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Optimal Investment Stopping Problem with Nonsmooth Utility in Finite Horizon

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Abstract

In this paper, we investigate an interesting and important stopping problem mixed with stochastic controls and a *nonsmooth* utility over a finite time horizon. The paper aims to develop new methodologies, which are significantly different from those of mixed dynamic optimal control and stopping problems in the existing literature, to figure out a manager's decision. We formulate our model to a free boundary problem of a fully *nonlinear* equation. By means of a dual transformation, however, we can convert the above problem to a new free boundary problem of a *linear* equation. Finally, using the corresponding inverse dual transformation, we apply the theoretical results established for the new free boundary problem to obtain the properties of the optimal strategy and the optimal stopping time to achieve a certain level for the original problem over a finite time investment horizon.

Keywords: Parabolic variational inequality; Free boundary; Optimal investment; Optimal stopping; Dual transformation.

Mathematics Subject Classification. 35R35; 60G40; 91B70; 93E20.

1 Introduction

Optimal stopping problems have important applications in many fields such as science, engineering, economics and, particularly, finance. The theory in this area has been well developed for stochastic dynamic systems over the past decades. In the field of financial investment, however, an investor frequently runs into investment decisions where investors stop investing

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in risky assets so as to maximize their expected utilities with respect to their wealth over a finite time investment horizon. These optimal stopping problems depend on underlying dynamic systems as well as investors' optimization decisions (controls). This naturally results in a mixed optimal control and stopping problem, and Ceci-Bassan (2004) is one of the typical representatives along this line of research. In the general formulation of such models, the control is mixed, composed of a control and a stopping time. The theory has also been studied in Bensoussan-Lions (1984), Elliott-Kopp (1999), Yong-Zhou (1999) and Fleming-Soner (2006), and applied in finance in Dayanik-Karatzas (2003), Henderson-Hobson (2008), Li-Zhou (2006), Li-Wu (2008, 2009), Shiryaev-Xu-Zhou (2008) and Jian-Li-Yi (2014).

In the finance field, finding an optimal stopping time point has been extensively studied for pricing American-style options, which allow option holders to exercise the options before or at the maturity. Typical examples that are applicable include, but are not limited to, those presented in Chang-Pang-Yong (2009), Dayanik-Karatzas (2003) and Rüschendorf-Urusov (2008). In the mathematical finance literature, choosing an optimal stopping time point is often related to a free boundary problem for a class of diffusions (see Fleming-Soner (2006) and Peskir-Shiryaev (2006)). In many applied areas, especially in more extensive investment problems, however, one often encounters more general controlled diffusion processes. In real financial markets, the situation is even more complicated when investors expect to choose as little time as possible to stop portfolio selection over a given investment horizon so as to maximize their profits (see Samuelson (1965), Karatzas-Kou (1998), Karatzas-Sudderth (1999), Karatzas-Wang (2000), Karatzas-Ocone (2002), Ceci-Bassan (2004), Henderson (2007), Li-Zhou (2006) and Li-Wu (2008, 2009)).

The initial motivation of this paper comes from our recent studies on choosing an optimal point at which an investor stops investing and/or sells all his risky assets (see Carpenter (2000) and Henderson-Hobson (2008)). The objective is to find an optimization process and stopping time so as to meet certaininvestment criteria, such as, the maximum of an expected nonsmooth utility value before or at the maturity. This is a typical yet important problem in thearea of financial investment. However, there are fundamental difficulties in handling such mixed controls and stopping problems. Firstly, our investment problem, which is signifcantly different from the classical American-style options, involves portfolio process in the objective over the entire time horizon. Secondly, itinvolves the portfolio in the drift and volatility terms of the dynamic systems so that the problem including multi-dimensional financial assets is more realistic than those addressed in finance literature (see Capenter (2000)). Therefore, it is difficult to solve these problems either analytically or numerically using current methods developed in the framework of studying American-style options. In our model, the corresponding HJB equation of the problem is formulated into a variational inequality of a fully *nonlinear* equation. We make a dual transformation for the problem to obtain a new free boundary problem with a *linear* equation. Tackling this new free boundary problem, we characterize the properties of the free boundary and optimal strategy for the original problem.

The main innovations of this paper include that: Firstly, we consider general non-smooth, non-concave utility function $g(x)$ and rigorously prove the limit of the value function when $t \to T$ is its concave hull $\varphi(x)$(see Theorem 2.1). Secondly, we prove the equivalence between the linear problem (3.4) and the original problem (2.12) under some easing restriction impose on $g(x)$. Thirdly, in a special model, we show a new method to study the free boundary while the exercise region is not connected (see (6.3)-(6.5) and Lemma 6.1) so that we can shed light on the monotonicity and differentiability of the free boundaries (see Figure 6.7-6.10.) under any cases of parameters.

In our previous works the closest one to this paper is Jian-Li-Yi (2014), where the utility function is smooth and concave. And the value function is continuous up to the terminal time T, moreover the exercise region is connected. So the problem in this paper is much difficult than the one in Jian-Li-Yi (2014).

The remainder of the paper is organized as follows. In Section 2, the mathematical formulation of the model is presented, and the corresponding HJB equation with certain boundary-terminal condition is posed. In Section 3, we make a dual transformation to convert the free boundary problem of a fully *nonlinear* PDE (2.12) to a new free boundary problem of a *linear* equation (3.4). Section 4 devotes to the study for variational inequality problem (3.4). In Section 5, using the corresponding inverse dual transformation, we construct the solution of the original problem (2.12). Section 6 gives a application of our results, moreover, under such a special model, we present the properties (including the monotonicity and differentiability) of its free boundaries under different cases. In Section 7, we present conclusions. Appendix gives the proof of Theorem 2.1. Appendix B gives the verification theorem to prove the solution of problem (2.12) is the value function defined in (2.3).

2 Model Formulation

2.1 The manager's problem

The manager operates in a complete, arbitrage-free, continuous-time financial market consisting of a riskless asset with instantaneous interest rate r and n risky assets. The risky asset prices S_i are governed by the stochastic differential equations

$$\frac{dS_{i,t}}{S_{i,t}} = (r + \mu_i)dt + \sigma_i dW_t^j, \quad \text{for } i = 1, 2, \cdots, n, \tag{2.1}$$

where the interest rate r, the excess appreciation rates μ_i, and the volatility vectors σ_i are constants, W is a standard n-dimensional Brownian motion. In addition, the covariance matrix $\sigma\sigma'$ is strongly nondegenerate.

A trading strategy for the manager is an n-dimensional process π_t whose i-th component, where $\pi_{i,t}$ is the holding amount of the i-th risky asset in the portfolio at time t. An admissible trading strategy π_t must be progressively measurable with respect to $\{\mathcal{F}_t\}$ such that $X_t \geq 0$. Note that $X_t = \pi_{0,t} + \sum\limits_{i=1}^{n} \pi_{i,t}$, where $\pi_{0,t}$ is the amount invested in the money. Hence, the wealth X_t evolves according to

$$dX_t = (rX_t + \mu'\pi_t)dt + \pi_t'\sigma dW_t,$$

the portfolio π_t is a progressively measurable and square integrable process with constraint $X_t \geq 0$ for all $t \geq 0$.

Now, we begin with any fix time t and suppose the wealth at the time t is x, then

$$\begin{aligned} &dX_s = (rX_s + \mu'\pi_s)ds + \pi_s'\sigma dW_s, \quad s \geq t, \\ &X_t = x, \end{aligned} \tag{2.2}$$

where $\pi_s,\ s \in [t, T]$ belongs to

$$\Pi_t := \{\pi_s \in L^2_{\mathcal{F}}([t, T]; \mathbb{R}) \mid X_s \geq 0,\ t \leq s \leq T\}.$$

The manager's dynamic problem is to choose an admissible trading strategy $\pi \in \Pi_t$ and a stopping time τ ($t \leq \tau \leq T$) to maximize his expected utility of the exercise wealth before or at the terminal time T:

$$V(x, t) = \sup_{\pi, \tau \geq t} \mathbb{E}_{t,x}[e^{-\beta(\tau - t)}g(X_\tau)] := \sup_{\pi, \tau \geq t} \mathbb{E}[e^{-\beta(\tau - t)}g(X_\tau)|X_t = x], \tag{2.3}$$

where β is the discounted factor. $g(x)$ is the utility function mapping from $[0, +\infty)$ onto $[0, +\infty)$.

In order to let (2.3) be well defined (be a finite function) and accordance with the actual case in finance, some constraints should be impose on $g(x)$. Without loss of generality, we suppose:

Condition I: The function $g(x)$ is non-negative increasing and satisfies

$$\lim_{x\to+\infty} g(x) = +\infty. \tag{2.4}$$

Condition II: There is a $\gamma \in (0,1)$ and $M > 0$ such that for all $x,\ y \geq 0$,

$$|g(x) - g(y)| \leq M\frac{1}{\gamma}|x-y|^{\gamma}. \tag{2.5}$$

which also implies a growth condition that

$$g(x) \leq g(0) + M\frac{1}{\gamma}x^{\gamma}. \tag{2.6}$$

Condition III: The function $g(x)$ is twice differentiable piecewise and

$$g''(x) > -\infty, \quad \forall x > 0, \tag{2.7}$$

which is equivalent to

$$g'(x-) \leq g'(x+), \quad \forall x > 0.$$

2.2 The boundary condition on $x = 0$

Here, we prove the boundary condition on $x = 0$. If $X_t = 0$, in order to keep $X_s \geq 0$, the only choice of π_s is 0 and thus $X_s \equiv 0,\ t \leq s \leq T$. Therefore

$$V(0,t) = \sup_{\pi,\tau\geq t} \mathbb{E}_{t,x}[e^{-\beta(\tau-t)}g(0)] = g(0).$$

Which means the optimal stopping time τ is the present moment t.

2.3 The terminal condition under $g(x)$ is Non-concave

If $g(x)$ is non-concave, denote $\varphi(x)$ as its concave hull, i.e. $\varphi(x)$ is the minimal concave function not less than $g(x)$ (See Figure 2.1).

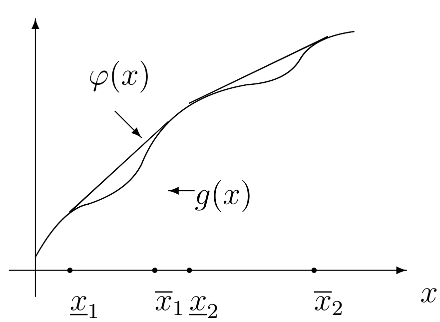

Fig 2.1 $\varphi(x)$.

