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Multi-Sourcing Supply Network Design: Two-Stage Chance-Constrained Model, Tractable Approximations, and Computational Results

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Abstract

In this paper, we study a multi-sourcing supply network design problem, in which each retailer faces uncertain demand and can source products from more than one distribution center (DC). The decisions to be simultaneously optimized include DC locations and inventory levels, which set of DCs serves each retailer, and the amount of shipments from DCs to retailers. We propose a nonlinear mixed integer programming model with a joint chance constraint describing a certain service level. Two approaches — a set-wise approximation and a Linear Decision Rule-based approximation — are constructed to robustly approximate the service level chance constraint with incomplete demand information. Both approaches yield sparse multi-sourcing distribution networks that effectively match uncertain demand using on-hand inventory, and hence successfully reach a high service level. We show through extensive numerical experiments that our approaches outperform other commonly adopted approximations of the chance constraint.

Keywords: Multi-Sourcing; Network Design; Chance Constraint Approximation

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1 Introduction

The globalization trend has significantly changed both the supply chain structures and the operations in many industries. Supply chain networks have become more complex and usually cover thousands of retailers located in different continents. Meanwhile, the demand faced by each retailer fluctuates, making it difficult to predict due to consumers' fast-changing tastes and fierce market competition. These issues raise challenges regarding how to design and manage distribution center networks to meet customer requirements in a timely and efficient manner.

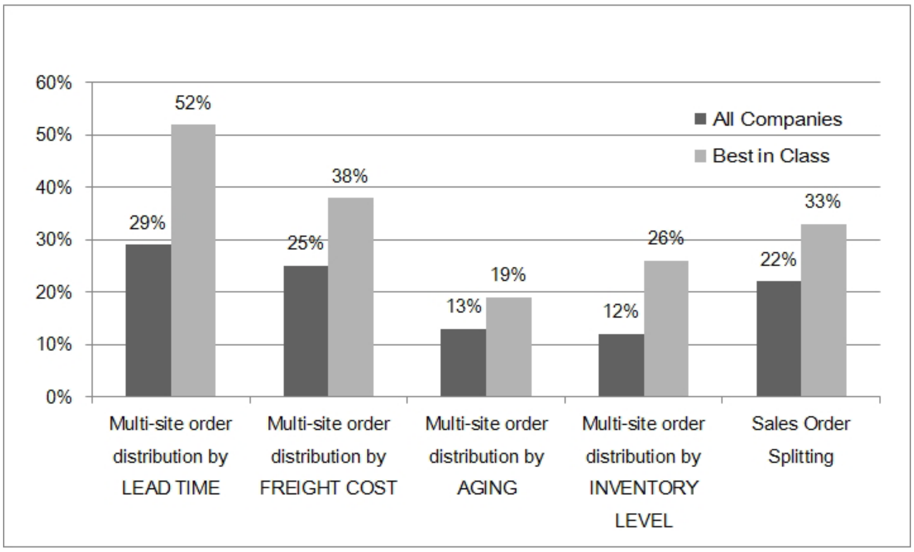

Figure 1: "Best-in-Class" companies are more capable of managing orders and distributions. Source: Aberdeen Group, April 2007

In 2007, Aberdeen Group conducted a survey on the operations of 146 multi-site distributors (cf. Aberdeen Group, 2007). They noted that almost two thirds of the companies surveyed could not effectively manage their distribution networks, and thereby simultaneously suffered high operational costs and low service levels. Only a few "Best-in-Class" companies can provide superior service levels without increasing costs. These successful companies share the following common characteristics:

- They tend to be equipped with more order-splitting capacity;
- They prefer "filling customer orders from the DCs that are overstocked";
- They build centralized computing systems to practice "Vendor Order Splitting".

Obviously, these "Best-in-Class" companies are able to apply the multi-sourcing strategy, which allows them to manage their distribution networks effectively.

The benefits of adopting the multi-sourcing strategy are further illustrated by the following example.

Example 1 *Figure 2 shows two distribution networks with 10 retailers labeled from A to J and 4 DCs labeled from DC1 to DC4. The locations of the retailers and the DCs are randomly generated in a unit square. The service radius of each DC is set to be 0.5, i.e., a DC can serve a retailer only if the distance between them is less than 0.5. The boundary of the service region for each DC is represented by a dotted circle. If a retailer is served by a DC, an arc links the DC and the retailer. Figure 2A is a single-sourcing network in which each retailer can only source products from a single DC to fulfill its demand, whereas Figure 2B corresponds to a multi-sourcing network in which a DC serves all the retailers within its service radius.*

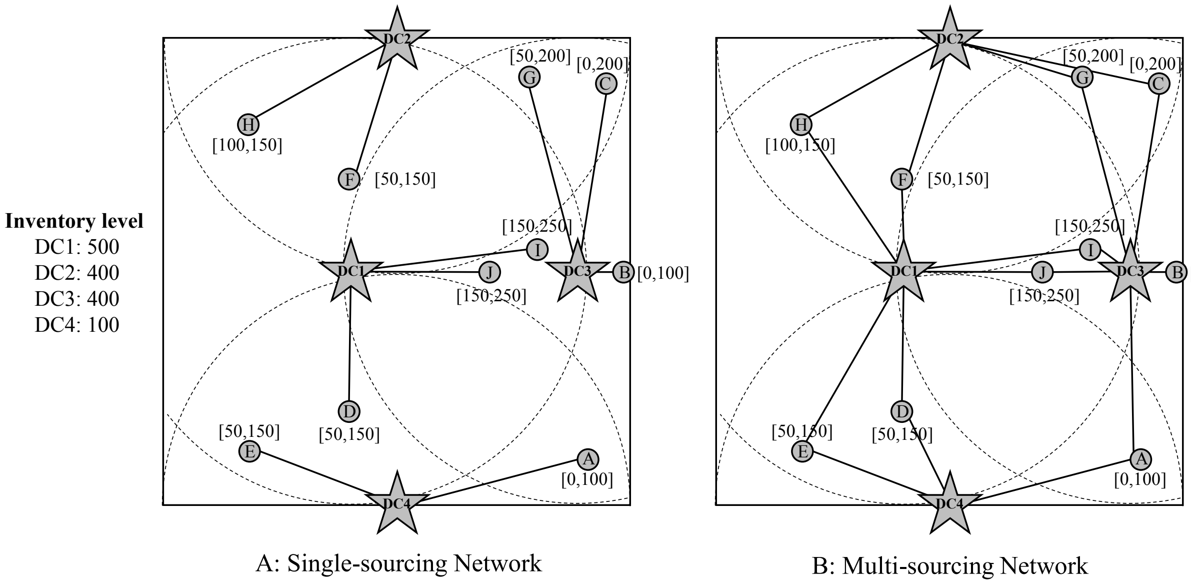

Figure 2: Example 1

Let the inventory levels at the 4 DCs be 500, 400, 400, and 100, respectively. Suppose the demand faced by any retailer follows a uniform distribution in the interval marked next to the retailer. We generate 100 demand scenarios to evaluate the performance of the two networks. For each scenario, the amount of satisfied demand can be obtained by solving a simple max-flow problem. The average service level is calculated as the average ratio of the satisfied demand to the total demand.

The simulation shows that the average service level of the multi-sourcing network is 99.99%, whereas that of the single-sourcing network is only 93.72%. To achieve the same service level as the multi-sourcing network, we need to increase the total inventory level of the single-sourcing network from what it currently is, 1400, to almost 1700, which is the sum of the demand upper bounds for all retailers.

The above example demonstrates that a properly designed multi-sourcing network achieves a high service level (close to 100%) with low inventory. Moreover, the effort required in managing the multi-sourcing network is moderate because the network only contains 19 links and each retailer is covered by, at most, 2 DCs.

Motivated by these observations, we explore how to design a multi-sourcing network, which requires careful handling of the following factors: (1) DC locations, (2) DC–retailer assignments, and (3) inventory level at each open DC. The first two factors determine the network structure, while the last one affects the service level. In this paper, using a chance constraint to ensure a certain service level, we propose a robust multi-sourcing supply network design model that simultaneously considers the above three factors. Moreover, the model possesses the following distinct advantages:

- Our model adopts a flexible order splitting strategy. A demand node can arbitrarily split its order and dynamically source the replenishment from the linked supply nodes according to the realized demand. This strategy is efficient and flexible as it can match random orders with fixed inventory in an online fashion and fully utilize the benefits of a multi-sourcing network.

- Our model does not require complete demand distribution. In practice, demand distribution is usually difficult to predict and the estimation is often unreliable. Instead, this model designs a robust distribution network based on mean and interval (i.e., the lower and upper bounds) demand, which can be obtained from historical data. The demand interval can also be adjusted to decision makers' risk preferences.

There is an extensive body of literature on location–inventory supply chain network design that aims at integrating the facility location, transportation, and inventory replenishment decisions into a distribution network design problem (see, e.g., Daskin et al. 2002, Shen et al. 2003, Teo and

Shu 2004, Shu et al. 2005, Aboolian et al. 2012, Li et al. 2013, and the references therein). These problems are NP-hard because they take a well-known NP-hard problem — the uncapacitated facility location problem (UFLP) — as a special case. Numerous studies focus on constructing effective approximation algorithms for the UFLP. For example, some constant factor approximation algorithms for the UFLP and its variants are proposed by Mahadian et al. (2006) and Zhang (2006). The aforementioned location–inventory supply chain network design models also inherit the single-sourcing characteristic from the UFLP, i.e., each retailer can only source products from a single DC. Single-sourcing networks possess practical advantages such as better quality control and reduced management and coordination complexity. However, as Aberdeen Group (2007) and Ozsen et al. (2009) show, multi-sourcing networks have become a better option in most contexts, due to state-of-the-art information technology.

