Dynamic asset-liability management in a Markov regime switching market with stochastic cash flow☆

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Abstract

In this paper we incorporate stochastic cash flow in the wealth dynamic process, and investigate an asset-liability management problem in a Markov regime switching market under multi-period mean-variance framework. The stochastic cash flow can be explained as capital additions or withdrawals during the investment process; for example, insurers may receive insurance premium and need paying for claim; pension fund may get contributions or issue distributions. In our model, the returns of assets and liabilities and the amounts of cash flows all depend on the stochastic market state which is assumed to follow a discrete-time Markov chain. By adopting the dynamic programming approach, the matrix theory and Lagrange dual principle, closed-form expressions for the efficient investment strategy and the mean-variance efficient frontier are derived.

Keywords: Stochastic cash flow; Asset-liability management; Multi-period mean-variance model; Markov regime switching; Efficient investment strategy

1. Introduction

Since introduced by Markowitz (1952), the mean–variance (M-V) portfolio selection problem has become one of the key research topics in finance. In M-V criteria, investors aim to determine the optimal investment strategies which minimizes the

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risk measured by the variance of the terminal wealth for a predetermined expected return of the terminal wealth or maximizes the expected return for a given risk level. In recent years, Li and Ng (2000) and Zhou and Li (2000) extend the Markowitz's M-V model to dynamic discrete-time and continuoustime settings, respectively; they derived the analytical optimal solutions by using an embedding technique The past ten years have witnessed numerous extensions of the M-V portfolio analysis in dynamic settings, see for example, Zhu et al. (2004), Bielecki et al. (2005); Xia and Yan (2006), Xiong and Zhou (2007), Basak and Chabakauri (2010), Fu et al. (2010), Chiu and Wong (2011), Cui et al (2014a, 2014b), and Yi et al. (2014).

Along another line, it is well known that asset-liability management (ALM) is essential for the success of many financial institutions, such as pension funds, insurance companies and banks. In ALM, the main concern is the surplus which is the net wealth (i.e., the asset value minuses the liability value). Accordingly, ALM is also known as surplus management. ALM problem has been receiving more and more attention, and the past decade has witnessed the increasing research on the ALM problem based on M-V criterion. Sharpe and Tint (1990) is the first one to study the ALM problem under Markowitz's M–V framework. Using M–V criterion, Leippold et al. (2004) study a multi-period ALM problem, where the liabilities are exogenous and uncontrollable. Chiu and Li (2006) investigate continuous-time M–V ALM problems, where the liability process is described by a geometric Brownian motion. Yi et al. (2008) extend the work of Leippold et al. (2004) to the cases of uncertain exit time. Leippold et al. (2011) and Yao et al. (2013a) consider the M–V portfolio selection problems with endogenous liabilities in multi-period and continuous-time settings, respectively, where both assets and liabilities are simultaneously portfolio optimized. Chiu and Wong (2012, 2013) investigate the continuous-time M–V ALM problem with cointegrated assets.

The works mentioned above all suppose that there is only one state of market mode. However, in the real world, the market might have a finite number of market states, such as "bullish" and "bearish" in the stock market, and could switch among them. The market state reflects the state of the underlying economy, the mood of investors, and other economic environments. Recently, there has been a growing interest of using Markov regime switching models in portfolio selection and ALM problems where the number of the market states is assumed to be finite and their transition follows a Markov chain. Çkmak and Öekici (2006) study a multi-period M–V portfolio selection problem in a Markov regime switching market. Costa and Araujo (2008) establish a more general multi-period M–V portfolio selection model with regime switching, where the intermediate variance and expectation of the portfolio are incorporated in the model. Chen and Yang (2011) considered the multi-period M–V ALM problem with Markor regime switching. Yao et al. (2013b) investigate an uncertain exit time multi-period M-V portfolio selection problem with endogenous liabilities in a Markov regime switching market. For more detailed discussion on the subject of regime switching, one is referred to Chen et al. (2008), Costa and de Oliveira (2012), and Wu and Zeng (2013), among others

