Investigation of Nonlinear Vehicle Suspension Systems Using a Frequency Domain Method

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Abstract
In this study, the frequency domain analysis method is introduced for the analysis and design of nonlinear vehicle suspension systems. The explicit relationship between system output spectrum and model parameters is derived. By using optimization method, the optimal stiffness and damping of the nonlinear vehicle suspension system can therefore be achieved. Comparison studies indicate that the nonlinear vehicle suspension systems demonstrate obviously advantageous dynamics performance than the corresponding linear counterparts over all frequency bands. The optimal damping characteristic is not only a velocity-dependent function but also dependent on displacement.

Key words: Frequency domain analysis, Vehicle suspension, Nonlinear system

1. Introduction

In vehicle suspension systems, an important characteristic of any spring is the fact that it can store energy. However, most springs, such as leaf spring, coil spring and rubber spring, cannot release energy in a desired way and thus wouldn't make sure the life of springs [1]. Therefore, damper is introduced in the suspension system. It can damp out the vertical motion and increase the spring's life. For most vehicle suspension dynamics analysis and calculations, the determination of damping coefficient is based on the assumption that the damping force is proportional to the damper piston velocity [1-2]. However, for some real vehicle suspension systems, the damping characteristic often reflects some nonlinear characteristic. For example, the damping characteristic curve for a suspension damper is a piecewise linear curve in [1]. Therefore, it is necessary to investigate a vehicle suspension system's nonlinearity.

There are various analysis methods especially for the nonlinear systems, such as the harmonic balance method and averaging method. In this study, the frequency domain analysis method based on Volterra series expansion will be introduced [3-9]. This method is to determine the generalized frequency response functions (GFRFs) for nonlinear system. By using this method, the system output spectrum polynomial can be derived to analyze the effect of nonlinear parameters on the output spectrum. The main advantage of this method is that it can explain how the system output frequency response is affected by the system nonlinear parameters. By using the numerical determination approach, the system output frequency response can be derived easily. This new method makes the nonlinear system analysis much more straightforward.

In this study, the key objective is to employ the existing knowledge of the vehicle suspension system together with nonlinear frequency domain analysis method to investigate how to suppress the vehicle suspension vibration. The paper will be structured as follows. Section 2 will give the system model of present study. In section 3, the frequency domain analysis method will be introduced for the determination of the system output spectrum and in section 4, the nonlinear optimal value will be obtained and some comparisons and discussions will be present in this part. Section 5 will give the conclusions.

2. System model

In the present study, the quarter-car (one degree of freedom) model was used to analyze vehicle suspension systems as shown in Fig. 1. In this figure, $m_s$represents the about quarter mass of the vehicle body. The vertical displacement of sprung mass is x_2 and the base excitation displacement is x_1 . The system has two nonlinear components, which are the nonlinear spring and nonlinear damper. The nonlinear spring force can be described as $F_k = kx + c_4x^3$and the nonlinear damping force can be written as $F_c = c\dot{x} + c_1\dot{x}^3 + c_2\dot{x}^2x + c_3\dot{x}x^2$.In this study, k is the stiffness, c is the damping, $c_{1,}c_{2,}c_{3,}c_4$ are the nonlinear coefficients and x is the relative displacement between sprung mass and the base excitation which can be defined as

$$x = x_2 - x_1 \tag{1}$$

Then the governing equation for this suspension system can be written as

$$m_s\ddot{x}_2 + F_k + F_c = 0 \tag{2}$$

Substitutex, F_k, F_c, Eq. (2) can be written as

$$m_s\ddot{x} + kx + c\dot{x} + c_1\dot{x}^3 + c_2\dot{x}^2x + c_3\dot{x}x^2 + c_4x^3 = -m_s\ddot{x}_1 \tag{3}$$

Assuming that the base excitation is considered to be a sinusoidal function which can be written as

$$x_1 = \overline{Y}\sin(\omega t) \tag{4}$$

Where ω is the frequency and $\overline{Y}$ is the magnitude of the base motion.

