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Optimal insurance design with a bonus*

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Abstract

This paper investigates an insurance design problem, in which a bonus will be given to the insured if no claim has been made during the whole lifetime of the contract, for an expected utility insured. In this problem, the insured has to consider the so-called optimal *action* rather than the contracted compensation (or indemnity) due to the existence of the bonus. For any pre-agreed bonus, the optimal insurance contract is given explicitly and shown to be either the full coverage contract when the insured pays high enough premium, or a deductible one otherwise. The optimal contract and bonus are also derived explicitly if the insured is allowed to choose both of them. The contract turns out to be of either zero reward or zero deductible. In all cases, the optimal contracts are universal, that is, they do not depend on specific form of the utility of the insured. A numerical example is also provided to illustrate the main theoretical results of the paper.

Keywords: optimal insurance design; bonus-malus system; insurance contract with bonus; personalized contract; expected utility.

1 Introduction

Risk sharing, also known as "risk distribution", is a method of managing or reducing risk exposure by spreading the burden of loss among each member of a group based on a predetermined formula. It can be mathematically formulated as a multi-objective optimization problem in which a Pareto optimality is sought with respect to each member's risk preference.

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In the context of insurance, the primary risk sharing problem is the designing of an insurance contract that achieves Pareto optimal for the (typically two) involved parties: the insurer and the insured. Specifically, given an upfront premium that the insured pays the insurer, the classical insurance design problem is to determine the (contracted) amount of loss $I(X)$ covered by the insurer - called compensation or indemnity - for a random loss X. In order to let the insurer have sufficient incentive to offer the contract, on top of the actuarial value of the contracted compensation, the premium should also cover a safety loading in addition - this is the so-called participation constraint of the insurer in the literature. Once the loss occurs, the insured will claim it and ask the insurer to cover the contracted amount of loss $I(X)$. Not only in theory but also in practice, the optimal designing of insurance contract is fundamentally important.

In the designing of an insurance contract, the insured's and the insurer's risk preferences manifestly play the key role. To model them, the classical expected utility theory (EUT), non-EUTs or mixed risk preferences have been considered in the insurance literature. The EUT models are vast, and in these models the insurer is often assumed to be risk-neutral while the insured is assumed to be risk-averse; see, e.g., Arrow [2, 3], Raviv [15], and Gollier and Schlesinger [11]. The optimal compensation usually turns out to be a deductible one in which the insurer covers the amount of loss exceeding a deductible level. Such theoretical result is consistent with most of the insurance contracts available in practice. However, the EUT has received many criticisms for its failure in describing numerous human behaviors or explaining experimental observations (see, e.g., Allais [1], Mehra and Prescott [13]), so that many non-EUTs have been introduced to overcome the drawback of the EUT. For instance, Quiggin [14] proposed the rank-dependent utility theory (RDUT); Tversky and Kahneman [16] proposed the cumulative prospect theory (CPT) (see [4] for an excellent survey). A number of papers have already studied insurance contract design problems in the RDUT or CPT frameworks; see, e.g., Chateauneuf, Dana, and Tallon [9], Carlier and Dana [8], Dana and Scarsini [10], Bernard, He, Yan, and Zhou [5], Xu, Zhou, and Zhuang [20]. At the meanwhile, other risk preferences including VaR and CTE have also been widely considered, see, e.g., [6] and [7].

At the meanwhile, in many standard insurance contracts today, the bonus-malus system is in place. The term bonus-malus is Latin for good-bad. This system records the insured's history (including both good and bad events) to determine her premium today. For instance, when the insured made a claim due to a car accident, her premium for the next contract may increase. This paper investigates an insurance design problem in which a bonus will be given to the insured if no claim has been made during the whole lifetime of the contract. This is a bonus-malus system problem. In such a system, the insured will compare the compensation with the potential bonus to be awarded by hiding her losses. This makes her to consider the so-called optimal *action* rather than the contracted compensation to optimize her risk preference. The problem is considered in the classical expected utility framework in this paper. The explicit contract is derived for each pre-agreed reward, either being the full coverage contract when the insured pays high enough premium, or being a deductible one otherwise. The optimal compensation and bonus are also derived when the insured is allowed to choose both of them. In all cases, the optimal contracts are universal, that is, they do not depend on specific form of the utility of the insured.

The rest of this paper is organized as follows. We mathematically formulate the problem in Section 2. We derive the optimal contract for any pre-agreed bonus in Section 3 and provide a numerical example to illustrate the theoretical results. Section 4 is devoted to the study of optimal personalized contract, i.e., a contract that allows the insured to choose both the compensation and bonus. We conclude the paper in Section 5.

2 Model formulation

In this section, we formulate an optimal insurance design problem in which a bonus will be given to the insured if no claim has been made during the whole lifetime of the contract.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. An insured, endowed with an initial wealth $\varpi > 0$, faces a random loss $X \geqslant 0$. She chooses an insurance contract, by paying a premium π to the insurer in return for a *compensation* (or *indemnity*) in the case of a loss, to protect herself from the loss. This contracted compensation is a function of the loss, denoted by $I(\cdot)$. In this paper, the compensation is also called *contract* as it clearly determines the essentials of the insurance contract. In our model, the insured will be paid a pre-agreed bonus θ if no claim has been made during the whole lifetime of the contract. It is this bonus feature that distinguishes our model from those in insurance design literature. Intuitively speaking, when facing a loss, the insured shall compare the instant loss with the potential bonus to decide whether to claim the loss. Such consideration leads her to take actions deviating from the contracted compensation $I(\cdot)$. We assume the insured will act as a function of the loss, denoted by $A(\cdot)$, called an *action*. We should note that any action is a consequence of some contracted compensation. In the absent of bonus, the action and the compensation are the same. In contrast, in our model, the insured will receive a bonus θ if no claim has been made, therefore we have the *realized compensation*

$$C(X) = \begin{Bmatrix} A(X), & A(X) > 0; \\ \theta, & A(X) = 0; \end{Bmatrix} = A(X) + \theta\, \mathbb{1}_{A(X)=0} \,.$$

This is the real amount that the insured will receive from the insurer. Its right hand side highlights the bonus feather of the model. As usual, here and hereafter, we use $\mathbb{1}_S$ to denote the indicator function of a sentence S, thus $\mathbb{1}_S$ equals 1 when S is true and equals zero otherwise.