However, there still exists a gap between the academic research and practice. The above mentioned literature does not consider the cash flows of investors (including individual investors and investment institutions). In real-world, investors might face the situations of capital injections or withdrawals during their investment processes. For example, households may need cash to maintain their lives or add their residual income into the investment; insurers can receive insurance premium and need paying for claim; pension fund may get contributions or issue distributions for their pension fund members; and in most case, the cash flows are stochastic. Hence, many investors, like households, insures, pension funds and banks, need taking into account their stochastic cash flows during their investments and ALM processes. In recently years, there are some authors consider the dynamic portfolio selection problem with uncontrolled cash flow. Under continuous-time expected utility maximization model, Munk and Sørensen (2010) consider an optimal asset allocation problem with stochastic income and interest rates. Wu and Li (2012) consider a multi-period M-V portfolio selection problem with regime switching and a stochastic cash flow. Using continuous-time M-V framework, Wu and Zeng (2013) study the optimal portfolio selection in a Lévy market with uncontrolled cash flows. Wu (2013) investigates an asset allocation problem with a stochastic cash flow in a Markov-switching jump–diffusion market. However, they don't consider the liability management at the same time. More Specifically, they only study the portfolio selection problem with cash flow, but they don't consider the surplus management (ALM) problem with cash flow. Recently, Yao et al. (2013c) study a multi-period M-V ALM problem with uncontrolled cash flow. But there is only one market state in the model, and the market state is deterministic.

As far as we know, there is no literature on dynamic M-V ALM in a Markov regime switching market with stochastic cash flow. In this paper, we will incorporate Markov regime switching and stochastic cash flow into a multi-period M–V ALM model, and derive the optimal investment strategy and the M-V efficient frontier. It is also different from Yao et al. (2013c) where the uncontrollable cash flow is putted into the liability process so that the cash flow can only affect the investment amount at the terminal time, we put in this paper the stochastic cash flow directly into the wealth process at each period. Our reason is stated as follows: in the real world, the investors (e.g., households and insurers) have income or expenditure, insurance premium or claim, at each period, not just at the terminal time. Namely, the income or payout can greatly affect the wealth level during the investment process. Hence it seems more reasonable to put stochastic cash flow in the wealth process rather than in the liability process.

From the mathematical point of view, consideration of a stochastic cash flow and Markov regime switching would makes the problem harder to be solved. In our model, there are three state variables, i.e., wealth, liability and market state, putting stochastic cash flow into the wealth dynamic process further increase the computational complexity in obtaining closed form solutions to the Bellman equation which comes from the dynamic programming approach. We will synthetically adopt the dynamic programming approach, the matrix theory and Lagrange dual principle to solve the multi-period M–V ALM problem

This paper proceeds as following. In Section 2, our multi-period M–V ALM problem and primary notations are described. In Section 3, the original problem is translated into a standard unconstrained stochastic optimal control problem by introducing a Lagrange multiplier, and the corresponding analytical solution is obtained. Section 4 is concerned with the explicit expressions of the optimal strategy and the efficient frontier for the original problem, respectively. Section 5 concludes this paper.

2. Model formulation

Suppose that there are m states in the stochastic market environment, and let $\Theta = \{1, 2, \cdots, m\}$ denote the market state set. The market state at time $k(k = 0, 1, \cdots, T)$ is denoted by ς_k, $\varsigma_k \in \Theta$. Assume that state process $\{\varsigma_k;\ k = 0, 1, \cdots, T\}$ is a time-varying Markov chain with regime space Θ. Throughout this paper, let $(\Omega, \mathcal{F}, \mathbf{P})$ be a fixed complete probability space. The Markov chain has transition probability matrix $\Pi(k) = (\pi_{ij}(k))_{m\times m}$, where $\pi_{ij}(k) = \Pr ob(\varsigma_{k+1} = j|\varsigma_k = i) \geq 0$ satisfying $\sum_{j=1}^{m} \pi_{ij}(k) = 1$, $i = 1,\ 2,\ \cdots,\ m$.

Suppose that the market consists of one risk-free asset and N risky assets whose returns depend on the state ς_k of the market. Denote by $s_k(\varsigma_k)$ (> 0) the deterministic return of the risk-free asset and by $e_k(\varsigma_k) = (e_k^1(\varsigma_k), e_k^2(\varsigma_k), \cdots, e_k^n(\varsigma_k))'$ the random return vector of the risky assets over period $k(k = 0, 1, \cdots, T-1)$ for given market state ς_k. Here A' represents the transpose of a matrix or a vector A. An investor, equipped with initial wealth x_0 and initial liability l_0, enters the market at time 0, and makes investments within T period. He (she) not only need to consider the investment strategy, but also consider the liability management. Let l_k be the liability value of the investor at the beginning of the kth period, Following Leippold et al. (2004), Ye et al. (2008) and Chen and Yang (2011), the liability is uncontrollable, and follows the dynamic process