Despite its successful implementation in industries, the literature on multi-sourcing supply chain network design is inadequate, as it has not kept pace with the interest and propagation of real-world installations. Thus, advances in the relevant research are particularly important in supporting the management of such systems. Recently, a few pioneers have begun to explore this area (cf. Ozsen et al. 2009 and Ağralı et al. 2012). Ozsen et al. (2009) propose the first multi-sourcing location–inventory model in the literature, which extends their earlier work (Ozsen et al. 2008) on the capacitated joint location–inventory problem. They motivate multi-sourcing by assuming that each facility is capacitated and analyze the potential savings that can be achieved. Using a set of computational experiments, they conclude that significant cost savings can be attained with only a few multi-sourced retailers. Based on the joint location–inventory model in Shen et al. (2003), Ağralı et al. (2012) also propose a multi-sourcing variant by considering continuous DC-retailer assignment variables and imposing an upper bound on the number of DCs to serve each retailer. The numerical results show that the cost of the single-sourcing supply chain can be 7% higher than that of its multi-sourcing counterpart. Note that the models in Ozsen et al. (2009) and Ağralı et al. (2012) fix the order-splitting pattern for all demand scenarios (static sourcing). Our model is different from these works as it allows for a flexible order-splitting pattern (dynamic sourcing). In addition, Ozsen et al. (2009) consider a capacitated location–inventory model, which requires a multi-sourcing network when a DC does not have enough capacity to fulfill a retailer's demand. In contrast, our model can obtain a multi-sourcing network with uncapacitated DCs.

Our model uses a joint chance constraint to meet a certain service level. We refer to the classical references, e.g., Prékopa (1995), Prékopa (2003), and Shapiro et al. (2009), on the topic of chance-constrained programming. Under a multivariate Gaussian distribution, the chance constraint considered in this paper could be treated, for instance, by the approach proposed in Henrion and Möller (2012), and the joint chance constraint obtained by applying the linear decision rule can be solved by the derivative formula in Ackooij et al. (2010). In addition, if a joint chance constraint is decomposed into a set of individual chance constraints, it can be trivially reformulated as explicit constraints (using quantiles) for many distributions (e.g., normal, exponential, or t-distributions). Unfortunately, none of these approaches can be applied to our model, as we do not assume any specific demand distribution. Instead, we approximate the chance constraint based on the idea of robust optimization, which has been demonstrated to be a very powerful technique for tackling mathematical programs under uncertainty. The important early works and recent developments have been contributed by Soyster (1973), Ben-Tal and Nemirovski (1998, 2000), Ben-Tal et al. (2004), Bertsimas and Sim (2003, 2004), Bertsimas et al. (2004), etc. Robust optimization has also been widely applied in supply chain management, cf. among others, Bertsimas and Thiele (2004), Erera et al. (2009), See and Sim (2010), Klabjan et al. (2013), and Shu and Song (2014).

The remainder of this paper is organized as follows. In section 2, we propose our robust multi-sourcing supply network design model. Two conservative approximations, i.e., the set-wise and Linear Decision Rule-based (LDR) approximations are presented in sections 3 and 4, respectively. Sections 5 and 6 demonstrate the effectiveness of the approximations through comparisons with the convex, Chebyshev, and sample average approximations using extensive computational experiments. Finally, we conclude the paper in section 7.

2 Model Formulation

Consider a set of demand nodes denoted by $\mathcal{R}$ and a set of potential supply nodes denoted by $\mathcal{W}$. A demand node stands for a retailer that faces uncertain demand and a supply node stands for a potential location to set up a DC. We want to (i) choose proper supply nodes from $\mathcal{W}$ to set up DCs, (ii) select DCs to serve each demand node, and (iii) determine the inventory level at each DC. The notations and decision variables are defined as follows.

Notations

c_{ij}	The fixed cost of setting up a link from supply node i to demand node (retailer) j, $\forall i \in \mathcal{W}$ and $j \in \mathcal{R}$
t_{ij}	The per unit delivery cost from supply node i to demand node j, $\forall i \in \mathcal{W}$ and $j \in \mathcal{R}$
f_i	The fixed cost of setting up a DC at supply node i, $\forall i \in \mathcal{W}$
h_i	The per unit inventory holding cost at supply node i, $\forall i \in \mathcal{W}$
$D_j(\omega)$	The demand realization of demand node j under the demand scenario ω, $\forall j \in \mathcal{R}$, where ω is in the set Ω
μ_j	The mean of the random demand faced by demand node j, $\forall j \in \mathcal{R}$
D_j^U	The upper bound of the random demand faced by demand node j, $\forall j \in \mathcal{R}$
D_j^L	The lower bound of the random demand faced by demand node j, $\forall j \in \mathcal{R}$

Decision Variables

I_i	The inventory level at supply node i, $\forall i \in \mathcal{W}$
X_{ij}	$X_{ij} = 1$ if supply node i is linked to (can serve) demand node j, and $X_{ij} = 0$ otherwise, $\forall i \in \mathcal{W}, j \in \mathcal{R}$
Y_i	$Y_i = 1$ if a DC is open at supply node i, and $Y_i = 0$ otherwise, $\forall i \in \mathcal{W}$
θ_{ij}	The amount of shipment from supply node i to demand node j, $\forall i \in \mathcal{W}, j \in \mathcal{R}$

Furthermore, we denote $\mathbf{I}$, $\mathbf{X}$, and $\mathbf{Y}$ as the vectors of I_i, X_{ij}, and Y_i, respectively. Any feasible $(\mathbf{I}, \mathbf{X}, \mathbf{Y})$ should satisfy $(\mathbf{I}, \mathbf{X}, \mathbf{Y}) \in \mathbf{\Xi}$ where

$$\mathbf{\Xi} := \Big\{ (\mathbf{I}, \mathbf{X}, \mathbf{Y}) \ \Big| \ X_{ij} \leq Y_i, I_i \leq MY_i, I_i \geq 0, Y_i \in \{0,1\}, X_{ij} \in \{0,1\}, \ \forall i \in \mathcal{W}, j \in \mathcal{R} \Big\}.$$

Here, M is a big constant (e.g., we can set $M = \sum_{j \in \mathcal{R}} D_j^U$). The first two constraints ensure that each demand node can only be served by open DCs and only open DCs can hold inventory. The other constraints are standard non-negative and binary constraints.

Given the decisions $(\mathbf{I}, \mathbf{X}, \mathbf{Y})$, we can define the following set of demand scenarios where all the demands can be fulfilled by on-hand inventory:

$$A := \left\{ \omega \in \Omega \ \middle| \ \exists \theta_{ij} : 0 \leq \theta_{ij} \leq M'_{ij} X_{ij}, \sum_{j \in \mathcal{R}} \theta_{ij} \leq I_i, \sum_{i \in \mathcal{W}} \theta_{ij} \geq D_j(\omega), \ \forall i \in \mathcal{W}, j \in \mathcal{R} \right\},$$

in which M'_{ij} is a big constant (e.g., we can set $M'_{ij} = D_j^U$). With the above notations, we formulate the robust multi-sourcing supply network design problem as follows:

$$Z_0^* = \min \left\{ \sum_{i \in \mathcal{W}} h_i I_i + \sum_{i \in \mathcal{W}} f_i Y_i + \sum_{i \in \mathcal{W}} \sum_{j \in \mathcal{R}} c_{ij} X_{ij} + \mathbf{E}_\omega \left[\mathcal{Q}(\mathbf{I}, \mathbf{X}, \omega) \right] \;\middle|\; \begin{array}{l} \text{Prob}(A) \geq 1 - \epsilon, \\ (\mathbf{I}, \mathbf{X}, \mathbf{Y}) \in \mathbf{\Xi} \end{array} \right\}, \quad (1)$$

where $\mathcal{Q}(\mathbf{I}, \mathbf{X}, \omega)$ is defined as

$$\mathcal{Q}(\mathbf{I}, \mathbf{X}, \omega) = \min \left\{ \sum_{i \in \mathcal{W}} \sum_{j \in \mathcal{R}} t_{ij} \theta_{ij} \;\middle|\; 0 \leq \theta_{ij} \leq M'_{ij} X_{ij}, \sum_{j \in \mathcal{R}} \theta_{ij} \leq I_i, \sum_{i \in \mathcal{W}} \theta_{ij} \geq D_j(\omega), \ \forall i \in \mathcal{W}, j \in \mathcal{R} \right\},$$

if $\omega \in A$; $\mathcal{Q}(\mathbf{I}, \mathbf{X}, \omega) = 0$ if $\omega \notin A$. In model (1), $\text{Prob}(A) \geq 1 - \epsilon$ is a joint chance constraint ensuring that all the demands must be satisfied with a probability of at least $1 - \epsilon$, where ϵ is a small non-negative number in $(0, 1)$. $1 - \epsilon$ can be interpreted as the service level of the distribution network. In the definitions of A and $\mathcal{Q}(\mathbf{I}, \mathbf{X}, \omega)$, the constraints guarantee that each supply node can only ship products to the linked demand nodes, the amount of shipment from each supply node cannot exceed its available inventory, and the demand at each demand node must be satisfied, respectively.

