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OPTIMALITY CONDITIONS AND A SMOOTHING TRUST REGION NEWTON METHOD FOR NONLIPSCHITZ OPTIMIZATION*

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Abstract. Regularized minimization problems with nonconvex, nonsmooth, perhaps non-Lipschitz penalty functions have attracted considerable attention in recent years, owing to their wide applications in image restoration, signal reconstruction, and variable selection. In this paper, we derive affine-scaled second order necessary and sufficient conditions for local minimizers of such minimization problems. Moreover, we propose a global convergent smoothing trust region Newton method which can find a point satisfying the affine-scaled second order necessary optimality condition from any starting point. Numerical examples are given to demonstrate the effectiveness of the smoothing trust region Newton method.

Key words. nonsmooth nonconvex optimization, smoothing methods, convergence, regularized optimization, penalty function, non-Lipschitz, trust region Newton method

AMS subject classifications. 90C30, 90C26, 65K05, 49M37

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1. Introduction.

In this paper, we consider the following nonsmooth unconstrained minimization problem:

$$\min_{x\in\Re^n} f(x) := \theta(x) + \lambda \sum_{i=1}^{m} \varphi(|d_i^T x|), \tag{1.1}$$

where $\theta : \Re^n \to \Re_+$, $\varphi : \Re_+ \to \Re_+$, $\lambda \in \Re_+$, and $d_i \in \Re^n$, $i = 1, \dots, m$.

We assume the objective function f has bounded level sets, the data fitting function θ is twice continuously differentiable, and the penalty function φ satisfies the following assumption.

Assumption 1.1.

(i) φ is differentiable in $(0, \infty)$, and φ' is locally Lipschitz continuous in $(0, \infty)$.

(ii) φ is continuous at 0 with $\varphi(0) = 0$, $\varphi'(0^+) > 0$, and $\varphi'(t) \geq 0$ for all $t > 0$.

The function φ may not be convex, differentiable, and perhaps not even Lipschitz. From (i) of Assumption 1.1, for any $x \in \Re^n$, if $d_i^T x \neq 0$, $i = 1, \dots, m$, then f is differentiable and f' is locally Lipschitz continuous in a neighborhood of x. Note that f may not be globally Lipschitz continuous in $\Re^n$. From (ii) of Assumption 1.1, $\varphi(t) = 0$ holds only at $t = 0$.

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Many widely used penalty functions in variable selection, image restoration, and signal reconstruction satisfy Assumption 1.1. For example,

$$\varphi_1(t) = \frac{\alpha t}{1+\alpha t}, \qquad \varphi_2(t) = \log(\alpha t + 1), \qquad \varphi_3(t) = t^q,$$
$$\varphi_4(t) = \lambda - (\lambda - t)_+^2/\lambda, \quad \varphi_5(t) = \int_0^t \min\left\{1, \frac{(\alpha - \tau/\lambda)_+}{\alpha - 1}\right\} d\tau, \quad \varphi_6(t) = \int_0^t \left(1 - \frac{\tau}{\alpha\lambda}\right)_+ d\tau,$$

where the parameters q and α are positive numbers. Especially, $\alpha > 2$ in φ_5 and $\alpha > 1$ in φ_6.

These penalty functions $\varphi_i : \Re_+ \to \Re_+, i = 1, \dots, 6$ are called fraction penalty [10, 35], Log-penalty [15], L_q penalty (or bridge penalty) [6, 18, 20, 24, 32], hard thresholding penalty [3, 25, 27], smoothly clipped absolute deviation penalty [15], and minimax concave penalty [47], respectively.

It is clear that the minimization problem (1.1) is nonsmooth, nonconvex, and perhaps not even Lipschitz. This problem presents an uniform framework for a large class of regularized minimization problems, for example, the L_2-L_q problem

$$\min_{x \in \Re^n} \|Ax - b\|_2^2 + \lambda \|x\|_q^q, \tag{1.2}$$

where $A \in \Re^{\ell \times n}, b \in \Re^{\ell}, \|x\|_q^q = \sum_{i=1}^n |x_i|^q, q \in (0,1)$. It is shown that problem (1.2) is strongly NP hard in [8]. The objective function in (1.2) is continuously differentiable at x if x has no zero components, but it is not Lipschitz continuous at x if x has zero components. Using a diagonal matrix $\text{diag}(x)$ as a scaling matrix, Chen, Xu, and Ye [9] present affine-scaled first order and second order necessary conditions for local minimizers of (1.2). However, such affine-scaled techniques can not be used for (1.1) with penalty functions which are not twice differentiable at some nonzero points, such as $\varphi_4, \varphi_5, \varphi_6$. In this paper, we use orthogonal decomposition and optimality conditions for LC^1 optimization[1] [23, 38] to present affine-scaled first order and second order necessary conditions for local minimizers of (1.1). Moreover, we present a sufficient optimality condition for local minimizers of (1.1). These optimality conditions provide important theoretical properties of (1.1) at its local minimizers.

Most existing algorithms for solving a nonsmooth, nonconvex, Lipschitz unconstrained optimization problem $\min_{x \in R^n} h(x)$ are designed to find a Clarke stationary point x^* satisfying the first order necessary optimality condition

$$0 \in \partial h(x^*), \qquad \partial h(x) \text{ is the Clarke subdifferential at } x^*$$

[2, 4, 7, 9, 10, 11, 28]. If h is an LC^1 function around x, then by Rademacher's theorem, h is almost everywhere twice differentiable in a neighborhood of x [11]. Let $\mathcal{D}_{\nabla h}$ be the set of points at which h is twice differentiable. The B-generalized Hessian is defined by

$$\partial_B^2 h(x) = \left\{ \lim_{z_k \to x, z_k \in \mathcal{D}_{\nabla h}} \nabla^2 h(z_k) \right\},$$

and the Clarke generalized Hessian is defined by $\partial^2 h(x) = \text{co}\partial_B^2 h(x)$, where co means the convex hull. We say x^* satisfies the second order necessary optimality condition [23] if

$$0 = \nabla h(x^*) \quad \text{and} \quad \forall\, a \in R^n, \text{ there is } V \in \partial^2 h(x^*) \text{ such that } a^T V a \geq 0,$$

[1] A function $h : \Re^{\ell} \to \Re$ is called an LC^1 or $C^{1,1}$ function around z if in a neighborhood of z, it is continuously differentiable, and its gradient ∇h is locally Lipschitz [23, 38].

where $\partial^2 h(x^*)$ is the generalized Hessian of h at x^* [11]. To the best of our knowledge, for the first time, this paper presents an algorithm which can find a point satisfying an affine-scaled second order necessary optimality condition for piecewise LC^1, perhaps non-Lipschitz optimization problems. This algorithm is based on smoothing approximations and trust region Newton methods [31]. We call this method a *smoothing trust region Newton method*.

Smoothing approximations for solving nonsmooth optimization problems have been studied for decades [2, 7, 10, 19, 33, 34, 41]. Trust region methods for solving nonsmooth optimization problems have also been studied for a long time (see [1, 12, 13, 14, 16, 17, 21, 26, 29, 30, 37, 39, 44, 45] and the references therein). However, there is little attention on combining smoothing approximations and trust region methods. Recently, Cartis, Gould, and Toint [5] presented a first order trust region algorithm for a special class of nonsmooth, nonconvex optimization problems and proved that the worst-case complexity of the algorithm is $O(\epsilon^{-2})$. However, the class of the nonsmooth, nonconvex optimization problems studied in [1, 5] are compositions of convex functions with smooth functions in the form $h(c(x))$, where $h : \Re^\ell \to \Re$ is nonsmooth and convex and $c : \Re^n \to \Re^\ell$ is smooth, which is not as general as the objective function f in this paper. It should be pointed out that our approach and that of [1] are very different, though both methods make use of smoothing functions. Bannert's method computes the trust region step by solving a nonsmooth trust region subproblem and uses a smoothing function as a merit function to overcome the Maratos effect. Our approach uses a smoothing function to approximate the nonsmooth function and construct smooth trust region subproblems depending on the smoothing function.

In our smoothing trust region Newton method, a sequence of parameterized smoothing functions is used to approximate the original nonsmooth function f. The main advantage of this method is to make the use of the efficient trust region Newton algorithm and code developed by Moré and Sorensen [31] for solving smooth subproblems. By updating the smoothing parameter, the smoothing trust region Newton method can find a point satisfying an affine-scaled second order necessary condition of the original nonsmooth optimization problem (1.1).

This paper is organized as follows. In section 2, we construct smoothing functions for the objective function f in (1.1) and present the smoothing trust region Newton method. In section 3, we present the affine-scaled first order and second order necessary conditions for local minimizers of (1.1). Moreover, we present a sufficient condition for local minimizers of (1.1). In section 4, we derive convergence of the smoothing trust region Newton method to points that satisfy the affine-scaled first order and second order necessary conditions. In section 5, we report numerical results for three often used testing problems to show the efficiency of the proposed method for solving (1.1).

Throughout this paper, we use $\varphi'(h(x))$ to denote $\varphi'(t)|_{t=h(x)}$ for a function $h : \Re^n \to \Re_+ = [0, \infty)$ and $\nabla\varphi(h(x)) = \varphi'(h(x))\nabla h(x)$.

2. Smoothing functions and a smoothing trust region Newton method. Smoothing methods for optimization problems have been studied for decades [1, 2, 7, 19, 33, 34, 41]. Smoothing methods use a sequence of parameterized smooth functions to approximate the original nonsmooth functions. The main advantage of the smoothing methods is to make the use of efficient algorithms for smooth optimization. By updating the smoothing parameter, the smoothing methods can solve the original nonsmooth optimization problems.

2.1. Selection of smoothing functions. To develop a smoothing trust region Newton method for (1.1), we construct a C^2 smoothing function $\tilde{f}(\cdot,\mu)$ for the objective function $f(\cdot)$ in (1.1). Since the first term θ of f is twice continuously differentiable, we only need to construct a C^2 smoothing function $\tilde{\varphi}(\cdot,\mu):\Re_+\to\Re_+$ for $\varphi(\cdot)$. In particular, we assume $\tilde{\varphi}(\cdot,\mu)$ satisfies the following assumption.

Assumption 2.1 (properties of smoothing function $\tilde{\varphi}$).

(1) For any $\mu>0$, $\tilde{\varphi}(\cdot,\mu)$ is twice continuously differentiable in $\Re_+$ and 0 is the unique minimizer of $\tilde{\varphi}(\cdot,\mu)$.

(2) For any $\mu>0$, $\tilde{\varphi}'(0,\mu)=0$, and for any $t>0$, $\lim_{\mu\downarrow 0}\tilde{\varphi}''(t,\mu)$ exists.

(3) For $t>0$ and $\{t_k\}\subset\Re_+$,

$$\lim_{\mu\downarrow 0, t_k\to t}\tilde{\varphi}'(t_k,\mu)=\varphi'(t)\quad\text{and}\quad\left\{\lim_{\mu\downarrow 0, t_k\to t}\tilde{\varphi}''(t_k,\mu)\right\}\subseteq\partial^2\varphi(t).$$

(4) For $t\ge 0$ and $0\le\mu_1\le\mu_2$,

$$\tilde{\varphi}(t,\mu_1)\le\tilde{\varphi}(t,\mu_2).$$

(5) For $t\ge 0$ and $\mu>0$,

$$0\le\tilde{\varphi}(t,\mu)-\varphi(t)\le\kappa(\mu), \tag{2.1}$$

where $\kappa(\mu):\Re_+\to\Re_+$ satisfies $\lim_{\mu\downarrow 0}\kappa(\mu)=0$ and $\kappa(\mu_1)\le\kappa(\mu_2)$ for $\mu_1\le\mu_2$.

From (2.1), we have that for $\{t_k\}\subset\Re_+$ and $t\in\Re_+$,

$$\lim_{\mu\downarrow 0, t_k\to t}\tilde{\varphi}(t_k,\mu)=\varphi(t).$$

For the six penalty functions given in section 1, we can easily construct their smoothing functions satisfying Assumption 2.1. In particular, we can use a smoothing function for the plus function $(t)_+$ to construct their smoothing functions, since we have

$$|t|=(-t)_++(t)_+,\quad\text{and}\quad\min(1,t)=1-(1-t)_+$$

and a smooth composition of smoothing functions is a smoothing function [7].