The insurer designs an insurance contract from the insured's point of view. For a potential loss X, the insured aims to choose an insurance contract (and hence the corresponding action) that provides the best tradeoff between the premium and the realized compensation based on her risk preference. In this paper, we consider an expected utility preference insured whose utility is $u(\cdot)$ mapping $\mathbb{R}$ to $\mathbb{R}^+$, so that her objective is to maximize

$$\mathbf{E}\Big[u\Big(\varpi - \pi - X + C(X)\Big)\Big] = \mathbf{E}\Big[u\Big(\varpi - \pi - X + A(X) + \theta\,\mathbb{1}_{A(X)=0}\Big)\Big].$$

On the other hand, the insurer is risk-neutral and the cost of offering the contract is proportional to the expectation of the realized compensation, so the premium to be charged for a realized compensation should satisfy the participation constraint

$$\pi \geqslant (1+\rho)\,\mathbf{E}[C(X)] = (1+\rho)\,\mathbf{E}\Big[A(X) + \theta\,\mathbb{1}_{A(X)=0}\Big],$$

where the constant $\rho \geqslant 0$ is the *safety loading coefficient* of the insurer.

It is natural to require any contracted compensation to satisfy

$$I(0) = 0, \quad 0 \leqslant I(x) \leqslant x, \quad \forall\, x \geqslant 0,$$

a constraint that has been imposed in most insurance design literature. In our framework, the action $A(\cdot)$ may be different from the contracted compensation $I(\cdot)$. But clearly in no situation, the insured can claim more than $I(\cdot)$. Hence it is natural to require

$$0 \leqslant A(x) \leqslant I(x), \quad \forall\, x \geqslant 0.$$

On the other hand, the insured will choose the best realized compensation (rather than the contracted compensation) in the presence of bonus, so the above constraint can be relaxed to

$$A(0) = 0, \quad 0 \leqslant A(x) \leqslant x, \quad \forall\, x \geqslant 0. \tag{2.1}$$

Once the best action has been found, one should recover a contract (namely contracted compensation) that will lead to this best action. At the meanwhile, we require the action to be globally increasing. Economically speaking, this means the insured's compensation is comonotone increasing with respect to the loss, asking more when a bigger loss occurs.

Mathematically speaking, we require

$$A(y) \leqslant A(x), \quad \forall\, y \leqslant x.$$

We can now formulate our bonus-malus system insurance design problem with a (pre-agreed) bonus $\theta \geqslant 0$ as

$$\begin{aligned} \max_{A(\cdot)\in\mathcal{A}} \quad & \mathbf{E}\Big[u\left(\varpi - \pi - X + A(X) + \theta\,\mathbb{1}_{A(X)=0}\right)\Big] \\ \text{subject to} \quad & (1+\rho)\,\mathbf{E}\Big[A(X) + \theta\,\mathbb{1}_{A(X)=0}\Big] \leqslant \pi, \end{aligned} \tag{2.2}$$

where the set of admissible actions is given by

$$\mathcal{A} = \{A(\cdot) : A(0) = 0, \quad A(y) \leqslant A(x) \leqslant x, \quad \forall\, 0 \leqslant y \leqslant x\}. \tag{2.3}$$

We denote by $F(\cdot)$ the probability distribution function of the potential loss X. For simplicity we assume that $F(\cdot)$ is strictly increasing and differentiable on $(0,+\infty)$ so that X has no atoms on $(0,+\infty)$. This assumption however allows the loss X to have a mass at 0, which is of course the most common case in insurance practice. Since $X \geqslant 0$, we have $F(0-) = \mathbf{P}(X < 0) = 0$. In addition, we also assume that the loss X has a finite expectation so that $\int_{[0,\infty)} x\,\mathrm{d}F(x) = \mathbf{E}[X] < \infty$. All these assumptions are technical and can be relaxed to more general cases without too much difficulties; this, however, is not the pursuit of the present paper.

Remark 2.1. *In contrast to Xu, Zhou and Zhuang [20], we* donot *require both the action and the real retention to be globally increasing. Different from Bernard, He, Yan, and Zhou [5] where a severe problem of moral hazard has arisen as their contract is not increasing with respect to the loss due to lack of the requirement, our optimal contract eventually turns out to satisfy the requirement automatically. The reason behind it is that we consider an EU preference insured rather than a RDUT one as in [5]. The moral hazard problem must be carefully treated if one considers a RDUT preference insured.*

3 Optimal action for a pre-agreed bonus

Our main target is to solve the optimal insurance design problem (2.2) for a pre-agreed bonus $\theta \geqslant 0$. To describe our main result more precisely we need some notation.

Define

$$G(\gamma) := \gamma F(\gamma) + \int_{(\gamma,\infty)} x\,\mathrm{d}F(x) - \gamma, \tag{3.1}$$

for $\gamma \geqslant 0$, with the convention that

$$G(+\infty) := 0. \tag{3.2}$$

Note

$$G(0) = \int_{(0,\infty)} x \, \mathrm{d}F(x) = \mathbf{E}[X] \,, \tag{3.3}$$

and

$$\begin{aligned} G(\gamma) &= \gamma F(\gamma) + \int_{(\gamma,\infty)} x \, \mathrm{d}F(x) - \gamma \\ &= \int_{(\gamma,\infty)} x \, \mathrm{d}F(x) - \gamma(1 - F(\gamma)) \\ &= \int_{(\gamma,\infty)} x \, \mathrm{d}F(x) - \int_{(\gamma,\infty)} \gamma \, \mathrm{d}F(x) \\ &= \int_{(\gamma,\infty)} (x - \gamma) \, \mathrm{d}F(x) \\ &= \int_{[0,\infty)} \max\{x - \gamma, 0\} \, \mathrm{d}F(x) \\ &= \mathbf{E}[(X - \gamma)_+] \,, \end{aligned} \tag{3.4}$$

which is indeed the ordinary deductible. By the monotone convergent theorem,

$$\lim_{\gamma \to +\infty} G(\gamma) = \lim_{\gamma \to +\infty} \mathbf{E}[(X - \gamma)_+] = \mathbf{E}\Big[\lim_{\gamma \to +\infty} (X - \gamma)_+\Big] = 0,$$

so we see that $G(\cdot)$ is a continuous and strictly decreasing bijective function mapping $[0, \infty]$ to $[0, \mathbf{E}[X]]$. Therefore, it has a unique continuous inverse function, denoted by $G^{-1}(\cdot)$, mapping $[0, \mathbf{E}[X]]$ to $[0, \infty]$.