Because $g(x)$ is an increasing continuous function, thus $\varphi(x)$ is also increasing and continuous, so $\{x > 0|\varphi(x) > g(x)\}$ is an open set which can be written as (in general case)

$$\{\varphi(x) > g(x)\} = \bigcup_{m=1}^{\infty} (\underline{x}_m, \overline{x}_m), \tag{2.8}$$

where $\{(\underline{x}_m, \overline{x}_m)\}_{m=1}^{\infty}$ are countable disjoint open intervals. Inner these intervals, $\varphi(x)$ is a linear function.

It is not hard to see that $\varphi(x)$ also satisfies (2.4)- (2.7) and $\varphi(0) = g(0)$.

Since the portfolio π_t is unconstrained, we point out that the terminal condition of $V(x,t)$ should be $\varphi(x)$ but not $g(x)$. In fact, in a short time, the asset price is behavior like a martingale. When time approaches the terminal date and the current asset price x located in $(\underline{x}_m, \overline{x}_m)(m \in \mathbb{Z})$, the investors could adopt such a strategy that buy much risk assets to enlarge the volatility and then X_s will rapidly touch $\underline{x}_m$ or $\overline{x}_m$ (with probability approximately equal to $\frac{x-\underline{x}_m}{\overline{x}_m-\underline{x}_m}$ and $\frac{\overline{x}_m-x}{\overline{x}_m-\underline{x}_m}$, respectively) and then keep it still (stop invest in risk assets). Therefore, the contribution of $\mathbb{E}_{t,x}g(X_T)$ to the value function is approximate to $\frac{x-\underline{x}_m}{\overline{x}_m-\underline{x}_m}g(\underline{x}_m) + \frac{\overline{x}_m-x}{\overline{x}_m-\underline{x}_m}g(\overline{x}_m) = \varphi(x)$. So the value function is not less than $\varphi(x)$ when $t \to T$. Under this idea, we could prove

Theorem 2.1 *Under* **Condition II**, $V(x,t)$ *defined in* (2.3) *satisfies*

$$\lim_{t \to T-} V(x,t) = \varphi(x). \tag{2.9}$$

The rigorous proof is presented in Appendix A.

2.4 The HJB equation

Using the theory of viscosity solution in differential equations(See e.g. Crandall and Lions(1983), Fleming and Soner(2006)), one can obtain the following HJB equation

$$\min\Big\{ -V_t - \max_{\pi}[\frac{1}{2}(\pi'\sigma\sigma'\pi)V_{xx} + \mu'\pi V_x] - rxV_x + \beta V,\ V - g(x)\Big\} = 0 \tag{2.10}$$

Note that the Hamiltonian operator

$$\max_{\pi}\Big[\frac{1}{2}(\pi'\sigma\sigma'\pi)V_{xx} + \mu'\pi V_x\Big] - rxV_x + rV$$

is singular if $V_{xx} > 0$, or $V_{xx} = 0$ and $V_x \neq 0$, thus, $V_{xx} \leq 0$. Moreover, if $V_x = 0$ holds on (x_0, t_0), then for all $x \geq x_0$, $V_x(x, t_0) \leq 0$, which contradicts $V(x,t) \geq g(x) \to +\infty,\ x \to +\infty$. The above analysis gives us reason to find the solution of (2.10) satisfies

$$V_x > 0, \quad V_{xx} < 0, \quad x > 0,\ 0 < t < T. \tag{2.11}$$

Note that the gradient of $\pi'\sigma\sigma'\pi$ with respect to π is

$$\nabla_\pi(\pi'\sigma\sigma'\pi) = 2\sigma\sigma'\pi.$$

Hence, the optimal

$$\pi^* = -(\sigma\sigma')^{-1}\mu\frac{V_x(x,t)}{V_{xx}(x,t)}.$$

Applying $V_{xx} < 0$, we have

$$V - g(x) \geqslant 0 \quad \text{if and only if} \quad V - \varphi(x) \geqslant 0.$$

Define $a^2 = \mu'(\sigma\sigma')^{-1}\mu$, then the variational inequality (2.10) is reduce to

$$\min\Big\{ -V_t + \frac{a^2}{2}\frac{V_x^2}{V_{xx}} - rxV_x + \beta V,\ V - \varphi(x)\Big\} = 0,$$

Hence, we formulate our problem into the following variational inequality problem

$$\begin{cases} \min\Big\{ -V_t + \frac{a^2}{2}\frac{V_x^2}{V_{xx}} - rxV_x + \beta V,\ V - \varphi(x)\Big\} = 0, \quad x > 0,\ 0 < t < T, \\ V(0,t) = g(0), \quad 0 < t < T, \\ V(x,T) = \varphi(x), \quad x > 0. \end{cases} \tag{2.12}$$

We will show that problem (2.12) has a solution $\widehat{V}(x,t)$ in the sense that

$$\widehat{V} \in C([0,+\infty) \times [0,T]), \tag{2.13}$$

$$\widehat{V}_x \in C((0,+\infty) \times (0,T)), \tag{2.14}$$

$$\widehat{V}_t \in C((0,+\infty) \times (0,T)), \tag{2.15}$$

$$\widehat{V}_{xx} \in L^\infty_{loc}((0,+\infty) \times (0,T)). \tag{2.16}$$

Also, in Appendix B we will present the verification theorem to ensure this solution is just V defined in (2.3).

2.5 The Höder continuity (w.r.t. x) of the value function

We first introduce the case of $g(x) = \frac{1}{\gamma}x^\gamma$, we can get the expression of solution "from scratch" that

$$V(x,t) = e^{B(T-t)}\frac{1}{\gamma}x^\gamma,$$

where

$$B = \max\Big\{\frac{a^2}{2}\frac{\gamma}{1-\gamma} + r\gamma - \beta,\ 0\Big\}. \tag{2.17}$$

that means if $\frac{a^2}{2}\frac{\gamma}{1-\gamma}+r\gamma>\beta$, $V(x,t)>g(x)=\frac{1}{\gamma}x^\gamma$ for all $t<T$, i.e. the optimal stopping time $\tau^*=T$; Otherwise, $\frac{a^2}{2}\frac{\gamma}{1-\gamma}+r\gamma\leq\beta$, $V(x,t)=\frac{1}{\gamma}x^\gamma$, the optimal stopping time is present moment, i.e. $\tau^*=t$.

Lemma 2.2 *Under* **Condition II**, *the value function defined in* (2.3) *satisfies*

$$|V(x,t)-V(y,t)|\leq Me^{B(T-t)}\frac{1}{\gamma}|x-y|^\gamma$$

for $M>0$ *determined by* (2.5).

Proof: Suppose (X_s,π_s) satisfies (2.2) and $\pi\in\Pi_t$ with initial condition $X_t=x$. For $y<x$, let $(Y_s,\pi_s^Y)=(\frac{y}{x}X_s,\frac{y}{x}\pi_s)$, $(Z_s,\pi_s^Z)=(\frac{x-y}{x}X_s,\frac{x-y}{x}\pi_s)$, then (Y_s,π_s^Y), (Z_s,π_s^Z) also satisfy the equation in (2.2) with initial conditions $Y_t=y$, $Z_t=x-y$, respectively, and π^Y, $\pi^Z\in\Pi_t$. For any stopping time $\tau\in[t,T]$, owing to (2.5), we have

$$\begin{aligned}
\mathbb{E}_{t,x}[e^{-\beta(\tau-t)}g(X_\tau)] &= \mathbb{E}_{t,x}[e^{-\beta(\tau-t)}g(Y_\tau)]+\mathbb{E}_{t,x}[e^{-\beta(\tau-t)}[g(X_\tau)-g(Y_\tau)]]\\
&\leq \mathbb{E}_{t,x}[e^{-\beta(\tau-t)}g(Y_\tau)]+M\mathbb{E}_{t,x}[e^{-\beta(\tau-t)}[\frac{1}{\gamma}(X_\tau-Y_\tau)^\gamma]]\\
&\leq \mathbb{E}_{t,x}[e^{-\beta(\tau-t)}g(Y_\tau)]+M\mathbb{E}_{t,x}[e^{-\beta(\tau-t)}\frac{1}{\gamma}Z_\tau^\gamma]\\
&\leq \sup_{\pi^Y,\tau}\mathbb{E}_{t,x}[e^{-\beta(\tau-t)}g(Y_\tau)]+M\sup_{\pi^Z,\tau}\mathbb{E}_{t,x}[e^{-\beta(\tau-t)}\frac{1}{\gamma}Z_\tau^\gamma],
\end{aligned}$$

the second term above is the value function of the case $g(x)=\frac{1}{\gamma}x^\gamma$, so

$$\mathbb{E}_{t,x}[e^{-\beta(\tau-t)}g(X_\tau)]\leq V(y,t)+Me^{B(T-t)}\frac{1}{\gamma}(x-y)^\gamma.$$

Taking supremum to the left, we have

$$V(x,t)\leq V(y,t)+Me^{B(T-t)}\frac{1}{\gamma}(x-y)^\gamma.$$

□

Thanks to Lemma 2.2, we have

$$V(x,t)\leq g(0)+Me^{B(T-t)}\frac{1}{\gamma}x^\gamma. \tag{2.18}$$

(2.18) gives a growth condition to problem (2.12).

3 Dual Problem

In this section, we will formulate the duel problem (3.4) subject to the original problem (2.12). After this section, we will begin with problem (3.4), using inverse dual transformation to construct the solution to problem (2.12).

3.1 The duel transformation of $\varphi(x)$

Firstly, we introduce the concept of dual transformation.

Definition 3.1 *If* $u : (0, +\infty) \to \mathbb{R}$ *is increasing, concave on* $(0, +\infty)$, *then the duel transformation of it is the function* $\widetilde{u} : (0, +\infty) \to \mathbb{R} \cup \{+\infty\}$ *that*

$$\widetilde{u}(y) = \sup_{x>0}(u(x) - xy), \quad y > 0.$$

The next proposition collects some results used in this section.