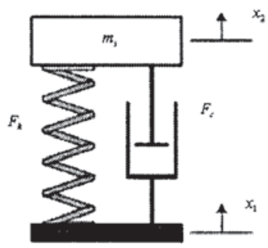

Fig. 1 A base excited suspension model

In order to conduct the analysis which is not specific to particular choices of system initial parameters, such as the sprung mass and the spring stiffness, the non-dimensional form of the governing equation can be derived as

$$\ddot{y}(\tau)+y(\tau)+\xi\dot{y}(\tau)+\xi_1\dot{y}(\tau)^3+\xi_2\dot{y}(\tau)^2y(\tau)+\xi_3\dot{y}(\tau)y(\tau)^2+\xi_4y(\tau)^3=\overline{Y}\sin(\Omega\tau) \tag{5}$$

where,

$$\begin{aligned}
&\tau = \omega_0 t, \omega_0 = \sqrt{k / m_s}, \Omega = \frac{\omega}{\omega_0}, \\
&Y = m_s\omega^2, z(\tau) = x(t) = x\left(\frac{\tau}{\omega_0}\right), y(\tau) = \frac{kz(\tau)}{Y}, \\
&\xi = \frac{c}{\sqrt{km_s}}, \xi_1 = \frac{c_1Y^2}{\sqrt{(km_s)^3}}, \xi_2 = \frac{c_2Y^2}{k^2m_s}, \xi_3 = \frac{c_3Y^2}{\sqrt{k^5m_s}}, \xi_4 = \frac{c_4Y^2}{k^3}
\end{aligned} \tag{6}$$

In this study, let the transmit force be the output and it can be written as

$$F=y(\tau)+\xi y(\tau)+\xi_1\dot{y}(\tau)^3+\xi_2\dot{y}(\tau)^2y(\tau)+\xi_3\dot{y}(\tau)y(\tau)^2+\xi_4 y(\tau)^3 \tag{7}$$

Therefore, the vehicle suspension system and the system output can be described as a non-dimensional model as

$$\begin{cases}\ddot{y}(\tau)+y(\tau)+\xi\dot{y}(\tau)+\xi_1\dot{y}(\tau)^3+\xi_2\dot{y}(\tau)^2y(\tau)+\xi_3\dot{y}(\tau)y(\tau)^2+\xi_4 y(\tau)^3=\bar{Y}\sin(\Omega\tau)\\ F=y(\tau)+\xi y(\tau)+\xi_1\dot{y}(\tau)^3+\xi_2\dot{y}(\tau)^2y(\tau)+\xi_3\dot{y}(\tau)y(\tau)^2+\xi_4 y(\tau)^3\end{cases} \tag{8}$$

3. Determination of the output spectrum

A brief review about the theory of nonlinear output spectrum is given firstly and then an efficient numerical approach is discussed for the determination and optimization of nonlinear output spectrum in terms of system parameters.

3.1 Nonlinear output spectrum: the theory

For nonlinear systems, the output $f(t)$ can be expressed by a Volterra functional polynomial of the input $u(t)$ [10] as

$$f(t)=\sum_{n=1}^{N} f_n(t) \tag{9}$$

where N is the maximum order of the system nonlinearity, and then the nth-order output of the system is given by

$$f_n(t)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h_n(\tau_{1,}\ldots\tau_n)\prod_{i=1}^{n}u(t-\tau_i)d\tau_i \tag{10}$$

where $h_n(\tau_1\ldots\tau_n)$ is the real function of $\tau_1\ldots\tau_n$ and is defined as the 'nth-order kernel' or 'nth-order impulse response' of the system. Then the system 'nth-order transfer function' (GFRF) can be derived by using the multi-dimensional Fourier transform of the nth-order impulse response, which can be written as

$$H_n(j\omega_1,\cdots j\omega_n)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h_n(\tau_1,\cdots\tau_n)\exp(-j(\omega_1\tau_1+\cdots+\omega_n\tau_n))d\tau_1\cdots d\tau_n \tag{11}$$