Let

$$\pi_\theta := \tfrac{\pi}{1+\rho} - \theta, \tag{3.5}$$

for $-\infty < \theta < \infty$. When $\frac{\pi}{1+\rho} - \mathbf{E}[X] \leqslant \theta \leqslant \frac{\pi}{1+\rho}$, one has

$$G(0) = \mathbf{E}[X] \geqslant \tfrac{\pi}{1+\rho} - \theta = \pi_\theta \geqslant 0 = G(+\infty),$$

so that we can define

$$\gamma_\theta := G^{-1}(\pi_\theta), \tag{3.6}$$

$$k_\theta := \gamma_\theta - \theta. \tag{3.7}$$

It is easy to see that γ_θ is a continuous and strictly increasing bijective function mapping $\left[\frac{\pi}{1+\rho} - \mathbf{E}[X], \frac{\pi}{1+\rho}\right]$ to $[0, \infty]$.

We are now ready to present our first main result.

Theorem 3.1 (Optimal insurance contract for a pre-agreed bonus $\theta \geqslant 0$). *The optimal insurance contract and the optimal action to the problem (2.2) are given as below.*

- *If $\frac{\pi}{1+\rho} \geqslant \mathbf{E}[X] + \theta$, then the optimal insurance contract is the full coverage contract with the bonus θ, and the optimal action is*

$$A(x) = \begin{cases} x, & x > \theta; \\ 0, & x \leqslant \theta. \end{cases} \tag{3.8}$$

- *If $\frac{\pi}{1+\rho} < \theta$, then there is no feasible action, so the problem is ill-posed.*

- *If $\frac{\pi}{1+\rho} = \theta$, then the optimal insurance contract is a deductible contract with the deductible $+\infty$ and the bonus θ. The case is not economic meaningful.*

- *If $\mathbf{E}[X] + \theta > \frac{\pi}{1+\rho} > \theta$, then the optimal insurance contract is the full coverage contract with the bonus θ, provided*

$$\theta F(\theta) + \int_{(\theta,\infty)} x \, \mathrm{d}F(x) \leqslant \tfrac{\pi}{1+\rho}, \tag{3.9}$$

(which can happen only when $\frac{\pi}{1+\rho} \geqslant \mathbf{E}[X]$); otherwise, the optimal insurance contract is a deductible contract with the deductible $k_\theta > 0$ and the bonus θ, and the optimal action is

$$A(x) = \begin{cases} x - k_\theta, & x > k_\theta + \theta; \\ 0, & x \leqslant k_\theta + \theta, \end{cases} \tag{3.10}$$

where k_θ is defined in (3.7).

Remark 3.1. *In the insurance literature, the optimal contract (3.8) is called as "Franchise Deductible". It can be regard as the sum of an ordinary deductible $(x-\theta)^+$ and a bonus θ.*

We are now going to prove this result.

3.1 Neodeductible contract

We call an action $A(\cdot)$ neodeductible if it can be expressed in the form of

$$A(x) = \begin{cases} x - k, & x > x_0; \\ 0, & x \leqslant x_0. \end{cases} \tag{3.11}$$

for some non-negative constants k and x_0. Here we allow $x_0 = +\infty$ so that $A(x) \equiv 0$ is regarded as a neodeductible action, which is doing nothing. The neodeductible action (3.11) reduces to the classical deductible compensation when $k = x_0$. Neodeductible action appears naturally when a bonus is presented in the insurance contract. Before giving an economic explanation for this fact, we first show that the optimal action to the problem (2.2), if it exists, must be neodeductible. The claim follows if we can find a feasible neodeductible action that gives a higher or the same performance as any given feasible action $A_0(\cdot)$ (namely, an action that satisfies the constraint of the problem (2.2)).

Let

$$x^* = \sup\{x \geqslant 0 \mid A_0(x) = 0\}.$$

Then it is nonnegative. Because any feasible action by (2.3) is increasing, one has

$$A_0(x) \begin{cases} > 0, & x > x^*; \\ = 0, & x \leqslant x^*. \end{cases} \tag{3.12}$$

If $x^* = +\infty$, then $A_0(x) \equiv 0$ and itself is a neodeductible action. So we only need to consider the case $0 \leqslant x^* < \infty$ below.

In view of (3.12), the objective of the problem (2.2) boils down to

$$\mathbf{E}\Big[u\Big(\varpi - \pi - X + \theta\Big)\,\mathbb{1}_{X \leqslant x^*}\Big] + \mathbf{E}\Big[u\Big(\varpi - \pi - X + A_0(X)\Big)\,\mathbb{1}_{X > x^*}\Big], \tag{3.13}$$

and the constraint to

$$(1 + \rho)\Big(\theta\mathbf{P}(X \leqslant x^*) + \mathbf{E}[A_0(X)\,\mathbb{1}_{X > x^*}]\Big) \leqslant \pi. \tag{3.14}$$

We define a constant c via the equation

$$\mathbf{E}\Big[\Big(\varpi - \pi - X + A_0(X)\Big)\,\mathbb{1}_{X > x^*}\Big] = c\mathbf{P}(X > x^*),$$

and define a function

$$L_1(x) = \begin{cases} \varpi - \pi - x + A_0(x), & x > x^*; \\ c, & x \leqslant x^*. \end{cases}$$

Then one has

$$\begin{aligned} \mathbf{E}[L_1(X)] &= \mathbf{E}[L_1(X)\,\mathbb{1}_{X>x^*}] + \mathbf{E}[L_1(X)\,\mathbb{1}_{X\leqslant x^*}] \\ &= \mathbf{E}\Big[\Big(\varpi - \pi - X + A_0(X)\Big)\,\mathbb{1}_{X>x^*}\Big] + \mathbf{E}[c\,\mathbb{1}_{X\leqslant x^*}] \\ &= c\mathbf{P}(X > x^*) + c\mathbf{P}(X \leqslant x^*) \\ &= c. \end{aligned}$$