For example, using the smoothing function

$$\hat{s}(t,\mu)=\frac{1}{2}(t+\sqrt{t^2+4\mu^2})$$

of $(t)_+$, we can define smoothing functions

$$s(t,\mu)=\sqrt{t^2+4\mu^2}\quad\text{and}\quad\check{s}(t,\mu)=1-\hat{s}(1-t,\mu)$$

of $|t|$ and $\min(1,t)$, respectively. For any $t\ne 0$, we have

$$\lim_{t_k\to t,\mu_k\downarrow 0}s'(t_k,\mu_k)=\lim_{t_k\to t,\mu_k\downarrow 0}\frac{t_k}{\sqrt{t_k^2+4\mu_k^2}}=\mathrm{sign}(t), \tag{2.2a}$$

$$\lim_{t_k\to t,\mu_k\downarrow 0}s''(t_k,\mu_k)=\lim_{t_k\to t,\mu_k\downarrow 0}\frac{4\mu_k^2}{(t_k^2+4\mu_k^2)^{\frac{3}{2}}}=0. \tag{2.2b}$$

For the six penalty functions

$$\varphi_1(|t|) = \frac{\alpha|t|}{1+\alpha|t|}, \quad \varphi_2(|t|) = \log(\alpha|t|+1), \quad \varphi_3(|t|) = |t|^q, \quad \varphi_4(|t|) = \lambda - \frac{(\lambda-|t|)_+^2}{\lambda},$$

$$\varphi_5(|t|) = \int_0^{|t|} \min\left\{1, \frac{(\alpha-\tau/\lambda)_+}{\alpha-1}\right\} d\tau, \qquad \varphi_6(|t|) = \int_0^{|t|} \left(1-\frac{\tau}{\alpha\lambda}\right)_+ d\tau\,,$$

we can use these three smoothing functions $s(\cdot,\mu), \hat{s}(\cdot,\mu), \check{s}(\cdot,\mu)$ to define their smoothing functions as follows:

$$\tilde{\varphi}_i(s(t,\mu),\mu) = \varphi_i(s(t,\mu)) \text{ for } i=1,2,3, \quad \tilde{\varphi}_4(s(t,\mu),\hat{\mu}) = \lambda - \frac{\hat{s}^2(\lambda - s(t,\mu),\hat{\mu})}{\lambda},$$

$$\tilde{\varphi}_5(s(t,\mu),\hat{\mu}) = \int_0^{s(t,\mu)} \check{s}\left(\frac{\hat{s}(\alpha-\tau/\lambda,\hat{\mu})}{\alpha-1},\hat{\mu}\right) d\tau,$$

$$\tilde{\varphi}_6(s(t,\mu),\hat{\mu}) = \int_0^{s(t,\mu)} \hat{s}\left(1-\frac{\tau}{\alpha\lambda},\hat{\mu}\right) d\tau,$$

where $\mu > 0, \hat{\mu} > 0$ are smoothing parameters. For simplicity, in the rest of this paper we set $\mu = \hat{\mu}$ and denote $\tilde{\varphi}(s(t,\mu),\mu) = \tilde{\varphi}(s(t,\mu))$. Since $s(\cdot,\mu), \hat{s}(\cdot,\mu), \check{s}(\cdot,\mu)$ are twice continuously differentiable, $\tilde{\varphi}_i(s(\cdot,\mu))$, $i=1,\dots,6$ are twice continuously differentiable. Hence we can define the smoothing function $\tilde{f}(\cdot,\mu)$ of the objective function $f(\cdot)$ in (1.1) and obtain the following smoothing optimization problem:

$$\min_{x\in\Re^n} \tilde{f}(x,\mu) := \theta(x) + \lambda \sum_{i=1}^m \tilde{\varphi}(s(d_i^T x,\mu))\,. \tag{2.3}$$

We can use other smoothing functions of $|t|$ to define smoothing functions $\tilde{\varphi}$ of φ that satisfy Assumption 2.1. For example, $s(t,\mu) = \mu\ln(2+e^{t/\mu}+e^{-t/\mu})$. See [7] for other examples.

The following properties of $\tilde{f}(x,\mu)$ will be used in the proof for the convergence of the smoothing trust region Newton method.

(i) $\tilde{f}(\cdot,\mu)$ is twice continuously differentiable,

$$\nabla\tilde{f}(x,\mu) = \nabla\theta(x) + \lambda\sum_{i=1}^m \tilde{\varphi}'(s(d_i^Tx,\mu))s'(d_i^Tx,\mu)d_i, \tag{2.4a}$$

$$\nabla^2\tilde{f}(x,\mu) = \nabla^2\theta(x) + \lambda\sum_{i=1}^m (\tilde{\varphi}''(s(d_i^Tx,\mu))(s'(d_i^Tx,\mu))^2 + \tilde{\varphi}'(s(d_i^Tx,\mu))s''(d_i^Tx,\mu))d_id_i^T. \tag{2.4b}$$

(ii) From (5) of Assumption 2.1, we know $\tilde{f}(x,\mu) \ge f(x)$ which implies that the level set of $\tilde{f}(\cdot,\mu)$ is a subset of the level set of f. Since f has bounded level sets, $\tilde{f}(\cdot,\mu)$ has also bounded level sets for any given $\mu > 0$.

(iii) From (4) of Assumption 2.1, for any μ_1,μ_2, if $\mu_1 \ge \mu_2 > 0$, $\tilde{f}(x,\mu_1) \ge \tilde{f}(x,\mu_2)$ for all $x \in \Re^n$.

2.2. A smoothing trust region Newton method for (1.1). In this subsection, we propose a smoothing trust region Newton method for solving nonsmooth problem (1).

ALGORITHM 1. SMOOTHING NEWTON METHOD.

Step 0. Given constants $0 < \eta_1 < \eta_2 < 1$, $0 < \gamma_1 < 1 < \gamma_2$, $0 < \underline{\Delta} < \overline{\Delta} < +\infty$, $0 < \nu < 1$, $\zeta > 0$, and initial value $\mu_0 > 0$, $x_0 \in \Re^n$,$\underline{\Delta} \le \Delta_0 < \overline{\Delta}$. Set $k = 0$.

Step 1. Compute p_k by approximately solving the trust region Newton model

$$\min \quad m_k(p) = \tilde{f}(x_k, \mu_k) + p^T \nabla \tilde{f}(x_k, \mu_k) + \frac{1}{2} p^T \nabla^2 \tilde{f}(x_k, \mu_k) p, \tag{2.5a}$$

$$\text{s.t.} \quad \|p\| \le \Delta_k. \tag{2.5b}$$

Step 2. If $m_k(p_k) = m_k(0)$, set $x_{k+1} = x_k$, $\Delta_{k+1} = \max\{\Delta_k, \underline{\Delta}\}$ and go to Step 3; otherwise, compute

$$\rho_k = \frac{\tilde{f}(x_k, \mu_k) - \tilde{f}(x_k + p_k, \mu_k)}{m_k(0) - m_k(p_k)}.$$

Set

$$\Delta_{k+1} = \begin{cases} \min\{\gamma_2 \Delta_k, \overline{\Delta}\} & \text{if } \rho_k \ge \eta_2 \text{ and } \|p_k\| = \Delta_k, \\ \gamma_1 \Delta_k & \text{if } \rho_k \le \eta_1, \\ \Delta_k, & \text{otherwise.} \end{cases}$$

If $\rho_k > \eta_1$, set $x_{k+1} = x_k + p_k$ and $\Delta_{k+1} = \max\{\underline{\Delta}, \Delta_{k+1}\}$. Otherwise, set $x_{k+1} = x_k$.

Step 3. If $\|\nabla \tilde{f}(x_k, \mu_k)\| \le \zeta \mu_k$ and $\Delta_k \ge \underline{\Delta}$, choose $\mu_{k+1} \le \nu \mu_k$; otherwise, set $\mu_{k+1} = \mu_k$. Set $k = k + 1$ and go to Step 1.

In the classic trust region method, there is usually no lower bound $\underline{\Delta}$ on the trust region radius. In recent literature [14, 26, 30] for nonsmooth optimization problems, a lower bound on the trust region radius is introduced to guarantee the global convergence to a stationary point or to have locally superlinear convergence under some regular conditions. Here, we impose a positive lower bound for the trust region radius on the successful step to ensure the global convergence of our algorithm to a point satisfying the second order optimality condition. The difference from existing methods is that in Algorithm 1, the trust region radius is updated when an iterate moves to a new point (i.e., $\rho_k > \eta_1$) or $m_k(p_k) = m_k(0)$ holds.

To exploit the second order information conveyed by $\nabla^2 \tilde{f}(x_k, \mu_k)$ sufficiently, we require the approximate solution p_k in Step 1 satisfying the following quality.

CONDITION 2.1 (accuracy of the solution of the trust region subproblem).

1. *There exists a constant $c_1 \in (0, 1]$ such that*

$$m_k(0) - m_k(p_k) \ge c_1(m_k(0) - m_k(p_k^*)) \tag{2.6}$$

for all k, where p_k^* is the exact solution of trust region subproblem (2.5).

2. *There exists a constant $c_2 \ge 1$, such that*

$$\|p_k\| \le c_2 \Delta_k \;\forall\, k.$$

Obviously, if p_k solves (2.5) exactly, i.e., $p_k = p_k^*$, Condition 2.1 holds. There are many practical algorithms which can obtain approximate solutions that satisfy Condition 2.1. For detailed discussions, please see Chapter 7 of [12]. In particular, the algorithm of [31], implemented as GQTPAR as part of the MINPACK software package, computes an approximate solution of the trust region subproblem by finding an accurate Lagrange multiplier. An approximate solution obtained by the algorithm of [31] satisfies Condition 2.1 with $c_1 = 1$ and $c_2 = 1 + \delta$, where $\delta \in (0, 1)$ is some tolerance parameter. For convex trust region subproblems, the truncated CG gradient method gives a solution satisfying Condition 2.1 with $c_1 = 0.5$ and $c_2 = 1$ [46]. It can be verified that any p_k satisfies inequality (2.6), also satisfies

$$m_k(0) - m_k(p_k) \geq \frac{c_1}{2} \|\nabla \tilde{f}(x_k, \mu_k)\| \min\left(\frac{\|\nabla \tilde{f}(x_k, \mu_k)\|}{\|\nabla^2 \tilde{f}(x_k, \mu_k)\|}, \triangle_k \right). \tag{2.7}$$

See Theorem 4.3 in [36]. It is worth noting that if the objective function of (2.5) is not convex, negative curvature directions must be explored in order for an algorithm to guarantee that Condition 2.1 will be satisfied.

Morè and Sorensen [31] show that if p_k satisfies Condition 2.1, the following inequality holds:

$$m_k(0) - m_k(p_k) \geq \frac{1}{2} c_1 (\|R_k p_k\|^2 + \beta_k \Delta_k^2), \tag{2.8}$$

where $\beta_k \geq 0$ and $R_k^T R_k$ is the Cholesky factorization of the positive semidefinite matrix $\nabla^2 \tilde{f}(x_k, \mu_k) + \beta_k I$ and $(\nabla^2 \tilde{f}(x_k, \mu_k) + \beta_k I) p_k^* = -\nabla \tilde{f}(x_k, \mu_k)$. We will use the algorithm proposed by Morè and Sorensen to solve the trust region subproblem in our numerical experiments.

3. Necessary and sufficient optimality conditions.

In this section, we present the first order necessary condition, second order necessary condition, and the sufficient conditions for local minimizers of (1.1). If $d_i = 0$ for all i, then $f = \theta$ which is twice continuously differentiable. This paper considers $d_i \neq 0$ for all i.

3.1. Necessary optimality conditions for (1.1).

For a given nonzero vector $\bar{x} \in \Re^n$, let

$$I_{\bar{x}} = \{i \in \{1, \dots, m\} \mid d_i^T \bar{x} = 0\} \quad \text{and} \quad J_{\bar{x}} = \{i \in \{1, \dots, m\} \mid d_i^T \bar{x} \neq 0\}.$$

Let $Y_{\bar{x}}$ be an $n \times (n - \ell)$ matrix whose columns form an orthonormal basis for $\{d_i \mid i \in I_{\bar{x}}\}$ and $Z_{\bar{x}}$ be an $n \times \ell$ matrix whose columns are an orthonormal basis for the null space of $\{d_i \mid i \in I_{\bar{x}}\}$. Then every $x \in \Re^n$ can be decomposed uniquely as $x = Y_{\bar{x}} y + Z_{\bar{x}} z$, where $y \in \Re^{n-\ell}$ and $z \in \Re^\ell$. Note that $\bar{x} \neq 0$ implies that $\{d_i, i \in I_{\bar{x}}\}$ cannot have n linearly independent vectors. Thus $\ell > 0$ and $Z_{\bar{x}}$ has ℓ orthonormal columns. The following equalities are often used in our analysis:

$$d_i^T Z_{\bar{x}} = 0 \; \forall \, i \in I_{\bar{x}} \quad \text{and} \quad \bar{x} = Z_{\bar{x}} \bar{z}, \tag{3.1}$$

where $\bar{z}$ is uniquely defined by $\bar{x}$ and the orthogonal decomposition as $\bar{z} = (Z_{\bar{x}}^T Z_{\bar{x}})^{-1} Z_{\bar{x}}^T \bar{x}$.

