in $\{s | u_{l_s}(\xi, p) = \|(u(\xi, p))_+\|_\infty, l_s \in \{l_1, \cdots, l_t\}\}$.

Remark 2 By simple calculation, it is easy to observe that, for any fixed (ξ, p), the solution set of y part of the LCP (5) is

$$S(\xi,p)=\left\{ y=(y_1,\cdots,y_{J+K}) : \begin{array}{l} y_j \geq 0, j=1,\cdots,J+K \\ \sum_{j\in\mathcal{J}} y_j = 1,\ if\ \|u(\xi,p)\|_\infty > 0 \\ \sum_{j\in\mathcal{J}} y_j \leq 1,\ if\ \|u(\xi,p)\|_\infty = 0 \\ y_j = 0,\ \forall j \notin \mathcal{J} \end{array} \right\}, \quad (27)$$

where

$$\mathcal{J} := \{ j : u_j(\xi,p) = \|u(\xi,p)\|_\infty,\ j \in \{1,\cdots,J+K\}\}.$$

For fixed (ξ, p), consider a simple example of the LCP (5) that $K + J = 5$, $u(\xi, p) = (5, 4, 3, 2, 5)$. By (27), $S(\xi, p) := \{(y_1, 0, 0, 0, y_5) : y_1 + y_5 = 1, y_1, y_5 \geq 0\}$. Then we consider sparse solutions with some orders in (27).

$S(\xi, p)$ has two sparse solutions: $(1, 0, 0, 0, 0)$ and $(0, 0, 0, 0, 1)$. Note that $(1, 0, 0, 0, 0)$ is the sparse solution with the order such that 1 is higher than 5 and $(0, 0, 0, 0, 1)$ is the sparse solution with the order such that 5 is higher than 1.

Moreover, if the order doesn't include 1 and 5, then every solution is a sparse solution w.r.t. the order.

Lemma 5 *For any optimal solution $p^* \in \mathcal{P}^*$ of problem (4), let $y^*(\xi, p^*)$ be any sparse solution w.r.t. $\{l_1, \ldots, l_K\}$ in $S(\xi, p^*)$ for almost all $\xi \in \Xi$, where $\{l_1, \ldots, l_K\}$ is the order such that $(p^* - c)_{l_1} \geq \ldots \geq (p^* - c)_{l_K}$. Then*

$$v^* = \mathbb{E}[y^*_{\mathcal{K}}(\xi, p^*)]^T (p^* - c).$$

Proof Let v^* be the optimal value of problem (4). Then there exists $y^*(\xi, p^*)$ such that

$$v^* = \mathbb{E}[y^*_{\mathcal{K}}(\xi, p^*)]^T (p^* - c).$$

Assume that $y^*(\xi, p^*)$ is not a sparse solution w.r.t. $\{l_1, \ldots, l_K\}$ in $S(\xi, p^*)$ for almost all $\xi \in \Xi$. Let $y(\xi, p^*)$ be any sparse solution w.r.t. $\{l_1, \ldots, l_K\}$ in $S(\xi, p^*)$ for almost all $\xi \in \Xi$. Since $(p^* - c)_{l_1} \geq \ldots \geq (p^* - c)_{l_K} \geq 0$,

$$\sum_{i=1}^{K} y_{l_i}(\xi, p^*)^T (p^* - c)_{l_i} \geq \sum_{i=1}^{K} y^*_{l_i}(\xi, p^*)^T (p^* - c)_{l_i},$$

and equation holds only in the case when there are some ties in the chain $(p^* - c)_{l_1} \geq \ldots \geq (p^* - c)_{l_K}$. Thus, we have

$$v^* = \mathbb{E}[y^*_{\mathcal{K}}(\xi, p^*)]^T (p^* - c) \leq \mathbb{E}[y_{\mathcal{K}}(\xi, p^*)]^T (p^* - c),$$

which implies $\mathbb{E}[y_{\mathcal{K}}(\xi, p^*)]^T (p^* - c) = v^*$. Moreover, for any other sparse solution w.r.t. $\{l_1, \ldots, l_K\}$ in $S(\xi, p^*)$ for almost all $\xi \in \Xi$, $y(\xi, p^*)$, it is clear that $y^*_{\mathcal{K}}(\xi, p^*) = y_{\mathcal{K}}(\xi, p^*)$, then

$$v^* = \mathbb{E}[y^*_{\mathcal{K}}(\xi, p^*)]^T (p^* - c) = \mathbb{E}[y_{\mathcal{K}}(\xi, p^*)]^T (p^* - c).$$

□

Lemma 6 *Suppose Assumption 1 holds. Then for any order* $\{l_1, \ldots, l_K\}$, $p^0 \in \mathbb{R}^K_+$ *and sequence* $\{\epsilon_k\} \downarrow 0$, *there exists a sequence* $\{p^{\epsilon_k}\} \to p^0$ *such that* $\mathbb{E}[y(\xi, p^0)] = \lim_{k\to\infty} \mathbb{E}[y^{\epsilon_k}(\xi, p^{\epsilon_k})]$ *and* $y(\xi, p^0)$ *is a sparse solution w.r.t.* $\{l_1, \ldots, l_K\}$ *in* $S(\xi, p^0)$ *for almost every* $\xi \in \Xi$.

Proof Since the vector of random variables ξ is finitely dimensional, similar to the proof for Lemma 4 and by [4, Theorem 1.4], we can find a compact set Ξ^τ such that $\mu(\Xi^\tau) \geq 1 - \tau$ for any $\tau \in (0, 1]$. Thus, it suffices to consider the case when $\xi \in \Xi^\tau$.

Without loss of generality, we consider $\{l_1, \ldots, l_K\} = \{1, \ldots, K\}$. Note that if $y(\xi, p^0)$ is a sparse solution w.r.t. $\{1, \ldots, K\}$, then $y(\xi, p^0)$ is a sparse solution w.r.t. $\{1, \ldots, t\}$ for all integer $t \in [1, K]$.