And when the system is subjected to a input such that

$$u(t)=\sum_{i=1}^{K}|A_i|\cos(\omega_i+\angle A_i) \tag{12}$$

According to [4], the system output spectrum can be written as

$$F(j\omega)=\sum_{n=1}^{N}\frac{1}{2^n}\sum_{\omega_1+\ldots+\omega_n=\omega}H_n(j\omega_1,\cdots,j\omega_n)A(\omega_{k_1})\cdots A(\omega_{k_n}) \tag{13}$$

where

$$A(\omega_{k_i})=|A_{k_i}|e^{j\angle A_{|k_i|}\cdot\mathrm{sgn}(k_i)}\text{ for }k_i\in\{\pm1,\cdots,\pm\bar{K}\},$$

$$\mathrm{sgn}(a)=\begin{cases}1 & a\geq 0\\ -1 & a<0\end{cases},\ \omega_{k_i}\in\{\pm\omega_1,\cdots,\pm\omega_{\bar{K}}\}$$

Therefore, the nonlinear system output spectrum can be obtained according to the above and the detailed steps can be found in [3].

3.2 Numerical determination of characteristic output spectrum function

In this part, a more efficient numerical approach will be introduced, which allows nonlinear output spectrum to be determined up to any high orders. Then the system output spectrum can be written into a more explicit polynomial form as follows [7, 8]

$$F(j\omega) = \sum_{n=1}^{N} CE(H_n(.))\varphi_n(j\omega)^T \tag{14}$$

where

$$\varphi_n(j\omega) = \frac{1}{\sqrt{N}(2\pi)^{n-1}} \int_{\omega_1+\cdots+\omega_n=\omega} f_n(j\omega_1,\cdots,j\omega_n)\prod_{i=1}^{n} U(j\omega_i)d\sigma_\omega \tag{15}$$

$CE(.)$ is a coefficient extraction operator which has two fundamental operation '$\oplus$' and'$\otimes$'. The detailed definitions were given in [5], and $CE(H_n(j\omega))$ is the parametric characteristics of the nth-order GFRF $H_n(.)$, which can be written as

$$CE(H_n(\cdot)) = C_{0,n} \oplus \left(\bigoplus_{q=1}^{n-1} \bigoplus_{p=1}^{n-q} C_{p,q} \otimes CE(H_{n-q-p+1}(.)) \right) \oplus \left(\bigoplus_{p=2}^{n} C_{p,0} \otimes CE(H_{n-p+1}(.)) \right) \tag{16}$$

Obviously, Eq. (14) can be written as

$$F(j\omega) = \psi\phi(j\omega)^T \tag{17}$$

where

$$\psi = \bigoplus_{n=1}^{N} CE(H_n(.)), \phi(j\omega) = [\phi_1(j\omega), \phi_2(j\omega), \cdots, \phi_n(j\omega)] \tag{18}$$

Therefore, the OFRF of the system with respect to nonlinear parameters $\xi_1, \xi_2, \xi_3, \xi_4$ can be obtained according to Eq. (17)

4. Optimization of nonlinear suspension systems

4.1 Computation of the nonlinear output spectrum

With the nonlinear output spectrum of the vehicle suspension system derived above, parameter optimization can then be conducted to find the optimal nonlinear parameters $\xi_1, \xi_2, \xi_3, \xi_4$. To understand the nonlinear output spectrum with respect to any nonlinear parameters of interest in the system, consider the output spectrum with respect to only two parameters ξ_1, ξ_2 which are defined as $\xi_1 \in [0,10], \xi_2 \in [0,10]$. Following the numerical approach above, it can be eventually obtained. The frequency Ω can be chosen at any values, for example here $\Omega = 1\text{rad/sec}$, the initial damping value is $\xi = 0.01$ and the magnitude of the system input is $\overline{Y} = 0.2$. Then the output spectrum can be obtained as