$$l_{k+1} = q_k(\varsigma_k) l_k \ , \tag{1}$$

where $q_k(\varsigma_k)$ is an exogenous random variable with its probability distributions depending on the market state ς_k. $q_k(\varsigma_k)$ can be understood as the random growth rate of the liability. We suppose that $q_k(\varsigma_k) > 0$ almost surely for all $k = 0, 1, \cdots, T-1$ and $\varsigma_k \in \Theta$. Let x_k and u_k^z $(z = 1, 2, \cdots, N)$ denote the value of wealth he holds and the amount invested in the zth risky asset at the beginning of the time period k, respectively. Then the amount invested in the risk-free asset is $x_k - \sum_{z=1}^{N} u_k^z$. We know that in real-world there would be a cash inflow or outflow during the investment process. For example, the insurers need to pay for claims; the pension fund members have to contribute or distribute during their accumulation or de-cumulation phase; and the individual investors have stochastic incomes and expenditures. Denote by $c_k(\varsigma_k)$ the stochastic cash flow for the investor over period k on market state ς_k. Then, incorporating the stochastic cash flow, the wealth dynamics can be written as (see We and Li (2012))

$$x_{k+1} = x_k s_k(\varsigma_k) + P_k'(\varsigma_k) u_k + c_k(\varsigma_k), \tag{2}$$

where

$$\begin{cases} P_k(\varsigma_k) = (e_k^1(\varsigma_k) - s_k(\varsigma_k), e_k^2(\varsigma_k) - s_k(\varsigma_k), \cdots, e_k^N(\varsigma_k) - s_k(\varsigma_k))' = e_k(\varsigma_k) - s_k \mathbf{1}_N, \\ u_k = (u_k^1, u_k^2, \cdots, u_k^N)', \ \mathbf{1}_N = (1, 1, \ \cdots, 1)' \in \mathrm{R}^N. \end{cases} \tag{3}$$

Let $\mathrm{E}[\cdot]$ and $\mathrm{Var}[\cdot]$ represent the expectation operator and variance operator, respectively. Denoted by $\mathrm{E}^{-1}[\cdot]$ and $\mathrm{Var}^{-1}[\cdot]$ the inverse matrices of $\mathrm{E}[\cdot]$ and $\mathrm{Var}[\cdot]$ respectively. Similar to Leippold et al. (2004, 2011), Ye et al. (2008), Chen and Yang (2011) and Yao et al. (2013b), this paper has the following assumptions.

Assumption 1. *The covariance matrix* $\mathrm{Var}[P_k(i)]$ *is positive definite for all* $i \in \Pi$ *and* $k = 0, 1, \cdots, T-1$.

Assumption 2. *Random sequence* $\Upsilon_k = (\varsigma_k, P_k'(\varsigma_k), q_k(\varsigma_k), c_k(\varsigma_k))$ *are statistically independent for* $k = 0, 1, \cdots, T-1$.

Assumption 3. $\mathrm{E}[P_k(i)] \neq \mathbf{0}_N$, *where* $\mathbf{0}_N$ *is a* N*-dimension zero vector, for* $i \in \Pi$ *and* $k = 0, 1, \cdots, T-1$.

Remark 1. Noting that $\mathrm{E}[P_k(i)P_k'(i)] = \mathrm{Var}[P_k(i)] + \mathrm{E}[P_k(i)]\mathrm{E}[P_k'(i)]$, Assumption 1 implies that $\mathrm{E}[P_k(i)P_k'(i)]$ is also positive definite. Assumption 2 is often used in the literature about multi-period M-V portfolio selection, see, Chen and Yang (2011), Costa and de Oliveira (2012) and Yao et al (2013), for example. Assumption 3 means that not all assets have zero risk premium.

Let $\wp_k$ be a family of filters, denoting the information available to the investor up to time k. We assume that the investor can observe the present asset value and the regime of the market state directly. Thus $\wp_k := \sigma\{(x_s, l_s, \varsigma_s)|0 \leq s \leq k\}$ is a σ-field. An investment strategy $u = \{u_k; k = 0, 1, \cdots, T-1\}$ is admissible if u_k is adapted to $\wp_k$. Denote by Γ_k the collection of all admissible investment strategy at the beginning of period k.