Applying Jensen's inequality to the concave utility function $u(\cdot)$, we see that

$$\begin{aligned} \mathbf{E}\Big[u\Big(\varpi - \pi - X + A_0(X)\Big)\,\mathbb{1}_{X>x^*}\Big] &= \mathbf{E}\Big[u\Big(L_1(X)\Big)\,\mathbb{1}_{X>x^*}\Big] \\ &= \mathbf{E}\Big[u\Big(L_1(X)\Big)\Big] - \mathbf{E}\Big[u\Big(L_1(X)\Big)\,\mathbb{1}_{X\leqslant x^*}\Big] \\ &\leqslant u\Big(\mathbf{E}[L_1(X)]\Big) - u(c)\mathbf{P}(X \leqslant x^*) \\ &= u(c) - u(c)\mathbf{P}(X \leqslant x^*) \\ &= u(c)\mathbf{P}(X > x^*). \end{aligned} \tag{3.15}$$

We next define an action

$$A(x) = \begin{cases} x - k, & x > x^*; \\ 0, & x \leqslant x^*, \end{cases}$$

where the constant k is determined by the identity

$$\mathbf{E}[A_0(X)\,\mathbb{1}_{X>x^*}] = \mathbf{E}[A(X)\,\mathbb{1}_{X>x^*}]\,. \tag{3.16}$$

Because

$$\mathbf{E}[(X-k)\,\mathbb{1}_{X>x^*}] = \mathbf{E}[A(X)\,\mathbb{1}_{X>x^*}] = \mathbf{E}[A_0(X)\,\mathbb{1}_{X>x^*}] \leqslant \mathbf{E}[X\,\mathbb{1}_{X>x^*}]\,,$$

we see that $k \geqslant 0$ and hence $A(\cdot)$ is a neodeductible action. Using (3.16), we have

$$\begin{aligned} (1+\rho)\Big(\theta\mathbf{P}(X \leqslant x^*) + \mathbf{E}[A(X)\,\mathbb{1}_{X>x^*}]\Big) &= (1+\rho)\Big(\theta\mathbf{P}(X \leqslant x^*) + \mathbf{E}[A_0(X)\,\mathbb{1}_{X>x^*}]\Big) \\ &\leqslant \pi, \end{aligned}$$

and so conclude that $A(\cdot)$ is a feasible neodeductible action of the problem (2.2). Using (3.16) again and by the definition of $A(\cdot)$, we now see that

$$\begin{aligned}
c\mathbf{P}(X > x^*) &= \mathbf{E}\Big[\Big(\varpi - \pi - X + A_0(X)\Big)\,\mathbb{1}_{X>x^*}\Big] \\
&= \mathbf{E}\Big[\Big(\varpi - \pi\Big)\,\mathbb{1}_{X>x^*}\Big] - \mathbf{E}[X\,\mathbb{1}_{X>x^*}] + \mathbf{E}[A_0(X)\,\mathbb{1}_{X>x^*}] \\
&= (\varpi - \pi)\mathbf{P}(X > x^*) - \mathbf{E}[X\,\mathbb{1}_{X>x^*}] + \mathbf{E}[A(X)\,\mathbb{1}_{X>x^*}] \\
&= (\varpi - \pi)\mathbf{P}(X > x^*) - \mathbf{E}\Big[\Big(X - A(X)\Big)\,\mathbb{1}_{X>x^*}\Big] \\
&= (\varpi - \pi)\mathbf{P}(X > x^*) - \mathbf{E}[k\,\mathbb{1}_{X>x^*}] \\
&= (\varpi - \pi - k)\mathbf{P}(X > x^*);
\end{aligned}$$

and it yields $c = \varpi - \pi - k$ and thus

$$\begin{aligned}
\mathbf{E}\Big[u\Big(\varpi - \pi - X + A(X)\Big)\,\mathbb{1}_{X>x^*}\Big] &= \mathbf{E}\Big[u\Big(\varpi - \pi - k\Big)\,\mathbb{1}_{X>x^*}\Big] \\
&= u(\varpi - \pi - k)\mathbf{P}(X > x^*) \\
&= u(c)\mathbf{P}(X > x^*).
\end{aligned}$$

Together with (3.15) this leads to

$$\mathbf{E}\Big[u\Big(\varpi - \pi - X + A_0(X)\Big)\,\mathbb{1}_{X>x^*}\Big] \leqslant \mathbf{E}\Big[u\Big(\varpi - \pi - X + A(X)\Big)\,\mathbb{1}_{X>x^*}\Big]\,;$$

and consequently,

$$\begin{aligned}
&\mathbf{E}\Big[u\Big(\varpi - \pi - X + A_0(X)\Big)\Big] \\
&= \mathbf{E}\Big[u\Big(\varpi - \pi - X + \theta\Big)\,\mathbb{1}_{X\leqslant x^*}\Big] + \mathbf{E}\Big[u\Big(\varpi - \pi - X + A_0(X)\Big)\,\mathbb{1}_{X>x^*}\Big] \\
&\leqslant \mathbf{E}\Big[u\Big(\varpi - \pi - X + \theta\Big)\,\mathbb{1}_{X\leqslant x^*}\Big] + \mathbf{E}\Big[u\Big(\varpi - \pi - X + A(X)\Big)\,\mathbb{1}_{X>x^*}\Big] \\
&= \mathbf{E}\Big[u\Big(\varpi - \pi - X + A(X)\Big)\Big]\,.
\end{aligned}$$

As desired, we have shown that the feasible neodeductible action $A(\cdot)$ gives a higher or the same performance as $A_0(\cdot)$.

Let us now give an economical explanation for the above result. When holding a classical deductible compensation without bonus

$$A_0(x) = \begin{cases} x - k, & x > k; \\ 0, & x \leqslant k; \end{cases}$$

and facing a loss X, the insured may claim the loss so as to receive a compensation $X - k$

from the insurer. Such an action is harmless to her performance if X is bigger than the deductible k, so she will do it whenever such a loss occurs. In contrast, when holding a compensation with a bonus, the insured shall compare the immediate compensation $X-k$ with the potential bonus θ so as to maximize her performance. If the compensation is too small compared to the bonus (namely, $X-k<\theta$), she will, instead of making a claim, hide her loss and wait for the bonus. Such a consideration pushes her to take the action

$$A(x)=\begin{cases}x-k, & x>k+\theta;\\ 0, & x\leqslant k+\theta,\end{cases}$$

a neodeductible action.

To solve the problem (2.2), we only need to focus on neodeductible actions. In the following section, we will seek for the best neodeductible action.

3.2 Optimal deductible for a pre-agreed bonus

For a pre-agreed bonus $0\leqslant\theta<\infty$, we are going to find the best neodeductible action of the form

$$A(x)=\begin{cases}x-k, & x>k+\theta;\\ 0, & x\leqslant k+\theta,\end{cases}$$

over all possible deductibles $0\leqslant k\leqslant\infty$.