For every integer $t \in [1, K]$, we define

$$\Xi^1_t := \left\{\xi \in \Xi^\tau, u_t(\xi, p^0) = \|(u(\xi, p^0))_+\|_\infty\right\},$$

and

$$\bar{\Xi}^1_t := \left\{\xi \in \Xi^\tau, u_t(\xi, p^0) = \|(u(\xi, p^0))_+\|_\infty = \max_{1\leq s\leq t-1}\{u_s(\xi, p^0)\}\right\}.$$

We will show that we can find $\{p^{\epsilon_k}\}$ such that for $\xi \in \Xi^1_s / \bar{\Xi}^1_s$,

$$\lim_{k\to\infty} y^{\epsilon_k}_s(\xi, p^{\epsilon_k}) = 1, \; s = 1, \ldots, K, \tag{28}$$

and for $\xi \in \Xi_\tau / \Xi^1_s$,

$$\lim_{k\to\infty} y^{\epsilon_k}_s(\xi, p^{\epsilon_k}) = 0, \; s = 1, \ldots, K. \tag{29}$$

In the following proof, we will construct K sequences $\{p^{\epsilon_k, K}\}, \ldots, \{p^{\epsilon_k, 1}\}$ with $\{p^{\epsilon_k, 1}\} = \{p^{\epsilon_k}\}$ that satisfy

$$p^{\epsilon_k, t}_i = p^{\epsilon_k, i}_i, \;\; i = t+1, \ldots, K,$$
$$p^{\epsilon_k, t}_i = p^0_i, \;\; i = 1, \ldots, t-1,$$

and $\{p^{\epsilon_k, t}_t\} \uparrow p^0_t$ as $\epsilon \to 0$ for all $t = 1, \ldots, K$. Then similar to the proof of Lemma 4, there exists a monotonically increasing sequence $\{p^{\epsilon_k, K}\}$ such that for all $\xi \in \Xi^1_K$,

$$\lim_{k\to\infty} y^{\epsilon_k}_K(\xi, p^{\epsilon_k, K}) = 1; \tag{30}$$

for all $\xi \in \Xi^\tau / \Xi^1_K$,

$$\lim_{k\to\infty} y^{\epsilon_k}_K(\xi, p^{\epsilon_k, K}) = 0, \tag{31}$$

and $\{p^{\epsilon_k, K}_K\} \uparrow p^0_K$ and $p^{\epsilon_k, K}_j = p^0_j$ for $j = 1, \ldots, K-1$.

The rest of the proof is given by backward induction.

Assume that for $s = t+1, \ldots, K$ and $\{\epsilon_k\} \to 0$, we can find $\{p^{\epsilon_k, t+1}\}$ such that for $\xi \in \Xi^1_s / \bar{\Xi}^1_s$, (28) holds; for $\xi \in \Xi_\tau / \Xi^1_s$, (29) holds, where $\{p^{\epsilon_k, t+1}\}$ is a monotonically increasing sequence such that $\{p^{\epsilon_k,t+1}\} \uparrow p^0$; and for $\xi \in \Xi_s^1 \cap \bar{\Xi}_i^1$ and $i = s+1, \ldots, K$

$$y_i^{\epsilon_k}(\xi, p^{\epsilon_k,t+1}) \to 0 \ \text{ as } k \to \infty.$$

We will prove there exists a sequence $\{p^{\epsilon_k,t}\}$ satisfying (28) and (29) for $s = t, \cdots, K$.

We consider three cases: $\xi \in \Xi_t^1$ in Case 1, $\xi \in \Xi^\tau/\Xi_t^1$ in Case 2, and $\xi \in \Xi_t^1 \cap \bar{\Xi}_s^1$ for $s = t+1, \ldots, K$ in Case 3.

Case 1: $\xi \in \Xi_t^1$. By the compactness of Ξ^τ, Ξ_t^1 is a compact set. Similar with the proof of Case A1 of Lemma 4, for a sequence $\{\epsilon_k\}$ defined in Case A1, we can find the maximal sequence $\{p^{\epsilon_k}\}$ such that

$$v_t(p^{\epsilon_k}) = \min_{\xi \in \Xi_t^1} u_t(\xi, p^{\epsilon_k}) \geq \epsilon_k, \quad \phi(\epsilon_k, p^{\epsilon_k}) = \min_{\xi \in \Xi_t^1} \alpha_2(\xi, p^{\epsilon_k}, \epsilon_k) \geq 0,$$

$$(p^0)_t \geq (p^{\epsilon_{k+1}})_t \geq (p^{\epsilon_k})_t, \quad p_j^{\epsilon_k} = p_j^{\epsilon_k,t+1}, \ j = t+1, \ldots, K,$$

and

$$p_j^{\epsilon_k} = p_j^0, j \neq t, \ldots, K.$$

Let $p^{\epsilon_k,t} = p^{\epsilon_k}$. We have the monotonically increasing sequence $\{p^{\epsilon_k,t}\} \uparrow p^0$ such that for $\xi \in \Xi_1^t$,

$$y^{\epsilon_k}(\xi, p^{\epsilon_k,t}) = \begin{cases} \frac{1+\epsilon_k u_j(\xi, p^{\epsilon_k,t})}{1+\epsilon_k^2} & \text{if} \quad j = t, \\ 0 & \text{otherwise,} \end{cases} \tag{32}$$