The multi-period M-V ALM problems is to find out the optimal investment strategy to minimize the risk of the final surplus, defined as $S_T = x_T - l_T$, under the condition that expected final surplus is given as d, where the risk is measured by variance. Specifically, the multi-period ALM model under the M-V framework can be formulated as follows

$$\begin{cases} \min\limits_{u \in \Gamma_0} \{\mathrm{Var}[S_T|\wp_0] = \mathrm{E}[S_T^2|\wp_0] - d^2\} \\ s.t. \quad \mathrm{E}[S_T|\wp_0] = d, (1) - (2), \end{cases} \tag{4}$$

where $\mathrm{E}[\cdot|\wp_0]$ and $\mathrm{Var}[\cdot|\wp_0]$ represent the expectation and variance operators conditional on $\wp_0$ The optimal solution $u^* = \{u_k^*; k = 0, 1, \cdots, T-1\}$ of Problem (3), is called the **efficient investment strategy**. The point $(\mathrm{Var}[S_T],\ d)$ in coordinate plane Variance-Mean correspond to an efficient strategy is called an efficient point. The collection of all efficient points is defined as the **efficient frontier**.

3. Transformation and solution to the problem

It is well known that the equality constraint $\mathrm{E}[S_T] = d$ in optimal control problem (4) can be dealt with by introducing a Lagrange multiplier μ. We can turn to solve the following unconstrained optimal control problem parameterized by μ

$$\begin{cases} \min\limits_{u \in \Gamma_0} \mathrm{E}[S_T^2|\wp_0] - d^2 + 2\mu(\mathrm{E}[S_T|\wp_0] - d) \\ s.t. \quad (1) - (2). \end{cases} \tag{5}$$

Since

$$\begin{aligned}\mathrm{E}[S_T^2|\wp_0] - d^2 + 2\mu(\mathrm{E}[S_T|\wp_0] - d) &= \mathrm{E}[(x_T - l_T)^2 + 2\mu(x_T - l_T)|\wp_0] - d^2 - 2\mu d \\ &= \mathrm{E}[x_T^2 + l_T^2 - 2x_T l_T + 2\mu x_T - 2\mu l_T|\wp_0] - d^2 - 2\mu d\end{aligned}$$

Therefore, the optimal control problem (5) is equivalent to

$$\begin{cases} \min\limits_{u\in\Gamma_0} \mathrm{E}[x_T^2 + l_T^2 - 2x_T l_T + 2\mu x_T - 2\mu l_T|\wp_0] \\ s.t. \quad (1)-(2) \end{cases} \tag{6}$$

In the following, we solve optimal control problem (6) by using dynamic programming approach.

Let $V_k(x_k, l_k, \varsigma_k)$ denote the optimal value function associated with the optimal problem (6) starting from time k with state: wealth x_k and liability. i.e.,

$$\begin{cases} V_k(x_k, l_k, \varsigma_k) = \min\limits_{u\in\Gamma_k} \mathrm{E}\left[x_T^2 + l_T^2 - 2x_T l_T + 2\mu x_T - 2\mu l_T |\, \wp_k\right] \\ s.t. \quad (1)-(2). \end{cases} \tag{7}$$

Noting that $\wp_k = \sigma\left\{(x_s, l_s, \varsigma_s)|\, 0 \le s \le t\right\}$ and by the Markov property, we have $\mathrm{E}\left[\cdot\,|\wp_k\right] = \mathrm{E}\left[\cdot\,|(x_k, l_k, \varsigma_k)\right]$. Then it follows that

$$\begin{cases} V_k(x_k, l_k, \varsigma_k) = \min\limits_{u\in\Gamma_k} \mathrm{E}\left[x_T^2 + l_T^2 - 2x_T l_T + 2\mu x_T - 2\mu l_T |\, (x_k, l_k, \varsigma_k)\right] \\ s.t. \quad (1)-(2). \end{cases} \tag{8}$$

Obviously, setting $t = 0$, then $V_0(x_0, l_0, \varsigma_0)$ and $(V_0(x_0, l_0, \varsigma_0) - d^2 - 2\mu d)$ are the optimal values of Problem (6) and (5), respectively.

For later use, for any $i \in \Pi$ and $k = 0, 1, \cdots, T-1$, let

$$\begin{cases} A_k(i) = \mathrm{E}[P_k'(i)]\mathrm{E}^{-1}[P_k(i)P_k'(i)]\mathrm{E}[P_k(i)], \\ B_k(i) = \mathrm{E}[q_k(i)P_k'(i)]\mathrm{E}^{-1}[P_k(i)P_k'(i)]\mathrm{E}[q_k(i)P_k(i)], \\ O_k(i) = \mathrm{E}[P_k'(i)]\mathrm{E}^{-1}[P_k(i)P_k'(i)]\mathrm{E}[q_k(i)P_k(i)], \\ J_k(i) = \mathrm{E}[q_k(i)P_k'(i)]\mathrm{E}^{-1}[P_k(i)P_k'(i)]\mathrm{E}[c_k(i)P_k(i)], \\ D_k(i) = \mathrm{E}[c_k(i)] - \mathrm{E}[P_k'(i)]\mathrm{E}^{-1}[P_k(i)P_k'(i)]\mathrm{E}[c_k(i)P_k(i)], \\ M_k(i) = \mathrm{E}[c_k^2(i)] - \mathrm{E}[c_k(i)P_k'(i)]\mathrm{E}^{-1}[P_k(i)P_k'(i)]\mathrm{E}[c_k(i)P_k(i)]. \end{cases} \tag{9}$$