In certain circumstances, e.g., disaster relief planning, the expected delivery cost $\mathbf{E}_\omega \left[\mathcal{Q}(\mathbf{I}, \mathbf{X}, \omega) \right]$ is approximately zero due to the extremely low probability of a disaster occurring. Furthermore, in these applications, the delivery has a tight time window, and so the links should be chosen based on the consideration of the distance, reliability, suitable transportation mode, etc. In contrast to the low frequency of actual delivery, the cost of setting up links becomes dominant. Hence, it is reasonable to approximate the total cost by ignoring the expected delivery cost $\mathbf{E}_\omega \left[\mathcal{Q}(\mathbf{I}, \mathbf{X}, \omega) \right]$. In this case, model (1) is reduced to

$$Z_1^* = \min \left\{ \sum_{i \in \mathcal{W}} h_i I_i + \sum_{i \in \mathcal{W}} f_i Y_i + \sum_{i \in \mathcal{W}} \sum_{j \in \mathcal{R}} c_{ij} X_{ij} \;\middle|\; \begin{array}{l} \text{Prob}(A) \geq 1 - \epsilon, \\ (\mathbf{I}, \mathbf{X}, \mathbf{Y}) \in \mathbf{\Xi} \end{array} \right\}. \quad (2)$$

In section 3, we ignore the expected delivery cost in the objective function and derive the set-wise approximation for model (2). This assumption of zero expected delivery cost is relaxed in section 4, where we show how to address model (1).

3 Set-wise Approximation for the Model without Delivery Cost

As a variant of the UFLP, model (2) is an NP-hard problem. In addition, evaluating a joint chance constraint is usually very difficult. For instance, Nemirovski and Shapiro (2006) show that assessing the distribution of a weighted aggregation of independently uniformly distributed variables is already NP-hard. Hence, we focus on constructing a conservative approximation for model (2).

First, we show that the chance constraint in model (2) can be reformulated as follows.

Proposition 1 *The chance constraint Prob$(A) \geq 1 - \epsilon$ is equivalent to*

$$\text{Prob}\left\{\sum_{j\in S} D_j(\omega) \leq \sum_{i\in\Gamma(\mathbf{X},S)} I_i,\ \forall S \subseteq \mathcal{R}\right\} \geq 1-\epsilon, \tag{3}$$

where $\Gamma(\mathbf{X},S) := \{i \in \mathcal{W} \mid X_{ij} = 1 \text{ for some } j \in S\}$ *for any* $\mathbf{X} \in \{0,1\}^{|\mathcal{W}|\times|\mathcal{R}|}$ *and* $S \subseteq \mathcal{R}$.

Proof. It is a variant of the max-flow min-cut theorem. □

The definition of $\Gamma(\mathbf{X},S)$ implies that any supply node in $\Gamma(\mathbf{X},S)$ can be used to serve some demand nodes in set S, i.e., $\Gamma(\mathbf{X},S)$ is the set of supply nodes linked to the demand nodes in set S. Consequently, constraint (3) indicates that for all subsets $S \subseteq \mathcal{R}$ of the demand nodes, the linked supply nodes should have sufficient inventory to satisfy the total demand of S with a joint probability of at least $1-\epsilon$.

Unfortunately, constraint (3) is still hard to solve because it needs to evaluate the probability that $2^{|\mathcal{R}|}$ inequalities hold simultaneously. To make the problem tractable, we propose a set-wise approximation that considers only one or a few subsets of $\mathcal{R}$ when determining the joint probability on the left side of (3). We can further show that for a single subset, the aggregated inventory level at its linked supply nodes can be determined by Hoeffding's inequality (cf. Hoeffding, 1963). These results lead to Theorem 1, which provides a conservative set-wise approximation for the chance constraint Prob$(A) \geq 1-\epsilon$.

Theorem 1 *Consider a system with a set of demand nodes ($\mathcal{R}$) and a set of potential supply nodes ($\mathcal{W}$). Each demand node $j \in \mathcal{R}$ faces a random independent demand $D_j(\omega)$ bounded in* $[D_j^L, D_j^U]$

with mean μ_j*. Given the sets* $S^1, ..., S^l \subseteq \mathcal{R}$*, if* $(\mathbf{I}, \mathbf{X}, \mathbf{Y}) \in \mathbf{\Xi}$ *satisfies*

$$\min \left\{ \begin{array}{l} \sum_{j \in S} D_j^U, \\ TI(\epsilon/l, S^1) - \sum_{j \in S^1 \setminus S} D_j^L + \sum_{j \in S \setminus S^1} D_j^U, \\ TI(\epsilon/l, S^2) - \sum_{j \in S^2 \setminus S} D_j^L + \sum_{j \in S \setminus S^2} D_j^U, \\ \dots \\ TI(\epsilon/l, S^l) - \sum_{j \in S^l \setminus S} D_j^L + \sum_{j \in S \setminus S^l} D_j^U \end{array} \right\} \leq \sum_{i \in \Gamma(\mathbf{X}, S)} I_i, \ \forall S \subseteq \mathcal{R}, \tag{4}$$

where

$$TI(\epsilon, S) := \sum_{j \in S} \mu_j + \sqrt{\frac{-(\ln \epsilon) \sum_{j \in S} (D_j^U - D_j^L)^2}{2}}$$

for any $\epsilon \in (0, 1)$ *and* $S \subseteq \mathcal{R}$*, then* $(\mathbf{I}, \mathbf{X}, \mathbf{Y})$ *is a feasible solution to model* (2)*.*

Proof. This theorem is proven in two steps: (i) For any scenario ω such that $\sum_{j \in S^\kappa} D_j(\omega) \leq TI(\epsilon/l, S^\kappa)$ for all $\kappa = 1, ..., l$, the solution $(\mathbf{I}, \mathbf{X}, \mathbf{Y})$ satisfying (4) can always fulfill the demand. (ii) The probability that $\sum_{j \in S^\kappa} D_j(\omega) \leq TI(\epsilon/l, S^\kappa)$ for all $\kappa = 1, ..., l$ is at least $1 - \epsilon$.

Step 1: $(\mathbf{I}, \mathbf{X}, \mathbf{Y})$ satisfying (4) can fulfill the demand if $\sum_{j \in S^\kappa} D_j(\omega) \leq TI(\epsilon/l, S^\kappa)$ for all $\kappa = 1, ..., l$.

By Proposition 1, it is equivalent to show that $Z(S) \leq \sum_{i \in \Gamma(\mathbf{X}, S)} I_i$ for any $S \subseteq \mathcal{R}$, where $Z(S)$ is defined as

$$\begin{aligned} Z(S) := \quad \max \quad & \textstyle\sum_{j \in S} D_j \\ \text{s.t.} \quad & \textstyle\sum_{j \in S^\kappa} D_j \leq TI(\epsilon/l, S^\kappa), \ \forall \kappa = 1, \dots, l, \\ & D_j^L \leq D_j \leq D_j^U, \quad \forall j \in \mathcal{R}, \\ & D_j \geq 0, \qquad\qquad \forall j \in \mathcal{R}. \end{aligned} \tag{5}$$

The first constraint in model (5) can be relaxed as

$$\sum_{j \in S \cap S^\kappa} D_j \leq TI(\epsilon/l, S^\kappa) - \sum_{j \in S^\kappa \setminus S} D_j \leq TI(\epsilon/l, S^\kappa) - \sum_{j \in S^\kappa \setminus S} D_j^L, \ \forall \kappa = 1, \dots, l,$$

and we get an upper bound of $Z(S)$:

$$\begin{aligned} Z^U(S) = \quad \max \quad & \textstyle\sum_{j \in S} D_j \\ \text{s.t.} \quad & \textstyle\sum_{j \in S \cap S^\kappa} D_j \leq TI(\epsilon/l, S^\kappa) - \sum_{j \in S^\kappa \setminus S} D_j^L, \ \forall \kappa = 1, \dots, l, \\ & D_j^L \leq D_j \leq D_j^U, \quad \forall j \in \mathcal{R}, \\ & D_j \geq 0, \qquad\qquad \forall j \in \mathcal{R}. \end{aligned}$$

The dual of $Z^U(S)$ is

$$\begin{aligned} Z^D(S) = \quad & \min \quad \textstyle\sum_{j\in S}\left(D_j^U\alpha_j - D_j^L\beta_j\right) + \sum_{\kappa=1}^{l}\left(TI(\epsilon/l, S^\kappa) - \sum_{j\in S^\kappa\setminus S} D_j^L\right)\theta_\kappa \\ & \text{s.t.} \quad \alpha_j - \beta_j + \textstyle\sum_{\kappa: j\in S\cap S^\kappa}\theta_\kappa \ge 1, \quad \forall j \in S, \\ & \qquad\ \ \alpha_j, \beta_j, \theta_\kappa \ge 0, \qquad\qquad\qquad \forall j \in S, \kappa = 1, \dots, l. \end{aligned}$$

Obviously, the following solutions are feasible to $Z^D(S)$:

- $\theta_\kappa^* = 0$, $\alpha_j^* = 1$ and $\beta_j^* = 0$, $\forall j \in S, \kappa = 1, \dots, l$. The corresponding objective value is $\sum_{j\in S} D_j^U$.
- $\theta_k^* = 1$ for some $k \in \{1, \dots, l\}$, $\theta_\kappa^* = 0$ for all $\kappa \neq k$, $\beta_j^* = 0$ for all $j \in S$, $\alpha_j^* = 0$ for all $j \in S \cap S^k$, and $\alpha_j^* = 1$ for all $j \in S \setminus S^k$. The corresponding objective value is $TI(\epsilon/l, S^k) - \sum_{j\in S^k\setminus S} D_j^L + \sum_{j\in S\setminus S^k} D_j^U$.