which means

$$\lim_{k \to \infty} y_t^{\epsilon_k}(\xi, p^{\epsilon_k,t}) = 1. \tag{33}$$

Note that $u_t(\xi, p^{\epsilon_k})$ decreases and $\epsilon_k \to 0$ with $k \to \infty$, $y_t^{\epsilon_k}(\xi, p^{\epsilon_k}) \to 1$ and $y_j^{\epsilon_k}(\xi, p^{\epsilon_k}) \to 0$ for $j \neq t$. Note also that for $i = t+1, \ldots, K$ and $\xi \in \Xi_t^1 \cap \bar{\Xi}_i^1$,

$$y_i^{\epsilon_k}(\xi, p^{\epsilon_k,t}) \to 0 \ \text{ as } k \to \infty.$$

Case 2: $\xi \in \Xi^\tau/\Xi_t^1$. Similar to the proof of Case A2 in Lemma 4, we can show that for the sequence $\{(\epsilon_k, p^{\epsilon_k,t})\}$ defined in Case 1 and all $\xi \in \Xi^\tau/\Xi_t^1$, with $k \to \infty$,

$$y_t^{\epsilon_k}(\xi, p^{\epsilon_k,t}) \to y_t(\xi, p^0) = 0. \tag{34}$$

Case 3: In this case, we consider all $\xi \in \Xi_s^1/\bar{\Xi}_s^1$ in the sequence $\{p^{\epsilon_k,t}\}$ defined above for $s = t+1, \ldots, K$. Note that $S_i(\xi, p^0) = \{0\}$ for all $\xi \in \Xi_s^1/\bar{\Xi}_s^1$ and $i = 1, \ldots, s-1$. We have $y_i^{\epsilon_k}(\xi, p^{\epsilon_k,t}) \to 0$ as $k \to \infty$ for $i = 1, \ldots, s-1$.

Since $p_l^{\epsilon_k,t} = p_l^{\epsilon_k,s}$, $l = 1, \ldots, t-1, s, \ldots, K$, for $i = s+1, \ldots, K, K+1, \ldots, K+J$ and $\xi \in \Xi_s^1/\bar{\Xi}_s^1$, it is clear that

$$\alpha_i(\xi, p^{\epsilon_k,t}, \epsilon_k) \geq \alpha_i(\xi, p^{\epsilon_k,s}, \epsilon_k) \geq 0,$$

$\|u(\xi, p^{\epsilon_k,t})\| \geq \epsilon_k$ and $\sum_1^{K+J} y_i^{\epsilon_k}(\xi, p^{\epsilon_k,t}) \to 1$ as $k \to \infty$. Then $y_i^{\epsilon_k}(\xi, p^{\epsilon_k,t}) \to 0$ as $k \to \infty$ for $i = s+1, \ldots, K, K+1, \ldots, K+J$ and $y_s^{\epsilon_k}(\xi, p^{\epsilon_k,t}) \to 1$ as $k \to \infty$.

Thus, for all integers $t \in [1, K]$, we have proved that there exists a sequence $\{p^{\epsilon_k,1}\} \to p^0$ with $k \to \infty$ such that for $\xi \in \Xi_t^1/\bar{\Xi}_t^1$,

$$\lim_{k\to\infty} y_t^{\epsilon_k}(\xi, p^{\epsilon_k,1}) = 1, \tag{35}$$

and for $\xi \in \Xi_\tau/\Xi_t^1$,

$$y_t^{\epsilon_k}(\xi, p^{\epsilon_k,1}) = 0. \tag{36}$$

Moreover, since $\bar{\Xi}_t^1 \subset \bigcup_{s=1}^{t-1}(\Xi_s^1/\bar{\Xi}_s^1)$ and $\bar{\Xi}_1^1 = \emptyset$, for $\xi \in \bar{\Xi}_t^1$,

$$\lim_{k\to\infty} y_t^{\epsilon_k}(\xi, p^{\epsilon_k,1}) = 0. \tag{37}$$

From (35)–(37), there exists a $y(\xi, p^0)$ such that

$$\lim_{k\to\infty} \mathbb{E}[y^{\epsilon_k}(\xi, p^{\epsilon_k})] = \mathbb{E}[y(\xi, p^0)],$$

where $\{p^{\epsilon_k}\} = \{p^{\epsilon_k,1}\}$ and $y(\xi, p^0)$ is a sparse solution w.r.t. $\{l_1, \ldots, l_K\}$ in $S(\xi, p^0)$ for almost every $\xi \in \Xi$. □

Now we are ready to prove the convergence of the SAA-regularized problem (9) to the original pricing problem (4).

Theorem 5 *Suppose the conditions of Lemma 6 hold, then* $\mathbb{D}(\mathcal{P}_\epsilon, \mathcal{P}^*) \to 0$ *as* $\epsilon \downarrow 0$.

By Lemma 6, the proof of Theorem 5 is similar to the proof of Theorem 2, we omit the details.

Next, we consider problems (8) and (9).

Theorem 6 *Suppose* Ξ *is a compact set, the random sample* $\{\xi^1, \ldots, \xi^N\}$ *of* ξ *is i.i.d. and Assumption 1 holds. Then for any* $\epsilon > 0$, $\nu_{\epsilon,N} \to \nu_\epsilon$ *and* $\mathbb{D}(\mathcal{P}_{\epsilon,N}, \mathcal{P}_\epsilon) \to 0$ *w.p.1 as* $N \to \infty$.

The proof of Theorem 6 is similar to the proof of Theorem 3, and hence, we omit the details.

Now we are ready to state the convergence of the solution set $\mathcal{P}_{\epsilon,N}$ of the SAA-regularized problem (9) to that of the multiproduct pricing problem (4) when $N \to \infty$ and $\epsilon \downarrow 0$.

Theorem 7 *Suppose the conditions of Theorem 5 and Theorem 6 hold. Then*

$$\lim_{\epsilon\downarrow 0} \lim_{N\to\infty} \mathbb{D}(\mathcal{P}_{\epsilon,N}, \mathcal{P}^*) = 0 \quad \text{w.p.1.} \tag{38}$$

The assertion follows directly from Theorem 5 and Theorem 6.