Proposition 3.2 $\widetilde{u}$ *is a decreasing function, convex on* $(0, +\infty)$, *and we have the conjugate relation*

$$u(x) = \inf_{y>0}(\widetilde{u}(y) + xy), \quad x > 0.$$

Denote $dom(\widetilde{u}) = \{y > 0 : \widetilde{u}(y) < +\infty\}$. *Suppose one of the two following equivalent conditions:*

(i) u is differentiable on $(0, +\infty)$

(ii) $\tilde{u}$ *is strictly convex on* $int(dom(\widetilde{u}))$

is satisfied, then the derivative u' *is a mapping from* $(0, +\infty)$ *into* $int(dom(\widetilde{u})) \neq \emptyset$ *and we have*

$$u'(x) = \arg\min_{y\geq 0}(\widetilde{u}(y) + xy), \quad \forall x > 0.$$

Moreover, we can define $\widetilde{u}'(y\pm) = \lim\limits_{z\to y\pm} \frac{\widetilde{u}(z)-\widetilde{u}(y)}{z-y}$, *then*

$$\widetilde{u}'(y-) \leq \widetilde{u}'(y+) \leq 0, \quad \forall y \in dom(\widetilde{u}),$$

and

$$\arg\max_{x\geq 0}(u(x) - xy) = \{x \geq 0 : u'(x) = y\} = [-\widetilde{u}'(y+), -\widetilde{u}'(y-)], \quad \forall y \in dom(\widetilde{u}).$$

If we further suppose that u is strictly concave, then $\widetilde{u}$ is differentiable with $\widetilde{u}'(y) = -(u')^{-1}(y)$.

Finally, under the additional conditions

$$u'(0) = +\infty, \quad u'(+\infty) = 0,$$

we have $int(dom(\widetilde{u})) = dom(\widetilde{u}) = (0, +\infty)$.

Proof: See Appendix B of Pham [25]. $\square$

Now, let us define the duel transformation of $\varphi(x)$ as

$$\psi(y) = \sup_{x>0}(\varphi(x) - xy), \quad y > 0.$$

(see Figure 3.1)

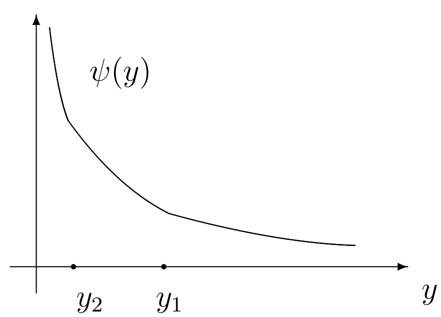

Fig 3.1 $\psi(y)$.

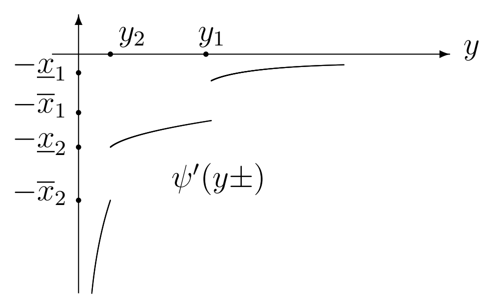

Fig 3.2 $\psi'(y\pm)$.

Then, by Proposition 3.2, $\psi(y)$ is a decreasing and convex function and

$$\varphi(x) = \inf_{y>0}(\psi(y) + xy).$$

Because $\varphi(x)$ is not strictly concave, thus $\psi(y)$ is not continuously differentiable. However, since $\psi(y)$ is convex, we can define

$$\psi'(y\pm) = \lim_{z \to y\pm} \frac{\psi(z) - \psi(y)}{z - y}.$$

Corresponding to the description of $\varphi(x)$ in (2.8) we can define

$$y_m = \varphi'(x), \quad x \in (\underline{x}_m, \overline{x}_m), \quad m = 1, 2, ...,$$

and we have

$$\psi'(y_m+) = -\underline{x}_m, \quad \psi'(y_m-) = -\overline{x}_m, \quad m = 1, 2, ...$$

(see Figure 3.2).

On the other hand, by (2.6), we have

$$\psi(y)=\sup_{x>0}(\varphi(x)-xy)\leq\sup_{x>0}\Big(g(0)+M\frac{1}{\gamma}x^{\gamma}-xy\Big)=g(0)+M^{\frac{1}{1-\gamma}}\frac{1-\gamma}{\gamma}y^{\frac{\gamma}{\gamma-1}},\qquad(3.1)$$

due to (2.4),

$$\psi(y)=\sup_{x>0}(\varphi(x)-xy)\geq\varphi(\frac{1}{y})-1\to+\infty,\quad y\to0.\qquad(3.2)$$

and we will use them latter.

3.2 The duel problem of (2.12)

Let us define dual transformation of $V(x,t)$. For any $t\in(0,T)$, let

$$v(y,t):=\max_{x\geq0}(V(x,t)-xy),\quad y>0.\qquad(3.3)$$

Then we can formulate the duel problem subject to (2.12) that

$$\begin{cases}\min\{-v_t-\dfrac{a^2}{2}y^2v_{yy}-(\beta-r)yv_y+\beta v,\ v-\psi\}=0,\quad(y,t)\in Q_y,\\ v(y,T)=\psi(y),\quad y>0,\end{cases}\qquad(3.4)$$

where, $Q_y=\{y>0,\ 0<t<T\}$. The derivation is left to the interested reader.

Remark 3.3 *The equation in (3.4) is degenerate on the boundary $y=0$. According to Fichera's theorem (see Oleĭnik and Radković [23]), we must not put the boundary condition on $y=0$.*

Owing to (2.18),

$$\begin{aligned}v(y,t)=\max_{x\geq0}(V(x,t)-xy)&\leq\max_{x\geq0}\Big(g(0)+Me^{B(T-t)}\frac{1}{\gamma}x^{\gamma}-xy\Big)\\&=g(0)+(Me^{B(T-t)})^{\frac{1}{1-\gamma}}\frac{1-\gamma}{\gamma}y^{\frac{\gamma}{\gamma-1}},\end{aligned}$$

which give a growth condition to problem (3.4).

4 The solution to (3.4)

In this section, we will prove the existence and uniqueness of problem (3.4) and some properties, in preparation for the construction of solution to problem (2.12).

Theorem 4.1 *Problem* (3.4) *has a unique solution* $v(y,t) \in W^{2,1}_{p,\,loc}(Q_y) \cap C(Q_y \cup \{t = 0, T\})$ *for any* $p > 2$. *Moreover, under* **Condition I** *and* **II**,

$$\psi(y) \le v \le M^{\frac{1}{1-\gamma}} \frac{1-\gamma}{\gamma} e^{\frac{B}{1-\gamma}(T-t)} y^{\frac{\gamma}{\gamma-1}} + g(0), \tag{4.1}$$

$$v_t \le 0, \tag{4.2}$$

$$v_y \le 0, \tag{4.3}$$

$$v_{yy} \ge 0, \tag{4.4}$$

where, M, B *are defined in* (2.5) *and* (2.17), *respectively.*

Proof: According to the results of existence and uniqueness of $W^{2,1}_p$ solutions[19], the solution of system (3.4) can be proved by a standard penalty method(see Friedman [10]). Furthermore, by Sobolev embedding theorem,

$$v_y \in C(Q_y), \tag{4.5}$$

and the method of [10] further gives

$$v_t \in C(Q_y), \tag{4.6}$$

here, we omit the details. The first inequality in (4.1) follows from (3.4) directly, we now prove the second inequality in (4.1).

Denote

$$w(y,t) = M^{\frac{1}{1-\gamma}} \frac{1-\gamma}{\gamma} e^{\frac{B}{1-\gamma}(T-t)} y^{\frac{\gamma}{\gamma-1}} + g(0),$$

then

$$\begin{aligned}
& -w_t - \frac{a^2}{2} y^2 w_{yy} - (\beta - r) y w_y + \beta w \\
= \;& M^{\frac{1}{1-\gamma}} \frac{1-\gamma}{\gamma} e^{\frac{B}{1-\gamma}(T-t)} y^{\frac{\gamma}{\gamma-1}} \Big(\frac{B}{1-\gamma} - \frac{a^2}{2} \Big(\frac{\gamma}{\gamma-1}\Big)\Big(\frac{\gamma}{\gamma-1} - 1\Big) - (\beta - r)\Big(\frac{\gamma}{\gamma-1}\Big) + \beta \Big) + \beta g(0) \\
= \;& M^{\frac{1}{1-\gamma}} \frac{1}{\gamma} e^{\frac{B}{1-\gamma}(T-t)} y^{\frac{\gamma}{\gamma-1}} \Big(B - \frac{a^2}{2} \frac{\gamma}{1-\gamma} - r\gamma + \beta \Big) + \beta g(0) \\
\ge \;& 0,
\end{aligned}$$

and $w(y,T) = M^{\frac{1}{1-\gamma}}\frac{1-\gamma}{\gamma}y^{\frac{\gamma}{\gamma-1}} + g(0) \geq \psi(y)$(see (3.1)). Using the comparison principle of variational inequality (see Friedman [10]), we know that w is a super solution of (3.4).

Next we prove (4.2). Let $\widetilde{v}(y,t) = v(y,t-\delta)$ for small $\delta > 0$, then $\widetilde{v}$ satisfies

$$\begin{cases} \min\{-\widetilde{v}_t - \frac{a^2}{2}y^2\widetilde{v}_{yy} - (\beta - r)y\widetilde{v}_y + \beta\widetilde{v},\ \widetilde{v} - \psi(y)\} = 0, & y > 0,\ \delta < t < T, \\ \widetilde{v}(y,T) \geq \psi(y), \quad y > 0. \end{cases}$$

Hence, by the comparison principle, we have $\widetilde{v} \geq v$, i.e. $v_t \leq 0$.

Define

$$\begin{aligned} \varepsilon\mathcal{R}_y &= \{(y,t) \in Q_y | v = \psi\}, \quad \textbf{exercise region}, \\ \mathcal{CR}_y &= \{(y,t) \in Q_y | v > \psi\}, \quad \textbf{continuation region}. \end{aligned}$$

Note that $(y_m)_{m=1,2...}$ are the discontinuous points of $\psi'(y)$ and $\psi''(y)$. Now, we claim $(y_m,t),\ m = 1,2...,\ t \in (0,T)$ could not be contained in $\varepsilon\mathcal{R}_y$. Otherwise, if $(y_m,t) \in \varepsilon\mathcal{R}_y$ for some m and $t < T$, then it belongs to the minimum points of $v - \psi(y)$, thus $v_y(y_m-,t) \leq \psi'(y_m-) < \psi'(y_m+) \leq v_y(y_m+,t)$, which implies v_y does not continue at the point (y_m,t), that contradicts (4.5).