Thus,

$$Z(S) \le Z^U(S) = Z^D(S) \le \min \left\{ \begin{array}{l} \sum_{j\in S} D_j^U, \\ TI(\epsilon/l, S^1) - \sum_{j\in S^1\setminus S} D_j^L + \sum_{j\in S\setminus S^1} D_j^U, \\ TI(\epsilon/l, S^2) - \sum_{j\in S^2\setminus S} D_j^L + \sum_{j\in S\setminus S^2} D_j^U, \\ \dots \\ TI(\epsilon/l, S^l) - \sum_{j\in S^l\setminus S} D_j^L + \sum_{j\in S\setminus S^l} D_j^U \end{array} \right\} \le \sum_{i\in\Gamma(\mathbf{X},S)} I_i,$$

where the last inequality is yielded by (4).

Step 2: Prob $\left\{\sum_{j\in S^\kappa} D_j(\omega) \le TI(\epsilon/l, S^\kappa), \forall \kappa = 1, ..., l\right\} \ge 1 - \epsilon$**.**
By Hoeffding's inequality (cf. Hoeffding, 1963), as the random demand of demand node j is independent and bounded in $[D_j^L, D_j^U]$ with mean μ_j for all $j \in \mathcal{R}$, the following inequality holds for any $\kappa = 1, ..., l$ and any positive t:

$$\text{Prob}\left\{\sum_{j\in S^\kappa} D_j(\omega) \le \sum_{j\in S^\kappa} \mu_j + t\right\} \ge 1 - \exp\left(-\frac{2t^2}{\sum_{j\in S^\kappa}(D_j^U - D_j^L)^2}\right).$$

The definition of $TI(\epsilon/l, S^\kappa)$ yields

$$\text{Prob}\left\{\sum_{j\in S^\kappa} D_j(\omega) \le TI(\epsilon/l, S^\kappa)\right\} \ge 1 - \frac{\epsilon}{l}, \ \forall \kappa = 1, \dots, l,$$

and hence $\text{Prob}\left\{\sum_{j\in S^\kappa} D_j(\omega) \leq TI(\epsilon/l, S^\kappa), \forall\kappa = 1, ..., l\right\} \geq 1-\epsilon$. □

Applying Theorem 1, we can obtain the set-wise approximation of model (2) by replacing the chance constraint $\text{Prob}(A) \geq 1-\epsilon$ with (4). Note that the complexity of constraint (4) increases with the increase of l. Hence, we consider the simplest case with $l = 1$, i.e., $S^1 = \mathcal{R}$, which yields the following corollary.

Corollary 1 *Consider a system with a set of demand nodes ($\mathcal{R}$) and a set of potential supply nodes ($\mathcal{W}$). Each demand node $j \in \mathcal{R}$ faces a random independent demand $D_j(\omega)$ bounded in $[D_j^L, D_j^U]$ with mean μ_j. If $(\mathbf{I}, \mathbf{X}, \mathbf{Y}) \in \mathbf{\Xi}$ satisfies*

$$\min\left\{\sum_{j\in S} D_j^U, TI(\epsilon, \mathcal{R}) - \sum_{j\notin S} D_j^L\right\} \leq \sum_{i\in\Gamma(\mathbf{X},S)} I_i,\ \forall S \subseteq \mathcal{R}, \tag{6}$$

then $(\mathbf{I}, \mathbf{X}, \mathbf{Y})$ is a feasible solution to model (2).

Interestingly, we note that inequality (6) in Corollary 1 is structurally the same as the "1-Expander" defined in the Process Flexibility study by Chou et al. (2011). The idea of Process Flexibility (cf. Sethi and Sethi, 1990; Jordan and Graves, 1995; Iravani et al., 2005; Chen et al., 2015; Wang and Zhang, 2015, etc.) is adopted in manufacturing or service systems to effectively match random supply and demand by ensuring that each production unit/server has the ability to produce/provide multiple products/services. The 1-Expander is an elegant and novel *sparse* Process Flexibility structure, which achieves the performance of full flexibility with only a small number of links. In Theorem 1, we show how to obtain the 1-Expander based on the set-wise approximation of the chance constraint. This provides a generalization and a new application of the 1-Expander structure.

4 LDR Approximation for the Model with Delivery Cost

Although model (2) can fit in certain contexts such as disaster relief operations, it has limitations when applied to other circumstances because it fails to consider the per unit delivery cost — an important cost component in general distribution network design. In this section, we study model (1), which incorporates delivery costs.

We use the linear decision rule to represent θ_{ij} as

$$\theta_{ij} = \sum_{k\in\mathcal{R}} a_{ij}^k D_k(\omega),$$

i.e., the amount of shipment from supply node i to demand node j is a linear combination of the realized demands at all demand nodes. Let $\mathbf{a}$ denote the vector of a_{ij}^k. $\mathbf{\Lambda}(\mathbf{X})$ is the feasible region of $\mathbf{a}$ given the network links $\mathbf{X}$. For example, we can set

$$\mathbf{\Lambda}(\mathbf{X}) = \left\{\mathbf{a} \,\middle|\, -MX_{ij} \le a_{ij}^k \le MX_{ij},\ \forall i \in \mathcal{W}, j \in \mathcal{R}, k \in \mathcal{R}\right\},$$

where a_{ij}^k is forced to zero for any $X_{ij} = 0$ so that a demand node can only be served by a linked supply node. Model (1) can then be reformulated as

$$\begin{aligned} Z_0^* = \quad & \min \quad \sum_{i\in\mathcal{W}} h_i I_i + \sum_{i\in\mathcal{W}} f_i Y_i + \sum_{i\in\mathcal{W}}\sum_{j\in\mathcal{R}} c_{ij}X_{ij} + \mathbf{E}_\omega\left[\sum_{i\in\mathcal{W}}\sum_{j\in\mathcal{R}}\sum_{k\in\mathcal{R}} t_{ij}a_{ij}^k D_k(\omega)\right] \\ & \text{s.t.} \quad \text{Prob}\left\{\begin{array}{ll} \sum_{j\in\mathcal{R}}\sum_{k\in\mathcal{R}} a_{ij}^k D_k(\omega) \le I_i, & \forall i \in \mathcal{W} \\ \sum_{i\in\mathcal{W}}\sum_{k\in\mathcal{R}} a_{ij}^k D_k(\omega) \ge D_j(\omega), & \forall j \in \mathcal{R} \end{array}\right\} \ge 1-\epsilon, \\ & \qquad \sum_{k\in\mathcal{R}} a_{ij}^k D_k(\omega) \ge 0, \quad \forall i \in \mathcal{W}, j \in \mathcal{R}, \omega \in \Omega, \\ & \qquad \mathbf{a} \in \mathbf{\Lambda}(\mathbf{X}),\ (\mathbf{I},\mathbf{X},\mathbf{Y}) \in \mathbf{\Xi}. \end{aligned} \tag{7}$$

Proposition 2 *Suppose that each demand node $j \in \mathcal{R}$ faces a random independent demand $D_j(\omega)$ bounded in $[D_j^L, D_j^U]$ with mean μ_j. Then, any $(\mathbf{I}, \mathbf{a})$ feasible to the following inequality system*

$$\begin{array}{ll} I_i \ge \sum_{k\in\mathcal{R}}\sum_{j\in\mathcal{R}} a_{ij}^k \mu_k + \sqrt{\frac{(\ln|\mathcal{W}| - \ln\epsilon)\sum_{k\in\mathcal{R}} A_{ik}^2}{2}}, & \forall i \in \mathcal{W}, \\ A_{ik} \ge \sum_{j\in\mathcal{R}} a_{ij}^k \left(D_k^U - D_k^L\right), & \forall i \in \mathcal{W}, k \in \mathcal{R}, \\ A_{ik} \ge \sum_{j\in\mathcal{R}} a_{ij}^k \left(D_k^L - D_k^U\right), & \forall i \in \mathcal{W}, k \in \mathcal{R}, \\ -\sum_{k\in\mathcal{R}} D_k^U \alpha_j^k + \sum_{k\in\mathcal{R}} D_k^L \beta_j^k \ge 0, & \forall j \in \mathcal{R}, \\ -\alpha_j^k + \beta_j^k \le \sum_{i\in\mathcal{W}} a_{ij}^k, & \forall j \in \mathcal{R}, k \in \mathcal{R}\setminus\{j\}, \\ -\alpha_j^j + \beta_j^j \le \sum_{i\in\mathcal{W}} a_{ij}^j - 1, & \forall j \in \mathcal{R}, \\ \alpha_j^k \ge 0,\ \beta_j^k \ge 0, & \forall j \in \mathcal{R}, k \in \mathcal{R}, \end{array}$$

satisfies the joint chance constraint in model (7)*, i.e.,*

$$\textit{Prob}\left\{\begin{array}{ll} \sum_{j\in\mathcal{R}}\sum_{k\in\mathcal{R}} a_{ij}^k D_k(\omega) \le I_i, & \forall i \in \mathcal{W} \\ \sum_{i\in\mathcal{W}}\sum_{k\in\mathcal{R}} a_{ij}^k D_k(\omega) \ge D_j(\omega), & \forall j \in \mathcal{R} \end{array}\right\} \ge 1-\epsilon. \tag{8}$$

Proof. Obviously, a sufficient condition to (8) is

$$\text{Prob}\left\{\sum_{j\in\mathcal{R}}\sum_{k\in\mathcal{R}} a_{ij}^k D_k(\omega) \le I_i\right\} \ge 1-\frac{\epsilon}{|\mathcal{W}|}, \quad \forall i\in\mathcal{W}, \tag{9}$$

$$\sum_{i\in\mathcal{W}}\sum_{k\in\mathcal{R}} a_{ij}^k D_k(\omega) \ge D_j(\omega), \quad \forall j\in\mathcal{R}, \omega\in\Omega. \tag{10}$$