Given such an action, the objective of the problem (2.2) reads

$$\begin{aligned}&\mathbf{E}\Big[u\Big(\varpi-\pi-X+A(X)+\theta\,\mathbb{1}_{A(X)=0}\Big)\Big]\\ =&\int_{[0,\infty)}u\Big(\varpi-\pi-x+A(x)+\theta\,\mathbb{1}_{A(x)=0}\Big)\,\mathrm{d}F(x)\\ =&\int_{[0,k+\theta]}u\Big(\varpi-\pi-x+\theta\Big)\,\mathrm{d}F(x)+\int_{(k+\theta,\infty)}u\Big(\varpi-\pi-k\Big)\,\mathrm{d}F(x)\\ =&\int_{[0,k+\theta]}u\Big(\varpi-\pi-x+\theta\Big)\,\mathrm{d}F(x)+u\Big(\varpi-\pi-k\Big)\Big(1-F(k+\theta)\Big),\end{aligned}\tag{3.17}$$

and the constraint reads

$$
\begin{aligned}
\pi &\geqslant (1+\rho)\,\mathbf{E}\Big[A(X) + \theta\,\mathbb{1}_{A(X)=0}\Big] \\
&= (1+\rho)\int_{[0,\infty)} \Big(A(x) + \theta\,\mathbb{1}_{A(x)=0}\Big)\,\mathrm{d}F(x) \\
&= (1+\rho)\left(\int_{[0,k+\theta]} \theta\,\mathrm{d}F(x) + \int_{(k+\theta,\infty)} (x-k)\,\mathrm{d}F(x)\right) \\
&= (1+\rho)\left(\theta F(k+\theta) + \int_{(k+\theta,\infty)} (x-k)\,\mathrm{d}F(x)\right) \\
&= (1+\rho)\left((\theta+k)F(k+\theta) + \int_{(k+\theta,\infty)} x\,\mathrm{d}F(x) - k\right) \\
&= (1+\rho)\left((\theta+k)F(k+\theta) + \int_{(k+\theta,\infty)} x\,\mathrm{d}F(x) - (k+\theta) + \theta\right), \\
&= (1+\rho)\left(G(\theta+k) + \theta\right). \qquad (3.18)
\end{aligned}
$$

The problem (2.2) hence reduces to

$$
\begin{aligned}
&\max_{k} \quad \int_{[0,k+\theta]} u\Big(\varpi - \pi - x + \theta\Big)\,\mathrm{d}F(x) + u\Big(\varpi - \pi - k\Big)\Big(1 - F(k+\theta)\Big) \qquad (3.19)\\
&\text{subject to} \quad G(\theta+k) \leqslant \tfrac{\pi}{1+\rho} - \theta = \pi_\theta, \quad k \geqslant 0.
\end{aligned}
$$

Set $\gamma = k + \theta$ and set

$$
J(\gamma) := \int_{[0,\gamma]} u\Big(\varpi - \pi - x + \theta\Big)\,\mathrm{d}F(x) + u\Big(\varpi - \pi - \gamma + \theta\Big)\Big(1 - F(\gamma)\Big).
$$

It is easy to see that if γ^* solves the problem

$$
\begin{aligned}
&\max_{\gamma} \quad J(\gamma) \qquad (3.20)\\
&\text{subject to} \quad G(\gamma) \leqslant \pi_\theta, \quad \gamma \geqslant \theta,
\end{aligned}
$$

then $k^* = \gamma^* - \theta$ solves the problem (3.19), vice versa. Our problem thus reduces to finding the optimal solution to the problem (3.20).

We first note that

$$
\begin{aligned}
J'(\gamma) &= u\left(\varpi - \pi - \gamma + \theta\right)F'(\gamma) - u'\left(\varpi - \pi - \gamma + \theta\right)\left(1 - F(\gamma)\right) \\
&\quad - u\left(\varpi - \pi - \gamma + \theta\right)F'(\gamma) \\
&= -u'\left(\varpi - \pi - \gamma + \theta\right)\left(1 - F(\gamma)\right) \\
&\leqslant 0
\end{aligned}
$$

for $\gamma > 0$, so solving the problem (3.20) reduces to finding the smallest γ that satisfies its constraints $G(\gamma) \leqslant \pi_\theta$ and $\gamma \geqslant \theta$. On the other hand, we also note from (3.4) that $G(\cdot)$ is continuous and strictly decreasing on $[0, \infty]$. Therefore

- When $\frac{\pi}{1+\rho} \geqslant \mathbf{E}[X] + \theta$, we have

  $$G(\gamma) \leqslant G(0) = \mathbf{E}[X] \leqslant \tfrac{\pi}{1+\rho} - \theta = \pi_\theta$$

  for any $0 \leqslant \gamma \leqslant +\infty$, so the constraint of the problem (3.20) reduces to $\gamma \geqslant \theta$ only; and hence its optimal solution is $\gamma^* = \theta$. Consequently, the optimal solution to the problem (3.19) is $k^* = \gamma^* - \theta = 0$ and the optimal action to the problem (2.2) is

  $$A(x) = \begin{cases} x, & x > \theta; \\ 0, & x \leqslant \theta. \end{cases} \tag{3.21}$$

  This action is a consequence of the contracted compensation

  $$I(x) = x,$$

  namely the full coverage contract. Economically speaking, the insured pays high enough premium so that the insurer can cover all the loss with the bonus θ (which is however no more than $\frac{\pi}{1+\rho} - \mathbf{E}[X]$).

- When $\frac{\pi}{1+\rho} < \theta$, we have

  $$G(\gamma) \geqslant G(+\infty) = 0 > \tfrac{\pi}{1+\rho} - \theta = \pi_\theta$$

  for any $0 \leqslant \gamma \leqslant +\infty$, so there is no γ satisfying the constraint of the problem (3.20); and hence the problem is ill-posed. Economically speaking, the insurance premium is too low (or equivalently, the desired bonus is too high) such that the participation constraint can not be satisfied and the insurer can not offer any contract.

- When $\frac{\pi}{1+\rho} = \theta$, we have

  $$G(\gamma) > G(+\infty) = 0 = \tfrac{\pi}{1+\rho} - \theta = \pi_\theta$$

  for any $0 \leqslant \gamma < +\infty$ and $G(+\infty) = 0 = \pi_\theta$. Hence $\gamma = +\infty$ is the unique feasible (and thus the optimal) solution to the problem (3.20); and this means $k^* = +\infty$ solves the problem (3.19) and the optimal action to the problem (2.2) is $A(x) \equiv 0$. Economically speaking, as the deductible is $+\infty$, no insured will buy such insurance contract and hence it is not an economic meaningful case.