Our necessary and sufficient optimality conditions for problem (1.1) is based on a reduced problem in $\Re^\ell$ with the following objective function:

$$v(z) = \theta(Z_{\bar{x}} z) + \lambda \sum_{i=1}^{m} \varphi(|d_i^T Z_{\bar{x}} z|).$$

Note that $\varphi(0) = 0$. From (3.1), $v(z)$ can be written as

$$v(z) = \theta(Z_{\bar{x}} z) + \lambda \sum_{i \in J_{\bar{x}}} \varphi(|d_i^T Z_{\bar{x}} z|).$$

Moreover, from the assumptions on θ and φ, the function v is an LC^1 function around $\bar{z} = Z_{\bar{x}}^T \bar{x} \in \Re^\ell$.

Consider the minimization problem

$$\min_{z \in \Re^\ell} v(z). \tag{3.2}$$

Lemma 3.1. *Suppose that v is an LC^1 function around $z \in R^\ell$.*

(1) *(Hiriart-Urruty, Strodiot, and Nguyen* [23]*) If z is a local minimizer of problem* (3.2), *then*

$$\nabla v(z) = 0 \quad \text{and} \quad \forall a \in \Re^\ell, \text{ there is a } V \in \partial^2 v(z) \text{ such that } a^T V a \geq 0.$$

(2) *(Qi* [38]*) If z satisfies*

$$\nabla v(z) = 0 \quad \text{and} \quad V \text{ is positive definite} \; \forall\, V \in \partial_B^2 v(z),$$

then z is a strict local minimizer of problem (3.2).

To present the necessary and sufficient optimality conditions for problem (1.1), we define the following function:

$$w(x) = \theta(x) + \lambda \sum_{i \in J_{\bar{x}}} \varphi(|d_i^T x|). \tag{3.3}$$

Since in a neighborhood of $\bar{x}$, $d_i^T x \neq 0$ for $i \in J_{\bar{x}}$, the function w is an LC^1 function around $\bar{x}$.

The B-generalized Hessian matrix of $\varphi(|d_i^T \bar{x}|)$ at $\bar{x}$ for $i \in J_{\bar{x}}$ is given by

$$\partial_B^2 \varphi(|d_i^T \bar{x}|) = \{M \mid \exists\, x_k \to \bar{x} \text{ with } \nabla^2 \varphi(|d_i^T x_k|) \text{ exist and } \nabla^2 \varphi(|d_i^T x_k|) \to M\}$$

and the Clarke generalized Hessian matrix is $\partial^2 \varphi(|d_i^T \bar{x}|) = \mathrm{co}\partial_B^2 \varphi(|d_i^T \bar{x}|)$. We define the C-generalized Hessian matrix of w at $\bar{x}$ as the following [11]:

$$\partial_C^2 w(\bar{x}) = \nabla^2 \theta(\bar{x}) + \lambda \sum_{i \in J_{\bar{x}}} \partial^2 \varphi(|d_i^T \bar{x}|). \tag{3.4}$$

Here we use the Minkowski addition of sets, which implies [40]

$$\partial_C^2 w(\bar{x}) = \mathrm{co}\{\nabla^2 \theta(\bar{x}) + \lambda \sum_{i \in J_{\bar{x}}} \partial_B^2 \varphi(|d_i^T \bar{x}|)\}. \tag{3.5}$$

At the point $\bar{x}$, from (3.1) we have $w(\bar{x}) = w(Z_{\bar{x}}\bar{z}) = v(\bar{z})$. We define

$$\partial_C^2 v(\bar{z}) = Z_{\bar{x}}^T \partial_C^2 w(\bar{x}) Z_{\bar{x}} = \{Z_{\bar{x}}^T H Z_{\bar{x}} \mid H \in \partial_C^2 w(\bar{x})\}.$$

Using (3.4) and (3.5), we can show

$$\begin{aligned} \partial_C^2 v(\bar{z}) &= Z_{\bar{x}}^T \nabla^2 \theta(\bar{x}) Z_{\bar{x}} + \lambda Z_{\bar{x}}^T \sum_{i \in J_{\bar{x}}} \partial^2 \varphi(|d_i^T \bar{x}|) Z_{\bar{x}} \\ &= \mathrm{co}\{Z_{\bar{x}}^T \nabla^2 \theta(\bar{x}) Z_{\bar{x}} + \lambda Z_{\bar{x}}^T \sum_{i \in J_{\bar{x}}} \partial_B^2 \varphi(|d_i^T \bar{x}|) Z_{\bar{x}}\}. \end{aligned} \tag{3.6}$$

Lemma 3.2. *If $\bar{z}$ is a local minimizer of problem* (3.2), *then*

$$\nabla v(\bar{z}) = 0 \quad \text{and} \quad \forall a \in \Re^\ell, \text{ there is a } V \in \partial_C^2 v(\bar{z}) \text{ such that } a^T V a \geq 0.$$

Proof. Let a be a fixed nonzero vector in $\Re^\ell$. Let $\mathcal{N}_{\bar z}$ be a neighborhood of $\bar z$ such that $|d_i^T Z_{\bar x} z| \neq 0$ for $z \in \mathcal{N}_{\bar z}$ and $i \in J_{\bar x}$. Let k_0 be a positive integer such that $z_k = \bar z + \frac{1}{k} a \in \mathcal{N}_{\bar z}$ for $k \geq k_0$. Using the second order Taylor expansion [23, Theorem 2.3] of φ in the $\mathcal{N}_{\bar z}$, we have

$$v(z_k) = v(\bar z) + \frac{1}{k} a^T \nabla v(\bar z) + \frac{1}{2k^2} a^T Z_{\bar x}^T \nabla^2 \theta(\hat x_k) Z_{\bar x} a + \lambda \sum_{i \in J_{\bar x}} \frac{1}{2k^2} a^T Z_{\bar x}^T M_{i,k} Z_{\bar x} a, \tag{3.7}$$

where $\hat x_k = Z_{\bar x} \hat z_k$, $M_{i,k} \in \partial^2 \varphi(|d_i^T \hat x_{i,k}|)$ with $\hat x_{i,k} = Z_{\bar x} \hat z_{i,k}$ and $\hat x_k, \hat x_{i,k} \in]\bar x, Z_{\bar x} z_k[$. Here $]\bar x, Z_{\bar x} z_k[$ means the line segment between the two points.

Since $\bar z$ is a local minimizer and $\nabla v(\bar z) = 0$, (3.7) implies that

$$\frac{1}{2k^2} a^T Z_{\bar x}^T \nabla^2 \theta(\hat x_k) Z_{\bar x} a + \lambda \frac{1}{2k^2} \sum_{i \in J_{\bar x}} a^T Z_{\bar x}^T M_{i,k} Z_{\bar x} a = v(z_k) - v(z) \geq 0 \tag{3.8}$$

for all sufficiently large k. Moreover, the assumption that φ' is locally Lipschitz continuous in $(0, \infty)$ implies that $\partial^2 \varphi(|d_i^T \hat x_{i,k}|)$ is bounded in $\mathcal{N}_{\bar z}$, and thus $\{M_{i,k}\}$ is bounded in $\mathcal{N}_{\bar z}$ and has a convergent subsequence. Let M_i be the limit of this subsequence for each $i \in J_{\bar x}$. By the upper-semicontinuous of $\partial^2 \varphi$, we have $M_i \in \partial^2 \varphi(|d_i^T \bar x|)$.

Moreover, (3.8) implies $a^T Z_{\bar x}^T (\nabla^2 \theta(\hat z_k) + \lambda \sum_{i \in J_{\bar x}} M_{i,k}) Z_{\bar x} a \geq 0$ for all sufficiently large k. Hence taking limit in it gives $a^T Z_{\bar x}^T (\nabla^2 \theta(\bar x) + \lambda \sum_{i \in J_{\bar x}} M_i) Z_{\bar x} a \geq 0$ and $Z_{\bar x}^T (\nabla^2 \theta(Z_{\bar x} z) + \lambda \sum_{i \in J_{\bar x}} M_i) Z_{\bar x} \in \partial_C^2 v(z)$. ∎

THEOREM 3.3 (second order necessary condition). *Under Assumption* 1.1, *if* $\bar x \in \Re^n$ *is a local minimizer of problem* (1.1), *then we have*

$$Z_{\bar x}^T \nabla w(\bar x) = 0 \tag{3.9}$$

and

$$\forall\, a \in \Re^\ell, \text{ there is an } H \in \partial_C^2 w(\bar x), \text{ such that } a^T Z_{\bar x}^T H Z_{\bar x} a \geq 0. \tag{3.10}$$

Proof. If $\bar x = 0$ is a local minimizer, then $w(\bar x) = \theta(\bar x)$ and $Z_{\bar x} = 0$. Thus (3.9) and (3.10) hold. Let $\bar x$ be an nonzero local minimizer of $f(x)$. Then there exists $\delta_{\bar x} > 0$, such that $f(x) \geq f(\bar x)$ for all x which satisfies $\|x - \bar x\| \leq \delta_{\bar x}$. From (3.1), we have

$$\begin{aligned}
f(\bar x) &= \min_x \left\{ \theta(x) + \lambda \sum_{i=1}^m \varphi(|d_i^T x|) : \|x - \bar x\| \leq \delta_{\bar x} \right\} \\
&= \min_{y,z} \left\{ \theta(Y_{\bar x} y + Z_{\bar x} z) + \lambda \sum_{i=1}^m \varphi(|d_i^T (Y_{\bar x} y + Z_{\bar x} z)|) : \|Y_{\bar x} y + Z_{\bar x} z - Z_{\bar x} \bar z\| \leq \delta_{\bar x} \right\} \\
&\leq \min_{z} \left\{ \theta(Y_{\bar x} 0 + Z_{\bar x} z) + \lambda \sum_{i=1}^m \varphi(|d_i^T (Y_{\bar x} 0 + Z_{\bar x} z)|) : y = 0, \|Z_{\bar x} z - Z_{\bar x} \bar z\| \leq \delta_{\bar x} \right\} \\
&= \min_{z} \left\{ \theta(Z_{\bar x} z) + \lambda \sum_{i=1}^m \varphi(|d_i^T Z_{\bar x} z|) : \|Z_{\bar x}(z - \bar z)\| \leq \delta_{\bar x} \right\} \\
&= \min_{z} \left\{ \theta(Z_{\bar x} z) + \lambda \sum_{i \in J_{\bar x}} \varphi(|d_i^T Z_{\bar x} z|) : \|Z_{\bar x}(z - \bar z)\| \leq \delta_{\bar x} \right\} \\
&= \min_z \{ v(z) : \|Z_{\bar x}(z - \bar z)\| \leq \delta_{\bar x} \}.
\end{aligned}$$

Using $\varphi(0)=0$ and (3.1) again, we obtain

$$v(\bar z)=\theta(Z_{\bar x}\bar z)+\lambda\sum_{i=1}^{m}\varphi(|d_i^T Z_{\bar x}\bar z|)=\theta(\bar x)+\lambda\sum_{i=1}^{m}\varphi(|d_i^T\bar x|)=f(\bar x). \tag{3.11}$$

Therefore, we find

$$v(\bar z)\le \min\{v(z):\|Z_{\bar x}(z-\bar z)\|\le \delta_{\bar x}\}.$$

Since $Z_{\bar x}$ is of full column rank, $\bar z$ is a local minimizer of $v(z)$. Based on Lemma 3.2, we have

$$\nabla v(\bar z)=0 \quad\text{and}\quad \forall\, a\in\Re^{\ell} \text{ there is an } \ell\times\ell \text{ matrix } V\in\partial_C^2 v(\bar z) \text{ such that } a^T V a\ge 0.$$

From $\nabla v(\bar z)=0$ and

$$\nabla v(\bar z)=Z_{\bar x}^T\nabla\theta(Z_{\bar x}\bar z)+\lambda\sum_{i\in J_{\bar x}} Z_{\bar x}^T d_i\varphi'(|d_i^T Z_{\bar x}\bar z|)\mathrm{sign}(d_i^T Z_{\bar x}\bar z)=Z_{\bar x}^T\nabla w(\bar x),$$

we derive the first order necessary optimality condition (3.9). Moreover, from (3.6), we know that for any $V\in\partial_C^2 v(\bar z)$, we can find a matrix $H\in\partial_C^2 w(\bar x)$, such that $V=Z_{\bar x}^T H Z_{\bar x}$. Therefore, the second order necessary optimality condition (3.10) is proved. ◻