By Hoeffding's inequality (cf. Hoeffding, 1963), (9) holds if

$$I_i \ge \sum_{k\in\mathcal{R}}\sum_{j\in\mathcal{R}} a_{ij}^k \mu_k + \sqrt{\frac{(\ln|\mathcal{W}|-\ln\epsilon)\sum_{k\in\mathcal{R}}\left(d_{ik}^U-d_{ik}^L\right)^2}{2}}, \quad \forall i\in\mathcal{W}, \tag{11}$$

where d_{ik}^U and d_{ik}^L are the maximum and minimum values of $\sum_{j\in\mathcal{R}} a_{ij}^k D_k(\omega)$ for all $i,k\in\mathcal{R}$, respectively. (11) can be reformulated as:

$$\begin{aligned}
&I_i \ge \sum_{k\in\mathcal{R}}\sum_{j\in\mathcal{R}} a_{ij}^k \mu_k + \sqrt{\frac{(\ln|\mathcal{W}|-\ln\epsilon)\sum_{k\in\mathcal{R}} A_{ik}^2}{2}}, && \forall i\in\mathcal{W},\\
&A_{ik} \ge \sum_{j\in\mathcal{R}} a_{ij}^k\left(D_k^U-D_k^L\right), && \forall i\in\mathcal{W}, k\in\mathcal{R},\\
&A_{ik} \ge \sum_{j\in\mathcal{R}} a_{ij}^k\left(D_k^L-D_k^U\right), && \forall i\in\mathcal{W}, k\in\mathcal{R}.
\end{aligned}$$

Constraint (10) is equivalent to the following LP with the optimal value $W_j^* \ge 0$ for all $j\in\mathcal{R}$:

$$\begin{aligned}
W_j^* = \quad &\min \quad \textstyle\sum_{k\ne j}\sum_{i\in\mathcal{W}} a_{ij}^k D_k + \left(\sum_{i\in\mathcal{W}} a_{ij}^j - 1\right) D_j\\
&\text{s.t.} \quad D_k^L \le D_k \le D_k^U,\ D_k \ge 0, \quad \forall k\in\mathcal{R}.
\end{aligned}$$

Consider the corresponding dual problem:

$$\begin{aligned}
DM_j:\quad &\max \quad -\textstyle\sum_{k\in\mathcal{R}} D_k^U\alpha_j^k + \sum_{k\in\mathcal{R}} D_k^L\beta_j^k\\
&\text{s.t.} \quad -\alpha_j^k+\beta_j^k \le \textstyle\sum_{i\in\mathcal{W}} a_{ij}^k, && \forall k\in\mathcal{R}\setminus\{j\},\\
&\qquad\ -\alpha_j^j+\beta_j^j \le \textstyle\sum_{i\in\mathcal{W}} a_{ij}^j - 1,\\
&\qquad\ \alpha_j^k \ge 0,\ \beta_j^k \ge 0, && \forall k\in\mathcal{R}.
\end{aligned}$$

By the strong duality theorem, $W_j^* \ge 0$ is equivalent to that a feasible solution to the dual maximization problem (DM_j) has a nonnegative objective value. □

Proposition 2 is crucial because it uses a tractable deterministic inequality system to approximate the joint chance constraint and avoid dealing with the random parameters.

Model (7) contains another constraint with random parameters, i.e.,

$$\sum_{k\in\mathcal{R}} a_{ij}^k D_k(\omega) \geq 0, \quad \forall i \in \mathcal{W}, j \in \mathcal{R}, \omega \in \Omega.$$

Similarly, we can rewrite it as a set of deterministic inequalities:

$$\begin{aligned} &-\sum_{k\in\mathcal{R}} D_k^U \eta_{ij}^k + \sum_{k\in\mathcal{R}} D_k^L \gamma_{ij}^k \geq 0, \qquad \forall i \in \mathcal{W}, j \in \mathcal{R}, \\ &-\eta_{ij}^k + \gamma_{ij}^k \leq a_{ij}^k, \eta_{ij}^k \geq 0, \gamma_{ij}^k \geq 0, \quad \forall i \in \mathcal{W}, j \in \mathcal{R}, k \in \mathcal{R}. \end{aligned}$$

Consequently, we obtain the following robust LDR approximation of model (1):

$$\begin{aligned} Z_0^* = \quad \min \quad & \sum_{i\in\mathcal{W}} h_i I_i + \sum_{i\in\mathcal{W}} f_i Y_i + \sum_{i\in\mathcal{W}} \sum_{j\in\mathcal{R}} c_{ij} X_{ij} + \sum_{i\in\mathcal{W}} \sum_{j\in\mathcal{R}} \sum_{k\in\mathcal{R}} t_{ij} a_{ij}^k \mu_k \\ \text{s.t.} \quad & I_i \geq \textstyle\sum_{k\in\mathcal{R}} \sum_{j\in\mathcal{R}} a_{ij}^k \mu_k + \sqrt{\frac{(\ln|\mathcal{W}| - \ln \epsilon) \sum_{k\in\mathcal{R}} A_{ik}^2}{2}}, && \forall i \in \mathcal{W}, \\ & A_{ik} \geq \textstyle\sum_{j\in\mathcal{R}} a_{ij}^k \left(D_k^U - D_k^L\right), && \forall i \in \mathcal{W}, k \in \mathcal{R}, \\ & A_{ik} \geq \textstyle\sum_{j\in\mathcal{R}} a_{ij}^k \left(D_k^L - D_k^U\right), && \forall i \in \mathcal{W}, k \in \mathcal{R}, \\ & -\textstyle\sum_{k\in\mathcal{R}} D_k^U \alpha_j^k + \sum_{k\in\mathcal{R}} D_k^L \beta_j^k \geq 0, && \forall j \in \mathcal{R}, \\ & -\alpha_j^k + \beta_j^k \leq \textstyle\sum_{i\in\mathcal{W}} a_{ij}^k, && \forall j \in \mathcal{R}, k \in \mathcal{R} \setminus \{j\}, \\ & -\alpha_j^j + \beta_j^j \leq \textstyle\sum_{i\in\mathcal{W}} a_{ij}^j - 1, && \forall j \in \mathcal{R}, \\ & -\textstyle\sum_{k\in\mathcal{R}} D_k^U \eta_{ij}^k + \sum_{k\in\mathcal{R}} D_k^L \gamma_{ij}^k \geq 0, && \forall i \in \mathcal{W}, j \in \mathcal{R}, \\ & -\eta_{ij}^k + \gamma_{ij}^k \leq a_{ij}^k, && \forall i \in \mathcal{W}, j \in \mathcal{R}, k \in \mathcal{R}, \\ & \alpha_j^k \geq 0, \ \beta_j^k \geq 0, && \forall j \in \mathcal{R}, k \in \mathcal{R}, \\ & \eta_{ij}^k \geq 0, \ \gamma_{ij}^k \geq 0, && \forall i \in \mathcal{W}, j \in \mathcal{R}, k \in \mathcal{R}, \\ & \mathbf{a} \in \mathbf{\Lambda}(\mathbf{X}), \ (\mathbf{I}, \mathbf{X}, \mathbf{Y}) \in \mathbf{\Xi}, \end{aligned} \tag{12}$$

which is a Second Order Conic Program with binary variables.

5 Numerical Study for the Model without Delivery Cost

This section evaluates the performance of the set-wise approximation with $l = 1$ by comparing it with three common approaches to handling chance constraints: the convex, Chebyshev, and sample average approximations. Because the set-wise approximation with $l = 1$ contains an exponential number of constraints, we first develop a cutting plane procedure to solve the corresponding optimization problem. The model formulations for the other three approaches are introduced subsequently. We then report the computational results that demonstrate the superior performance of the set-wise approximation with $l = 1$.

5.1 Cutting Plane Procedure for the Set-wise Approximation with $l = 1$

Consider the set-wise approximation with $l = 1$, i.e., model (2) with (6) replacing the chance constraint $\text{Prob}(A) \geq 1 - \epsilon$. To represent the term $\sum_{i \in \Gamma(\mathbf{X}, S)} I_i$ as a linear expression of the decision variables $\mathbf{I}$ and $\mathbf{Y}$, we introduce a new decision variable $q_{i,S}$ for any $i \in \mathcal{W}$ and $S \subseteq \mathcal{R}$, denoting an upper bound of the amount of shipment from supply node i to all the demand nodes in set S. Constraint (6) can be replaced by

$$\begin{aligned} &\min\left\{\textstyle\sum_{j \in S} D_j^U, TI(\epsilon, \mathcal{R}) - \sum_{j \notin S} D_j^L\right\} \leq \textstyle\sum_{i \in \mathcal{W}} q_{i,S}, \quad && \forall S \subseteq \mathcal{R}, \\ &0 \leq q_{i,S} \leq I_i, \ q_{i,S} \leq M \textstyle\sum_{j \in S} X_{ij}, && \forall i \in \mathcal{W}, S \subseteq \mathcal{R}. \end{aligned} \tag{13}$$

As there are an exponential number of constraints in (13), we adopt the cutting plane procedure to solve it iteratively. In each cutting plane iteration, we consider the following separation subproblem to determine whether the solution in the current iteration is optimal:

$$Z_{sep}^* := \min_{S: S \subseteq \mathcal{R}} \left\{ \sum_{i \in \Gamma(\mathbf{X}, S)} I_i - \min\left\{ \sum_{j \in S} D_j^U, TI(\epsilon, \mathcal{R}) - \sum_{j \notin S} D_j^L \right\} \right\}.$$

Obviously, the separation subproblem Z_{sep}^* can be reformulated as

$$\min_{\substack{z \geq 0 \\ y_i, x_j \in \{0,1\}}} \left\{ \sum_{i \in \mathcal{W}} I_i y_i - \sum_{j \in \mathcal{R}} D_j^U x_j + z \;\middle|\; \begin{array}{l} z \geq \sum_{j \in \mathcal{R}} D_j^U x_j - TI(\epsilon, \mathcal{R}) + \sum_{j \in \mathcal{R}} D_j^L (1 - x_j), \\ y_i \geq x_j, \quad \forall i \in \mathcal{W}, j \in \mathcal{R} \text{ such that } X_{ij} = 1 \end{array} \right\}, \tag{14}$$

which is readily solvable by the CPLEX MIP Solver. If $Z_{sep}^* \geq 0$, the current solution is optimal and the cutting plane procedure should be terminated. Otherwise, we identify a valid inequality in the form of (13) associated with S^* (the optimal solution to the separation subproblem in the current iteration), add it into the master problem, and start the next iteration.