Proposition 1. *For any $i \in \Pi$ and $k = 0, 1, \cdots, T$, we have $0 < A_k(i) < 1$.*

PROOF. For any $i \in \Pi$ and $k = 0, 1, \cdots, T$, according to Remark 1, $\mathrm{E}[P_{T-1}(i)P_{T-1}'(i)]$ is positive definite under Assumption 1, then so is $\mathrm{E}^{-1}[P_{T-1}(i)P_{T-1}'(i)]$. By Assumption 3, $\mathrm{E}[P_k(i)] \neq \mathbf{0}_N$. Then $A_k(i) = \mathrm{E}[P_k'(i)]\mathrm{E}^{-1}[P_k(i)P_k'(i)]\mathrm{E}[P_k(i)] > 0$.

In the following, we prove $A_k(i) < 1$. To this end, let $Q_k(i) = \mathrm{E}[P'_k(i)]\mathrm{Var}^{-1}[P(i)_k]\mathrm{E}[P_k(i)]$. By Assumptions 1 and 3, $\mathrm{Var}^{-1}[P_k(i)]$ is positive definite and $\mathrm{E}[P_k(i)] \neq \mathbf{0}_N$, then $Q_k(i) > 0$. Noting that $\mathrm{E}[P_k(i)]\mathrm{E}[P'_k(i)] = \mathrm{E}[P_k(i)P'_k(i)] - \mathrm{Var}[P_k(i)]$, we obtain

$$
\begin{aligned}
&A_k(i)Q_k(i)\\
=\,&\mathrm{E}[P'_k(i)]\mathrm{E}^{-1}[P_k(i)P'_k(i)]\mathrm{E}[P_k(i)]\mathrm{E}[P'_k(i)]\mathrm{Var}^{-1}[P_k(i)]\mathrm{E}[P_k(i)]\\
=\,&\mathrm{E}[P'_k(i)]\mathrm{E}^{-1}[P_k(i)P'_k(i)](\mathrm{E}[P_k(i)P'_k(i)] - \mathrm{Var}[P_k(i)])\mathrm{Var}^{-1}[P_k(i)]\mathrm{E}[P_k(i)]\\
=\,&\mathrm{E}[P'_k](i)\mathrm{Var}^{-1}[P_k(i)]\mathrm{E}[P_k(i)] - \mathrm{E}[P'(i)_k]\mathrm{E}^{-1}[P_k(i)P'_k(i)]\mathrm{E}[P_k(i)]\\
=\,&Q_k(i) - A_k(i)
\end{aligned}
$$

Therefore, it follows that $A_k(i) = \frac{Q_k(i)}{1+Q_k(i)} < 1$. This completes the proof of the proposition. □

To obtain the explicit expression of $V_k(x_k, l_k, \varsigma_k)$, for any $i \in \Pi$ and $k = 0, 1, \cdots, T$, we construct series $w_k(i)$, $\lambda_k(i)$, $n_k(i)$, $\gamma_k(i)$, $h_k(i)$, $\phi_k(i)$, $f_k(i)$,$g_k(i)$, $\psi_k(i)$ and $\theta_k(i)$ satisfying the following recurrence relations and boundary conditions.

$$w_k(i) = \overline{w_{k+1}(i)}\,(1 - A_k(i))\, s_k^2(i),\ w_T(i) = 2, \tag{10}$$

$$\lambda_k(i) = \overline{\lambda_{k+1}(i)}\,(\mathrm{E}[q_k(i)] - O_k(i))\, s_k(i),\ \lambda_T(i) = -2, \tag{11}$$

$$n_k(i) = \overline{n_{k+1}(i)}\,(1 - A_k(i))\, s_k(i),\ n_T(i) = 2, \tag{12}$$

$$\gamma_k(i) = \overline{\gamma_{k+1}(i)}\mathrm{E}[q_k^2(i)] - \frac{1}{2}\frac{(\overline{\lambda_{k+1}(i)})^2}{\overline{w_{k+1}(i)}}B_k(i),\ \gamma_T(i) = 1, \tag{13}$$