Here, we present the proof of (4.3) and (4.4). Recall that

$$\psi' \leq 0, \quad \psi'' \geq 0, \quad y \neq y_m,\ m = 1,2....$$

Note that

$$v_y = \psi' \leq 0, \quad (y,t) \in \varepsilon\mathcal{R}_y. \tag{4.7}$$

Taking the derivative for the following equation

$$-v_t - \frac{a^2}{2}y^2 v_{yy} - (\beta - r)yv_y + \beta v = 0 \quad \text{in} \quad \mathcal{CR}_y$$

with respect to y leads to

$$-\partial_t v_y - \frac{a^2}{2}y^2\partial_{yy}v_y - (a^2 + \beta - r)y\partial_y v_y + rv_y = 0 \quad \text{in} \quad \mathcal{CR}_y. \tag{4.8}$$

Note that $v_y = \psi' \leq 0$ on $\partial(\mathcal{CR}_y)$, where $\partial(\mathcal{CR}_y)$ is the boundary of $\mathcal{CR}_y$ in the interior of Q_y, using the maximum principle we obtain

$$v_y \leq 0, \quad (y,t) \in \mathcal{CR}_y \tag{4.9}$$

(4.7) and (4.9) yields (4.3).

In addition, $\psi''(y) \geq 0$ and $v(y,t) \geq \psi(y)$ implies

$$\lim_{\mathcal{CR}_y \ni y \to \partial(\mathcal{CR}_y)} v_{yy}(y,t) \geq 0.$$

and $v_{yy}(y,T) = \psi'' \geq 0$. Taking the derivative for equation (4.8) with respect to y, we have

$$-\partial_t v_{yy} - \frac{a^2}{2} y^2 \partial_{yy} v_{yy} - 2a^2 y \partial_y v_{yy} + (r - a^2) v_{yy} = 0 \quad \text{in} \quad \mathcal{CR}_y.$$

Using the maximum principle, we obtain

$$v_{yy} \geq 0 \quad \text{in} \quad \mathcal{CR}_y. \tag{4.10}$$

Together with $v_{yy} \geq \psi''(y) \geq 0$ in $\varepsilon\mathcal{R}_y$ we have (4.4). □

5 The solution to original problem (2.12)

Thanks to the properties of solution to problem (3.4), we could construct the solution to (2.12). Before that, we should research a spacial free boundary line

$$f(t) := \inf\{y > 0 | v(y,t) = g(0)\}.$$

We will prove that $f(t) = \widehat{V}_x(0,t)$ in Theorem 5.4, where $\widehat{V}$ is the solution to (2.12) defined in (5.8).

Define

$$k := \inf\{y > 0 | \psi(y) = g(0)\}.$$

Note that $\psi(y) = g(0)$, i.e. $\max\limits_{x \geq 0}(\varphi(x) - xy) = \varphi(0)$ is equivalent to $y \geq \varphi'(0)$, thus

$$k = \varphi'(0),$$

and $v(y,t) \geq \psi(y)$ implies

$$f(t) \geq k.$$

Lemma 5.1 *Under* **Condition I-III**, *the solution to* (3.4) *satisfies*

$$v_{yy} > 0, \quad 0 < y < f(t), \quad 0 < t < T. \tag{5.1}$$

Proof: Apply strong maximum principle,

$$v_{yy} > 0 \quad in \quad \mathcal{CR}_y,$$

Since $v-\psi$ takes minimal value 0 in $\varepsilon\mathcal{R}_y$, thus

$$v_{yy} \geq \psi'' \quad \text{in} \quad \varepsilon\mathcal{R}_y.$$

For $(y,t) \in \varepsilon\mathcal{R}_y \cap \{0<y<f(t)\}$, by the definition of $f(t)$ and k, we have $\varepsilon\mathcal{R}_y \cap \{k<y<f(t)\} = \emptyset$, so $(y,t) \in \varepsilon\mathcal{R}_y \cap \{0<y<k\}$. Since $y \neq y_m \ \forall m \in \mathbb{Z}_+$(see the prove of Theorem 4.1),

$$v_{yy} \geq \psi''(y) = -[-\psi'(y)]' = -[(\varphi')^{-1}(y)]' = -\frac{1}{\varphi''(-\psi'(y))} = -\frac{1}{g''(-\psi'(y))} > 0,$$

the last inequality is due to **Condition III**. Then (5.1) follows. $\square$

Lemma 5.2 *Under* **Condition I-III**, *the solution to* (3.4) *satisfies*

$$\lim_{y\to 0+} v_y(y,t) = -\infty, \quad 0<t<T, \tag{5.2}$$

$$\lim_{y\to f(t)-} v_y(y,t) = 0, \quad 0<t<T. \tag{5.3}$$

Proof: For any $t \in (0,T)$, it is not hard to see that $\lim\limits_{y\to 0+} v(y,t) \geq \lim\limits_{y\to 0+} \psi(y) = +\infty$(see (3.2)), and by $v_{yy} \geq 0$, thus for some fix $y_0>0$,

$$v_y(y,t) \leq \frac{v(y_0,t)-v(y,t)}{y_0-y} \to -\infty, \quad y \to 0+.$$

Let us prove (5.3). If $f(t) < +\infty$, the results follows from the continuity of v_y and $v_y(y,t) = \psi'(y) = 0$ for $y > f(t)$; If $f(t) = +\infty$, owing to $v_{yy} > 0$ for any $y > 0$, there exists

$$v_y(y,t) \geq \frac{v(y,t)-v(\frac{y}{2},t)}{\frac{y}{2}}.$$

Using (4.1),

$$v_y(y,t) \geq \frac{\psi(y)-g(0)-M^{\frac{1}{1-\gamma}}e^{A(T-t)}\frac{1-\gamma}{\gamma}\left(\frac{y}{2}\right)^{\frac{\gamma}{\gamma-1}}}{\frac{y}{2}} \geq -Cy^{\frac{1}{\gamma-1}} \to 0, \quad y\to+\infty.$$

Combine with $v_y \leq 0$, we obtain (5.3). $\square$

Thanks to Lemma 5.1 and Lemma 5.2, we can define a transformation

$$y = J(x,t) = \begin{cases} (v_y(\cdot,t))^{-1}(-x), & \text{for} \quad x>0; \\ f(t), & \text{for} \quad x=0, \end{cases}, \ 0<t<T, \tag{5.4}$$

then we have

$$J(x,t) \in C((0,+\infty)\times(0,T)). \tag{5.5}$$

Lemma 5.3 *The function $J(x,t)$ defined in* (5.4) *satisfies*

$$\lim_{x\to 0+} J(x,t) = f(t), \quad 0<t<T, \tag{5.6}$$

$$\lim_{x\to +\infty} J(x,t) = 0, \quad 0<t<T. \tag{5.7}$$

PROOF: (5.6) and (5.7) are the results of (5.3) and (5.2), respectively. $\square$

Now, we set

$$\widehat{V}(x,t) = \min_{y\geq 0}(v(y,t)+xy). \tag{5.8}$$

From Lemma 5.1 and Lemma 5.2, it is easily seen that $J(x,t)\in \arg\min_{y>0}(v(y,t)+xy)$ for all $(x,t)\in Q_x$, thus

$$\widehat{V}(x,t) = v(J(x,t),t) + xJ(x,t), \quad (x,t)\in Q_x. \tag{5.9}$$

Theorem 5.4 $\widehat{V}$ *which is defined in* (5.9) *is the solution to* (2.12) *satisfying* (2.13)-(2.16) *and the following estimations*

$$\varphi(x) \leq \widehat{V}(x,t) \leq g(0) + Me^{B(T-t)}\frac{1}{\gamma}x^{\gamma}, \tag{5.10}$$

$$\widehat{V}_t \leq 0, \tag{5.11}$$

$$\widehat{V}_x > 0, \tag{5.12}$$

$$\widehat{V}_{xx} < 0. \tag{5.13}$$

Moreover,

$$\lim_{x\to 0+}\widehat{V}_x(x,t) = f(t), \quad \lim_{x\to +\infty}\widehat{V}_x(x,t) = 0, \quad \forall t\in(0,T).$$

PROOF: Firstly, due to the first inequlity in (4.1),

$$\widehat{V}(x,t) = \min_{y>0}(v(y,t)+xy) \geq \min_{y>0}(\psi(y)+xy) \geq \varphi(x).$$

Due to the second inequlity in (4.1),

$$\widehat{V}(x,t) \leq \min_{y>0}\left(M^{\frac{1}{1-\gamma}}\frac{1-\gamma}{\gamma}e^{\frac{B}{1-\gamma}(T-t)}y^{\frac{\gamma}{\gamma-1}} + g(0) + xy\right) = g(0) + Me^{B(T-t)}\frac{1}{\gamma}x^{\gamma}.$$

In addition

$$\widehat{V}_x(x,t) = v_y(J(x,t),t)J_x(x,t) + xJ_x(x,t) + J(x,t) = J(x,t) \geq 0, \tag{5.14}$$

$$\widehat{V}_{xx}(x,t) = J_x(x,t) = \partial_x[(v_y(\cdot,t))^{-1}(x)] = \frac{-1}{v_{yy}(J(x,t),t)} < 0, \tag{5.15}$$

$$\widehat{V}_t(x,t) = v_y(J(x,t),t)J_t(x,t) + v_t(J(x,t),t) + xJ_t(x,t) = v_t(J(x,t),t) \leq 0, \tag{5.16}$$

So

$$\lim_{x\to 0+}\widehat{V}_x(x,t)=\lim_{x\to 0+}J(x,t)=f(t),\quad 0<t<T.$$
$$\lim_{x\to +\infty}\widehat{V}_x(x,t)=\lim_{x\to +\infty}J(x,t)=0,\quad 0<t<T.$$

Together with (5.5), (5.8), (5.14), (5.15), (5.16), (4.6) and (5.1) imply (2.13)-(2.16).

Now, we verify $\widehat{V}$ is the solution to (2.12). Firstly, taking limits $x\to 0+$ to (5.10), the boundary condition $\widehat{V}(0+,t)=g(0)$ holds.