5.2 Convex Approximation

In this subsection, we formulate the convex approximation for model (2) based on the work of Nemirovski and Shapiro (2006).

Consider the decision variable $q_{i,S}$ for any $i \in \mathcal{W}$ and $S \subseteq \mathcal{R}$ in (13). Constraint (3) obtained in Proposition 1 can be equivalently written as

$$\begin{aligned} &\text{Prob}\left\{-\textstyle\sum_{i \in \mathcal{W}} q_{i,S} + \sum_{j \in \mathcal{R}} D_j(\omega) \mathbb{I}_{j \in S} \leq 0, \ \forall S \subseteq \mathcal{R}, S \neq \emptyset\right\} \geq 1 - \epsilon, \\ &0 \leq q_{i,S} \leq I_i, \ q_{i,S} \leq M \textstyle\sum_{j \in S} X_{ij}, \qquad \forall i \in \mathcal{W}, S \subseteq \mathcal{R}, S \neq \emptyset, \end{aligned} \tag{15}$$

where $\mathbb{I}_{j\in S} \in \{0,1\}$ indicates whether $j \in S$.

Following the procedures in Nemirovski and Shapiro (2006), the joint chance constraint in (15) is satisfied once

$$\text{Prob}\left\{-\sum_{i\in\mathcal{W}} q_{i,S} + \sum_{j\in\mathcal{R}} D_j(\omega)\mathbb{I}_{j\in S} \le 0\right\} \ge 1 - \frac{\epsilon}{2^{|\mathcal{R}|}-1}, \quad \forall S \subseteq \mathcal{R}, S \ne \emptyset. \tag{16}$$

We then apply the Bernstein approximation of ambiguous chance constraint proposed by Nemirovski and Shapiro (2006), and obtain the convex approximation of constraint (16) as follows:

$$f_{S0} + \sum_{j\in\mathcal{R}} \bar{\mu}_j f_{Sj} + \sqrt{2\ln\left(\frac{2^{|\mathcal{R}|}-1}{\epsilon}\right)\sum_{j\in\mathcal{R}} f_{Sj}^2} \le 0, \quad \forall S \subseteq \mathcal{R}, S \ne \emptyset,$$

where

$$\bar{\mu}_j := \frac{2\left(\mu_j - D_j^L\right)}{D_j^U - D_j^L} - 1, \qquad f_{S0} := \sum_{j\in\mathcal{R}} \frac{D_j^U + D_j^L}{2}\mathbb{I}_{j\in S} - \sum_{i\in\mathcal{W}} q_{i,S}, \qquad f_{Sj} := \frac{D_j^U - D_j^L}{2}\mathbb{I}_{j\in S}.$$

The convex approximation of constraint (16) is further simplified to

$$\sum_{j\in S}\mu_j + \sqrt{\frac{\ln(2^{|\mathcal{R}|}-1) - \ln\epsilon}{2}\sum_{j\in S}\left(D_j^U - D_j^L\right)^2} - \sum_{i\in\mathcal{W}} q_{i,S} \le 0, \quad \forall S \subseteq \mathcal{R}, S \ne \emptyset.$$

Obviously, the convex approximation also transforms the joint chance constraint into an exponential number of constraints, which share the same structure as those in the set-wise approximation. As a result, we can solve the convex approximation problem iteratively using the cutting plane procedure developed for the set-wise approximation. The only difference is that the following separation subproblem, instead of model (14), is solved in each cutting plane iteration:

$$\min_{\substack{z\ge 0\\ y_i,x_j\in\{0,1\}}} \left\{\sum_{i\in\mathcal{W}} I_i y_i - \sum_{j\in\mathcal{R}} \mu_j x_j - z \;\middle|\; \begin{array}{l} z^2 \le \frac{\ln(2^{|\mathcal{R}|}-1)-\ln\epsilon}{2}\sum_{j\in\mathcal{R}}(D_j^U - D_j^L)^2 x_j, \\ y_i \ge x_j, \quad \forall i \in \mathcal{W}, j \in \mathcal{R} \text{ such that } X_{ij} = 1 \end{array}\right\}.$$

5.3 Chebyshev Approximation

According to the Chebyshev bounds presented in section 9.4 of Birge and Louveaux (1997), the Chebyshev approximation of constraint (16) is formulated as

$$\sum_{j\in S}\mu_j + \sqrt{\left(\frac{2^{|\mathcal{R}|}-1}{\epsilon} - 1\right)\sum_{j\in S}\sigma_j^2} \le \sum_{i\in\mathcal{W}} q_{i,S}, \quad \forall S \subseteq \mathcal{R}, S \ne \emptyset, \tag{17}$$

where σ_j^2 denotes the variance of the uncertain demand $D_j(\omega)$.

Again, the resulting formulation contains an exponential number of constraints. Thus, it can be solved using the cutting plane procedure we propose for the set-wise and convex approximations. The corresponding separation subproblem is

$$\min_{\substack{z\geq 0\\ y_i,x_j\in\{0,1\}}} \left\{ \sum_{i\in\mathcal{W}} I_i y_i - \sum_{j\in\mathcal{R}} \mu_j x_j - z \;\middle|\; \begin{array}{l} z^2 \leq \left(\frac{2^{|\mathcal{R}|}-1}{\epsilon} - 1\right) \sum_{j\in\mathcal{R}} \sigma_j^2 x_j, \\ y_i \geq x_j, \quad \forall i\in\mathcal{W}, j\in\mathcal{R} \text{ such that } X_{ij}=1 \end{array} \right\}.$$

5.4 Sample Average Approximation (SAA)

Consider N samples of demand realization denoted by ω^n where $n = 1, ..., N$. Let $D_j(\omega^n)$ denote the demand realization of demand node j in sample ω^n, θ_{ij}^n is the amount of shipment from supply node i to demand node j in sample ω^n, and z^n represents whether sample ω^n satisfies the conditions of event A in the chance constraint of model (2). The SAA formulation of model (2) can be written as

$$\begin{array}{lll} \min & \sum_{i\in\mathcal{W}} h_i I_i + \sum_{i\in\mathcal{W}} f_i Y_i + \sum_{i\in\mathcal{W}} \sum_{j\in\mathcal{R}} c_{ij} X_{ij} & \\ \text{s.t.} & \sum_{n=1}^{N} z^n \geq N(1-\epsilon), & \\ & 0 \leq \theta_{ij}^n \leq M'_{ij} X_{ij}, & \forall i\in\mathcal{W}, j\in\mathcal{R}, n = 1, ..., N, \\ & \sum_{j\in\mathcal{R}} \theta_{ij}^n \leq I_i, & \forall i\in\mathcal{W}, n = 1, ..., N, \\ & \sum_{i\in\mathcal{W}} \theta_{ij}^n \geq z^n D_j(\omega^n), & \forall j\in\mathcal{R}, n = 1, ..., N, \\ & z^n \in \{0,1\}, & \forall n = 1, ..., N, \\ & (\mathbf{I}, \mathbf{X}, \mathbf{Y}) \in \mathbf{\Xi}. & \end{array} \tag{18}$$

5.5 Computational Results

In this subsection, we generate random instances to evaluate the performance of the set-wise approximation with $l = 1$, and compare it with the convex, Chebyshev, and sample average approximations.

We view a combination of $(|\mathcal{W}|, |\mathcal{R}|, 1-\epsilon)$ as a test class. This study considers the 12 test classes displayed in Table 1. For each test class, 40 instances are generated using the following procedure. The locations of the demand nodes and the potential supply nodes are uniformly distributed over $[0, 100] \times [0, 100]$. The per-unit inventory holding costs and the fixed location costs are randomly generated, respectively, by $h_i \sim U(8, 10)$ and $f_i \sim U(100, 200)$, where $U(a, b)$ represents a uniform distribution in $[a, b]$. The fixed cost c_{ij} of the link between supply node i and demand node j is assumed to be proportional to the corresponding euclidean distance in the 2-D plane. The mean μ_j and variance σ_j^2 of demand are generated by $\mu_j \sim U(100, 200)$ and $\sigma_j^2 \sim U(300, 500)$, respectively. The support of demand is assumed to be $[\mu_j - 1.96\sigma_j, \mu_j + 1.96\sigma_j]$, i.e., $D_j^L = \mu_j - 1.96\sigma_j$ and $D_j^U = \mu_j + 1.96\sigma_j$. Note that $\mu_j \geq 100$ and $\sigma_j^2 \leq 500$ ensure $\mu_j - 1.96\sigma_j \geq 0$.