$$f_k(i) = \overline{f_{k+1}(i)}\mathrm{E}[q_k(i)] - \frac{\overline{\lambda_{k+1}(i)}\,\overline{n_{k+1}(i)}}{\overline{w_{k+1}(i)}}O_k(i),\ f_T(i) = -2, \tag{14}$$

$$\phi_k(i) = \phi_{k+1}(i) - \frac{1}{2}\frac{(\overline{n_{k+1}(i)})^2}{\overline{w_{k+1}(i)}}A_k(i),\ \phi_T(i) = 0, \tag{15}$$

$$h_k(i) = \overline{h_{k+1}(i)}\,(1 - A_k(i))\, s_k(i) + \overline{w_{k+1}(i)}D_k(i)s_k(i),\ h_T(i) = 0, \tag{16}$$

$$g_k(i) = \overline{g_{k+1}(i)}\mathrm{E}[q_k(i)] + \overline{\lambda_{k+1}(i)}\left(\mathrm{E}[c_k(i)q_k(i)] - J_k(i) - \frac{\overline{h_{k+1}(i)}}{\overline{w_{k+1}(i)}}O_k\right),\ g_T(i) = 0, \tag{17}$$

$$\psi_k(i) = \overline{\psi_{k+1}(i)} + \frac{1}{2}\overline{w_{k+1}(i)}M_k(i) + \overline{h_{k+1}(i)}D_k(i) - \frac{(\overline{h_{k+1}(i)})^2}{2\overline{w_{k+1}(i)}}A_k(i),\ \psi_T(i) = 0, \tag{18}$$

$$\theta_k(i) = \overline{\theta_{k+1}(i)} + \overline{n_{k+1}(i)}D_k(i) - \frac{\overline{h_{k+1}(i)}\,\overline{n_{k+1}(i)}}{\overline{w_{k+1}(i)}}A_k(i),\ \theta_T(i) = 0. \tag{19}$$

where

$$\begin{cases} \overline{w_{k+1}(i)} = \sum_{j=1}^{m} w_{k+1}(j)\pi_{ij}(k), & \overline{\lambda_{k+1}(i)} = \sum_{j=1}^{m} \lambda_{k+1}(j)\pi_{ij}(k) \\ \overline{n_{k+1}(i)} = \sum_{j=1}^{m} n_{k+1}(j)\pi_{ij}(k), & \overline{\gamma_{k+1}(i)} = \sum_{j=1}^{m} \gamma_{k+1}(j)\pi_{ij}(k), \\ \overline{h_{k+1}(i)} = \sum_{j=1}^{m} h_{k+1}(j)\pi_{ij}(k), & \overline{\phi_{k+1}(i)} = \sum_{j=1}^{m} \phi_{k+1}(j)\pi_{ij}(k), \\ \overline{f_{k+1}(i)} = \sum_{j=1}^{m} f_{k+1}(j)\pi_{ij}(k), & \overline{\psi_{k+1}(i)} = \sum_{j=1}^{m} \psi_{k+1}(j)\pi_{ij}(k), \\ \overline{\gamma_{k+1}(i)} = \sum_{j=1}^{m} \gamma_{k+1}(j)\pi_{ij}(k), & \overline{g_{k+1}(i)} = \sum_{j=1}^{m} g_{k+1}(j)\pi_{ij}(k). \end{cases} \tag{20}$$

Proposition 2. *For any $i \in \Pi$ and $k = 0, 1, \cdots, T$, we have $w_k(i) > 0$.*

Proof. We prove the proposition by mathematical induction. For $k = T$, according to (10), we have $w_T(i) = 2 > 0$ for all $i \in \Pi$.

Now, we prove $w_k(i) > 0$ for any $i \in \Pi$, under the assumption that $w_{k+1}(i) > 0$ for any $i \in \Pi$. Since $\pi_{ij}(k) \geq 0$ for $i, j \in \Pi$, $\sum_{j=1}^{S} \pi_{ij}(k) = 1$ for $i \in \Pi$. Then, we have $\overline{w_{k+1}(i)} = \sum_{j=1}^{S} w_{k+1}(j)\pi_{ij}(k) > 0$ for any $i \in \Pi$. By Proposition 1, it follows that $(1 - A_k(i))\, s_k^2(i) > 0$ for any $i \in \Pi$. Therefore, according to (10), for all $i \in \Pi$, we have

$$w_k(i) = \overline{w_{k+1}(i)}\,(1 - A_k(i))\, s_k^2(i) > 0.$$

By the Principle of Mathematical Induction, the proposition is proved. □

Theorem 1. *For simplicity, let $x = x_k$, $y = l_k$ and $i = \varsigma_k$. Then for $k = 0, 1, \cdots, T$ and $i \in \Pi$, the optimal value function of Problem (6) is given by*