- When $\mathbf{E}[X] + \theta > \frac{\pi}{1+\rho} > \theta$, we see from (3.6) that

$$G(\gamma) \leqslant \pi_\theta \quad \text{if and only if} \quad \gamma \geqslant \gamma_\theta. \tag{3.22}$$

  Hence the constraint of the problem (3.20) boils down to $\gamma \geqslant \max\{\theta, \gamma_\theta\}$; and its optimal solution is thus $\gamma^* = \max\{\theta, \gamma_\theta\}$. As a consequence, the optimal solution to the problem (3.19) is $k^* = \gamma^* - \theta = \max\{0, \gamma_\theta - \theta\}$ and the optimal action to the problem (2.2) is

$$A(x) = \begin{cases} x - k^*, & x > k^* + \theta; \\ 0, & x \leqslant k^* + \theta. \end{cases} \tag{3.23}$$

  This is a neodeductible action and a consequence of the contracted compensation

$$I(x) = \begin{cases} x - k^*, & x > k^*; \\ 0, & x \leqslant k^*. \end{cases} \tag{3.24}$$

  This contract can be the full coverage one (if $k^* = 0$) or a deductible one (if $k^* > 0$). We will discuss them in the following sections.

We have until now proved Theorem 3.1 except for the last case. The proof will be completed in the following section.

3.2.1 Full coverage contract

To finish the proof of Theorem 3.1, we need to identify under which condition the contract (3.24) is the full coverage contract.

The contract (3.24) is the full coverage contract if and only if $k^* = 0$, that is, $\theta \geqslant \gamma_\theta$. This however by (3.22) is equivalent to $G(\theta) \leqslant \pi_\theta$, namely

$$\theta F(\theta) + \int_{(\theta,\infty)} x \, \mathrm{d}F(x) - \theta \leqslant \tfrac{\pi}{1+\rho} - \theta$$

which is obviously equivalent to the desired inequality (3.9). This is the necessary and sufficient condition for the contract being full coverage.

Denote by $H(\theta)$ the left hand side of (3.9). We see from (3.4) that

$$
\begin{aligned}
H(\theta) &= \theta F(\theta) + \int_{(\theta,\infty)} x \,\mathrm{d}F(x) \\
&= G(\theta) + \theta \\
&= \int_{[0,\infty)} \max\{x-\theta, 0\} \,\mathrm{d}F(x) + \theta \\
&= \int_{[0,\infty)} \max\{x-\theta, 0\} \,\mathrm{d}F(x) + \int_{[0,\infty)} \theta \,\mathrm{d}F(x) \\
&= \int_{[0,\infty)} \max\{x, \theta\} \,\mathrm{d}F(x)
\end{aligned}
$$

is a strictly increasing function. Moreover,

$$H(0) = \int_{[0,\infty)} x \,\mathrm{d}F(x) = \mathbf{E}[X],$$

and by the monotone convergent theorem,

$$H(+\infty) = \lim_{\theta\to+\infty} H(\theta) = \lim_{\theta\to+\infty} \int_{[0,\infty)} \max\{x,\theta\} \,\mathrm{d}F(x) = +\infty.$$

Therefore, the inequality (3.9) can only happens when $\pi \geqslant (1+\rho)\,\mathbf{E}[X]$ as $H(\theta) \geqslant H(0) = \mathbf{E}[X]$.

If the contract (3.24) is not the full coverage contract, then $k^* > 0$ so that $k^* = \gamma_\theta - \theta = k_\theta$. And consequently the action (3.23) boils down to (3.10). This completes the proof of Theorem 3.1.

The economic meaning is very clear: if the insured pays high enough insurance premium (namely $\pi \geqslant (1+\rho)\,\mathbf{E}[X]$), she can be offered the full coverage insurance contract with a reasonable bonus (up to some amount determined by her insurance premium, namely $H^{-1}\left(\frac{\pi}{1+\rho}\right)$); otherwise no full coverage contract with a nonnegative bonus can be offered by the insurer.

In the case $H(\theta) > \frac{\pi}{1+\rho}$, the insurer can only offer a deductible contract with the bonus θ. We are interested in the relationship between the deductible and the bonus in such situation. This will be discussed in the following section.

3.2.2 Deductible contract

As is discussed in the previous section, we only need to consider the case

$$\mathbf{E}[X] + \theta > \tfrac{\pi}{1+\rho} > \theta, \quad H(\theta) > \tfrac{\pi}{1+\rho}, \tag{3.25}$$

which is hence assumed in this section.

In this case, the optimal action is (3.10) where $k_\theta = \gamma_\theta - \theta > 0$ and γ_θ is determined by $\gamma_\theta = G^{-1}(\pi_\theta)$ that is equivalent to

$$G(\gamma_\theta) = \pi_\theta.$$

Differentiation on both sides of the last equation with respect to θ gives

$$G'(\gamma_\theta)\gamma'_\theta = \pi'_\theta = -1. \tag{3.26}$$

At the same time

$$G'(\gamma) = F(\gamma) + \gamma F'(\gamma) - \gamma F'(\gamma) - 1 = F(\gamma) - 1 \in (-1, 0), \tag{3.27}$$

so we obtain $\gamma'_\theta > 1$; and consequently,

$$k'_\theta = \gamma'_\theta - 1 > 0. \tag{3.28}$$

Therefore, the optimal deductible should increase if one wants a higher bonus but does not want to pay more premium.

Now let us study an extreme case. Note

$$\lim_{\theta \uparrow \frac{\pi}{1+\rho}} G(\gamma_\theta) = \lim_{\theta \uparrow \frac{\pi}{1+\rho}} \pi_\theta = 0,$$

so we conclude by the monotonicity of $G(\cdot)$ and (3.2) that

$$\lim_{\theta \uparrow \frac{\pi}{1+\rho}} \gamma_\theta = +\infty;$$

and consequently,

$$\lim_{\theta \uparrow \frac{\pi}{1+\rho}} k_\theta = \lim_{\theta \uparrow \frac{\pi}{1+\rho}} (\gamma_\theta - \theta) = +\infty. \tag{3.29}$$

It says that the insured has to bear extremely large loss if she wants to have a very big bonus but not to pay more premium.