Remark 3.1. (1) If φ is twice continuously differentiable in $(0,\infty)$, for example, the fraction penalty φ_1, the Log-penalty φ_2, and the L_q penalty φ_3, then the function w is twice continuously differentiable in a neighborhood of $\bar x$ and

$$\nabla^2 w(\bar x)=\nabla^2\theta(\bar x)+\lambda\sum_{i\in J_{\bar x}} d_i d_i^T\varphi''(|d_i^T\bar x|).$$

(2) If $d_i=e_i$, e_i is the ith column of the identity matrix for all $i\in J_{\bar x}$, then all matrices in $\partial d_i\varphi'(|x_i|)$ are diagonal and all their entries are zero except the diagonal entry at the ith column. Let $\sigma_i=\min\{V_{ii}\,:\, V\in\partial d_i\varphi'(|\bar x_i|)\}$ for $i\in J_{\bar x}$ and $\sigma_i=0$ for $i\in I_{\bar x}$. If the matrix

$$Z_{\bar x}^T\nabla^2\theta(\bar x)Z_{\bar x}+\lambda Z_{\bar x}^T\,\mathrm{diag}(\sigma_1,\dots,\sigma_n)Z_{\bar x}$$

is positive semidefinite, then condition (3.10) holds. Moreover, if $\varphi(|x_i|)$ is twice differentiable at $x_i\ne 0$, for example, φ_1, φ_2, and φ_3, then we have

$$\nabla^2 w(x)=\nabla^2\theta(x)+\lambda\sum_{i\in J_{\bar x}} e_i e_i^T\varphi''(|x_i|).$$

3.2. Sufficient optimality conditions. Now we present a sufficient condition for minimizers of problem (1.1).

THEOREM 3.4 (second order sufficient condition). *Suppose Assumption* 1.1 *holds and $\varphi'(0^+)=+\infty$. For a given nonzero vector $\bar x\in\Re^n$, if*

$$Z_{\bar x}^T\nabla w(\bar x)=0 \quad\textit{and}\quad Z_{\bar x}^T H Z_{\bar x} \textit{ is positive definite } \forall H\in\partial_C^2 w(\bar x), \tag{3.12}$$

then $\bar x$ is a strict local minimizer of problem (1.1).

Proof. *Case* 1: $I_{\bar{x}} = \emptyset$. In this case, $Z_{\bar{x}}$ is an $n \times n$ nonsingular matrix and $f = w$ is an LC^1 function around $\bar{x}$. Hence condition (3.12) becomes

$$\nabla f(\bar{x}) = 0 \text{ and } H \text{ is positive definite } \forall H \in \partial_C^2 f(\bar{x}).$$

From (2) of Lemma 3.1 and $\partial_B^2 f(\bar{x}) \subseteq \partial_C^2 f(\bar{x})$ [11], $\bar{x}$ is a strict local minimizer of $f(x)$.

Case 2: $I_{\bar{x}} \neq \emptyset$. From the equality in (3.12),

$$0 = Z_{\bar{x}}^T \nabla w(\bar{x}) = Z_{\bar{x}}^T \Big(\nabla\theta(Z_{\bar{x}}\bar{z}) + \lambda \sum_{i \in J_{\bar{x}}} d_i \varphi'(|d_i^T Z_{\bar{x}}\bar{z}|) \mathrm{sign}(d_i^T Z_{\bar{x}}\bar{z})\Big) = \nabla v(\bar{z}).$$

It follows from (3.6) that

$$\forall V \in \partial_C^2 v(\bar{z}), \text{ there is a matrix } H \in \partial_C^2 w(\bar{x}) \text{ such that } V = Z_{\bar{x}}^T H Z_{\bar{x}}.$$

According to the second part of (3.12) we know V is positive definite. Then from (2) of Lemma 3.1 and $\partial_B^2 v(\bar{z}) \subseteq \partial_C^2 v(\bar{z})$ [11], $\bar{z} = Z_{\bar{x}}^T \bar{x}$ is a strict local minimizer of v.

Assume to the contrary that $\bar{x}$ is not a strict local minimizer of problem (1.1). Then we can find a sequence of nonzero vectors $\{x_l\}$ satisfying

$$f(\bar{x} + x_l) \leq f(\bar{x}) \quad \text{and} \quad \lim_{l \to \infty} x_l = 0. \tag{3.13}$$

Let $x_l = Y_{\bar{x}} y_l + Z_{\bar{x}} z_l$ for all l. Then from (3.11), we have

$$f(\bar{x} + x_l) - f(\bar{x}) = \theta(Z_{\bar{x}}\bar{z} + Y_{\bar{x}} y_l + Z_{\bar{x}} z_l) + \lambda \sum_{i=1}^m \varphi(|d_i^T (Z_{\bar{x}}\bar{z} + Y_{\bar{x}} y_l + Z_{\bar{x}} z_l)|) - v(\bar{z}).$$

If $d_i^T Y_{\bar{x}} y_l = 0$ for all $i \in I_{\bar{x}}$ and all sufficiently large l, then from $Y_{\bar{x}} y_l \in R^{n-l}$ and $\{d_i, i \in I_{\bar{x}}\}$ having $n - l$ linearly independent vectors, we have $Y_{\bar{x}} y_l = 0$. Hence, from the above equality we get

$$f(\bar{x} + x_l) - f(\bar{x}) = \theta(Z_{\bar{x}}(\bar{z} + z_l)) + \lambda \sum_{i \in J_{\bar{x}}} \varphi(|d_i^T Z_{\bar{x}}(\bar{z} + z_l)|) - v(\bar{z}) = v(\bar{z} + z_l) - v(\bar{z})$$

for all sufficiently large l. Since $\bar{z}$ is a strict local minimizer of v, for all sufficiently large l, $v(\bar{z} + z_l) > v(\bar{z})$, and consequently, $f(\bar{x} + x_l) > f(\bar{x})$ for all sufficiently large l, which contradicts (3.13). Hence there must exist a subsequence of $\{y_l\}$ such that for each l, $d_i^T Y_{\bar{x}} y_l \neq 0$ for some $i \in I_{\bar{x}}$. By passing on the subsequence, without loss of generality, we assume $\sum_{i \in I_{\bar{x}}} |d_i^T Y_{\bar{x}} y_l| \neq 0$ for all sufficiently large l.

By the definition of the function v, $\bar{x} = Z_{\bar{x}}\bar{z}$ and $d_i^T Z_{\bar{x}} = 0$ for $i \in I_{\bar{x}}$, we have

$$\begin{aligned}
&f(\bar{x} + x_l) - f(\bar{x}) \\
&\quad = \theta(\bar{x} + x_l) + \lambda \sum_{i=1}^m \varphi(|d_i^T (\bar{x} + x_l)|) - v(\bar{z}) \\
&\quad = -v(\bar{z}) + \lambda \sum_{i \in I_{\bar{x}}} \varphi(|d_i^T Y_{\bar{x}} y_l|) \\
&\qquad + \theta(\bar{x} + Y_{\bar{x}} y_l + Z_{\bar{x}} z_l) + \lambda \sum_{i \in J_{\bar{x}}} \varphi(|d_i^T (\bar{x} + Y_{\bar{x}} y_l + Z_{\bar{x}} z_l)|) \\
&\quad = v(\bar{z} + z_l) - v(\bar{z}) + \lambda \sum_{i \in I_{\bar{x}}} \varphi(|d_i^T Y_{\bar{x}} y_l|) + \theta(\bar{x} + Y_{\bar{x}} y_l + Z_{\bar{x}} z_l) - \theta(\bar{x} + Z_{\bar{x}} z_l)
\end{aligned}$$

$$+ \lambda \sum_{i \in J_{\bar{x}}} \big(\varphi(|d_i^T (\bar{x} + Y_{\bar{x}} y_l + Z_{\bar{x}} z_l)|) - \varphi(|d_i^T (\bar{x} + Z_{\bar{x}} z_l)|)\big). \tag{3.14}$$

Since θ is twice continuously differentiable and φ is LC^1 around $|d_i^T\bar{x}|$, $i \in J_{\bar{x}}$, (3.14) implies that there is a positive constant c such that

$$f(\bar{x}+x_l) - f(\bar{x}) \geq v(\bar{z}+z_l) - v(\bar{z}) + \lambda \sum_{i\in I_{\bar{x}}} \varphi(|d_i^T Y_{\bar{x}} y_l|) - \|Y_{\bar{x}} y_l\| c.$$

Now we show that there is $c_0 > 0$ such that

$$\max_{i\in I_{\bar{x}}} |d_i^T Y_{\bar{x}} y_l| \geq c_0 \|Y_{\bar{x}} y_l\|. \tag{3.15}$$

Let $q_l = Y_{\bar{x}} y_l / \|Y_{\bar{x}} y_l\|_2$. By the definition of $Y_{\bar{x}}$, we know that $q_l \in \text{span}\{d_i, i \in I_{\bar{x}}\}$. Hence there is $a_l \in R^{|I_{\bar{x}}|}$ such that $q_l = Da_l$. Let $D \in R^{n\times|I_{\bar{x}}|}$ whose columns are $d_i, i \in I_{\bar{x}}$. Consider

$$\begin{aligned}\max_{i\in I_{\bar{x}}} |d_i^T q_l| = \|D^T q_l\|_\infty \geq\ & \min_p\{\|D^T p\|_\infty : \|p\|_2 = 1,\, p = Da,\ a \in R^{|I_{\bar{x}}|}\} \\ =:\ & \|D^T D\bar{a}\|_\infty =: c_0.\end{aligned}$$

Since $\|D\bar{a}\|_2 = 1$, we have $\bar{a}^T D^T D\bar{a} = 1$, which implies $D^T D\bar{a} \neq 0$. Hence we have $c_0 > 0$ and (3.15).

From (ii) of Assumption 1.1 on φ, $\varphi'(0^+) = +\infty$, and $\lim_{l\to\infty} d_i^T Y_{\bar{x}} y_l = 0$ for $i \in I_{\bar{x}}$, there is sufficiently small $\varepsilon > 0$ such that

$$\begin{aligned}&\lambda \sum_{i\in I_{\bar{x}}} \varphi(|d_i^T Y_{\bar{x}} y_l|) - \|Y_{\bar{x}} y_l\| c \\ &\quad\geq \lambda(\varphi(c_0\|Y_{\bar{x}} y_l\|) - \varphi(0)) - \|Y_{\bar{x}} y_l\| c \\ &\quad\geq (\lambda\varphi'(\varepsilon)c_0 - c)\|Y_{\bar{x}} y_l)\| > 0 \text{ for all sufficiently large } l. \end{aligned}\tag{3.16}$$

Hence $f(\bar{x}+x_l) > f(\bar{x})$ for all sufficiently large l, which contracts the assumption in (3.13). This completes the proof. ∎

Remark 3.2. From (3.16), we can see that the assumption $\varphi'(0^+) = +\infty$ in Theorem 3.4 can be relaxed to $\varphi'(0^+) > c/(\lambda c_0)$, where c is a Lipschitz constant of function w and c_0 is defined in (3.15).

For problem (1.1) with $\varphi(d_i^T x) = |x_i|^q, i = 1, \dots m$ and $m = n$, the sufficient optimality condition (3.12) can be simplified as

$$X\nabla\theta(x) + \lambda q|x|^q = 0 \text{ and } \left(X(\nabla^2\theta(x))X + \lambda q(q-1)|X|^q\right)_{J_{\bar{x}},J_{\bar{x}}} \text{ is positive definite,} \tag{3.17}$$

where $X = \text{diag}(x)$.

3.3. Optimality conditions of smoothing problems. The following proposition presents the convergence of the smoothing problem (2.3) to the original problem (1.1) as the smoothing parameter $\mu \downarrow 0$, regarding the first order and second order necessary conditions for local minimizers as well as global minimizers.

PROPOSITION 3.5. *Under Assumptions* 1.1 *and* 2.1, *for any sequence* $\{\mu_k\}$ *that satisfies* $\mu_k > 0$ *and* $\lim_{k\to\infty} \mu_k = 0$, *the following statements hold.*

(1) *Let* $\{x_{\mu_k}\}$ *be a sequence of vectors satisfying the first order necessary condition of* (2.3) *with* $\mu = \mu_k$, *and then any accumulation point of* $\{x_{\mu_k}\}$ *satisfies the first order necessary condition* (3.9).

(2) *Let* $\{x_{\mu_k}\}$ *be a sequence of vectors satisfying the second order necessary condition of* (2.3) *with* $\mu = \mu_k$, *and then any accumulation point of* $\{x_{\mu_k}\}$ *satisfies the second order necessary condition* (3.9) *and* (3.10).