Up to now, convergence of the SAA methods for the multiproduct pricing problem (4) has not been established due to the difficulties in dealing with the expected values of the set-valued solution mapping in the problem. In this paper, for the first time in literatures, we establish the convergence of the SAA methods for the multiproduct pricing problem (4) via the regularization approach regarding global solutions. It is worth noting that problem (4) and the SAA-regularized problem (9) are nonconvex and nonsmooth. We believe that the convergence regarding stationary points via the regularization approach is promising and interesting, and list it as our further research work.

4 Numerical Examples

In this section, we consider several examples of the price optimization problem (4) and their corresponding SAA-regularized problem (9). We use the Matlab package `HANSO` [7,21] to solve the SAA-regularized problem

$$\max_p f_N^\epsilon(p) := \frac{1}{N}\sum_{i=1}^N (y_{\mathcal{K}}^\epsilon(\xi^i, p))^T(p-c), \tag{39}$$

where the closed-form formula of $y^\epsilon(\xi^i, p)$ is derived in (11) and (12) and $y_{\mathcal{K}}^\epsilon(\xi, p)$ denotes the components of $y^\epsilon(\xi, p)$ associated with firm F. The numerical analysis with `HANSO` was executed in MATLAB 8.0 on a IBM Notebook PC with Windows 7 operating system, and Intel Core i5 processor. For Example 2, we also compare the performance of `HANSO` for solving problem (39) and `SNOPT` [19] coded in `AMPL` [16] for solving SAA problem with stochastic regularized LCP constraints:

$$\begin{aligned} &\max_{p, y(\xi^i), \gamma(\xi^i)} \frac{1}{N}\sum_{i=1}^N (y_{\mathcal{K}}(\xi^i))^T(p-c) \\ &\text{s.t.} \qquad \text{for } i = 1, \dots, N: \\ &\qquad 0 \le \begin{pmatrix} y(\xi^i) \\ \gamma(\xi^i) \end{pmatrix} \perp M^\epsilon \begin{pmatrix} y(\xi^i) \\ \gamma(\xi^i) \end{pmatrix} + \begin{pmatrix} -u(\xi^i, p) \\ 1 \end{pmatrix} \ge 0. \end{aligned} \tag{40}$$

Example 2 (with discrete distribution for ξ). In this example, there are total three products with $K = 1$ and $J = 2$. Hence, firm F decides the price for one product (the first product). The utility of a consumer with preference $\xi = (\xi_1, \xi_2, \xi_3)$ for purchasing product j (= 1, 2, 3) is

$$u_j(\xi, p_j) = \xi_1 + \xi_2 x_j - \xi_3 p_j, \tag{41}$$

where x_j and p_j are product characteristic and price for product j. We assume that the random vector for consumers' preference $\xi = (\xi_1, \xi_2, \xi_3)$ has a discrete distribution of three preference types specified below

$$\xi = \begin{cases} (\xi_1^1, \xi_2^1, \xi_3^1) = (3,3,1), \text{ with probability } \frac{1}{4}; \\ (\xi_1^2, \xi_2^2, \xi_3^2) = (2,2,1), \text{ with probability } \frac{1}{2}; \\ (\xi_1^3, \xi_2^3, \xi_3^3) = (1,1,2), \text{ with probability } \frac{1}{4}. \end{cases}$$

Let $x_1 = 5$ and $c_1 = 5$ be the observed product characteristic and marginal cost, respectively, for firm F's product; firm F needs to decide the price p_1. The observed characteristic and observed price for the other two products are $(x_2, p_2) = (3, 3)$ and $(x_3, p_3) = (1, 0.5)$.

Analysis of Example 2. With a slight abuse of the notation, we denote by $U(p_1)$ a matrix with the component $U_{ij}(p_1)$ being the utility of a type-i consumer purchasing product j when firm F's product is priced at p_1; that is,

$$U_{ij}(p_1) = u_j(\xi^i, p_j), \qquad \text{for } i, j = 1, \ldots, 3,$$

where $u_j(\xi^i, p_j)$ is specified in (40). With $x = (x_1, x_2, x_3) = (5, 3, 1)$ and $(p_2, p_3) = (3, 0.5)$, we have

$$U(p_1) = \begin{pmatrix} u_1(\xi^1, p_1) & 9 & 5.5 \\ u_1(\xi^2, p_1) & 5 & 3.5 \\ u_1(\xi^3, p_1) & -2 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} y^1(p_1) \\ y^2(p_1) \\ y^3(p_1) \end{pmatrix} = \begin{pmatrix} \mathbb{1}[u_1(\xi^1, p_1) \geq \max\{9, 5.5\}] \\ \mathbb{1}[u_1(\xi^2, p_1) \geq \max\{5, 3.5\}] \\ \mathbb{1}[u_1(\xi^3, p_1) \geq \max\{-2, 1\}] \end{pmatrix}.$$

Then firm F's profit maximization problem becomes

$$\max_{p_1} \left[\frac{1}{4} y^1(p_1) + \frac{1}{2} y^2(p_1) + \frac{1}{4} y^3(p_1) \right] (p_1 - c_1). \tag{42}$$

Moreover, $u_1(\xi^i, p_1)$ is strictly monotonically decreasing w.r.t. p_1, and hence, $y^i(p_1)$ is monotonically decreasing w.r.t. p_1. It is easy to derive that

$$y^1(p_1) = \begin{cases} 1, & p_1 \leq 9; \\ 0, & p_1 > 9; \end{cases} \quad y^2(p_1) = \begin{cases} 1, & p_1 \leq 7; \\ 0, & p_1 > 7; \end{cases} \quad \text{and } y^3(p_1) = \begin{cases} 1, & p_1 \leq 5; \\ 0, & p_1 > 5. \end{cases}$$

By the analysis above, one can show that for the profit maximization problem (42), $p_1^* = 7$ is the global optimal solution with optimal profit 1.5 and $\hat{p}_1 = 9$ is a local optimal solution with profit 1.