In the case

$$\mathbf{E}[X] > \tfrac{\pi}{1+\rho},$$

the insured can choose a contract without any bonus but with a deductible

$$k_0 = \gamma_0 = G^{-1}(\pi_0) = G^{-1}\left(\tfrac{\pi}{1+\rho}\right) > G^{-1}\left(\mathbf{E}[X]\right) = 0. \tag{3.30}$$

This deductible is monotonic decreasing with respect to the premium and it becomes zero when $\pi \geqslant (1+\rho)\,\mathbf{E}[X]$.

3.3 Numerical example: The bonus and the optimal deductible

In this section, we give a numerical example to verify the theoretical results obtained thus far.

We assume the loss X follows an exponential distribution with mean 4 and the safety loading of the insurer is $\rho = 20\%$.

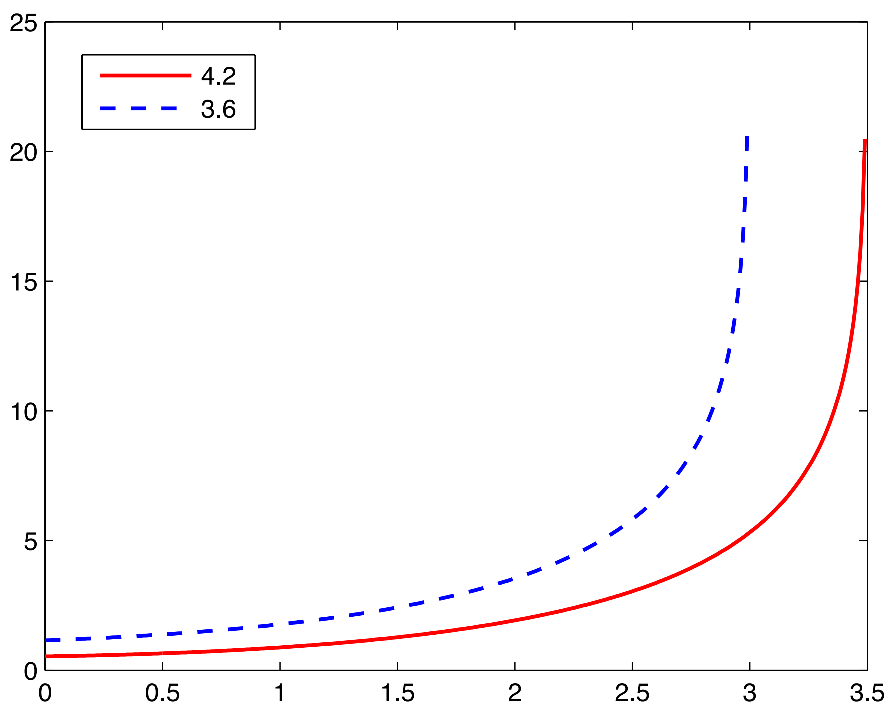

Figure 1: The bonus θ and the optimal deductible k_θ.

Figure 1 illustrates the relationship between the bonus θ and the optimal deductible k_θ. The horizontal axis and vertical axis stand for the bonus and the optimal deductible respectively. The red curve illustrates the case $\pi = 4.2$, while the blue dot one illustrates the case $\pi = 3.6$. We see from Figure 1 that the optimal deductible is strictly increasing with respect to the bonus (as predicted by (3.28)) and it goes to infinity as the bonus approaches to $\frac{\pi}{1+\rho}$ (as predicted by (3.29)). Moreover, the optimal deductible will reduce if one pays a higher premium.

Until now, we have only discussed the optimal contract design problem for a pre-agreed bonus. What happens if the insured is allowed to choose the bonus as well? In another words, what is the best point on the $\theta - k_\theta$ curve for the insured? This will be addressed in the following section.

4 Optimal personalized contract

One should note the important fact that the optimal contract obtained thus far, for a pre-agreed bonus, *does not* dependent on the insured's utility function. We are going to determine the optimal contract if the insured is allowed to choose the bonus, or equivalently, the deductible. Surprisingly, it turns out that the optimal contract does not depended on the risk preference as well.

We only need to consider the case

$$0 \leqslant \theta < \tfrac{\pi}{1+\rho}, \tag{4.1}$$

otherwise no contract can be offered by the insurer.

If the insured is allowed to choose the bonus θ, then her target becomes maximizing

$$\hat{J}(\theta) := \mathbf{E}\Big[u\left(\varpi - \pi - X + A(X) + \theta\, \mathbb{1}_{A(X)=0}\right)\Big],$$

over all θ satisfying (4.1). Here $A(\cdot)$ is the optimal action given in Theorem 3.1, which has difference expressions in different regions. Let us consider the problem case by case.

- When $\pi \geqslant (1+\rho)\,\mathbf{E}[X]$, we see from (3.21) and (3.23) that the optimal action is

$$A(x) = \begin{cases} x - k_\theta, & x > k_\theta + \theta, \quad \tfrac{\pi}{1+\rho} - \mathbf{E}[X] < \theta < \tfrac{\pi}{1+\rho}; \\ 0, & x \leqslant k_\theta + \theta, \quad \tfrac{\pi}{1+\rho} - \mathbf{E}[X] < \theta < \tfrac{\pi}{1+\rho}; \\ x, & x > \theta, \quad 0 \leqslant \theta \leqslant \tfrac{\pi}{1+\rho} - \mathbf{E}[X]\,; \\ 0, & x \leqslant \theta, \quad 0 \leqslant \theta \leqslant \tfrac{\pi}{1+\rho} - \mathbf{E}[X]\,. \end{cases} \tag{4.2}$$

– If $0 \leqslant \theta \leqslant \frac{\pi}{1+\rho} - \mathbf{E}[X]$, we have

$$\begin{aligned}\hat{J}(\theta) &= \mathbf{E}\Big[u\Big(\varpi - \pi - X + \theta\Big)\,\mathbb{1}_{X\leqslant\theta}\Big] + \mathbf{E}\Big[u\Big(\varpi - \pi\Big)\,\mathbb{1}_{X>\theta}\Big] \\ &= \int_{[0,\theta]} u\Big(\varpi - \pi - x + \theta\Big)\,\mathrm{d}F(x) + u\Big(\varpi - \pi\Big)\Big(1 - F(\theta)\Big).\end{aligned}$$

Differentiation both sides yields

$$\begin{aligned}\hat{J}'(\theta) &= \int_{[0,\theta]} u'\Big(\varpi - \pi - x + \theta\Big)\,\mathrm{d}F(x) + u\Big(\varpi - \pi\Big)F'(\theta) - u\Big(\varpi - \pi\Big)F'(\theta) \\ &= \int_{[0,\theta]} u'\Big(\varpi - \pi - x + \theta\Big)\,\mathrm{d}F(x) > 0\end{aligned}$$

for $\theta > 0$. Intuitively speaking, the contract in this situation is of full coverage, so the insured prefers the bonus as high as possible.