(3) *Let* $\{x_{\mu_k}\}$ *be a sequence of global minimizers of the smooth approximation* (2.3) *with* $\mu = \mu_k$*, and then any accumulation point of* $\{x_{\mu_k}\}$ *is a global minimizer of* (1.1).

Proof. Suppose that $\{x_{\mu_k}\}$ has a convergent subsequence with an accumulation point $\bar{x}$. By working on the subsequence, we may assume that $\{x_{\mu_k}\}$ converges to $\bar{x}$ in this proof for simplicity.

(1) Because x_{μ_k} satisfies the first order necessary condition of the smooth minimization (2.3), we have

$$\nabla \tilde{f}(x_{\mu_k}, \mu_k) = \nabla\theta(x_{\mu_k}) + \lambda \sum_{i=1}^{m} d_i \tilde{\varphi}'(s(d_i^T x_{\mu_k}, \mu_k)) s'(d_i^T x_{\mu_k}, \mu_k) = 0. \tag{3.18}$$

Multiplying $Z_{\bar{x}}^T$ on both sides of the above equation, we get

$$Z_{\bar{x}}^T \nabla\theta(x_{\mu_k}) + \lambda \sum_{i \in J_{\bar{x}}} Z_{\bar{x}}^T d_i \tilde{\varphi}'(s(d_i^T x_{\mu_k}, \mu_k)) s'(d_i^T x_{\mu_k}, \mu_k) = 0$$

due to the fact that $Z_{\bar{x}}^T d_i = 0$ for all $i \notin J_{\bar{x}}$. From (3) of Assumption 2.1, for all $i \in J_{\bar{x}}$, we have

$$\lim_{k\to\infty} \tilde{\varphi}'(s(d_i^T x_{\mu_k}, \mu_k)) s'(d_i^T x_{\mu_k}, \mu_k) = \varphi'(|d_i^T \bar{x}|)\mathrm{sign}(d_i^T \bar{x}).$$

Consequently, we have

$$\begin{aligned} 0 &= \lim_{k\to\infty} (Z_{\bar{x}}^T \nabla\theta(x_{\mu_k}) + \lambda \sum_{i \in J_{\bar{x}}} Z_{\bar{x}}^T d_i \tilde{\varphi}'(s(d_i^T x_{\mu_k}, \mu_k)) s'(d_i^T x_{\mu_k}, \mu_k)) \\ &= Z_{\bar{x}}^T \nabla\theta(\bar{x}) + \lambda \sum_{i \in J_{\bar{x}}} Z_{\bar{x}}^T d_i \varphi'(|d_i^T \bar{x}|)\mathrm{sign}(d_i^T \bar{x}) = Z_{\bar{x}}^T \nabla w(\bar{x}), \end{aligned}$$

i.e., $\bar{x}$ satisfies the first order necessary condition (3.9).

(2) Suppose $\bar{x}$ does not satisfy the second order necessary condition (3.9) and (3.10) of problem (1.1). It follows from Theorem 3.3 that there exists a nonzero vector a such that

$$a^T Z_{\bar{x}}^T H Z_{\bar{x}} a < 0 \qquad \forall H \in \partial_C^2 w(\bar{x}).$$

From (2.2b), (2.4b) and (3) of Assumption 2.1, we know there is $\hat{H} \in \partial_C^2 w(\bar{x})$ such that

$$\begin{aligned} a^T Z_{\bar{x}}^T (\nabla^2\theta(\bar{x}) + \lambda \sum_{i \in J_{\bar{x}}} d_i d_i^T \lim_{k\to\infty} (\tilde{\varphi}''(s(d_i^T x_{\mu_k}, \mu_k))(s'(d_i^T x_{\mu_k}, \mu_k))^2 \\ + \tilde{\varphi}'(s(d_i^T x_{\mu_k}, \mu_k)) s''(d_i^T x_{\mu_k}, \mu_k))) Z_{\bar{x}} a = a^T Z_{\bar{x}}^T \hat{H} Z_{\bar{x}} a, \end{aligned}$$

which implies that

$$\begin{aligned} a^T Z_{\bar{x}}^T (\nabla^2\theta(x_{\mu_k}) + \lambda \sum_{i \in J_{\bar{x}}} d_i d_i^T (\tilde{\varphi}''(s(d_i^T x_{\mu_k}, \mu_k))(s'(d_i^T x_{\mu_k}, \mu_k))^2 \\ + \tilde{\varphi}'(s(d_i^T x_{\mu_k}, \mu_k)) s''(d_i^T x_{\mu_k}, \mu_k))) Z_{\bar{x}} a < 0 \end{aligned} \tag{3.19}$$

for sufficiently large k. However, because x_{μ_k} satisfies the second order necessary condition of the smooth minimization (2.3), x_{μ_k} satisfies (3.18) and the matrix

$$\nabla^2 \tilde{f}(x_{\mu_k}, \mu_k) = \nabla^2\theta(x_{\mu_k}) + \lambda \sum_{i=1}^m (\tilde{\varphi}''(s(d_i^T x_{\mu_k}, \mu_k))(s'(d_i^T x_{\mu_k}, \mu_k))^2 \\ + \tilde{\varphi}'(s(d_i^T x_{\mu_k}, \mu_k))s''(d_i^T x_{\mu_k}, \mu_k))d_i d_i^T$$

is positive semidefinite. Consequently, we obtain

$$\begin{aligned} 0 &\le a^T Z_{\bar{x}}^T \nabla^2 \tilde{f}(x_{\mu_k}, \mu_k) Z_{\bar{x}} a \\ &= a^T Z_{\bar{x}}^T (\nabla^2\theta(x_{\mu_k}) + \lambda \sum_{i=1}^m (\tilde{\varphi}''(s(d_i^T x_{\mu_k}, \mu_k))(s'(d_i^T x_{\mu_k}, \mu_k))^2 \\ &\quad + \tilde{\varphi}'(s(d_i^T x_{\mu_k}, \mu_k))s''(d_i^T x_{\mu_k}, \mu_k))d_i d_i^T) Z_{\bar{x}} a \\ &= a^T Z_{\bar{x}}^T (\nabla^2\theta(x_{\mu_k}) + \lambda \sum_{i\in J_{\bar{x}}} (\tilde{\varphi}''(s(d_i^T x_{\mu_k}, \mu_k))(s'(d_i^T x_{\mu_k}, \mu_k))^2 \\ &\quad + \tilde{\varphi}'(s(d_i^T x_{\mu_k}, \mu_k))s''(d_i^T x_{\mu_k}, \mu_k))d_i d_i^T) Z_{\bar{x}} a, \end{aligned}$$

which contradicts (3.19). The contradiction shows that $\bar{x}$ satisfies the second order necessary condition (3.9) and (3.10) of nonsmooth minimization (1.1).

(3) Let x^* be a global minimizer of (1.1); we know

$$\begin{aligned} \theta(x_{\mu_k}) + \lambda \sum_{i=1}^m \varphi(|d_i^T x_{\mu_k}|) &\le \theta(x_{\mu_k}) + \lambda \sum_{i=1}^m \tilde{\varphi}(s(d_i^T x_{\mu_k}, \mu_k)) \\ &\le \theta(x^*) + \lambda \sum_{i=1}^m \tilde{\varphi}(s(d_i^T x^*, \mu_k)), \end{aligned}$$

where the first inequality is from (4) of Assumption 2.1. Let $k \to \infty$, and we have

$$\theta(\bar{x}) + \lambda \sum_{i=1}^m \varphi(|d_i^T \bar{x}|) \le \theta(x^*) + \lambda \sum_{i=1}^m \varphi(|d_i^T x^*|).$$

Hence $\bar{x}$ is a global minimizer of (1.1). □

4. Convergence analysis. In this section, we will present convergence analysis of the smoothing trust region Newton method for problem (1.1).

LEMMA 4.1. *Consider the iterates $\{x_k\}$ and $\{\mu_k\}$ generated by Algorithm 1. Define the index set*

$$\mathcal{K} := \{k \mid \|\nabla \tilde{f}(x_k, \mu_k)\| \le \zeta\mu_k \text{ and } \Delta_k \ge \underline{\Delta}\}. \tag{4.1}$$

If $\mathcal{K}$ is an infinite set, then

$$\liminf_{k\to\infty} \|\nabla \tilde{f}(x_k, \mu_k)\| = 0.$$

Proof. Let $\mathcal{K} = \{k_j, \quad j = 1, 2, \ldots\}$ with $0 \le k_1 < k_2 < k_3 < \cdots$. From $0 < \nu < 1$ and

$$0 < \mu_{k_j} \le \nu\mu_{k_j - 1} = \nu\mu_{k_{j-1}} \le \nu^2\mu_{k_j - 2} = \nu^2\mu_{k_{j-2}} \le \cdots \le \nu^{j-1}\mu_{k_1}$$

we get $\lim_{j\to\infty}\mu_{k_j}=0$ if $\mathcal{K}$ is an infinite set. This implies that $\lim_{k\to\infty}\mu_k=0$ because our algorithm generates a monotonically decreasing sequence $\{\mu_k\}_{k=1}^{\infty}$. Therefore, the following inequalities

$$\|\nabla\tilde f(x_{k_j},\mu_{k_j})\|\le\zeta\mu_{k_j}\le\zeta\nu^{j-1}\mu_{k_0}$$

give that $\lim_{j\to\infty}\|\nabla\tilde f(x_{k_j},\mu_{k_j})\| = 0$. Consequently, $\liminf_{k\to\infty}\|\nabla\tilde f(x_k,\mu_k)\| = 0$. ∎

LEMMA 4.2. *Suppose f has bounded level sets and Assumptions* 1.1, 2.1 *hold. Consider the iterates $\{x_k\}$ and $\{\mu_k\}$ generated by applying Algorithm* 1 *to problem* (1.1). *Then $\mathcal{K}$ defined in* (4.1) *is an infinite set.*

Proof. Suppose $\mathcal{K}$ is finite. Then based on the definition of $\mathcal{K}$ and the design of Algorithm 1, there exists a nonnegative integer $\hat K$, such that for all nonnegative integers j, $\mu_{\hat K+j}=\mu_{\hat K}$ and

$$\|\nabla\tilde f(x_{k+j},\mu_{\hat K})\|>\zeta\mu_{\hat K}\ \text{or}\ \Delta_{k+j}<\underline{\Delta}. \tag{4.2}$$

Note that $\tilde f(\cdot,\mu_{\hat K})$ has bounded level sets. Denote

$$\hat\Omega:=\{x\mid\tilde f(x,\mu_{\hat K})\le\tilde f(x_{\hat K},\mu_{\hat K})\}\ \text{and}\ \hat\Omega(\overline{\Delta})=\{x\,|\,\|x-y\|\le\overline{\Delta}\ \text{for some}\ y\in\hat\Omega\}.$$

From the design of Algorithm 1 and Assumption 2.1, we know

$$\tilde f(x_{\hat K},\mu_{\hat K})\ge\tilde f(x_{\hat K+j},\mu_{\hat K+j})=\tilde f(x_{\hat K+j},\mu_{\hat K})\ \text{for all nonegative integers}\ j.$$

So $\{x_{\hat K+j}\}\subseteq\hat\Omega$. Since $\tilde f(\cdot,\mu_{\hat K})$ is twice continuously differentiable on $\Re^n$, we have

$$M_{\hat K}=\max_{x\in\hat\Omega(\overline{\Delta})}\|\nabla^2\tilde f(x,\mu_{\hat K})\|<+\infty.$$

From the classic convergent result for the smooth optimization (e.g., see Theorem 4.6 in [36]), we can claim that

$$\lim_{j\to\infty}\nabla\tilde f(x_{\hat K+j},\mu_{\hat K})=0. \tag{4.3}$$

Since $\zeta\mu_{\hat K}>0$, there exists a positive integer $\hat J$, such that

$$\|\nabla\tilde f(x_{\hat K+j},\mu_{\hat K})\|<\zeta\mu_{\hat K}\ \forall\, j\ge\hat J. \tag{4.4}$$

Case 1: $\|\nabla\tilde f(x_{\hat K+\hat J},\mu_{\hat K})\|\ne0$. In this case, it follows from Condition 2.1 that $m_{\hat K+\hat J}(p_{\hat K+\hat J})\ne m_{\hat K+\hat J}(0)$. If $\rho_{\hat K+\hat J}>\eta_1$, we have that $\Delta_{\hat K+\hat J+1}\ge\underline{\Delta}$. According to (4.4), we have

$$\|\nabla\tilde f(x_{\hat K+\hat J+1},\mu_{\hat K})\|<\zeta\mu_{\hat K}.$$

This means that

$$\Delta_{\hat K+\hat J+1}\ge\underline{\Delta}\ \text{and}\ \|\nabla\tilde f(x_{\hat K+\hat J+1},\mu_{\hat K})\|<\zeta\mu_{\hat K},$$

which contracts (4.2). Therefore, $\rho_{\hat K+\hat J}<\eta_1$, which implies

$$x_{\hat K+\hat J+1}=x_{\hat K+\hat J}\ \text{and}\ \Delta_{\hat K+\hat J+1}\le\gamma_1\Delta_{\hat K+\hat J}.$$