Since p_2 and p_3 are fixed in this example, the formulation of regularized problem (40) for Example 2 is:

$$\begin{aligned}
\max_{p_1, y_{ij}, \gamma_i} \ & (\tfrac{1}{4}y_{11} + \tfrac{1}{2}y_{21} + \tfrac{1}{4}y_{31})(p_1 - c_1) \\
\text{s.t.} \ & \text{for } i = 1, \ldots, 3, \ j = 1, \ldots, 3: \\
& \gamma_i + \epsilon y_{i1} - (\xi_1^i + \xi_2^i x_1 - \xi_3^i p_1) \geq 0 \\
& 1 - \textstyle\sum_{j=1}^3 y_{ij} + \epsilon \gamma_i \geq 0 \\
& \gamma_i - \max\{(\xi_1^i + \xi_2^i x_2 - \xi_3^i p_2) - \epsilon y_{i2}, (\xi_1^i + \xi_2^i x_3 - \xi_3^i p_3) - \epsilon y_{i3}, 0\} \geq 0 \\
& y_{ij}(\gamma_i + \epsilon y_{ij} - (\xi_1^i + \xi_2^i x_j - \xi_3^i p_j)) = 0 \\
& \gamma_i (1 - \textstyle\sum_{j=1}^3 y_{ij} + \epsilon \gamma_i) = 0 \\
& y_{ij} \geq 0,
\end{aligned} \tag{43}$$

where its original problem is:

$$\begin{aligned}
\max_{p_1, y_{ij}, \gamma_i} \ & (\tfrac{1}{4}y_{11} + \tfrac{1}{2}y_{21} + \tfrac{1}{4}y_{31})(p_1 - c_1) \\
\text{s.t.} \ & \text{for } i = 1, \ldots, 3, \ j = 1, \ldots, 3: \\
& \gamma_i - (\xi_1^i + \xi_2^i x_1 - \xi_3^i p_1) \geq 0 \\
& 1 - \textstyle\sum_{j=1}^3 y_{ij} \geq 0 \\
& \gamma_i - \max\{(\xi_1^i + \xi_2^i x_2 - \xi_3^i p_2), (\xi_1^i + \xi_2^i x_3 - \xi_3^i p_3), 0\} \geq 0 \\
& y_{ij}(\gamma_i - (\xi_1^i + \xi_2^i x_j - \xi_3^i p_j)) = 0 \\
& \gamma_i (1 - \textstyle\sum_{j=1}^3 y_{ij}) = 0 \\
& y_{ij} \geq 0.
\end{aligned} \tag{44}$$

Numerical results of Example 2 Table 1 shows the numerical results. We solve the SAA-regularized problems (39) and (40) using `HANSO` and `AMPL/SNOPT`, respectively, for Example 2 with different ϵ ranging from 0.1 to 0.0001. We set $p_1 = 6$ as the starting point. Figure 1 shows the convergence of the objective function when $\epsilon \to 0$.

The results show that the two methods solve the problem effectively and converge to their true counterpart. Specifically,

$$\{(p_1^\epsilon, \{y^\epsilon(\xi^i, p_1^\epsilon)\}_{i=1}^3, \{\gamma^\epsilon(\xi^i)\}_{i=1}^3)\} \to (p_1^*, \{y(\xi^i, p_1^*)\}_{i=1}^3, \{\gamma(\xi^i)\}_{i=1}^3),$$

where $(p_1^\epsilon, \{y^\epsilon(\xi^i, p_1^\epsilon)\}_{i=1}^3, \{\gamma^\epsilon(\xi^i)\}_{i=1}^3)$ is the solution of the regularized problem (43) and $(p_1^*, \{y(\xi^i, p_1^*)\}_{i=1}^3, \{\gamma(\xi^i)\}_{i=1}^3)$ is the solution of the original problem (44) and $y(\xi^i, p_1^*)$ is a sparse solution of w.r.t. 1 in $S(\xi^i, p_1^*)$ for $i = 1, 2, 3$. This is the type of convergence stated in Theorem 5. It is important to note that there exist other sequences $\{(\epsilon, \hat{p}_1^\epsilon)\} \to (0, p_1^*)$ such that $\{\mathbb{E}[y^\epsilon(\xi, \hat{p}_1^\epsilon)]\} \nrightarrow \mathbb{E}[y(\xi, p_1^*)]$; for example, when $\hat{p}_\epsilon = p_1^*$, $\{\mathbb{E}[y_1^\epsilon(\xi, \hat{p}_1^\epsilon)]\}$ pointwise converges to 0.5 while $\mathbb{E}[y_1(\xi, p_1^*)] = 0.75$.

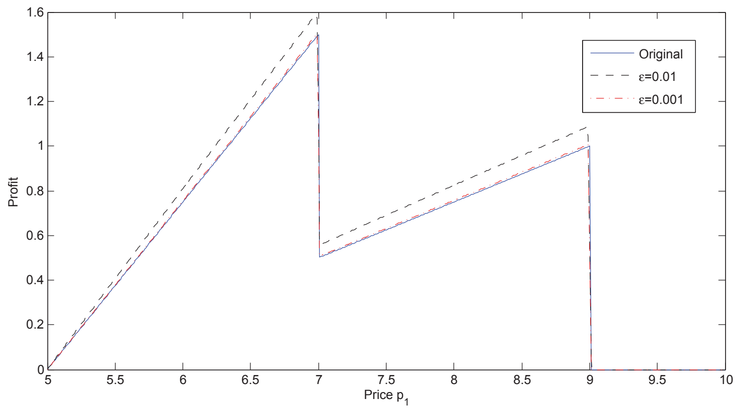

Fig. 1 The graph of original objective function and its approximations with $\epsilon = 0.01$ and 0.001.