$$\begin{aligned} V_k(x, l, i) = {} & \tfrac{1}{2} w_k(i)x^2 + \lambda_k(i)xl + \gamma_k(i)l^2 + h_k(i)x + g_k(i)l \\ & + \mu n_k(i)x + f_k(i)\mu l + \psi_k(i) + \theta_k(i)\mu + \phi_k(i)\mu^2, \end{aligned} \tag{21}$$

where $w_k(i)$, $\lambda_k(i)$, $n_k(i)$, $\gamma_k(i)$, $h_k(i)$, $\phi_k(i)$, $f_k(i)$,$g_k(i)$, $\psi_k(i)$ and $\theta_k(i)$ are defined by (10)-(19).

Proof. Following Costa and Araujo (2008), Chen and Yang (2011) and Yao et al. (2013b), by the dynamic programming principle, the optimal value function of Problem (6), $V_k(x_k, l_k, \varsigma_k)$, satisfies the following Bellman equation

$$\begin{cases} \quad V_k(x_k, l_k, \varsigma_k) \\ = \min\limits_{u_k} \mathrm{E}\left[V_{k+1}(x_{k+1}, l_{k+1}, \varsigma_{k+1}) \middle| (x_k, l_k, \varsigma_k)\right] \\ = \min\limits_{u_k} \sum_{j=1}^{S} \mathrm{E}\left[V_{k+1}(x_k s_k(\varsigma_k) + P_k'(\varsigma_k)u_k + c_k(\varsigma_k), q_k(\varsigma_k)l_k, \varsigma_{k+1}) | \varsigma_{k+1} = j\right] \Pr ob(\varsigma_{k+1} = j | \varsigma_k) \\ = \min\limits_{u_k} \sum_{j=1}^{m} \mathrm{E}\left[V_{k+1}\left(x_k s_k(\varsigma_k) + P_k'(\varsigma_k)u_k + c_k(\varsigma_k), q_k(\varsigma_k)l_k, j\right)\right]\pi_{\varsigma_k j}(k), \\ V_T(x_T, l_T, \varsigma_T) = x_T^2 + l_T^2 - 2x_T l_T + 2\mu x_T - 2\mu l_T. \end{cases} \tag{22}$$

We prove this theorem by mathematical induction on k.

For $k = T$, on one hand, by the boundary conditions of (10)-(19), we have

$$\begin{aligned}&\tfrac{1}{2}w_T(i)x^2 + \lambda_T(i)xl + \gamma_T(i)l^2 + h_T(i)x + g_T(i)l\\&+\mu n_T(i)x + f_T(i)\mu l + \psi_T(i) + \theta_T(i)\mu + \phi_T(i)\mu^2\\&= x^2 + l^2 - 2xl + 2\mu x - 2\mu l.\end{aligned}$$

On the other hand, it is known from the boundary condition of Bellman equation (8) that $V_T(x, l, i) = x^2 + l^2 - 2xl + 2\mu x - 2\mu l$. Therefore, (21) holds for $k = T$.

Now suppose that (21) holds for $k + 1$, i.e.,

$$\begin{aligned}V_{k+1}(x, l, i) &= \tfrac{1}{2}w_{k+1}(i)x^2 + \lambda_{k+1}(i)xl + \gamma_{k+1}(i)l^2 + h_{k+1}(i)x + g_{k+1}(i)l\\&\quad+\mu n_{k+1}(i)x + f_{k+1}(i)\mu l + \psi_{k+1}(i) + \theta_{k+1}(i)\mu + \phi_{k+1}(i)\mu^2\\&= \tfrac{1}{2}w_{k+1}(i)x^2 + \lambda_{k+1}(i)xl + \gamma_{k+1}(i)l^2 + \left(h_{k+1}(i) + \mu n_{k+1}(i)\right)x\\&\quad+\left(g_{k+1}(i) + \mu f_{k+1}(i)\right)l + \psi_{k+1}(i) + \theta_{k+1}(i)\mu + \phi_{k+1}(i)\mu^2\end{aligned} \tag{23}$$