The above arguments are still true when $\hat{J}$ is replaced by $\hat{J}+1$. Thus, by induction,

$$\rho_{\hat{K}+j} < \eta_1, \quad x_{\hat{K}+j} = x_{\hat{K}+\hat{J}}, \text{ and } \Delta_{\hat{K}+j} \le \gamma_1^{j-\hat{J}} \Delta_{\hat{K}+\hat{J}} \ \forall j > \hat{J}. \tag{4.5}$$

Since $\tilde{f}(\cdot, \mu_{\hat{K}})$ has bounded level sets, denote

$$\bar{\Omega} = \{x \mid \tilde{f}(x, \mu_{\hat{K}}) \le \tilde{f}(x_{\hat{K}+\hat{J}}, \mu_{\hat{K}})\} \text{ and } \bar{\Omega}(\overline{\Delta}) = \{x | \|x-y\| \le \overline{\Delta} \text{ for some } y \in \bar{\Omega}\}.$$

Together with the continuity of $\nabla^2 \tilde{f}(\cdot, \mu_{\hat{K}})$, we have

$$\bar{M} = \max_{x \in \bar{\Omega}(\overline{\Delta})} \{\|\nabla^2 \tilde{f}(x, \mu_{\hat{K}})\|^2\} < +\infty.$$

Similar to the proof of Theorem 4.5 in [36], because $\tilde{f}(\cdot, \mu_{\hat{K}})$ is twice continuously differentiable, from Taylor's theorem we can get

$$\begin{aligned} |m_k(p_k) - \tilde{f}(x_k + p_k, \mu_{\hat{K}})| &\le \frac{1}{2}\|p_k\|^2 \max_{0 \le \tau \le 1} \|\nabla^2 f(x_{\hat{K}+\hat{J}}) - \nabla^2 \tilde{f}(x_{\hat{K}+\hat{J}} + \tau p_k, \mu_{\bar{K}})\| \\ &\le \frac{1}{2}(M + \bar{M})\|p_k\|^2, \end{aligned}$$

where $M = \|\nabla^2 f(x_{\hat{K}+\hat{J}})\|$. From Assumption 2.1 and (2.7), we have for all $k > \hat{K}+\hat{J}$ that

$$m_k(0) - m_k(p_k) \ge \frac{c_1}{2}\|\nabla \tilde{f}(x_{\hat{K}+\hat{J}}, \mu_{\hat{K}})\| \min\left(\frac{\|\nabla \tilde{f}(x_{\hat{K}+\hat{J}}, \mu_{\hat{K}})\|}{M}, \triangle_k\right).$$

Since $\|\nabla \tilde{f}(x_{\hat{K}+\hat{J}}, \mu_{\hat{K}})\| \ne 0$ for all $k > \hat{K}+\hat{J}$, it follows that

$$\begin{aligned} |\rho_k - 1| &= \left| \frac{m_k(p_k) - \tilde{f}(x_{\hat{K}+\hat{J}} + p_k, \mu_{\hat{K}})}{m_k(0) - m_k(p_k)} \right| \\ &\le \frac{(M+\bar{M})c_2^2\Delta_k^2}{c_1\|\nabla \tilde{f}(x_{\hat{K}+\hat{J}}, \mu_{\hat{K}})\| \min\left(\|\nabla \tilde{f}(x_{\hat{K}+\hat{J}}, \mu_{\hat{K}})\|/M, \Delta_k\right)}. \end{aligned}$$

The above inequality and the last inequality in (4.5) imply that $\lim_{k\to\infty} \rho_k = 1$, which contradicts the first inequality in (4.5).

Case 2: $\|\nabla \tilde{f}(x_{\hat{K}+\hat{J}}, \mu_{\hat{K}})\| = 0$ and $\nabla^2 \tilde{f}(x_{\hat{K}+\hat{J}}, \mu_{\hat{K}})$ is positive semidefinite. In this case, $m_{\hat{K}+\hat{J}}(p_{\hat{K}+\hat{J}}) = m_{\hat{K}+\hat{J}}(0)$. Then $x_{\hat{K}+\hat{J}} = x_{\hat{K}+\hat{J}+1}$ and $\Delta_{\hat{K}+\hat{J}+1} \ge \underline{\Delta}$. So

$$\Delta_{\hat{K}+\hat{J}+1} \ge \underline{\Delta} \quad \text{and} \quad \|\nabla \tilde{f}(x_{\hat{K}+\hat{J}+1}, \mu_{\hat{K}})\| = 0 < \zeta\mu_{\hat{K}},$$

which contracts (4.2).

Case 3: $\|\nabla \tilde{f}(x_{\hat{K}+\hat{J}}, \mu_{\hat{K}})\| = 0$ and $\nabla^2 \tilde{f}(x_{\hat{K}+\hat{J}}, \mu_{\hat{K}})$ has a negative eigenvalue. In this case, the exact solution $p^*_{\hat{K}+\hat{J}}$ of (2.5) must be on the boundary of the feasible region. Hence from $\triangle_k > 0$ and Condition 2.1, we have $m_{\hat{K}+\hat{J}}(p_{\hat{K}+\hat{J}}) \ne m_{\hat{K}+\hat{J}}(0)$.

If $\rho_{\hat{K}+\hat{J}+j} \le \eta_1$ for all nonnegative integers j, according to Algorithm 1, we have

$$x_{\hat{K}+\hat{J}+j} = x_{\hat{K}+\hat{J}} \quad \text{and} \quad \Delta_{\hat{K}+\hat{J}+j} = \gamma_1 \Delta_{\hat{K}+\hat{J}+j-1} = \gamma_1^j \Delta_{\hat{K}+\hat{J}}. \tag{4.6}$$

Then, $m_{\hat{K}+\hat{J}+j}(\cdot) = m_{\hat{K}+\hat{J}}(\cdot)$ for all nonnegative integers j. Because $\nabla^2 \tilde{f}(x_{\hat{K}+\hat{J}}, \mu_{\hat{K}})$ has a negative eigenvalue, we have $\beta > 0$, where $-\beta$ is the smallest eigenvalue of

$\nabla^2 \tilde{f}(x_{\hat{K}+\hat{J}}, \mu_{\hat{K}})$. Then $\beta_{\hat{K}+\hat{J}+j} \geq \beta > 0$. This, together with Condition 2.1, implies that

$$\begin{aligned} m_{\hat{K}+\hat{J}}(0) - m_{\hat{K}+\hat{J}}(p_{\hat{K}+\hat{J}+j}) &\geq \frac{1}{2} c_1 (\|R_{\hat{K}+\hat{J}+j} p_{\hat{K}+\hat{J}+j}\|^2 + \beta_{\hat{K}+\hat{J}+j} \Delta^2_{\hat{K}+\hat{J}+j}) \\ &\geq \frac{c_1}{2} \beta_{\hat{K}+\hat{J}+j} \Delta^2_{\hat{K}+\hat{J}+j} \geq \frac{c_1}{2} \beta \Delta^2_{\hat{K}+\hat{J}+j} \geq \frac{c_1 \beta}{2 c_2^2} \|p_{\hat{K}+\hat{J}+j}\|^2. \end{aligned}$$

From Taylor's theorem we have

$$\begin{aligned} &m_{\hat{K}+\hat{J}}(p_{\hat{K}+\hat{J}+j}) - \tilde{f}(x_{\hat{K}+\hat{J}} + p_{\hat{K}+\hat{J}+j}, \mu_{\hat{K}}) \\ &\quad = \frac{1}{2} p^T_{\hat{K}+\hat{J}+j} (\nabla^2 \tilde{f}(x_{\hat{K}+\hat{J}}, \mu_{\hat{K}}) - \nabla^2 \tilde{f}(x_{\hat{K}+\hat{J}} + \tau p_{\hat{K}+\hat{J}+j}, \mu_{\hat{K}})) p_{\hat{K}+\hat{J}+j} \end{aligned}$$

for some $\tau \in [0, 1]$, which yields

$$\begin{aligned} &|m_{\hat{K}+\hat{J}}(p_{\hat{K}+\hat{J}+j}) - \tilde{f}(x_{\hat{K}+\hat{J}+j} + p_{\hat{K}+\hat{J}+j}, \mu_{\hat{K}})| \\ &\quad \leq \frac{1}{2} \|p_{\hat{K}+\hat{J}+j}\|^2 \|\nabla^2 \tilde{f}(x_{\hat{K}+\hat{J}}, \mu_{\hat{K}}) - \nabla^2 \tilde{f}(x_{\hat{K}+\hat{J}} + \tau p_{\hat{K}+\hat{J}+j}, \mu_{\hat{K}})\|. \end{aligned}$$

Consequently,

$$|\rho_{\hat{K}+\hat{J}+j} - 1| \leq \frac{c_2^2}{\beta c_1} \|\nabla^2 \tilde{f}(x_{\hat{K}+\hat{J}}, \mu_{\hat{K}}) - \nabla^2 \tilde{f}(x_{\hat{K}+\hat{J}} + \tau p_{\hat{K}+\hat{J}+j}, \mu_{\hat{K}})\|. \tag{4.7}$$

Moreover, (4.6) implies that $\{\Delta_{\hat{K}+\hat{J}+j}\}$ converges to zero as j goes to infinity and hence $\{\|p_{\hat{K}+\hat{J}+j}\|\}$ also converges to zero. Thus, relation (4.7) and the uniform continuity of $\nabla^2 \tilde{f}(\cdot, \mu_{\hat{K}})$ on Ω imply that $\rho_{\hat{K}+\hat{J}+j} > \eta_1$ for all sufficiently large j, which contracts to our assumption $\rho_{\hat{K}+\hat{J}+j} \leq \eta_1$ for all positive integers j. This contradiction shows that there must be a positive integer $\tilde{j}$ such that $\rho_{\hat{K}+\hat{J}+\tilde{j}} > \eta_1$. According to Algorithm 1, $\Delta_{\hat{K}+\hat{J}+\tilde{j}+1} \geq \underline{\Delta}$. Thus, for the positive integer $\hat{K} + \hat{J} + \tilde{j} + 1$ we have

$$\Delta_{\hat{K}+\hat{J}+\tilde{j}+1} \geq \underline{\Delta} \text{ and } \|\nabla \tilde{f}(x_{\hat{K}+\hat{J}+\tilde{j}+1}, \mu_{\hat{K}})\| < \zeta \mu_{\hat{K}},$$

which contracts (4.2) again.

In all the cases, we find the contradiction to the assumption that $\mathcal{K}$ is finite. Therefore, $\mathcal{K}$ is an infinite set. □

LEMMA 4.3. *Suppose assumptions of Lemma* 4.2 *hold. Consider the iterates* $\{x_k\}$ *and* $\{\mu_k\}$ *generated by applying Algorithm* 1 *to problem* (1.1). *For any infinite subsequence* $\mathcal{K}_2 \subset \mathcal{K}$ *at which* $\lim_{k\to\infty, k\in\mathcal{K}_2} x_k$ *exist, we have*

$$\lim_{k\to\infty, k\in\mathcal{K}_2} \beta_k = 0, \tag{4.8}$$

where β_k *satisfies* (2.8).