When we use $p_1 = 8$ as the starting value, the results by `HANSO` and `AMPL/SNOPT` may converge to $p_1 = 9$, which is a local optimal solution.

Table 1 Numerical results of Example 2 with different ϵ

Solver \ ϵ		0.1	0.01	0.001	0.0001	True Value
`HANSO`	Opt. Solution p_1^ϵ	6.85	6.9895	6.9990	6.9998	7
	Opt. Value	2.356	1.5965	1.5097	1.5009	1.5
	$\mathbb{E}[(y^\epsilon(p_1^\epsilon,\xi))_1]$	1.2735	0.8025	0.7540	0.7505	0.75
`AMPL/SNOPT`	Opt. Solution p_1^ϵ	6.85	6.9895	6.999	6.9999	7
	Opt. Value	2.356	1.5966	1.5097	1.5009	1.5
	$\mathbb{E}[(y^\epsilon(p_1^\epsilon,\xi))_1]$	1.2735	0.8025	0.75525	0.7505	0.75
Pointwise Convergence	True Solution p_1^*	7	7	7	7	7
	$\mathbb{E}[(y^\epsilon(p_1^*,\xi))_1]$	0.8929	0.5400	0.5040	0.5004	0.5

Example 3 (Continuous Distribution Case) The set up in this example is similar to that of Example 2, except that consumers' preference ξ has continuous distribution. There are three products with $K = 1$ and $J = 2$. The utility function of a consumer with preference $\xi = (\xi_1, \xi_2)$ for purchasing product j is

$$u_j(\xi, p) = 1 + \xi_1 x_j - \xi_2 p_j, \tag{45}$$

where $x = (x_1, x_2, x_3) = (5, 3, 1)$ denotes the characteristics of three products; $c = (c_1, c_2, c_3) = (5, 2, 0.3)$, $p = (p_1, p_2, p_3) = (p_1, 3, 0.5)$ denote the marginal cost and price of three products. Firm F sells product 1 and needs to decide its price p_1. A consumer's preference $\xi = (\xi_1, \xi_2) : \Omega \to \Xi \subseteq \mathbb{R}^2$ is a random vector with $\xi_1 \sim U[1, 5]$, $\xi_2 \sim U[1, 3]$; that is, ξ_1 and ξ_2 are independent and uniformly distributed random variables in the interval $[1, 5]$ and $[1, 3]$ respectively.

Analysis of Example 3. By Lemma 5, a consumer with $\xi = (\xi_1, \xi_2)$ will choose product 1 if

$$1 + 5\xi_1 - p_1\xi_2 \geq \max\{1 + 3\xi_1 - 3\xi_2, 1 + \xi_1 - \frac{1}{2}\xi_2, 0\},$$

or equivalently,

$$\xi_2 \leq \frac{2}{p_1 - 3}\xi_1, \quad \xi_2 \leq \frac{4}{p_1 - \frac{1}{2}}\xi_1 \text{ and } \xi_2 \leq \frac{1 + 5\xi_1}{p_1}.$$

This implies that $\frac{4}{p_1 - \frac{1}{2}}\xi_1 \geq \frac{2}{p_1-3}\xi_1$, $\frac{1+5\xi_1}{p_1} \geq \frac{2}{p_1-3}\xi_1$ for $p_1 \geq 5.5$ and $\frac{2}{p_1-3}\xi_1 \leq 3$ for all $\xi_1 \in [1, 5]$ and $p_1 \geq \frac{19}{3}$.

Based on the analysis above, we can derive the market share for product 1:

$$\begin{aligned} &\mathbb{E}[\mathbb{1}(1 + 5\xi_1 - p_1\xi_2 \geq \max\{1 + 3\xi_1 - 3\xi_2, 1 + \xi_1 - \tfrac{1}{2}\xi_2, 0\})] \\ &= \int_{\frac{p_1-3}{2}}^{5} \frac{1}{4} \int_{1}^{\frac{2}{p_1-3}\xi_1} \frac{1}{2} d\xi_2 d\xi_1 = \frac{(p_1 - 13)^2}{32(p_1 - 3)}, \end{aligned} \tag{46}$$

when $p_1 \geq \frac{19}{3}$. Then firm F's pricing problem is equivalent to

$$\max_{p_1} \ f(p_1) := \frac{(p_1 - 13)^2}{32(p_1 - 3)}(p_1 - c_1), \tag{47}$$

when $p_1^* \geq \frac{19}{3}$. We use the Matlab solver `fminunc` to solve problem (47) and obtain a local optimal solution $p_1^* = 6.7016 > \frac{19}{3}$ with the objective value $f^* = 0.5699$. Moreover, since $\nabla^2_{p_1} f \leq 0$ for all $p_1 \leq 8.848$, f is a concave function when $p_1 \leq 8.848$. Moreover, from Figure 2, it is obvious that p_1^* is a global optimal solution of the original problem over $[5, 10]$. Figure 2 also shows the convergence between the original problem and the regularized problem when $\epsilon \to 0$. The regularized problem here is the SAA-regularized problem with sample size 10000.

Numerical results of Example 3. In the numerical experiment, we generate and solve 20 sample problems for each ϵ and sample size N. Tables 2 and 3 report the mean of optimal solutions, optimal values computed by `HANSO` and CPU time needed for different sample size with $\epsilon = 0.01$ and $\epsilon = 0.001$, respectively.

Figures 3 – 6 show the convergence of optimal solutions and optimal values with increasing sample size when $\epsilon = 0.01$ and $\epsilon = 0.001$ respectively.