$$\begin{aligned} F(j\Omega)\big|_{\Omega=1} &= (1.5599\text{e}-001+1.6160\text{e}-001\text{i})+(-2.4599\text{e}-002+5.7895\text{e}-004\text{i})*\xi_1 \\ &+(4.8410\text{e}-002-3.1037\text{e}-002\text{i})*\xi_2+(2.5315\text{e}-003-2.1792\text{e}-003\text{i})*\xi_1^2 \\ &+(-1.2208\text{e}-002+1.0914\text{e}-002\text{i})*\xi_1*\xi_2+(-3.4105\text{e}-003-1.5597\text{e}-003\text{i})*\xi_2^2 \\ &+(1.5519\text{e}-004-9.5445\text{e}-005\text{i})*\xi_1^3+(2.9229\text{e}-004-6.2336\text{e}-005\text{i})*\xi_1^2*\xi_2 \\ &+(2.0404\text{e}-003-1.8717\text{e}-003\text{i})*\xi_1*\xi_2^2+(-5.2480\text{e}-004+1.0759\text{e}-003\text{i})*\xi_2^3 \\ &+(-2.4643\text{e}-005+2.5870\text{e}-005\text{i})*\xi_1^4+(4.5495\text{e}-005-6.8368\text{e}-005\text{i})*\xi_1^3*\xi_2 \\ &+(-1.1712\text{e}-004+1.1884\text{e}-004\text{i})*\xi_1^2*\xi_2^2+(-4.6285\text{e}-005+4.7699\text{e}-005\text{i})*\xi_1*\xi_2^3 \\ &+(4.5823\text{e}-005-7.5763\text{e}-005\text{i})*\xi_2^4 \end{aligned} \tag{19}$$

Eq. (19) can be plotted in a three dimension figure as

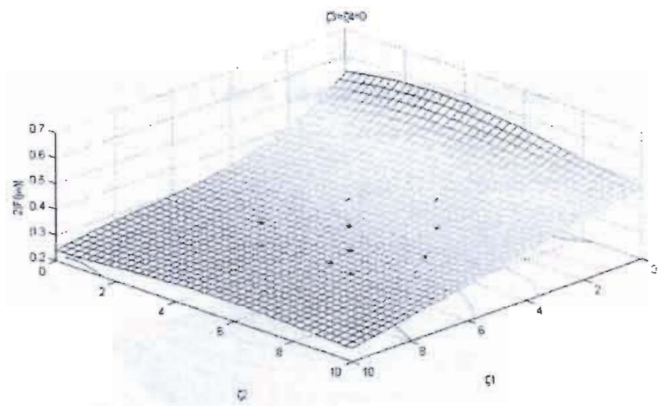

Fig. 2 Nonlinear output spectrum with respect to ξ_1, ξ_2 ($\xi_3 = \xi_4 = 0$the stars were obtained by the theoretical computation)

From Fig. 2, it can be seen that the output spectrum is typical nonlinear function of nonlinear parametersξ_1, ξ_2. The polynomial provides a straightforward and powerful insight into the analytical relationship between system output response and system parameters.

4.2 Parameter optimization

According to Eq. (17), the polynomial with respect to all the nonlinear parameters can be derived. To this end, the following points should be noted.

a. Matrix ψ should cover a large range to make sure the optimal results can be found as mentioned above. However, usually when the range of the nonlinear parameter is too large, the matrix inverse in would be ill-conditioned. To solve this problem, the range of the variables can also be divided into several parts. For example, the range of [0,100] can be divided into[0,1], [1,10]and[10,100]. Using this subsection method, together with the scaling method discussed in [7], the matrix will be easy to be non-singular and the accuracy of the solution can be guaranteed.

b. The optimal solution should be the optimal one within all the sub-ranges.