Proof. For any $k \in \mathcal{K}$, from the construction of $\mathcal{K}$, only two situations can happen:

(i) $m_k(p_k) \neq m_k(0)$ and $\rho_k > \eta_1$. In this case, since $m_k(0) - m_k(p_k) > 0$ and $\tilde{f}(x, \mu)$ is an increasing function in μ, we have

$$\begin{aligned} \tilde{f}(x_k, \mu_k) - \tilde{f}(x_{k+1}, \mu_{k+1}) &\geq \tilde{f}(x_k, \mu_k) - \tilde{f}(x_{k+1}, \mu_k) \\ &= \tilde{f}(x_k, \mu_k) - \tilde{f}(x_k + p_k, \mu_k) > \eta_1 (m_k(p_k) - m_k(0)). \end{aligned}$$

(ii) $m_k(p_k) = m_k(0)$. In this case,

$$\tilde{f}(x_k, \mu_k) - \tilde{f}(x_{k+1}, \mu_{k+1}) = \tilde{f}(x_k, \mu_k) - \tilde{f}(x_k, \mu_{k+1}) \geq 0 = \eta_1(m_k(p_k) - m_k(0)).$$

Hence, in both cases, we have that

$$\Delta_k \geq \underline{\Delta} \text{ and } \tilde{f}(x_k, \mu_k) - \tilde{f}(x_{k+1}, \mu_{k+1}) \geq \eta_1(m_k(p_k) - m_k(0)) \qquad \forall k \in \mathcal{K}.$$

It follows from (2.8) that

$$m_k(0) - m_k(p_k) \geq \frac{1}{2} c_1(\|R_k p_k\|^2 + \beta_k \Delta_k^2) \geq \frac{1}{2} c_1 \beta_k \Delta_k^2.$$

Therefore, for all $k \in \mathcal{K}$, we have that

$$\tilde{f}(x_k, \mu_k) - \tilde{f}(x_{k+1}, \mu_{k+1}) \geq \frac{1}{2}\eta_1 c_1 \beta_k \Delta_k^2 \geq \frac{1}{2}\eta_1 c_1 \beta_k \underline{\Delta}^2. \tag{4.9}$$

Sort the index in $\mathcal{K}_2$ in ascending order and denote the jth element as k_j, and from (4.9) we have that

$$\begin{aligned}\tilde{f}(x_{k_j}, \mu_{k_j}) - \tilde{f}(x_{k_{j+1}}, \mu_{k_{j+1}}) &= \sum_{i=0}^{k_{j+1}-k_j-1} (\tilde{f}(x_{k_j+i}, \mu_{k_j+i}) - \tilde{f}(x_{k_j+i+1}, \mu_{k_j+i+1})) \\ &\geq \tilde{f}(x_{k_j}, \mu_{k_j}) - \tilde{f}(x_{k_j+1}, \mu_{k_j+1}) \geq \frac{1}{2}\eta_1 c_1 \beta_{k_j} \underline{\Delta}^2\end{aligned}$$

for any $j \geq 1$. Consequently, (4.8) holds because $\lim_{j\to\infty} x_{k_j}$ exists and $\underline{\Delta} > 0$. ∎

THEOREM 4.4 (global convergence to second order stationary points). *Suppose that the assumptions of Lemma* 4.2 *hold. Apply Algorithm* 1 *to problem* (1.1). *Then any accumulation point of* $\{x_k\}$ *satisfies the second order necessary condition* (3.9), (3.10).

Proof. From Lemma 4.2, we know that $\mathcal{K}$ is an infinite set, $\lim_{k\to\infty} \mu_k = 0$, and

$$\lim_{k\to\infty, k\in\mathcal{K}} \|\nabla \tilde{f}(x_k, \mu_k)\| = \lim_{k\to\infty, k\in\mathcal{K}} \|\nabla\theta(x_k) + \lambda \sum_{i=1}^{m} d_i \tilde{\varphi}'(s(d_i^T x_k, \mu_k)) s'(d_i^T x_k, \mu_k)\| = 0.$$

Moreover, from (4.9), we have $\tilde{f}(x_k, \mu_k) \geq \tilde{f}(x_{k+1}, \mu_{k+1}), k \in \mathcal{K}$. Since $\tilde{f}$ has bound level sets, $\{x_k\}$ is in a compact set. Hence $\{x_k\}_{k\in\mathcal{K}}$ has at least one accumulating point. For any accumulation point $\bar{x}$ of $\{x_k\}_{k\in\mathcal{K}}$, there exists a subset $\mathcal{K}_2$ such that $\lim_{k\to\infty, k\in\mathcal{K}_2} x_k = \bar{x}$. Then we have

$$\begin{aligned}0 &= \lim_{k\to\infty, k\in\mathcal{K}_2} \left\| Z_{\bar{x}}^T [\nabla\theta(x_k) + \lambda \sum_{i=1}^{m} d_i \tilde{\varphi}'(s(d_i^T x_k, \mu_k)) s'(d_i^T x_k, \mu_k)] \right\| \\ &= \lim_{k\to\infty, k\in\mathcal{K}_2} \left\| Z_{\bar{x}}^T \nabla\theta(x_k) + \lambda \sum_{i\in J_{\bar{x}}} Z_{\bar{x}}^T d_i \tilde{\varphi}'(s(d_i^T x_k, \mu_k)) s'(d_i^T x_k, \mu_k) \right\|.\end{aligned}$$

Consequently, due to (3) of Assumption 2.1 and the fact that $\nabla\theta(\cdot)$ is continuously differentiable on $\Re^n$,

$$\begin{aligned}0 &= \lim_{k\to\infty, k\in\mathcal{K}_2} (Z_{\bar{x}}^T \nabla\theta(x_k) + \lambda \sum_{i\in J_{\bar{x}}} Z_{\bar{x}}^T d_i \tilde{\varphi}'(s(d_i^T x_k, \mu_k)) s'(d_i^T x_k, \mu_k)) \\ &= Z_{\bar{x}}^T \nabla\theta(\bar{x}) + \lambda \sum_{i\in J_{\bar{x}}} Z_{\bar{x}}^T d_i \varphi'(|d_i^T \bar{x}|) \mathrm{sign}(d_i^T \bar{x}) = Z_{\bar{x}}^T \nabla w(\bar{x}),\end{aligned}$$

which means $\bar{x}$ satisfies condition (3.9).

To show $\bar{x}$ satisfies condition (3.10), we need to prove that for any $a \in \Re^n$, there exists a matrix $H \in \partial_C^2 w(\bar{x})$, such that $a^T Z_{\bar{x}}^T H Z_{\bar{x}} a \geq 0$. We prove this result by contradiction. Suppose there exists a nonzero vector a such that

$$a^T Z_{\bar{x}}^T H Z_{\bar{x}} a < 0 \qquad \forall H \in \partial_C^2 w(\bar{x}). \tag{4.10}$$

Since $\nabla^2 \tilde{f}(x_k, \mu_k) + \beta_k I$ is positive semidefinite, we have

$$\begin{aligned}
0 &\leq a^T Z_{\bar{x}}^T (\nabla^2 \tilde{f}(x_k, \mu_k) + \beta_k I) Z_{\bar{x}} a \\
&= a^T Z_{\bar{x}}^T (\nabla^2 \theta(x_k) + \lambda \sum_{i=1}^m d_i d_i^T (\tilde{\varphi}''(s(d_i^T x_k, \mu_k))(s'(d_i^T x_k, \mu_k))^2 \\
&\quad + \tilde{\varphi}'(s(d_i^T x_k, \mu_k)) s''(d_i^T x_k, \mu_k)) + \beta_k I) Z_{\bar{x}} a \\
&= a^T Z_{\bar{x}}^T \nabla^2 \theta(x_k) Z_{\bar{x}} a + \lambda \sum_{i \in J_{\bar{x}}} a^T Z_{\bar{x}}^T d_i d_i^T Z_{\bar{x}} a (\tilde{\varphi}''(s(d_i^T x_k, \mu_k))(s'(d_i^T x_k, \mu_k))^2 \\
&\quad + \tilde{\varphi}'(s(d_i^T x_k, \mu_k)) s''(d_i^T x_k, \mu_k)) + \beta_k \|Z_{\bar{x}} a\|^2.
\end{aligned}$$

Based on Lemma 4.3, we know that $\{\beta_k\}_{k \in \mathcal{K}_2}$ goes to zero as k goes to infinity. Thus, by letting k go to infinity in $\mathcal{K}_2$, we obtain that

$$\begin{aligned}
&\lambda \sum_{i \in J_{\bar{x}}} a^T Z_{\bar{x}}^T d_i d_i^T Z_{\bar{x}} a \lim_{k \to \infty, k \in \mathcal{K}_2} (\tilde{\varphi}''(s(d_i^T x_k, \mu_k))(s'(d_i^T x_k, \mu_k))^2 \\
&\quad + \tilde{\varphi}'(s(d_i^T x_k, \mu_k)) s''(d_i^T x_k, \mu_k)) + a^T Z_{\bar{x}}^T \nabla^2 \theta(\bar{x}) Z_{\bar{x}} a \geq 0,
\end{aligned} \tag{4.11}$$

where $\lim_{k \to \infty, k \in \mathcal{K}_2} (\tilde{\varphi}''(s(d_i^T x_k, \mu_k))(s'(d_i^T x_k, \mu_k))^2 + \tilde{\varphi}'(s(d_i^T x_k, \mu_k)) s''(d_i^T x_k, \mu_k)) \in \partial^2 \varphi(|d_i^T \bar{x}|), i \in J_{\bar{x}}$ by (3) of Assumption 2.1.

Note that $a^T Z_{\bar{x}}^T d_i d_i^T Z_{\bar{x}} a \geq 0$, $i \in J_{\bar{x}}$, and $J_{\bar{x}} \subseteq J_{x_k}$. From (4.11) we derive that there is $H \in \partial_C^2 w(\bar{x})$ such that $a^T Z_{\bar{x}}^T H Z_{\bar{x}} a \geq 0$, which contradicts (4.10). This contradiction shows that $\bar{x}$ satisfies (3.10). $\square$

COROLLARY 4.5. *Suppose that φ is twice continuously differentiable in $\Re$ except at 0, and $\varphi'(0^+) = +\infty$. If $\bar{x}$ is an accumulation point of $\{x_k\}_{k \in \mathcal{K}}$ at which*

$$Z_{\bar{x}}^T \nabla^2 \theta(\bar{x}) Z_{\bar{x}} + \lambda \sum_{i \in J_{\bar{x}}} Z_{\bar{x}}^T d_i d_i^T Z_{\bar{x}} \varphi''(|d_i^T \bar{x}|) \tag{4.12}$$

is nonsingular, then $\bar{x}$ is a strict local minimizer of (1.1).

Proof. Since φ is twice continuously differentiable except at 0, we know

$$\partial_C^2 w(\bar{x}) = \partial^2 w(\bar{x}) = \partial_B^2 w(\bar{x}) = \left\{ \nabla^2 \theta(\bar{x}) + \sum_{i \in J_{\bar{x}}} d_i d_i^T \varphi''(|d_i^T \bar{x}|) \right\}.$$

From Theorem 4.4, we know that the matrix in (4.12) is positive semidefinite. Our nonsingularity assumption implies that the matrix is positive definite. Thus, it follows from Theorem 3.4 that $\bar{x}$ is a strict local minimizer. $\square$

5. Numerical experiments. In this section, we test the effectiveness of Algorithm 1, smoothing trust region Newton method. We implemented Algorithm 1 in MATLAB and called the Fortran subroutine GQTPAR [31] to solve the trust region subproblem (2.5). GQTPAR is implemented based on the approach of Morè and Sorensen for the trust region subproblem, which is guaranteed to produce a nearly optimal solution of (2.5) satisfying Condition 2.1 (see sections 3 and 4 of [31]).

In theory, we have proved that when the solution found for the trust region subproblem (2.5) satisfies Condition 2.1 with the smoothing parameter μ_k approaching zero, the iterates generated by Algorithm 1 would globally converge to a point satisfying the second order necessary condition (3.9), (3.10) in Theorem 4.4. In our numerical tests, we terminated the iterates when the smoothing parameter is small enough (i.e., the smoothing function approximates the original objective function well enough), and the corresponding smoothing minimization problem is solved accurately enough. In particular, the termination criterions for Algorithm 1 are

$$\mu_k \leq \bar{\mu} \text{ and } \|\nabla \tilde{f}(x_k, \mu_k)\| \leq \bar{\mu} \tag{5.1}$$

for a given tolerance $\bar{\mu} > 0$.

For the experiments described in this section, the values of parameters in Algorithm 1 are chosen as follows. The initial point is $x_0 = 0$; the initial trust region radius is $\Delta_0 = 1$; the parameters for adjusting trust region radius are $\eta_1 = 0.1$, $\eta_2 = 0.9$, $\gamma_1 = 0.5$, $\gamma_2 = 2$, $\underline{\Delta} = 10^{-4}$, and $\overline{\Delta} = 10^{12}$; the initial smoothing parameter is $\mu_0 = 0.01$; the reducing rate for the smoothing parameter is $\nu = 0.1$; the tolerance for termination is $\bar{\mu} = 10^{-4}$; and $\zeta = 1$. The norm $\|\cdot\|$ is the Euclidean norm.

We test Algorithm 1 with the six penalty functions $\varphi_1, \ldots, \varphi_6$ in section 1 and their smoothing functions $\tilde{\varphi}_1, \ldots, \tilde{\varphi}_6$ in subsection 2.1. We select the value of λ in the interval $[5, 70]$ with a step length 0.5. On the value of α for φ_1, φ_2 we adopt the commonly used value $\alpha = 1.0$; for the SCAD penalty function φ_5, we adopt $\alpha = 3.7$ based on the suggestion of Fan and Li [15]; and for φ_6, we choose $\alpha = 2.7$ which performs reasonably well for our experiments in this paper. For the function φ_3, we use $q = 0.3, 0.4, \ldots, 0.9$ in Example 5.1 and $q = 0.5, 1.0$ in Examples 5.2–5.3.