– If $\frac{\pi}{1+\rho} - \mathbf{E}[X] < \theta < \frac{\pi}{1+\rho}$, we have

$$\begin{aligned}\hat{J}(\theta) &= \mathbf{E}\Big[u\Big(\varpi - \pi - X + \theta\Big)\,\mathbb{1}_{X\leqslant k_\theta+\theta}\Big] + \mathbf{E}\Big[u\Big(\varpi - \pi - k_\theta\Big)\,\mathbb{1}_{X>k_\theta+\theta}\Big] \\ &= \int_{[0,k_\theta+\theta]} u\Big(\varpi - \pi - x + \theta\Big)\,\mathrm{d}F(x) + u\Big(\varpi - \pi - k_\theta\Big)\Big(1 - F(k_\theta + \theta)\Big).\end{aligned}$$

Differentiation both sides yields

$$\begin{aligned}\hat{J}'(\theta) &= \int_{[0,k_\theta+\theta]} u'\Big(\varpi - \pi - x + \theta\Big)\,\mathrm{d}F(x) + u\Big(\varpi - \pi - k_\theta\Big)F'(k_\theta + \theta)(k'_\theta + 1) \\ &\quad - u\Big(\varpi - \pi - k_\theta\Big)F'(k_\theta + \theta)(k'_\theta + 1) - u'\Big(\varpi - \pi - k_\theta\Big)\Big(1 - F(k_\theta + \theta)\Big)k'_\theta \\ &= \int_{[0,k_\theta+\theta]} u'\Big(\varpi - \pi - x + \theta\Big)\,\mathrm{d}F(x) - u'\Big(\varpi - \pi - k_\theta\Big)\Big(1 - F(k_\theta + \theta)\Big)k'_\theta \\ &\leqslant \int_{[0,k_\theta+\theta]} u'\Big(\varpi - \pi - (k_\theta + \theta) + \theta\Big)\,\mathrm{d}F(x) - u'\Big(\varpi - \pi - k_\theta\Big)\Big(1 - F(k_\theta + \theta)\Big)k'_\theta \\ &= u'\Big(\varpi - \pi - k_\theta\Big)F(k_\theta + \theta) - u'\Big(\varpi - \pi - k_\theta\Big)\Big(1 - F(k_\theta + \theta)\Big)k'_\theta \\ &= u'\Big(\varpi - \pi - k_\theta\Big)\Big(F(k_\theta + \theta) - \Big(1 - F(k_\theta + \theta)\Big)k'_\theta\Big)\end{aligned}$$

for $\theta > 0$, where we used the concavity of the utility function $u(\cdot)$ to obtain the inequality. But we see from (3.26), (3.27) and (3.28) that

$$\begin{aligned}\big(1-F(k_\theta+\theta)\big)k'_\theta &= \big(1-F(k_\theta+\theta)\big)(\gamma'_\theta-1)\\ &= \big(1-F(k_\theta+\theta)\big)\left(\frac{-1}{G'(\gamma_\theta)}-1\right)\\ &= \big(1-F(k_\theta+\theta)\big)\left(\frac{-1}{F(k_\theta+\theta)-1}-1\right)\\ &= F(k_\theta+\theta).\end{aligned}$$

Therefore, we deduce $\hat{J}'(\theta)\leqslant 0$.

From the above discussion, we conclude that the optimal contract should be the full coverage one with the optimal bonus

$$\theta^* = \tfrac{\pi}{1+\rho} - \mathbf{E}[X]\,.$$

- When $\pi < (1+\rho)\,\mathbf{E}[X]$, the optimal contract must be a deductible one. We see from (3.23) that the optimal action is

$$A(x) = \begin{cases} x-k_\theta, & x>k_\theta+\theta, \quad 0\leqslant\theta<\tfrac{\pi}{1+\rho};\\ 0, & x\leqslant k_\theta+\theta, \quad 0\leqslant\theta<\tfrac{\pi}{1+\rho}. \end{cases} \tag{4.3}$$

  Same as the previous case, one can show that $\hat{J}(\theta)$ is decreasing on $\left[0,\tfrac{\pi}{1+\rho}\right)$, thus the optimal bonus is $\theta^*=0$ and correspondingly $k_{\theta^*}=G^{-1}\left(\tfrac{\pi}{1+\rho}\right)$ by (3.30).

We summarize the above results in the following

Theorem 4.1 (Optimal personalized contract). *The optimal contract is the full coverage contract with the bonus* $\frac{\pi}{1+\rho}-\mathbf{E}[X]$ *if* $\pi\geqslant(1+\rho)\,\mathbf{E}[X]$. *Otherwise, it is a deductible contract with zero bonus and the deductible* $G^{-1}\left(\frac{\pi}{1+\rho}\right)$, *where* $G(\cdot)$ *is defined by* (3.1).

We see that the optimal contract has either a zero deductible or a zero bonus.

5 Concluding remarks

In all cases, the optimal contract turns out to be universal, meaning that it does not depend on specific form of the utility function of the insured. This makes it very easy to implement the contract in practice.

In this paper, we have assumed that the insurer knows the insured's action and realized compensation so as to determine the participation constraint. If the insurer has no such knowledge, then the participation constraint will be

$$\pi \geqslant (1+\rho)\,\mathbf{E}[I(X)]\,.$$

This has been imposed in most insurance design literature. Such change of the participation constraint in our problem will make it very hard to tackle. On the other hand, this paper has considered a risk-averse EUT preference insured. It is clearly very important to study the problem for RDUT preference insureds. The method used in this paper may not be suitable to solve the problem in that case. We believe the quantile formulation should be adopted to tackle the problem; see, e.g., He and Zhou [12], Xu and Zhou [19], Xu [17], Hou and Xu [18]. The aforementioned problems will be addressed in forthcoming works.

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