Moreover, two new variables a, b can be introduced which are defined as

$$\xi_1 = a, \xi_2 = 3ab, \xi_3 = 3ab^2, \xi_4 = ab^3 \tag{20}$$

Therefore, Eq. (7) can be written as

$$F = y(\tau) + \xi\dot{y}(\tau) + a[\dot{y}(\tau) + by(\tau)]^3 \tag{21}$$

Then the system output spectrum is a function with respect to two variables a, b and can be obtained. In this study, the range of a, b is [0,10] and [−1,0] respectively. The system output frequency response with respect to nonlinear variables was obtained in this study.

$$\begin{aligned}
F(j\Omega)\big|_{\Omega=1} &= (1.2297\text{e-}001 + 2.6101\text{e-}001i) + (1.4043\text{e-}002 - 2.1368\text{e-}002i)*a \\
&+ (3.1627\text{e-}002 + 3.7759\text{e-}003i)*3ab + (-7.5741\text{e-}003 + 2.3285\text{e-}002i)*3ab^2 \\
&+ (4.8602\text{e-}003 + 1.7763\text{e-}001i)*ab^3 + (-1.8905\text{e-}003 + 1.9025\text{e-}003i)*a^2 \\
&+ (-2.8603\text{e-}003 + 9.4444\text{e-}004i)*3a^2b + (6.9260\text{e-}005 - 8.4367\text{e-}004i)*3a^2b^2 \\
&+ (-1.2002\text{e-}002 + 1.3644\text{e-}002i)*a^2b^3 + (1.5270\text{e-}003 - 9.8513\text{e-}004i)*9a^2b^2 \\
&+ (1.6447\text{e-}003 - 2.7494\text{e-}004i)*9a^2b^3 + (-2.2671\text{e-}003 + 2.4648\text{e-}002i)*3a^2b^4 \\
&+ (5.7254\text{e-}003 - 3.0388\text{e-}003i)*9a^2b^4 + (3.5859\text{e-}003 + 2.4658\text{e-}002i)*3a^2b^5 \\
&+ (-7.3119\text{e-}002 + 2.0120\text{e-}001i)*a^2b^6
\end{aligned} \tag{22}$$

Note that the parameter here is dimensionless. In this study, the optimal value for the non-dimensional system with initial parameters $\bar{Y} = 0.2$ and $\xi = 0.01$ can be derived as

$$a = 3.89, b = -0.77, 2|F(j\omega)| = 0.251 \quad (23)$$

And the system output magnitude with respect to a and b will be given in Fig. 3.

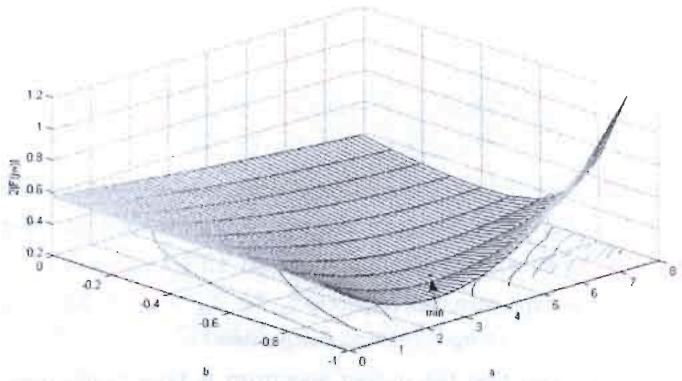

Fig. 3 Nonlinear output spectrum with respect to *a*, *b*

4.3 Comparison with linear systems

For pure linear system, the spring force is $F_k = kx$ and the damper force is $F_c = c\dot{x}$. Then the non-dimensional linear system governing equation can be written as

$$\ddot{y}(\tau) + \xi\dot{y}(\tau) + y(\tau) = \overline{Y}\sin(\Omega\tau) \quad (24)$$

The definitions of $\Omega, \tau, \xi, y(\tau)$ can be found in Eq. (6). Similarly, let the transmit force be the output and the system transfer function can be written

$$H(j\Omega) = \frac{\xi(j\Omega)+1}{(j\Omega)^2 + \xi(j\Omega)+1} \quad (25)$$

According to the [2], the magnitude of the transfer function is

$$|H(j\Omega)| = \sqrt{\frac{1+(\xi\Omega)^2}{(1-\Omega^2)^2 + (\xi\Omega)^2}} \quad (26)$$