Secondly, thanks to (4.1), which implies $v(y,t)\geq\psi(y)$, Note that

$$\widehat{V}(x,t)=\min_{y\geq 0}(v(y,t)+xy)\leq v(\varphi'(x),t)+x\varphi'(x).$$

Let $t\to T-$ we get

$$\limsup_{t\to T-}\widehat{V}(x,t)\leq\lim_{t\to T-}v(\varphi(x),t)+x\varphi(x)=\psi(\varphi'(x))+x\varphi'(x)=\varphi(x).\tag{5.17}$$

together with $\widehat{V}(x,t)\geq\varphi(x)$, the terminal condition

$$\lim_{t\to T-}\widehat{V}(x,t)=\varphi(x)$$

is met. Thirdly, we come to verify the variational inequality in (2.12). Due to (5.14), (5.15) and (5.16),

$$\Big(-\widehat{V}_t+\frac{a^2}{2}\frac{\widehat{V}_x^2}{\widehat{V}_{xx}}-rx\widehat{V}_x+\beta\widehat{V}\Big)(x,t)=\Big(-v_t-\frac{a^2}{2}y^2v_{yy}-(\beta-r)yv_y+\beta v\Big)(J(x,t),t)\geq 0,\tag{5.18}$$

together with the first ineqality in (5.10), we have

$$\min\Big\{-\widehat{V}_t+\frac{a^2}{2}\frac{\widehat{V}_x^2}{\widehat{V}_{xx}}-rx\widehat{V}_x+\beta\widehat{V},\ \widehat{V}-\varphi\Big\}\geq 0\quad\text{in}\quad Q_x.$$

It remains to prove that

$$\widehat{V}(x,t)>\varphi(x)\Rightarrow\Big(-\widehat{V}_t+\frac{a^2}{2}\frac{\widehat{V}_x^2}{\widehat{V}_{xx}}-rx\widehat{V}_x+\beta\widehat{V}\Big)(x,t)=0.\tag{5.19}$$

Before that we first claim

$$\widehat{V}(x,t)>\varphi(x)\Rightarrow v(J(x,t),t)>\psi(J(x,t)).\tag{5.20}$$

If $v(J(x,t),t)=\psi(J(x,t))$, then

$$x=-v_y(J(x,t),t)=-\psi'(J(x,t)),$$

so

$$J(x,t) = (\psi')^{-1}(-x),$$

$$\widehat{V}(x,t) = v(J(x,t),t) + xJ(x,t) = \psi((\psi')^{-1}(-x)) + x(\psi')^{-1}(-x) = \varphi(x).$$

So we have (5.20).

Combining with (5.20), the variational inequality in (3.4) and (5.18) yields

$$\begin{aligned}
&\widehat{V}(x,t) > \varphi(x) \\
\Rightarrow\ & v(J(x,t),t) > \psi(J(x,t)) \\
\Rightarrow\ & \Big(- v_t - \tfrac{a^2}{2} y^2 v_{yy} - (\beta - r) y v_y + \beta v \Big)(J(x,t),t) = 0 \\
\Rightarrow\ & \Big(- \widehat{V}_t + \tfrac{a^2}{2} \tfrac{\widehat{V}_x^2}{\widehat{V}_{xx}} - r x \widehat{V}_x + \beta \widehat{V} \Big)(x,t) = 0.
\end{aligned}$$

Therefore, $\widehat{V}(x,t)$ satisfies the variational inequality in (2.12). So far, we have proved $\widehat{V}(x,t)$ is the solution to (2.12). □

6 An Example: Stopping Problem of a Call Option with Risk Averse Utility

Consider the manager's wealth at the terminal date T is the payoff of a call option on the assets plus a constant $K > 0$, that includes fixed compensation and personal wealth, suppose the strike price is $b > 0$, so the wealth at T is

$$W_T = (X_T - b)^+ + K.$$

The manager chooses an investment policy to maximize his expected utility of wealth at T. His utility function U on behavior of risk averse, is strictly increasing, strictly concave function

$$U(W) = \frac{1}{\gamma} W^\gamma$$

with $0 < \gamma < 1$.

So in this model,

$$g(x) := U((x-b)^+ + K) = \frac{1}{\gamma}((x-b)^+ + K)^\gamma.$$

It is straightforward to verify that it meets **Condition I-III**. Its concave hull

$$\varphi(x) = \begin{cases} kx + \frac{1}{\gamma} K^\gamma, & 0 < x < \widehat{x}, \\ \frac{1}{\gamma}(x - b + K)^\gamma, & x \geq \widehat{x}, \end{cases}$$

where k and $\widehat{x}$ satisfy

$$\begin{cases} k\widehat{x} + \frac{1}{\gamma}K^{\gamma} = \frac{1}{\gamma}(\widehat{x} - b + K)^{\gamma}, \\ k = (\widehat{x} - b + K)^{\gamma-1}. \end{cases} \tag{6.1}$$

see Figure 6.1.

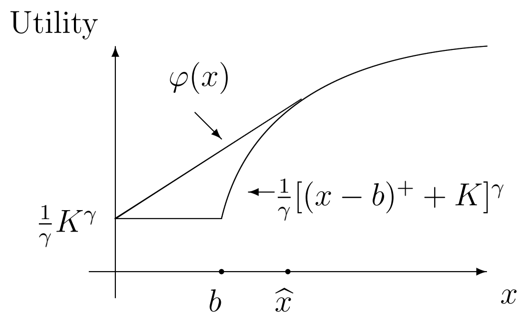

Fig 6.1 $\varphi(x)$.

6.1 The free boundary to (3.4)

The duel transformation of $\varphi(x)$:

$$\psi(y) = \max_{x \geq 0}(\varphi(x) - xy), \quad y > 0,$$

can be obtain by the following derivation:

The optimal x to fix y, which we denote by $x_{\varphi}(y)$, is

$$x_{\varphi}(y) = \begin{cases} y^{\frac{1}{\gamma-1}} - (K - b), & \text{for } 0 < y < k, \\ \in [0, \widehat{x}], & \text{for } y = k, \\ 0, & \text{for } y > k. \end{cases}$$

So we have

$$\begin{aligned} \psi(y) &= \varphi(x_{\varphi}(y)) - x_{\varphi}(y)y \\ &= \begin{cases} \frac{1-\gamma}{\gamma} y^{\frac{\gamma}{\gamma-1}} + (K - b)y, & \text{for } 0 < y < k, \\ \frac{1}{\gamma}K^{\gamma}, & \text{for } y \geq k, \end{cases} \end{aligned} \tag{6.2}$$

(see Fig. 6.2).

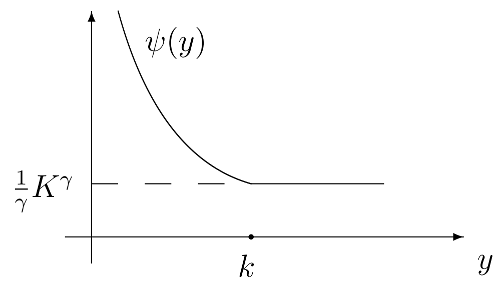

Fig. 6.2. $\psi(y)$

Note that k is the discontinuous point of ψ' and ψ'', thus $(k,t)\in\varepsilon\mathcal{R}_y$. (see the proof of Theorem 4.1). Now we define the free boundaries to (3.4) that

$$q(t)=\inf\{y\in[0,k]|v(y,t)=\psi(y)\},\quad 0<t<T, \tag{6.3}$$

$$p(t)=\sup\{y\in[0,k]|v(y,t)=\psi(y)\},\quad 0<t<T, \tag{6.4}$$

$$f(t)=\inf\{y\in[k,+\infty)|v(y,t)=\psi(y)\},\quad 0<t<T. \tag{6.5}$$

Owing to $\partial_t(v(y,t)-\psi(y))=v_t\le 0$, functions $q(t)$ and $f(t)$ are decreasing in t and $p(t)$ is increasing in t.

Substituting the first expression in (6.2) into the equation in (3.4) yields

$$\begin{aligned}
&-\partial_t\psi-\frac{a^2}{2}y^2\partial_{yy}\psi-(\beta-r)y\partial_y\psi+\beta\psi\\
=\;&\frac{a^2}{2}\Big(\frac{\gamma}{\gamma-1}-1\Big)y^{\frac{\gamma}{\gamma-1}}-(\beta-r)y\Big[-y^{\frac{1}{\gamma-1}}+(K-b)\Big]+\beta\Big[\frac{1-\gamma}{\gamma}y^{\frac{\gamma}{\gamma-1}}+(K-b)y\Big]\\
=\;&\Big(\frac{\beta-r\gamma}{\gamma}-\frac{a^2}{2}\frac{1}{1-\gamma}\Big)y^{\frac{\gamma}{\gamma-1}}+r(K-b)y,\quad y<k,
\end{aligned} \tag{6.6}$$

and noting that

$$-\partial_t\psi-\frac{a^2}{2}y^2\partial_{yy}\psi-(\beta-r)y\partial_y\psi+\beta\psi=\frac{\beta}{\gamma}K^\gamma>0,\quad y>k.$$

Denote the right hand side of (6.6) by $\Psi(y)$. i.e.

$$\Psi(y):=\Big(\frac{\beta-r\gamma}{\gamma}-\frac{a^2}{2}\frac{1}{1-\gamma}\Big)y^{\frac{\gamma}{\gamma-1}}+r(K-b)y.$$

It is not hard to see that

$$\varepsilon\mathcal{R}_y\subset[\{\Psi(y)\ge 0,\ y<k\}\cup(k,+\infty)]\times(0,T). \tag{6.7}$$

Lemma 6.1 *The set $\varepsilon\mathcal{R}_y$ is expressed as*

$$\varepsilon\mathcal{R}_y = \{(y,t) \in Q_y | q(t) \le y \le p(t)\} \cup \{(y,t) \in Q_y | y \ge f(t)\}. \tag{6.8}$$

Proof: By the definitions of $q(t),\ p(t)$ and $f(t)$, we get

$$\varepsilon\mathcal{R}_y \subset \{(y,t) \in Q_y | q(t) \le y \le p(t)\} \cup \{(y,t) \in Q_y | y \ge f(t)\}.$$

Now, we prove

$$\Omega := \{(y,t) \in Q_y | q(t) \le y \le p(t)\} \subset \varepsilon\mathcal{R}_y. \tag{6.9}$$

Since $\{(q(t),t),\ (p(t),t)\} \cap Q_y \subset \varepsilon\mathcal{R}_y \cap \{y < k\} \subset \{\Psi \ge 0\}$ and $\{\Psi \ge 0\}$ is a connected region, we have

$$\Omega \subset \{\Psi \ge 0\}.$$

Assume that (6.9) is false, since $\mathcal{CR}_y$ is an open set, there exists its open subset $\mathcal{N}$ such that $\mathcal{N} \subset \Omega$ and $\partial_p\mathcal{N} \subset \varepsilon\mathcal{R}_y$, where $\partial_p\mathcal{N}$ is the parabolic boundary of $\mathcal{N}$. Thus,

$$\begin{cases} -v_t - \dfrac{a^2}{2} y^2 v_{yy} - (\beta - r) y v_y + \beta v = 0 \quad \text{in } \mathcal{N}, \\ -\psi_t - \dfrac{a^2}{2} y^2 \psi_{yy} - (\beta - r) y \psi_y + \beta \psi \ge 0 \quad \text{in } \mathcal{N}, \\ v = \psi \quad \text{on} \quad \partial_p\mathcal{N}. \end{cases} \tag{6.10}$$

By the comparison principle, $v \le \psi$ in $\mathcal{N}$, which implies $\mathcal{N} = \emptyset$.