Example 5.1 (prostate cancer). The prostate cancer data comes from a study in [42] that examined the correlation between the level of prostate specific antigen and a number of clinical measures. The data set is downloaded from the website http://stat.stanford.edu/˜tibs/ElemStatLearn/data.html. It consists of the medical records of 97 patients who were about to receive a radical prostatectomy. Each record contains eight predictors (lcavol, lweight, age, lbph, svi, lcp, gleason, pgg45) and one outcome (lpsa). These 97 records were further divided into two parts: a training set with 67 observations and a test set with 30 observations. More detailed explanation for the data set can be found in [22, 42].

Let the prediction error be the mean squared errors (MSEs) over the test set. In this experiment, we want to find fewer main factors with smaller prediction error by fitting a linear model $\theta(x) = \|Ax - b\|^2$. We first run the experiment with the six penalty functions $\varphi_1, \ldots, \varphi_6$. For the L_q penalty function φ_3, we choose two commonly used values: $q = 1$ (lasso) and $q = 0.5$ ($\frac{1}{2}$-norm). Let $\bar{x}$ be the solution obtained by our algorithm; for all $i = 1, \ldots, n$, if the ith component $\bar{x}_i \leq 10\bar{\mu}$, we truncated this component of $\bar{x}$ as 0, i.e., $\bar{x}_i := 0$. The MSE values in Table 5.1 were computed before such components were set to zero.

Seven sets of experiment results with the corresponding penalty functions and values of parameters are reported in Table 5.1. From this table we can see that our algorithm can find sparse solutions on the prostate cancer dataset with the six different penalty functions. Among these results, the model using the L_q penalty function φ_3 with $q = 0.5$ gives the best solution. In fact, compared with the results reported in [9, 22], we successfully found a better solution in the sense that our solution is sparser and has lower MSE.

Since the L_q penalty φ_3 with $q = 0.5$ performs best on this dataset, we concentrate on φ_3 for the rest part of this test. Generally speaking, for the L_q penalty, the smaller

TABLE 5.1
Results for prostate cancer.

	φ_1	φ_2	$\varphi_3(q = 1.0)$	$\varphi_3(q = 0.5)$	φ_4	φ_5	φ_6
λ	14.5	14.5	14.5	8.0	7.5	14.5	14.5
α	1.0	1.0	(-)	(-)	(-)	3.7	2.7
	0.607	0.596	0.549	0.646	0.559	0.549	0.551
	0.210	0.220	0.216	0.275	0.215	0.216	0.216
	0	0	0	0	0	0	0
	0.088	0.083	0.091	0	0.087	0.091	0.091
	0.133	0.150	0.158	0.128	0.152	0.158	0.157
	0	0	0	0	0	0	0
	0	0	0	0	0	0	0
	0.042	0.046	0.061	0	0.055	0.061	0.060
MSE	0.446	0.444	0.451	0.428	0.450	0.451	0.451

TABLE 5.2
Results for prostate cancer data with L_q penalty function φ_3 and $\lambda = 8$.

	$q = 0.9$	$q = 0.8$	$q = 0.7$	$q = 0.6$	$q = 0.5$	$q = 0.4$	$q = 0.3$
	0.566	0.583	0.609	0.620	0.646	0.656	0.654
	0.226	0.226	0.223	0.229	0.275	0.278	0.284
	0	0	0	0	0	0	0
	0.132	0.123	0.114	0.098	0	0	0
	0.188	0.184	0.191	0.178	0.128	0.119	0.129
	0	0	0	0	0	0	0
	0	0	0	0	0	0	0
	0.075	0.053	0	0	0	0	0
MSE	0.453	0.446	0.436	0.433	0.428	0.431	0.427

the value of q is, the more similar to the 0-norm. Since the utilization of 0-norm is the "best" choice from the modeling point of view, it is reasonable to use small values of q. Thus we fix the parameter $\lambda = 8$ and vary q from 0.9 to 0.3 with a step length 0.1. The corresponding results are listed in Table 5.2. From Table 5.2 we can see that in general when q is reduced, Algorithm 1 finds a sparser solution with smaller prediction error. Furthermore, for all these solutions, besides the termination conditions (5.1) being satisfied, the second order sufficient conditions (3.17) are also satisfied numerically, i.e., $\|X\nabla\theta(x) + \lambda q|x|^q\| \le 10\bar{\mu}$ and the matrix

$$\left(X(\nabla^2\theta(x))X + \lambda q(q-1)|X|^q\right)_{J_{\bar{x}},J_{\bar{x}}}$$

is positive definite, which verifies our theoretical results in the previous sections.

Example 5.2 (linear regression). Consider the data model

$$b = a^T x + \sigma\epsilon,$$

where $x = (3, 1.5, 0, 0, 2, 0, 0, 0)^T$, $\epsilon \sim N(0, 1)$, and the input a is an eight-dimensional vector from multivariate normal distribution with covariance between a_i and a_j being $0.5^{|i-j|}(1 \le i, j \le 8)$. Linear regression (i.e., $\theta(x) = \|Ax - b\|^2$) with different kinds of penalty terms can be used to estimate x from the sampled data set. The data model was first given by Tibshirani in [43] and used as a test problem in many papers. Here we follow the experiment setting in [15]: First, let $n = 40$ and $\sigma = 3$; then σ is reduced to 1; finally the sample size n is increased to 60. For each pair (n, σ), 100 datasets are randomly generated and all the results are based on the average of 100 runs.

TABLE 5.3
Results of linear regression.

Method	Param.	MRME (%)	Avg. no. of 0 coefficients	
			Correct	Incorrect
	n = 40, $\sigma = 3$			
φ_1	$\lambda = 70.0$, $\alpha = 1.0$	0.40	4.75	0.29
φ_2	$\lambda = 70.0$, $\alpha = 1.0$	0.40	4.78	0.30
φ_3	$\lambda = 49.0$, $q = 0.5$	0.30	4.81	0.31
φ_3	$\lambda = 50.0, q = 1.0$	0.81	4.12	0.11
φ_4	$\lambda = 26.0$	0.69	4.09	0.07
φ_5	$\lambda = 57.0$, $\alpha = 3.7$	0.68	4.10	0.06
φ_6	$\lambda = 47.5$, $\alpha = 2.7$	0.76	4.05	0.04
	n = 40, $\sigma = 1$			
φ_1	$\lambda = 41.5$, $\alpha = 1.0$	0.31	5.00	0
φ_2	$\lambda = 40.0$, $\alpha = 1.0$	0.36	5.00	0
φ_3	$\lambda = 19.0$, $q = 0.5$	0.13	5.00	0
φ_3	$\lambda = 20.0, q = 1.0$	0.73	4.04	0
φ_4	$\lambda = 8.5$	0.48	4.13	0
φ_5	$\lambda = 17.5$, $\alpha = 3.7$	0.67	4.05	0
φ_6	$\lambda = 19.5$, $\alpha = 2.7$	0.91	4.18	0
	n = 60, $\sigma = 1$			
φ_1	$\lambda = 44.5$, $\alpha = 1.0$	0.21	5.00	0
φ_2	$\lambda = 44.5$, $\alpha = 1.0$	0.47	5.00	0
φ_3	$\lambda = 20.0$, $q = 0.5$	0.11	5.00	0
φ_3	$\lambda = 22.5, q = 1.0$	0.76	4.17	0
φ_4	$\lambda = 11.5$	0.42	4.25	0
φ_5	$\lambda = 24.5$, $\alpha = 3.7$	0.79	4.15	0
φ_6	$\lambda = 22.5$, $\alpha = 2.7$	0.89	4.21	0

To measure the sparsity of the solution, the average of zero coefficients is reported in Table 5.3, in which the column labeled "Correct" presents the average restricted only to the true zero coefficients, and the column labeled "Incorrect" depicts the average of coefficients erroneously set to 0. To measure the zero element numerically, denote the estimator found by our algorithm as $\bar{x}$ and the true solution as x^*. For any component $\bar{x}_i \leq 10\bar{\mu}$, $i \in \{1, \dots, n\}$, if the component $x_i^* = 0$, we say this component is correctly recognized as zero and add one to the value of "C"; otherwise, we say this component is incorrectly recognized as zero and add one to the value of "IC". Furthermore, define the model error $ME(\bar{x})$ by $ME(\bar{x}) = (\bar{x} - x^*)^T \Sigma (\bar{x} - x^*)$, where Σ denotes the variance/covariance matrix of the regressors. Let x_{LS} be the least squares estimator, and then the relative model error (RME) of $\bar{x}$ is

$$RME(\bar{x}) = \frac{(\bar{x} - x^*)^T \Sigma (\bar{x} - x^*)}{(x_{LS} - x^*)^T \Sigma (x_{LS} - x^*)}.$$

Obviously, for two different estimators, when the values of "C" and "IC" are the same, the one with the smaller value of RME is better. So to reflect the quality of the solution found by our algorithm, in Table 5.3 we report the median RME (MRME) [15] for each 100 randomly generated datasets at the same time. The results in Table 5.3 show that our algorithm can find sparse solutions with small model error, and the quality of the solutions is improved with the increasing of dataset size or the decreasing of noise level.

Example 5.3 (logistic regression). In this example, we want to test our algorithm with a general regression function $\theta(x)$ instead of the linear least square term $\|Ax - b\|^2$ in the objective function (1.1). We use a data model given by Fan and Li in [15]: 100 datasets is simulated consisting of 200 observations from the model $b \sim$

TABLE 5.4
Results of logistic regression.

Method	Param.	MRME (%)	Avg. no. of 0 coefficients	
			Correct	Incorrect
φ_1	$\lambda = 11.0,\ \alpha = 1.0$	0.02	4.97	0.02
φ_2	$\lambda = 10.5,\ \alpha = 1.0$	0.03	4.94	0.00
φ_3	$\lambda = 7.5,\ \ q = 0.5$	0.01	5.00	0.04
φ_3	$\lambda = 21.5, q = 1.0$	0.47	4.91	0.03
φ_4	$\lambda = 9.5$	0.50	4.90	0.04
φ_5	$\lambda = 21.0,\ \alpha = 3.7$	0.45	4.84	0.01
φ_6	$\lambda = 20.0,\ \alpha = 2.7$	0.42	4.94	0.00

Beroulli$\{p(a^T x)\}$, where $p(u) = \exp(u)/(1 + \exp(u))$. The first six components of a and x are the same as those in Example 5.2. The last two components of a are independently identically distributed as a Bernoulli distribution with probability of success 0.5. Based on the above distribution of data, the logistic regression model is

$$\theta(x) = \sum_{i=1}^{n} \ln \frac{e^{-b_i(x^T a_i)}}{1 + e^{x^T a_i}}.$$

Following the experiment setting in [15], we set the sample size n and the noise level σ as 200 and 1, respectively. The values of "C", "IC", and "MRME" based on the average of 100 runs are listed in Table 5.4, from which we can see that for a general function $\theta(\cdot)$, Algorithm 1 can also find sparse solutions with small prediction error.

6. Conclusion. In this paper, we give affine-scaled second order necessary and sufficient conditions for local minimizers of a special class of non-Lipschitz optimization problems and propose a smoothing trust region Newton method for solving such problems. Global convergence results of our algorithm indicate that our method can find a point satisfying the affine-scaled second order necessary optimality condition from any starting point. We also present numerical results which demonstrate the effectiveness of our method.

The condition

$$\|\nabla \tilde{f}(x_k, \mu_k)\| \le \zeta \mu_k \tag{6.1}$$

in Step 3 of Algorithm 1 plays an important role for the convergence theorems and numerical experiments. Condition (6.1) and stopping condition (5.1) are very strict for smoothing algorithms if we choose a very small initial smoothing parameter μ_0. Note that (6.1) is not a stopping condition for our trust region Newton method. We use (6.1) as an updating condition to monitor when the smoothing parameter should be updated. In contrast with other smoothing algorithms that solve a smoothing problem with a fixed smoothing parameter, the trust region Newton method updates the smoothing parameter in the iterations to find a stationary point of the original problem. The reasons that we can use (5.1) and (6.1) in our algorithm and numerical experiment might be that we use the trust region Newton method for the smoothing problem (as the trust region Newton method generally can find a very high accurate approximation solution) and update the smoothing parameter with an adapted scheme and a good initial μ_0. In a practical implementation of the trust region Newton method for large scale problems, the stopping condition (5.1) can be modified. For example, the algorithm is terminated when either condition (5.1) holds or a certain number of iterations is reached.

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