From Eq. (26), it can be shown that when the input frequency $\Omega = 1\text{rad/sec}$, the transfer function is a decreasing function and the minimum is 1 when the linear damping ξ is as large as possible. However, the recommended linear vehicle suspension damping ratio usually gets the value of 0.25[2]. Therefore, according to the definition in Eq. (6), in the present study the damping value will be $\xi = 0.5$. In this study, the recommended linear damping will be used to compare with the nonlinear optimal damping. To be noted that, the system transmissibility for the three different systems are used to compare the effectiveness. According to [9], the system transmissibility can be defined as the system output spectrum divided by input spectrum. Then the system transmissibility for the system with respect to different system values can be shown in table1. In table1, linear 1, linear 2, nonlinear represent the initial linear system, recommended linear system and nonlinear optimal system, respectively.

Table 1 the system transmissibility

Parameters	Linear 1	Linear 2	Nonlinear
ξ	0.01	0.5	0.01
a	0	0	3.89
b	0	0	-0.77
$2\|F(j\Omega)\|/\overline{Y}$	100.005	2.2361	1.2550

In order to verify whether the nonlinear optimal value has a positive effect on the system output spectrum in the whole frequency range, the system transmissibility for the three different systems was obtained and can be shown in Fig. 4

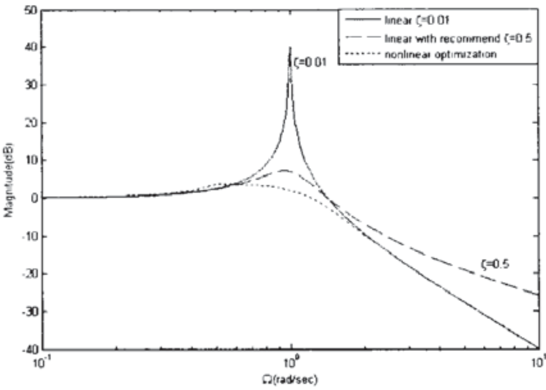

Fig. 4 The force transmissibility for different systems

By further investigating Fig.4, here are three conclusions:

1. It is obvious that when the nonlinear parameters get an optimal value, the system transmissibility at the frequency of $\Omega = 1$is excellent than the initial linear system. In Fig.4, the peak at the frequency of $\Omega = 1$means the maximum transmissibility. It can be shown that the nonlinear optimal system can get the smallest transmissibility at the frequency of$\Omega = 1$, which means that when the system under the same input signal, the system output spectrum with the nonlinear optimal value will get the lowest output spectrum. Therefore, it can be concluded that the system vibration is well suppressed at the system resonant frequency when the nonlinear system get the optimal value.

2. From Fig.4, it is also clearly that the optimal nonlinear value will be helpful in the high frequency range. According to [2], in the high frequency range the system damping value should be as small as possible in order to suppress vibration. Fig. 4 provides information that the system transmissibility obtained by the nonlinear optimal value can match very well with the curve obtained by the initial damping value. For the case curve obtained by$\xi = 0.5$, it can suppress vibration in the low frequency range. However, it can increase the system vibration significantly in the high frequency range. Therefore, it can be concluded that in the high frequency range, the nonlinear optimal value is much more competitive than the recommended linear value.

3. In the low frequency, the nonlinear optimal value cannot minimize the force transmissibility. From Fig. 4, it can be shown that the system transmissibility obtained by the nonlinear optimal value reached a small peak at the frequency of$\Omega = 0.5$. This is because that in the present study, the nonlinear stiffness was introduced in this study. In this study the system stiffness was decreased and therefore the resonance frequency of the nonlinear system will be changed. However, this will be helpful for the suspension design as the vehicle suspension system resonance frequency will be changed by introducing the nonlinear terms, which can be used to design the suspensions system in order to avoid some important frequency points, such as the human sensitive frequency or the engine resonant frequency.