Example 4 In this example, there are 18 products with $K = 3$ and $J = 15$. The utility of a consumer with preference $\xi = (\xi_1, \xi_2)$ for product j is:

$$u_j(\xi, p_j) = \beta_0 + (\beta_1 + \theta_1\xi_1)x_j - \exp(\theta_2\xi_2)p_j,$$

where x_j and p_j denote the observed characteristic and price of product j.

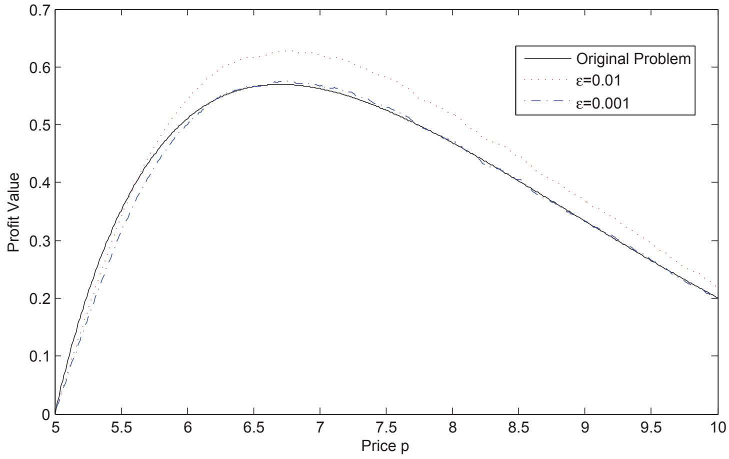

Fig. 2 The graph of original objective function and its approximations with $\epsilon = 0.01$ and 0.001

Table 2 Mean of optimal solutions, optimal values and CPU time with $\epsilon = 0.01$

Sample Size	Optimal Solution	Optimal Value	CPU time (s)
100	6.9874	0.6546	26.4330
500	6.7502	0.6350	106.1024
1000	6.7386	0.6426	159.5934
5000	6.7295	0.6298	1384.2
10000	6.6929	0.6305	2832.2

Table 3 Mean of optimal solutions, optimal values and CPU time with $\epsilon = 0.001$

Sample Size	Optimal Solution	Optimal Value	CPU time (s)
100	7.2380	0.6108	29.2109
500	6.9454	0.5782	69.1229
1000	6.7408	0.5817	120.8095
5000	6.7260	0.5777	534.4333
10000	6.6713	0.5796	1706.4

We choose $\beta_0 = 2$, $\beta_1 = 1$, $\theta_1 = \theta_2 = 1$ and generate three groups of i.i.d. samples of $\xi = (\xi_1, \xi_2)$ from the standard normal distribution with different sample size from 500-5000. For each groups, we test our model with different ϵ ranging from 0.1 to 0.0001. The characteristics and costs of all products and the price of products $j \in \{1, \dots, 15\}$ are given.

Numerical results of Example 4

Tables 4-6 report the numerical results of Example 4, which show the efficiency of our algorithm for solving larger scale problems.

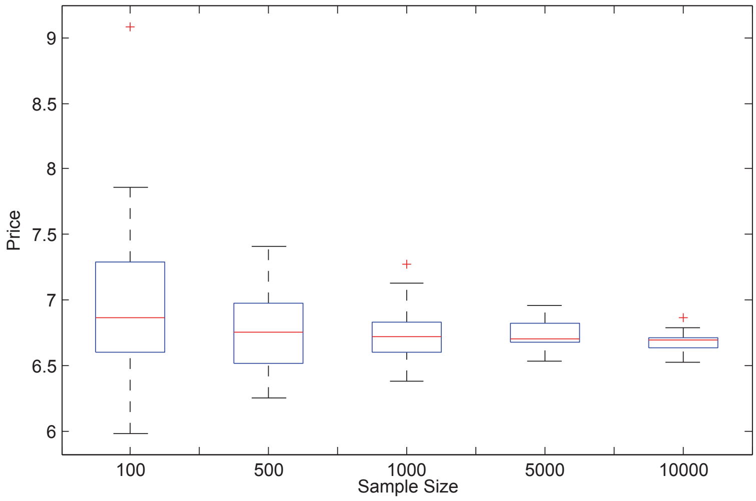

Fig. 3 Convergence of optimal solutions with increasing sample size, $\epsilon = 0.01$.

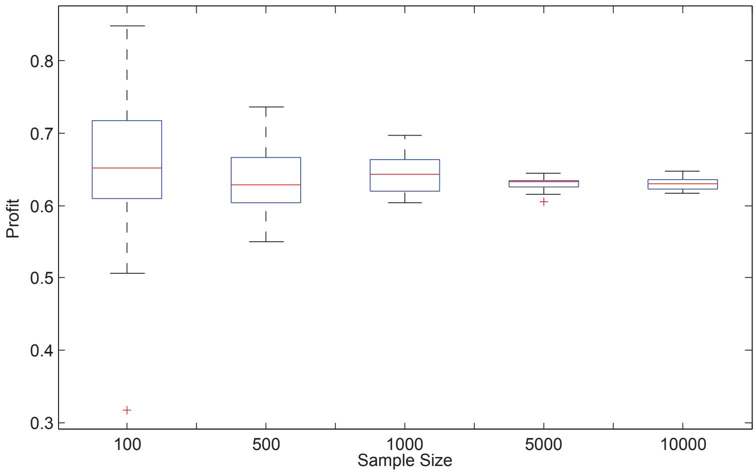

Fig. 4 Convergence of optimal values with increasing sample size, $\epsilon = 0.01$.