Similar proof yields

$$\{(y,t) \in Q_y | y \ge f(t)\} \subset \varepsilon\mathcal{R}_y.$$

Therefore, the desired result (6.8) holds. □

Thanks to Lemma 6.1, $q(t)$, $p(t)$ and $f(t)$ are three free boundaries of (3.4).

Theorem 6.2 *The free boundaries $q(t),\ p(t)$ and $f(t) \in C^\infty(0,T)$ and have the following classification*

Case I: $\beta \ge \frac{a^2}{2}\frac{\gamma}{1-\gamma} + r\gamma$**,** $\Psi(k) \ge 0$**.**

$$q(t) \equiv 0 \le p(t) \le p(T-) = k = f(T-) \le f(t),$$

see Fig 6.3.

Case II: $\beta \ge \frac{a^2}{2}\frac{\gamma}{1-\gamma} + r\gamma$**,** $\Psi(k) < 0$**.**

If $\beta > \frac{a^2}{2}\frac{\gamma}{1-\gamma} + r\gamma$,

$$q(t) \equiv 0 \le p(t) \le p(T-) = \left(\frac{-r(K-b)}{\frac{\beta - r\gamma}{\gamma} - \frac{a^2}{2}\frac{1}{1-\gamma}}\right)^{\gamma-1} < k = f(T-) \le f(t),$$

see Fig 6.4.

If $\beta = \frac{a^2}{2}\frac{\gamma}{1-\gamma} + r\gamma$,

$$\varepsilon\mathcal{R}_y \cap \Big((0,k)\times(0,T)\Big) = \emptyset, \quad k = f(T-) \le f(t),$$

see Fig 6.6.

Case III: $\beta < \frac{a^2}{2}\frac{\gamma}{1-\gamma} + r\gamma$**,** $\Psi(k) > 0$**.**

$$\left(\frac{-r(K-b)}{\frac{\beta - r\gamma}{\gamma} - \frac{a^2}{2}\frac{1}{1-\gamma}}\right)^{\gamma-1} = q(T-) \le q(t) \le p(t) \le p(T-) = k = f(T-) \le f(t),$$

see Fig 6.5.

Case IV: $\beta < \frac{a^2}{2}\frac{\gamma}{1-\gamma} + r\gamma$**,** $\Psi(k) \le 0$**.**

$$\varepsilon\mathcal{R}_y \cap (0,k)\times(0,T) = \emptyset, \quad k = f(T-) \le f(t),$$

see Fig 6.6.

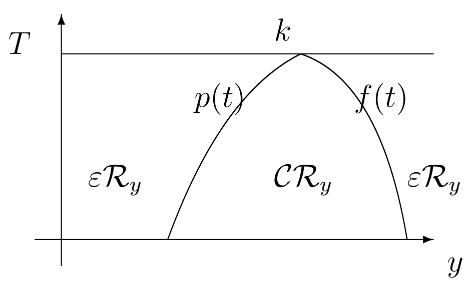

Fig 6.3. $\beta \ge \frac{a^2}{2}\frac{\gamma}{1-\gamma} + r\gamma$, $\Psi(k) \ge 0$.

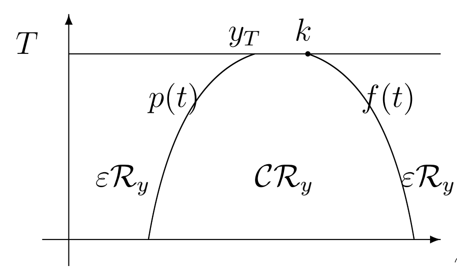

Fig 6.4. $\beta > \frac{a^2}{2}\frac{\gamma}{1-\gamma} + r\gamma$, $\Psi(k) < 0$.

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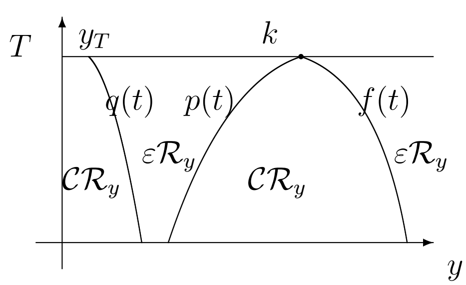

Fig 6.5. $\beta < \frac{a^2}{2}\frac{\gamma}{1-\gamma} + r\gamma$, $\Psi(k) > 0$.

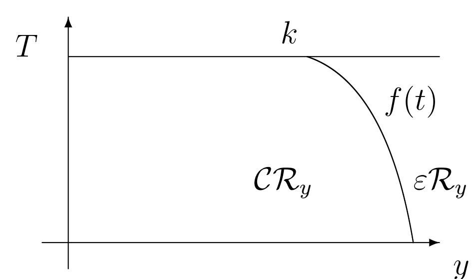

Fig 6.6. $\beta < \frac{a^2}{2}\frac{\gamma}{1-\gamma} + r\gamma$, $\Psi(k) \le 0$,
or $\beta = \frac{a^2}{2}\frac{\gamma}{1-\gamma} + r\gamma$, $\Psi(k) < 0$.

Proof: By the method of [10], we could prove $q(t),\ p(t),\ f(t) \in C^\infty(0,T)$, we omit the details.

Here, we only prove the results in **Case II**, the remaining situations are similar. If $\beta > \frac{a^2}{2}\frac{\gamma}{1-\gamma} + r\gamma$ and $\Psi(k) < 0$, then $K < b$, Denote $y_T = \left(\frac{-r(K-b)}{\frac{\beta-r\gamma}{\gamma} - \frac{a^2}{2}\frac{1}{1-\gamma}}\right)^{\gamma-1}$, then $\Psi(k) < 0$ implies $y_T < k$. By (6.7) and $\{\Psi \geq 0\} = (0, y_T]$,

$$\varepsilon\mathcal{R}_y \subset \Big((0, y_T] \cup (k, \infty)\Big) \times (0, T).$$

thus

$$0 \leq q(t) \leq p(t) \leq y_T < k \leq f(t).$$

Now, we prove $q(t) \equiv 0$. Set $\mathcal{N} := \{(y,t)|0 < y \leq q(t),\ 0 < t < T\}$. It follows from (6.10) that we have $v \leq \psi$ in $\mathcal{N}$. By the definition of $q(t)$, $\mathcal{N} = \emptyset$ as well as $q(t) \equiv 0$.

Here, we aim to prove $f(T-) := \lim\limits_{t\uparrow T} f(t) = k$. Otherwise, if $k < f(T-)$, then there exists a contradiction that

$$\begin{aligned} 0 &= -v_t - \frac{a^2}{2}y^2 v_{yy} - (\beta - r)yv_y + \beta v \\ &= -v_t - \frac{a^2}{2}y^2\psi_{yy} - (\beta - r)y\psi_y + \beta\psi = -\partial_t v + \frac{1}{\gamma}K^\gamma > 0, \quad k < y < f(T-),\ t = T. \end{aligned}$$

So $f(T-) = k$. The proof of $p(T-) = y_T$ is similar that if $p(T-) < y_T$, there exists contradiction

$$\begin{aligned} 0 &= -v_t - \frac{a^2}{2}y^2 v_{yy} - (\beta - r)yv_y + \beta v \\ &= -v_t - \frac{a^2}{2}y^2\psi_{yy} - (\beta - r)y\psi_y + \beta\psi = -\partial_t v + \Psi(y) > 0, \quad p(T-) < y < y_T,\ t = T. \end{aligned}$$

If $\beta = \frac{a^2}{2}\frac{\gamma}{1-\gamma} + r\gamma$ and $\Psi(k) < 0$, then $K < b$ as well as $\Psi(y) < 0$ for all $0 < y < k$, thus $(0, k] \times (0, T) \subset \mathcal{CR}_y$, so $q(t),\ p(t)$ do not exist. $\square$

6.2 The free boundary to original problem (2.12)

Now, we discuss the free boundary of (2.12). Define

$$\begin{aligned} \varepsilon\mathcal{R}_x &= \{\widehat{V} = \varphi\}, \quad \textbf{exercise region}, \\ \mathcal{CR}_x &= \{\widehat{V} > \varphi\}, \quad \textbf{continuation region}. \end{aligned}$$

And

$$Q(t) = \sup\{x > 0|\widehat{V}(x,t) = \varphi(x)\}, \quad 0 < t < T,$$
$$P(t) = \inf\{x > 0|\widehat{V}(x,t) = \varphi(x)\}, \quad 0 < t < T.$$

On the two free boundaries $y = q(t)$ and $y = p(t)$,

$$v(y,t) = \frac{1-\gamma}{\gamma} y^{\frac{\gamma}{\gamma-1}} + (K-b)y,$$
$$v_y(y,t) = -y^{\frac{1}{\gamma-1}} + (K-b).$$

Note that

$$x = -v_y(y,t). \tag{6.11}$$

Then the corresponding two free boundaries of (2.12) are

$$Q(t) = -v_y(q(t),t) = q(t)^{\frac{1}{\gamma-1}} - (K-b),$$
$$P(t) = -v_y(p(t),t) = p(t)^{\frac{1}{\gamma-1}} - (K-b).$$

Moreover

$$Q'(t) = \frac{1}{\gamma-1} q(t)^{\frac{1}{\gamma-1}-1} q'(t) \geq 0,$$
$$P'(t) = \frac{1}{\gamma-1} p(t)^{\frac{1}{\gamma-1}-1} p'(t) \leq 0,$$

and

$$Q(T-) = q(T-)^{\frac{1}{\gamma-1}} - (K-b),$$
$$P(T-) = p(T-)^{\frac{1}{\gamma-1}} - (K-b).$$

On the other hand, by (5.14) and (5.6),

$$\widehat{V}_x(0,t) = J(0,t) = (v(\cdot,t))^{-1}(0) = f(t).$$

The above analysis conclude that

Theorem 6.3 *The two free boundaries of (2.12): $Q(t)$, $P(t) \in C^\infty(0,T)$ and $Q'(t) \geq 0$, $P'(t) \leq 0$, $\widehat{V}_x(0,t) = f(t)$. Moreover, they have the following classification.*
Case I: $\beta \geq \frac{a^2}{2}\frac{\gamma}{1-\gamma} + r\gamma$**,** $\Psi(k) \geq 0$**.**

$$Q(t) \equiv +\infty, \quad k^{\frac{1}{\gamma-1}} - (K-b) = P(T-) \leq P(t),$$

i.e.

$$Q(t) \equiv +\infty, \quad \widehat{x} = P(T-) \leq P(t),$$

see Fig 6.7.