In this study, the area which was combined by the three output spectrum curves, $x = 0.1, x = 10$and $y = 0$ can be calculated to show the effect of optimal nonlinearity in the whole frequency range. The detailed steps of calculating the output spectrums can be seen in [1, 2]. To be noted that the unit of ylabel is N and the unit of xlabel is rad/sec, so in this study the unit of the area here is $N.rad/sec$. The areas of different systems can be seen in table 2.

Table 2 the area of different systems

system	area($N.rad/sec$)
linear ζ=0.01	1.3091
linear with recommend ζ=0.5	0.71116
nonlinear optimal system	0.47281

From table 2 it can be show that though optimal system will get a larger transmissibility in the frequency of $\Omega = 0.5$, the area obtained by the nonlinear system is the smallest one in the above three different systems. Therefore, the optimal nonlinear system is much more competitive than the linear system as it can successfully suppress vibration than the linear system in the full frequency range.

4.4 The nonlinear damping characteristics

Based on the above discussion, the system damping force can be obtained in this study. By further analyzing Eq. (7), the nonlinear damping force with respect to the displacement and velocity can be written as

$$F_d = 0.01\dot{y} + 3.89\dot{y}^3 - 8.9589\dot{y}^2 y + 6.9161\dot{y}y^2 \tag{27}$$

And the nonlinear spring force with respect to the displacement and velocity can be written as

$$F_s = y - 1.7759y^3 - 8.9589\dot{y}^2 y + 6.9161\dot{y}y^2 \tag{28}$$

In this study, the range of velocity $\dot{y}$ is chosen to be $[-1.5,1.5]$ and the range of displacement y is chosen to be $[-0.04,0.04]$, the non-dimensional units of displacement and velocity are m^2/s^2 and m^2/s^3. To be noted that both of these are the non-dimensional ranges, the real range can be obtained when given a specific system model parameters. In this study, the damping force was taken as an example to further investigate and it can be shown in Fig. 5.

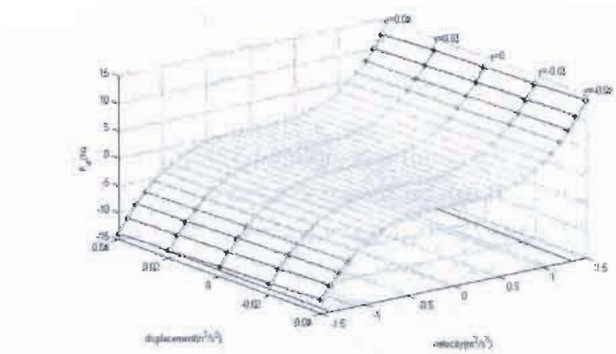

Fig. 5 suspension damping force

Fig. 5 is the suspension damping characteristic under different displacements and velocities. It can be concluded that in this study, the damping force is not only a function of velocity but also affected by the displacement. From Fig.5, it can be shown that when the relative displacement gets different value, the curve of the damping force will change. In some real systems, for example, the viscoelastic damper is not only affected by the piston velocity but also exits a complex nonlinear function between the damping force and the deformation [11]. In this study, the damping characteristic obtained by frequency domain method can reflect the detailed relationship between the force and velocity-displacement. Similarly, Eq. (28) shows that the velocity also has an influence on the spring force. For different values of velocity, the spring versus displacement curve is different. Therefore, the damping force and spring force obtained in this study is more realistic and can demonstrate the nature of suspension system.

5. Conclusion

A recently-developed frequency domain method for the analysis and design of nonlinear systems is applied in this study to the analysis and optimal design of nonlinear damping and stiffness of vehicle suspension systems. Nonlinear characteristic output spectrum function was derived to reveal the effect of nonlinear model parameters on system output vibration response. The optimal solution could be obtained, which was used to compare with corresponding linear systems. The results indicate that system vibration suppression performance can be much better with optimal nonlinear damping and stiffness characteristics than with only linear counterparts.

Future work will focus on two degree of freedom suspension systems, considering the nonlinear effects incurred by wheel tires.

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