For each instance, we solve the models based on the set-wise, convex, Chebyshev, and sample average approximations. The results are reported in Table 1. The column titled "Model" indicates the approximation approach used to obtain the values in the corresponding row. The CPU time, the number of DCs open, and the number of selected links are shown in the columns titled "CPU Time", "#DC", and "#Arc", respectively. Recall that 40 instances are generated for each test class. All these values correspond to the averages of the 40 instances. We also compare the total inventory and total cost of the convex, Chebyshev, and sample average approximations with the counterparts of the set-wise approximation in the columns titled "ΔInv" and "ΔCost", which present the relative increments of the average (denoted by "AV") and the worst-case (denoted by "WC") scenarios, respectively.

Two performance measures, fill rate and chance, are used to compare the effectiveness of the various approximations. The fill rate is defined as the percentage of demand satisfied by on-hand inventory, which is an important measurement of service level. Specifically, given the solution $(\mathbf{I}, \mathbf{X}, \mathbf{Y}) \in \Xi$, the total amount of satisfied demand can be obtained by solving the following maximum flow problem:

$$Z'(\omega) = \max_{\theta_{ij}} \left\{ \sum_{i\in\mathcal{W}} \sum_{j\in\mathcal{R}} \theta_{ij} \;\middle|\; 0 \leq \theta_{ij} \leq M'_{ij} X_{ij}, \sum_{j\in\mathcal{R}} \theta_{ij} \leq I_i, \sum_{i\in\mathcal{W}} \theta_{ij} \leq D_j(\omega), \ \forall i \in \mathcal{W}, j \in \mathcal{R} \right\}.$$

Then, the fill rate is calculated as

$$\text{Fill Rate} := \mathbf{E}_\omega \left[Z'(\omega) \Big/ \sum_{j\in\mathcal{R}} D_j(\omega) \right].$$

The chance refers to the probability that the realized demand is satisfied by on-hand inventory, i.e.,

$$\text{Chance} := \text{Prob} \left\{ Z'(\omega) = \sum_{j\in\mathcal{R}} D_j(\omega) \right\}.$$

In the numerical experiments, both the fill rate and the chance for each instance are estimated based on 10,000 realizations of $\{D_j(\omega) | \forall j \in \mathcal{R}\}$. For each test class, we report the average fill rate and chance of the 40 instances in the columns titled "Fill Rate" and "Chance" in Table 1, respectively.

Table 1 indicates that the proposed set-wise approximation is capable of providing solutions with reasonable computational efforts. For example, the average computational time to solve the set-wise approximations with 50 potential supply nodes and 100 demand nodes is no more than 13 minutes. Another important observation is that the number of selected links in the network is very close to the number of demand nodes, which implies that the network obtained is very sparse. The chance to satisfy demand using on-hand inventory is always higher than the prespecified level $1-\epsilon$ and the fill rate is almost 100% for all the considered instances. Thus, we can safely conclude that a sparse multi-sourcing network can lead to a very high service level. Next, we compare the set-wise approximation results with those obtained from other approaches to further validate the effectiveness of the set-wise approximation.

The convex approximation provides a similar service level with higher inventory levels and total costs, which indicates that it is more conservative than the set-wise approximation. For example, in the test class (5,10,0.9), the total inventory and total cost of the convex approximation are, on average, 5.0% (i.e., ΔInv-AV) and 5.5% (i.e., ΔCost-AV) higher than the counterparts of the set-wise approximation, and the values of ΔInv-WC and ΔCost-WC are 4.2% and 4.7%, respectively. The convex approximation is worse off when the problem size grows. In particular, when we have 50 supply nodes and 100 demand nodes, it requires roughly 12% more inventory and 7.5% higher total cost. The conservativeness of convex approximation is also shown in the network structure. It yields denser networks except for the test classes (50,100,0.9) and (50,100,0.95), where the number of selected links is only 0.1% lower than that of the set-wise approximation. Furthermore, the convex approximation generally requires much more computational effort. In particular, only 57.5% of the instances with $|\mathcal{W}| = 50$ and $|\mathcal{R}| = 100$ can be solved within an hour.

The Chebyshev approximation is even more conservative as its total inventory and total cost are extremely high. For example, in the test class (5,10,0.9), the values of ΔInv-AV and ΔCost-AV of the Chebyshev approximation are 142.2% and 106.3%, and those of ΔInv-WC and ΔCost-WC are 135.8% and 102.3%, respectively. As the problem size increases, ΔInv-AV (resp. ΔInv-WC, ΔCost-AV, ΔCost-WC) rises to 2509.9% (resp. 3242.9%, 1939.2%, 2689.3%) in the class (10, 20, 0.9). Constraint (17) in the Chebyshev approximation indicates that the required inventory grows

Table 1: Computational results for the model without delivery cost

$\|\mathcal{W}\|$	$\|\mathcal{R}\|$	$1-\epsilon$	Model	CPU Time (s)	#DC	#Arc	ΔInv(%) AV	ΔInv(%) WC	ΔCost(%) AV	ΔCost(%) WC	Fill Rate (%)	Chance (%)
5	**10**	**0.9**	Set-wise	70.5	2.0	10.4	-	-	-	-	100.00	99.74
			Convex	336.0	2.0	11.0	5.0	4.2	5.5	4.7	100.00	100.00
			Chebyshev	145.4	1.0	10.1	142.2	135.8	106.3	102.3	100.00	100.00
			SAA-100	117.2	1.7	10.0	-4.8	-5.7	-4.4	-4.3	99.92	87.11
		0.95	Set-wise	58.0	2.0	10.2	-	-	-	-	100.00	99.79
			Convex	320.7	2.0	11.0	4.5	3.8	5.5	5.0	100.00	100.00
			Chebyshev	83.0	1.0	10.0	201.3	192.8	150.5	144.4	100.00	100.00
			SAA-100	35.3	1.7	10.1	-5.2	-6.5	-4.2	-4.1	99.96	92.08
8	**15**	**0.9**	Set-wise	208.4	2.2	15.4	-	-	-	-	100.00	99.85
			Convex*1	347.8	2.2	16.1	5.9	6.0	5.9	5.4	100.00	100.00
			Chebyshev	15.3	1.0	15.2	688.8	644.9	520.7	497.2	100.00	100.00
			SAA-100	214.0	2.1	15.0	-4.6	-5.7	-4.4	-3.5	99.94	86.22
		0.95	Set-wise	168.1	2.2	15.4	-	-	-	-	100.00	99.90
			Convex*1	392.9	2.2	16.1	5.9	7.4	5.8	5.3	100.00	100.00
			Chebyshev	6.4	1.0	15.2	971.9	916.5	732.1	703.2	100.00	100.00
			SAA-100	155.7	2.0	15.1	-4.6	-5.1	-4.3	-3.3	99.97	92.09
10	**20**	**0.9**	Set-wise	223.9	2.5	21.1	-	-	-	-	100.00	99.99
			Convex*1	666.7	2.4	21.2	7.6	7.2	6.2	6.4	100.00	100.00
			Chebyshev	32.0	1.0	20.0	2509.9	3242.9	1939.2	2689.3	100.00	100.00
			SAA-100	524.0	2.2	20.1	-3.8	-5.3	-4.4	-3.6	99.95	86.06
		0.95	Set-wise	184.2	2.5	20.6	-	-	-	-	100.00	99.94
			Convex*1	461.7	2.4	21.2	6.4	4.4	5.9	6.1	100.00	100.00
			Chebyshev	11.3	1.0	20.0	3607.8	4515.4	2773.4	3551.8	100.00	100.00
			SAA-100	417.9	2.2	20.1	-4.5	-5.6	-4.5	-3.9	99.97	91.76
20	**40**	**0.9**	Set-wise	299.3	4.2	41.5	-	-	-	-	100.00	99.99
			Convex*4	891.3	3.6	42.7	7.5	7.4	6.7	5.3	100.00	100.00
			SAA-50*11	2895.7	3.5	40.9	-6.0	-7.4	-6.9	-6.4	99.95	80.08
		0.95	Set-wise	284.6	4.2	41.6	-	-	-	-	100.00	99.98
			Convex*4	1014.6	3.6	43.4	7.0	10.7	6.6	5.4	100.00	100.00
			SAA-50*3	1200.4	3.7	40.9	-6.0	-6.2	-7.1	-6.1	99.98	88.54
30	**60**	**0.9**	Set-wise	510.6	5.3	62.2	-	-	-	-	100.00	99.95
			Convex*5	1083.5	3.5	63.3	7.7	11.0	6.5	7.1	100.00	100.00
			SAA-20	406.9	4.7	61.0	-7.0	-7.8	-8.8	-8.2	99.93	67.98
		0.95	Set-wise	400.7	5.2	61.9	-	-	-	-	100.00	100.00
			Convex*7	1071.8	3.4	62.7	8.3	10.6	6.8	7.4	100.00	100.00
			SAA-20	261.6	4.8	60.8	-6.9	-7.7	-8.6	-8.1	99.95	71.87
50	**100**	**0.9**	Set-wise	729.8	6.0	100.4	-	-	-	-	100.00	100.00
			Convex*17	1756.8	3.6	100.3	11.5	12.3	8.0	7.1	100.00	100.00
			SAA-20*10	2983.4	6.0	103.2	-8.0	-7.0	-9.7	-8.4	99.94	66.55
		0.95	Set-wise	692.8	6.1	100.4	-	-	-	-	100.00	100.00
			Convex*17	1709.7	3.6	100.3	11.6	12.4	8.0	7.2	100.00	100.00
			SAA-20*5	2457.5	6.0	102.2	-7.8	-7.3	-10.0	-8.4	99.95	67.04

*1-17 The number in the superscript represents the number of instances unsolved within one hour.

exponentially with respect to the number of demand nodes $|\mathcal{R}|$. Such a tremendous amount of inventory leads to a trivial network structure; that is, serving all the demand nodes using the supply node with the lowest inventory holding cost. These properties of the optimal solution, i.e., high inventory and straightforward network structure, explain the short CPU time of the Chebyshev approximation to some extent. Due to its poor performance, the results of the Chebyshev approximation are omitted in Table 1 for the test classes with more than 10 potential supply nodes and 20 demand nodes.