Case II: $\beta \geq \frac{a^2}{2}\frac{\gamma}{1-\gamma} + r\gamma$**,** $\Psi(k) < 0$**.**

If $\beta > \frac{a^2}{2}\frac{\gamma}{1-\gamma} + r\gamma$,

$$Q(t) \equiv +\infty, \quad y_T = \left(\frac{-r(K-b)}{\frac{\beta - r\gamma}{\gamma} - \frac{a^2}{2}\frac{1}{1-\gamma}}\right) - (K-b) = P(T-) < P(t),$$

see Fig 6.8.

If $\beta = \frac{a^2}{2}\frac{\gamma}{1-\gamma} + r\gamma$,

$$\varepsilon\mathcal{R}_x = \emptyset,$$

see Fig 6.10.

Case III: $\beta < \frac{a^2}{2}\frac{\gamma}{1-\gamma} + r\gamma$**,** $\Psi(k) > 0$**.**

$$k^{\frac{1}{\gamma-1}} - (K-b) = P(T-) \leq P(t) \leq Q(t) \leq Q(T-) = \left(\frac{-r(K-b)}{\frac{\beta - r\gamma}{\gamma} - \frac{a^2}{2}\frac{1}{1-\gamma}}\right) - (K-b),$$

see Fig 6.9.

Case IV: $\beta < \frac{a^2}{2}\frac{\gamma}{1-\gamma} + r\gamma$**,** $\Psi(k) \leq 0$**.**

$$\varepsilon\mathcal{R}_x = \emptyset,$$

see Fig 6.10.

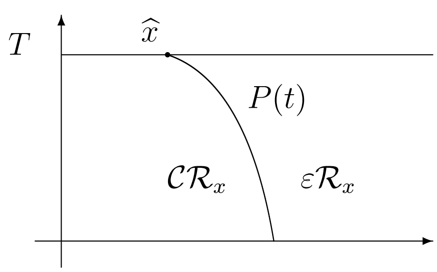

Fig 6.7. $\beta \geq \frac{a^2}{2}\frac{\gamma}{1-\gamma} + r\gamma$, $\Psi(k) \geq 0$.

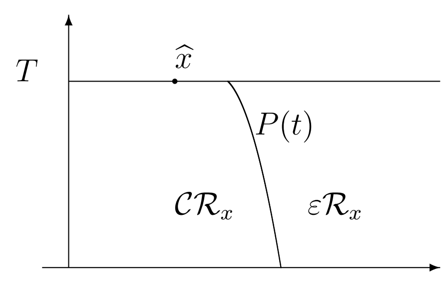

Fig 6.8. $\beta > \frac{a^2}{2}\frac{\gamma}{1-\gamma} + r\gamma$, $\Psi(k) < 0$.

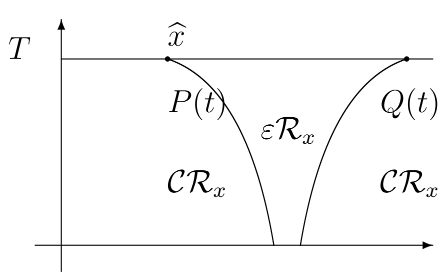

Fig 6.9. $\beta < \frac{a^2}{2}\frac{\gamma}{1-\gamma} + r\gamma$, $\Psi(k) > 0$.

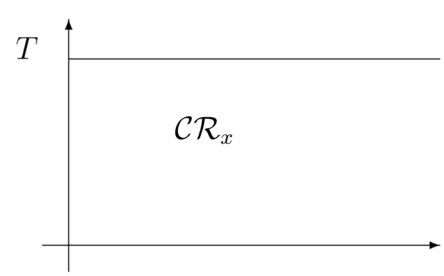

Fig 6.10. $\beta < \frac{a^2}{2}\frac{\gamma}{1-\gamma} + r\gamma$, $\Psi(k) \leq 0$,
or $\beta = \frac{a^2}{2}\frac{\gamma}{1-\gamma} + r\gamma$, $\Psi(k) < 0$.

7 Conclusions

In this paper we adopt dual method of partial differential equations, to obtain the existence of solution to fully nonlinear problem (2.12) under the utility function $g(x)$ is non-smooth, non-concave (under **Condition I-III**). Meanwhile, we present a new method to study the free boundaries while the exercise region is not connected (see (6.3)-(6.5) and Lemma 6.1) so that we can shed light on the behaviors of the free boundaries for a fully nonlinear variational inequality without any restrictions on parameters (see Figure 5.1-5.4.). The financial meaning is that if at time t, investor's wealth x is located in $\mathcal{CR}_x$, then he should continue to invest; and if investor's wealth x is located in $\varepsilon\mathcal{R}_x$, then he should stop to investing.

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Appendix A

In this section, we prove Theorem 2.1. It can be accomplished by the following two lemmas.

Lemma A.1 *Under* **Condition II**, *the value function defined in* (2.3) *satisfies*

$$\limsup_{t\to T-} V(x,t) \le \varphi(x). \tag{A.1}$$

Proof: Define

$$\zeta_s = e^{-(r+\frac{1}{2}\mu'(\sigma'\sigma)^{-1}\mu)s-\mu'\sigma^{-1}W_s},$$

then

$$d\zeta_s = \zeta_s[-rds - \mu'\sigma^{-1}dW_s]$$

and

$$\begin{aligned} d(\zeta_s X_s) &= \zeta_s dX_s + X_s d\zeta_s + d\zeta_s dX_s \\ &= \zeta_s[(rX_s + \mu'\pi_s)ds + \pi_s'\sigma dW_s - rX_s ds - \mu'\sigma^{-1}X_s dW_s - (\mu'\sigma^{-1})(\pi_s'\sigma)'ds] \\ &= \zeta_s[\pi_s'\sigma - \mu'\sigma^{-1}X_s]dW_s. \end{aligned} \tag{A.2}$$

Thus, $\zeta_s X_s$ is a martingale. For any admissible π, by Jensen's inequality, we have

$$\mathbb{E}_{t,x}\varphi\Big(\frac{\zeta_T}{\zeta_t}X_T\Big) \le \varphi\Big(\mathbb{E}_{t,x}\Big(\frac{\zeta_T}{\zeta_t}X_T\Big)\Big) = \varphi(x).$$

Then

$$\limsup_{t\to T-}\sup_{\pi}\mathbb{E}_{t,x}\varphi\Big(\frac{\zeta_T}{\zeta_t}X_T\Big) \le \varphi(x). \tag{A.3}$$

We come to prove

$$\lim_{t\to T-}\sup_{\pi}\mathbb{E}_{t,x}\Big|\varphi(X_T) - \varphi\Big(\frac{\zeta_T}{\zeta_t}X_\tau\Big)\Big| = 0. \tag{A.4}$$

For any admissible π, since

$$|\varphi(x) - \varphi(y)| \le C|x-y|^\gamma,$$

where, $C = M/\gamma$,

$$\mathbb{E}_{t,x}\Big|\varphi(X_T) - \varphi\Big(\frac{\zeta_T}{\zeta_t}X_T\Big)\Big| \le C\mathbb{E}_{t,x}\Big(\Big(\frac{\zeta_T}{\zeta_t}X_T\Big)^\gamma\Big|\frac{\zeta_t}{\zeta_T} - 1\Big|^\gamma\Big).$$

Using Hölder inequality, we obtain

$$\begin{aligned}\mathbb{E}_{t,x}\Big|\varphi(X_T)-\varphi\Big(\frac{\zeta_T}{\zeta_t}X_T\Big)\Big| &\le C\Big(\mathbb{E}_{t,x}\Big(\frac{\zeta_T}{\zeta_t}X_T\Big)\Big)^{\gamma}\Big(\mathbb{E}_{t,x}\Big|\frac{\zeta_t}{\zeta_T}-1\Big|^{\frac{\gamma}{1-\gamma}}\Big)^{1-\gamma}\\ &\le Cx^{\gamma}\Big(\mathbb{E}_{t,x}\Big|\frac{\zeta_t}{\zeta_T}-1\Big|^{\frac{\gamma}{1-\gamma}}\Big)^{1-\gamma}.\end{aligned}$$

Hence,

$$\lim_{t\to T-}\sup_{\pi}\mathbb{E}_{t,x}\Big|\varphi(X_T)-\varphi\Big(\frac{\zeta_T}{\zeta_t}X_T\Big)\Big|\le Cx^{\gamma}\lim_{t\to T-}\Big(\mathbb{E}_{t,x}\Big|\frac{\zeta_t}{\zeta_T}-1\Big|^{\frac{\gamma}{1-\gamma}}\Big)^{1-\gamma}=0.$$

Therefore,

$$\begin{aligned}\limsup_{t\to T-}V(x,t) &= \limsup_{t\to T-}\sup_{\pi}\mathbb{E}_{t,x}\Big[e^{-r(\tau-t)}g(X_\tau)\Big]\\ &\le \limsup_{t\to T-}\sup_{\pi}\mathbb{E}_{t,x}\varphi(X_\tau)\\ &\le \limsup_{t\to T-}\sup_{\pi}\mathbb{E}_{t,x}\varphi\Big(\frac{\zeta_\tau}{\zeta_t}X_\tau\Big)+\lim_{t\to T-}\sup_{\pi}\mathbb{E}_{t,x}\Big|\varphi(X_\tau)-\varphi\Big(\frac{\zeta_\tau}{\zeta_t}X_\tau\Big)\Big|\\ &\le \varphi(x).\end{aligned}$$

□

Lemma A.2 *Under* **Condition II**, *the value function defined in* (2.3) *satisfies*

$$\liminf_{t\to T-}V(x,t)\ge\varphi(x). \tag{A.5}$$

PROOF: For fix $t<T$, if $x\in\{\varphi(x)=g(x)\}$, then

$$V(x,t)\ge g(x)=\varphi(x).$$

So $\liminf_{t\to T-}V(x,t)\ge\varphi(x)$.