5 Conclusion

In this paper, we consider a firm's multiproduct pricing problem which is based on the pure characteristics demand model. Existing optimization methods with the SAA become challenging for solving such problems. Recently, Pang et. al. [25] proposed an MPLCC approach for the pure characteristics demand model with a finite number of observations; Chen et. al. [8] present an SAA-regularized method for the generalized method of moments estimate problem of the model when the MPLCC approach is applied. It is interesting to investigate whether the similar method can be applied to the pure charac-

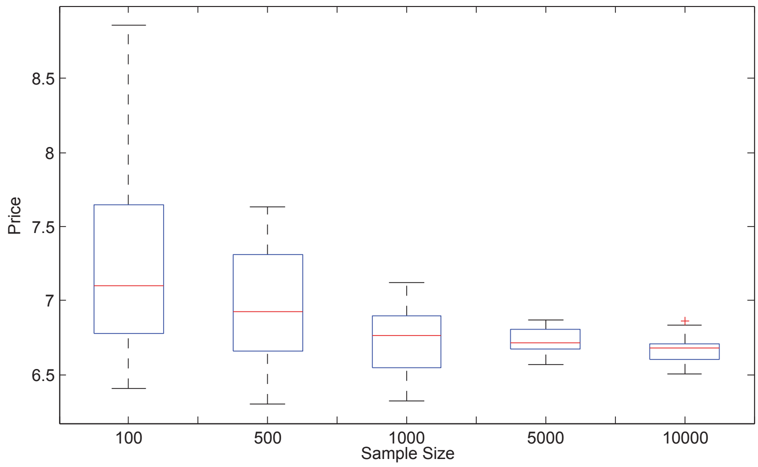

Fig. 5 Convergence of optimal solutions with increasing sample size, $\epsilon = 0.001$.

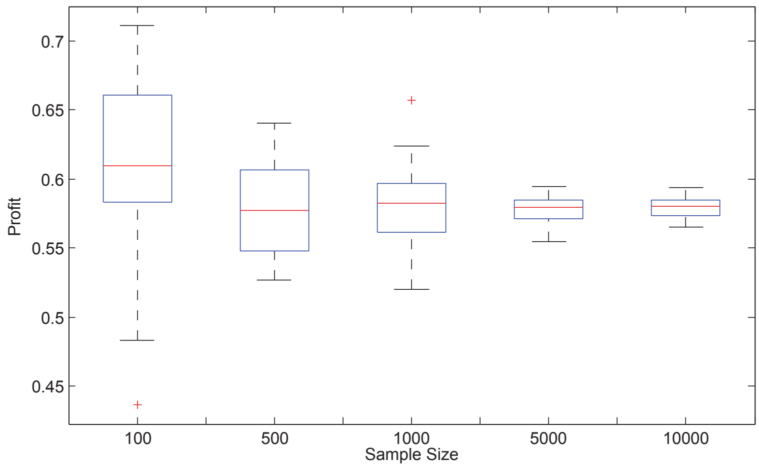

Fig. 6 Convergence of optimal objective values with increasing sample size, $\epsilon = 0.001$.

Table 4 Numerical results of Example 4 with sample size 500

ϵ	Calculated Price	Profit	Market Share	CPU time (s)
$\epsilon = 0.1$	(1.0946, 1.9049, 0.78841)	0.1581	(0.2160, 0, 0.1960)	211.9261
$\epsilon = 0.01$	(1.1128, 1.4726, 0.8211)	0.1586	(0.2440, 0, 0.1340)	145.7416
$\epsilon = 0.001$	(1.1271, 1.6657, 0.7990)	0.1594	(0.1940, 0, 0.2000)	71.9952

teristics demand model based multiproduct pricing problem. However, since $S(\xi, p)$ is a set-valued mapping which may not be continuous and one of key conditions in the convergence analysis established in [8] may not hold (see comments at the end of subsection 3.1), the results in the existing literature

Table 5 Numerical results of Example 4 with sample size 1000

ϵ	Calculated Price	Profit	Market Share	CPU time (s)
$\epsilon = 0.1$	(1.0957, 2.1942, 0.7727)	0.1555	(0.2160, 0, 0.1940)	236.3183
$\epsilon = 0.01$	(1.1076, 1.0667, 0.8073)	0.1557	(0.2340, 0, 0.1470)	315.2014
$\epsilon = 0.001$	(1.0840, 1.3514, 0.7893)	0.1565	(0.2490, 0, 0.1550)	291.2174

Table 6 Numerical results of Example 4 with sample size 5000

ϵ	Calculated Price	Profit	Market Share	CPU time (s)
$\epsilon = 0.1$	(1.0939, 1.0177, 0.7778)	0.1524	(0.2070, 0, 0.1740)	592.0729
$\epsilon = 0.01$	(1.1137, 1.0493, 0.7949)	0.1537	(0.2074, 0, 0.1764)	2531.8
$\epsilon = 0.001$	(1.1168, 1.5442, 0.7984)	0.1540	(0.2072, 0, 0.1740)	2887.1

can not guarantee the convergence between the multiproduct pricing problem and its SAA-regularized problem.

Our main contribution is to develop the SAA-regularized method for multiproduct pricing problems based on the pure characteristics demand model. We show that (i) there exists a sequence $\{(\epsilon_k, p^{\epsilon_k})\} \to (0, p^0)$ such that the y part of the solution of the regularized LCP (7) converges to a sparse solution with a specified order in $S(\xi, p^0)$ (see Definition 2) for any fixed p^0 and almost all $\xi \in \Xi$; (ii) if p^* is an optimal solution of the multiproduct pricing problem (4) and $y^*(\xi, p^*)$ is any sparse solution with a specified order in $S(\xi, p^*)$, $\mathbb{E}[y^*(\xi, p^*)]^T(p^* - c)$ will achieve the optimal value. We have proved the convergence of the SAA-regularized problem of the multiproduct pricing problem. Moreover, by using the solution's closed-form of the regularized LCP (7) and Matlab solver "Hanso", we have solved the multiproduct pricing problem based on the pure characteristics demand model effectively.

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