Recall that the SAA formulation requires N samples of demand realization as input. In this numerical study, we consider the SAA formulations with $N = 100$, 50, and 20, which are denoted by SAA-100, SAA-50, and SAA-20, respectively. All the samples are generated by a truncated normal distribution with mean (location) μ_j, variance (squared scale) σ_j^2, and support $[\mu_j - 1.96\sigma_j, \mu_j + 1.96\sigma_j]$ for all $j \in \mathcal{R}$. Thus, they are generated by a distribution with the same mean and support as the set-wise approximation. For each combination of $|\mathcal{W}|$ and $|\mathcal{R}|$, we choose the maximum N in $\{100, 50, 20\}$ such that the corresponding SAA formulations for at least 80% of the instances can be solved within an hour. From Table 1, we observe that the SAA solution has the lowest total inventory and total cost among all the approximations. This is expected because the SAA requires the complete demand distribution to draw N samples of demand realization and only considers these samples in optimization. In most cases, its multi-sourcing network is also slightly sparser than those of the set-wise and convex approximations. However, the SAA shows no advantage in CPU time, especially for problems with larger size. For the test class (50,100,0.9), even with $N = 20$ samples, its average computational time is more than 4 times of that of the set-wise approximation. More importantly, the service level of the SAA is significantly lower than those of the other approximations. In particular, the chance of the SAA is always lower than the required service level $1 - \epsilon$ in all test classes, especially for the test classes with SAA-50 or SAA-20. For example, the chance of SAA-50 in the test class (20,40,0.9) is 80.08% and that of SAA-20 in the test class (50,100,0.9) is merely 66.55%. We also observe that the absolute value of ΔCost-WC of the SAA is always smaller than that of ΔCost-AV of the SAA for all test classes, which indicates that the cost advantage of the SAA weakens in the worst-case situations. This is consistent with the intuition that the set-wise approximation, as a robust approach, has better worst-case than average performance.

In summary, the proposed set-wise approximation achieves a high service level comparable to those of the convex and Chebyshev approximations, but requires significantly lower inventory and cost. The SAA, on the other hand, may yield a solution with lower inventory and cost, but it cannot reach the prespecified service level. Furthermore, the set-wise approximation is also computationally more efficient than the convex approximation and the SAA. Therefore, we can safely conclude that the set-wise approximation outperforms the other three approaches when considering CPU time, total inventory, total cost, and service level.

6 Numerical Study for the Model with Delivery Cost

In this section, we compare the performance of the LDR approximation, i.e., model (12), with the SAA approach. The SAA formulation for the model with delivery cost, i.e., model (1), has the same constraints as model (18), but the objective function is replaced with

$$\min \quad \sum_{i\in\mathcal{W}} h_i I_i + \sum_{i\in\mathcal{W}} f_i Y_i + \sum_{i\in\mathcal{W}}\sum_{j\in\mathcal{R}} c_{ij} X_{ij} + \frac{1}{N}\sum_{n=1}^{N}\sum_{i\in\mathcal{W}}\sum_{j\in\mathcal{R}} t_{ij}\theta_{ij}^{n}.$$

Note that the convex and Chebyshev approximations cannot be applied to the model with delivery cost, because these approaches cannot handle the expected delivery cost in the objective function.

Following the settings in subsection 5.5, we generate 40 instances for each of the 12 test classes. Furthermore, the unit delivery cost t_{ij} is set to the distance between supply node i and demand node j multiplied by a random disturbance. The computational results are presented in Table 2, which are the counterparts of those in Table 1.

Table 2 shows that the LDR approximation can be solved very efficiently. For most test classes, the average computational time of the LDR approximation is less than one minute. Even for the test classes with the largest scale, i.e., $|\mathcal{W}| = 50$ and $|\mathcal{R}| = 100$, the corresponding LDR approximation can be solved within 3 minutes. The multi-sourcing network obtained by the LDR approximation is very sparse, which is similar to the results of the set-wise approximation. In most cases, less than 10% of the demand nodes are multi-sourced. Meanwhile, the service level of the LDR approximation is very high as its chance is always above the preset service level $1-\epsilon$ and is almost equal to 100%.

The SAA requires much more computational effort than the LDR approximation. Its computational time can easily exceed four times that of the LDR approximation even for the test classes

Table 2: Computational results for the model with delivery cost

$\|\mathcal{W}\|$	$\|\mathcal{R}\|$	$1-\epsilon$	Model	CPU Time(s)	#DC	#Arc	ΔInv(%) AV	ΔInv(%) WC	ΔCost(%) AV	ΔCost(%) WC	Fill Rate(%)	Chance (%)
			SAA-100	38.4	2.1	11.1	-8.3	-7.7	-4.4	6.3	99.93	89.74
		0.95	LDR	0.7	1.8	10.4	-	-	-	-	100.00	100.00
			SAA-100	23.7	2.2	11.2	-8.7	-6.7	-4.3	6.0	99.95	92.98
8	**15**	**0.9**	LDR	1.1	2.3	15.2	-	-	-	-	100.00	100.00
			SAA-100	169.2	2.5	16.6	-8.5	-8.6	-6.3	-6.1	99.94	89.47
		0.95	LDR	1.9	2.3	15.3	-	-	-	-	100.00	100.00
			SAA-100	82.4	2.6	16.8	-8.9	-9.3	-6.2	-0.7	99.96	93.08
10	**20**	**0.9**	LDR	2.4	2.5	20.4	-	-	-	-	100.00	100.00
			SAA-100	419.8	3.0	22.3	-8.0	-8.5	-5.4	-1.8	99.95	89.61
		0.95	LDR	2.3	2.5	20.6	-	-	-	-	100.00	100.00
			SAA-100	189.6	3.1	22.5	-8.1	-7.8	-4.9	-2.0	99.97	94.02
20	**40**	**0.9**	LDR	4.5	3.6	41.7	-	-	-	-	100.00	100.00
			SAA-50	465.4	4.3	43.9	-7.6	-6.8	-5.0	-3.3	99.95	86.71
		0.95	LDR	5.4	3.6	42.1	-	-	-	-	100.00	100.00
			SAA-50	170.0	4.6	44.2	-7.7	-6.7	-4.4	-0.4	99.98	92.95
30	**60**	**0.9**	LDR	27.1	4.2	62.9	-	-	-	-	100.00	100.00
			SAA-20	393.4	5.5	64.7	-7.3	-7.2	-4.7	-4.2	99.94	77.28
		0.95	LDR	45.7	4.2	63.0	-	-	-	-	100.00	100.00
			SAA-20	132.7	5.7	65.5	-7.6	-7.9	-4.4	-3.3	99.95	81.85
50	**100**	**0.9**	LDR	171.4	5.7	107.9	-	-	-	-	100.00	100.00
			SAA-20	1052.4	7.0	106.7	-6.8	-7.2	-5.2	-3.7	99.95	75.97
		0.95	LDR	178.5	5.5	109.1	-	-	-	-	100.00	100.00
			SAA-20	411.8	7.2	108.8	-6.9	-6.8	-4.7	-4.7	99.97	83.10

where the SAA-20 is adopted. Moreover, the chance of the SAA is much lower than that of the LDR approximation and always below the prespecified level of $1-\epsilon$. In the test classes that adopt SAA-20, the chance is at least 11.9% lower than $1-\epsilon$.

Although the SAA requires longer computational time and results in unacceptable service levels, it generally performs better than the LDR approximation in terms of the total inventory and total cost. As in subsection 5.5, this advantage could be explained by the fact that the SAA requires the complete demand distribution and provides solutions with lower service levels. We also observe that the cost saving of the SAA in the worst-case scenario is less significant than in the average case. Especially for the test classes (5,10,0.9) and (5,10,0.95), ΔCost-WC is even greater than zero, which means that the LDR approximation outperforms the SAA in the worst-case scenario. This observation further validates that the LDR approximation is robust and performs relatively better in the worst-case scenario.

Another observation is that in both average and worst-cases scenarios, ΔInv is slightly lower than ΔCost, i.e., the SAA solution saves a little more inventory than cost. Note that the multi-sourcing network of the SAA is slightly denser than that of the LDR approximation, as the former generally has more open DCs and selected links. A denser network is more flexible in multi-sourcing and hence requires less inventory. However, it incurs a higher setup cost of DCs and links, which leads to lower savings in cost, i.e., a higher ΔCost.

To conclude the comparison between the LDR approximation and the SAA, the former always achieves the required service level and needs much less computational time than the latter. The SAA is advantageous in terms of the total inventory and total cost, but it cannot meet the prespecified service level. Note that knowing the complete demand distribution is a prominent reason why the total inventory and total cost of the SAA are lower than those of the LDR approximation. Given that the LDR approximation only requires the means and supports of the demand distribution, it is an efficient and effective approach to solve the multi-sourcing network design problem with delivery cost.

7 Conclusions

In this paper, we explore how to design a robust multi-sourcing supply network by introducing a location–inventory model with a service level chance constraint. The model considers the complex trade-offs among location, linkage, transportation, and inventory holding costs. We propose two safe approximations of the model that only require the means and supports of the uncertain demands. This work also provides a more general understanding of the 1-Expander structure defined in Chou et al. (2011). Furthermore, the effectiveness of the proposed approximations are validated through extensive numerical comparisons with commonly adopted approaches to approximate chance constraints.

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