Otherwise, if $x\in(\underline{x}_m,\overline{x}_m)$ for a $m\in\mathbb{Z}$, choose π_s to let the coefficient of (A.2) that

$$\frac{\zeta_s}{\zeta_t}[\pi_s'\sigma-\mu'\sigma^{-1}X_s]=(\pi_s^N)':=N\chi_{\left\{\underline{x}_m<\frac{\zeta_s}{\zeta_t}X_s<\overline{x}_m\right\}}I_n',\quad\forall\ N>0,$$

where I_n is an n-dimensional unit column vector. Let $X_s^N=\frac{\zeta_s}{\zeta_t}X_s$. Then using (A.2) we have

$$dX_s^N=(\pi_s^N)'dW_s,\quad t\le s\le T.$$

It is not hard to obtain

$$\underline{x}_m\le X_s^N\le\overline{x}_m,\quad t\le s\le T,$$

and since

$$\begin{aligned}\{\underline{x}_m < X_T^N < \overline{x}_m\} &= \{\underline{x}_m < X_s^N = x + NI_n'(W_s - W_t) < \overline{x}_m,\ t \le s \le T\} \\ &\subset \{\underline{x}_m < x + NI_n'(W_T - W_t) < \overline{x}_m\},\end{aligned}$$

we have

$$\mathbb{P}(\underline{x}_m < X_T^N < \overline{x}_m) \le \mathbb{P}(\underline{x}_m < x + NI_n'(W_T - W_t) < \overline{x}_m) \to 0, \quad N \to \infty.$$

So

$$\underline{x}_m \mathbb{P}(X_T^N = \underline{x}_m) + \overline{x}_m \mathbb{P}(X_T^N = \overline{x}_m) \to \mathbb{E}_{t,x} X_T^N = x, \quad N \to \infty.$$

Therefore,

$$\lim_{N\to\infty} \mathbb{P}(X_T^N = \underline{x}_m) = \frac{\overline{x}_m - x}{\overline{x}_m - \underline{x}_m}, \quad \lim_{N\to\infty} \mathbb{P}(X_T^N = \overline{x}_m) = \frac{x - \underline{x}_m}{\overline{x}_m - \underline{x}_m}.$$

As a result,

$$\lim_{N\to\infty} \mathbb{E}_{t,x} g(X_T^N) = \frac{\overline{x}_m - x}{\overline{x}_m - \underline{x}_m} g(\underline{x}_m) + \frac{x - \underline{x}_m}{\overline{x}_m - \underline{x}_m} g(\overline{x}_m) = \varphi(x).$$

So

$$\sup_\pi \mathbb{E}_{t,x}\Big(g\Big(\frac{\zeta_T}{\zeta_t} X_T\Big)\Big) \ge \lim_{N\to\infty} \mathbb{E}_{t,x} g(X_T^N) = \varphi(x).$$

Meanwhile, similar to (A.4), we have

$$\lim_{t\to T-} \sup_\pi \mathbb{E}_{t,x}\Big| g(X_T) - g\Big(\frac{\zeta_T}{\zeta_t} X_T\Big)\Big| = 0.$$

Hence,

$$\begin{aligned}\liminf_{t\to T-} V(x,t) &\ge \liminf_{t\to T-} \sup_\pi \mathbb{E}_{t,x}\Big(g(X_T)\Big) \\ &\ge \liminf_{t\to T-} \sup_\pi \mathbb{E}_{t,x}\Big(g\Big(\frac{\zeta_T}{\zeta_t} X_T\Big)\Big) - \lim_{t\to T-} \sup_\pi \mathbb{E}_{t,x}\Big| g(X_T) - g\Big(\frac{\zeta_T}{\zeta_t} X_T\Big)\Big| \\ &\ge \varphi(x).\end{aligned}$$

$\square$

Appendix B

In this section, we will prove $\widehat{V}(x,t)$ (the solution of (2.12)) is the value function $V(x,t)$ (defined in (2.3)). To this end, we first introduce the following so called dynamic programming principle. For any stopping time θ ($t \le \theta \le T$),

$$V(x,t) = \sup_{\pi,\tau\ge t} \mathbb{E}_{t,x}\Big[e^{-\beta(\tau-t)}g(X_\tau)\chi_{\{\theta>\tau\}} + e^{-\beta(\theta-t)}V(X_\theta,\theta)\chi_{\{\theta\le\tau\}}\Big]. \tag{B.1}$$

Lemma B.1 *$V(x,t)$ which is defined in* (2.3) *satisfies*

$$V(x,t) \ge \sup_{\pi,\tau\ge t} \mathbb{E}_{t,x}\Big[e^{-\beta(\tau-t)}g(X_\tau)\chi_{\{\tau<T\}} + e^{-\beta(T-t)}\varphi(X_T)\chi_{\{\tau=T\}}\Big].$$

PROOF: For any (X_s, π_s) satisfies (2.2) and $\pi \in \Pi_t$. Choose $\theta = T-\varepsilon$ in (B.1) we have

$$V(x,t) \ge \mathbb{E}_{t,x}\Big[e^{-\beta(\tau-t)}g(X_\tau)\chi_{\{T-\varepsilon>\tau\}} + e^{-\beta(T-\varepsilon-t)}V(X_{T-\varepsilon},T-\varepsilon)\chi_{\{T-\varepsilon\le\tau\}}\Big]. \tag{B.2}$$

Letting $\varepsilon \to 0$,

$$V(x,t) \ge \mathbb{E}_{t,x}\Big[e^{-\beta(\tau-t)}g(X_\tau)\chi_{\{\tau<T\}} + e^{-\beta(T-t)}V(X_{T-},T-)\chi_{\{\tau=T\}}\Big].$$

By using Theorem 2.1, we have

$$V(x,t) \ge \mathbb{E}_{t,x}\Big[e^{-\beta(\tau-t)}g(X_\tau)\chi_{\{\tau<T\}} + e^{-\beta(T-t)}\varphi(X_T)\chi_{\{\tau=T\}}\Big].$$

Since π and τ are arbitrary, we complete the proof. $\square$

Theorem B.2 *Suppose $\widehat{V}(x,t)$ is the solution to problem* (2.12) *satisfies* (2.13)-(2.16), *$\widehat{V}(x,t)$ is the value function defined in* (2.3), *then*

$$\widehat{V}(x,t) = V(x,t).$$

PROOF: For any admissible $\pi \in \Pi_t$, suppose X_s satisfies (2.2), by Itô formula,

$$\begin{aligned} d[e^{-\beta s}\widehat{V}(X_s,s)] &= e^{-\beta s}\Big[-\beta\widehat{V}(X_s,s) + \widehat{V}_t(X_s,s) + (rX_s + \mu'\pi_s)\widehat{V}_x(X_s,s) \\ &\quad + \frac{1}{2}\pi_s'(\sigma\sigma')\pi_s\widehat{V}_{xx}(X_s,s)\Big]ds + e^{-\beta s}\widehat{V}_x(X_s,s)\pi_s'\sigma dW_s. \end{aligned}$$

Thus, for any stopping time τ ($t \le \tau \le T$),

$$\begin{aligned}
\widehat{V}(x,t) &= \mathbb{E}_{t,x}[e^{-\beta(\tau-t)}\widehat{V}(X_\tau,\tau)] + \mathbb{E}_{t,x}\Big[\int_t^\tau e^{-\beta(s-t)}\Big(\beta\widehat{V} - \widehat{V}_t - (rX_s+\mu\pi_s)\widehat{V}_x \\
&\quad -\frac{1}{2}\pi_s'(\sigma\sigma')\pi_s\widehat{V}_{xx}\Big)(X_s,s)ds\Big] + \mathbb{E}_{t,x}\Big[\int_t^\tau e^{-\beta s}\widehat{V}_x(X_s,s)\pi_s'\sigma dW_s\Big] \\
&\ge \mathbb{E}_{t,x}[e^{-\beta(\tau-t)}\widehat{V}(X_\tau,\tau)] + \mathbb{E}_{t,x}\Big[\int_t^\tau e^{-\beta(s-t)}\Big(-\widehat{V}_t - \sup_\pi\Big(\frac{1}{2}\pi'(\sigma\sigma')\pi\widehat{V}_{xx} + \mu'\pi\widehat{V}_x\Big) \\
&\quad -rX_s\widehat{V}_x + \beta\widehat{V}\Big)(X_s,s)ds\Big] \\
&\ge \mathbb{E}_{t,x}[e^{-\beta(\tau-t)}\varphi(X_\tau)] \\
&\ge \mathbb{E}_{t,x}[e^{-\beta(\tau-t)}g(X_\tau)].
\end{aligned}$$

Since π and τ are arbitrary, we have $\widehat{V}(x,t) \ge V(x,t)$.

On the other hand, define

$$\widehat{\pi}(x,t) := -(\sigma\sigma')^{-1}\mu\frac{\widehat{V}_x(x,t)}{\widehat{V}_{xx}(x,t)}.$$

Let X_s^* be the solution of the following SDE,

$$\begin{cases} dX_s = (rX_s + \mu'\widehat{\pi}(X_s,s))ds + \widehat{\pi}(X_s,s)'\sigma dW_s, \\ X_t = x, \end{cases}$$

and let

$$\tau^* = \inf\{s\in[t,T] \mid \widehat{V}(X_s^*,s) = \varphi(X_s^*)\}, \quad \pi_s^* = \widehat{\pi}(X_s^*,s).$$

By Itô formula,

$$\begin{aligned}
\widehat{V}(x,t) &= \mathbb{E}_{t,x}[e^{-\beta(\tau^*-t)}\widehat{V}(X_{\tau^*}^*,\tau^*)] + \mathbb{E}_{t,x}\Big[\int_t^{\tau^*} e^{-\beta(s-t)}\Big(\beta\widehat{V} - \widehat{V}_t - (rX_s^*+\mu\pi_s^*)\widehat{V}_x \\
&\quad -\frac{1}{2}\pi_s^{*\prime}(\sigma\sigma')\pi_s^*\widehat{V}_{xx}\Big)(X_s^*,s)ds\Big] + \mathbb{E}_{t,x}\Big[\int_t^{\tau^*} e^{-\beta s}\widehat{V}_x(X_s^*,s)\pi_s^{\varepsilon\prime}\sigma dW_s\Big] \\
&= \mathbb{E}_{t,x}[e^{-\beta(\tau^*-t)}\varphi(X_{\tau^*}^*)].
\end{aligned}$$

Note that by the result in Section 6, $\varepsilon\mathcal{R}_x \subset \{x \ge \widehat{x}\}$, i.e. $\{\widehat{V}(x,t) = \varphi(x)\} = \{\widehat{V}(x,t) = g(x)\}$ for any $t < T$, thus

$$\begin{aligned}
\widehat{V}(x,t) &\le \mathbb{E}_{t,x}[e^{-\beta(\tau^*-t)}\varphi(X_{\tau^*}^*)\chi_{\{\tau<T\}} + \varphi(X_T^*)\chi_{\{\tau=T\}}] \\
&= \mathbb{E}_{t,x}[e^{-\beta(\tau^*-t)}g(X_{\tau^*}^*)\chi_{\{\tau<T\}} + \varphi(X_T^*)\chi_{\{\tau=T\}}].
\end{aligned}$$

Thanks to Lemma B.1, we have $\widehat{V}(x,t) \le V(x,t)$. $\square$