Then for k, According to the Bellman equation (22) and noting that $x = x_k$, $y = l_k$ and $i = \varsigma_k$, we obtain

$$\begin{aligned}&V_k(x, l, i) = \min_{u_k} \sum_{j=1}^{m} \mathrm{E}\left[V_{k+1}\left(xs_k(i) + P_k'(i)u_k + c_k(i), q_k(i)l, j\right)\right]\pi_{ij}(k)\\&= \min_{u_k} E\Big\{\tfrac{1}{2}\overline{w_{k+1}(i)}(xs_k(i) + P_k'(i)u_k + c_k(i))^2 + \overline{\lambda_{k+1}(i)}(xs_k(i) + P_k'(i)u_k + c_k(i))\\&\quad\times q_k(i)l + \overline{\gamma_{k+1}(i)}q_k^2(i)l^2 + \left(\overline{h_{k+1}(i)} + \mu\overline{n_{k+1}(i)}\right)(xs_k(i) + P_k'(i)u_k + c_k(i))\\&\quad+\left(\overline{g_{k+1}(i)} + \mu\overline{f_{k+1}(i)}\right)q_k(i)l + \overline{\psi_{k+1}(i)} + \overline{\theta_{k+1}(i)}\mu + \overline{\phi_{k+1}(i)}\mu^2\Big\}\\&= \tfrac{1}{2}\overline{w_{k+1}(i)}s_k^2(i)x^2 + \tfrac{1}{2}\overline{w_{k+1}(i)}\mathrm{E}[c_k^2(i)] + \overline{w_{k+1}(i)}xs_k(i)\mathrm{E}[c_k(i)] + \overline{\lambda_{k+1}(i)}\mathrm{E}[q_k(i)]s_k(i)xl\\&\quad+\overline{\lambda_{k+1}(i)}\mathrm{E}[c_k(i)q_k(i)]l + \overline{\gamma_{k+1}(i)}\mathrm{E}[q_k^2(i)]l^2 + \left(\overline{h_{k+1}(i)} + \mu\overline{n_{k+1}(i)}\right)(s_k(i)x + \mathrm{E}[c_k(i)])\\&\quad+\left(\overline{g_{k+1}(i)} + \mu\overline{f_{k+1}(i)}\right)\mathrm{E}[q_k(i)]l + \overline{\psi_{k+1}(i)} + \overline{\theta_{k+1}(i)}\mu + \overline{\phi_{k+1}(i)}\mu^2\\&\quad+\min_{u_k}\Big\{\tfrac{1}{2}\overline{w_{k+1}(i)}u_k'\mathrm{E}[P_k(i)P_k'(i)]u_k + \Big(\overline{w_{k+1}(i)}\mathrm{E}[P_k'(i)]s_k(i)x\\&\quad+\overline{\lambda_{k+1}(i)}\mathrm{E}[q_k(i)P_k'(i)]l + \overline{h_{k+1}(i)}\mathrm{E}[P_k'(i)] + \overline{w_{k+1}(i)}\mathrm{E}[c_k(i)P_k'(i)]\Big)u_k\Big\}.\end{aligned} \tag{24}$$

According Proposition 2, it follows that $w_{k+1}(j) > 0$ for all $j \in \Pi$ then $\overline{w_{k+1}(i)} = \sum_{j=1}^{m} w_{k+1}(j)\pi_{ij}(k) > 0$. In addition, under Assumption 1, we have that $\mathrm{E}\left[P_k(i)P_k'(i)\right]$ is positive definite. Therefore, the first order condition (also is sufficient condition) about u_k in (24) gives the optimal strategy

$$\begin{aligned}u_k^* = -\mathrm{E}^{-1}[P_k(i)P_k'(i)]\Big(&\mathrm{E}[P_k(i)]s_k(i)x + \tfrac{\overline{\lambda_{k+1}(i)}}{\overline{w_{k+1}(i)}}\mathrm{E}[q_k(i)P_k(i)]l\\&+\tfrac{\overline{h_{k+1}(i)}+\mu\overline{n_{k+1}(i)}}{\overline{w_{k+1}(i)}}\mathrm{E}[P_k(i)] + \mathrm{E}[c_k(i)P_k(i)]\Big).\end{aligned} \tag{25}$$

Substituted above formula (25) back into (24), it follows that

$$
\begin{aligned}
&V_k(x,l,i) = \tfrac{1}{2}\overline{w_{k+1}(i)}s_k^2(i)x^2 + \tfrac{1}{2}\overline{w_{k+1}(i)}\mathrm{E}[c_k^2(i)] + \overline{w_{k+1}(i)}xs_k(i)\mathrm{E}[c_k(i)] \\
&+\overline{\lambda_{k+1}(i)}\mathrm{E}[q_k(i)]s_k(i)xl + \overline{\lambda_{k+1}(i)}\mathrm{E}[c_k(i)q_k(i)]l + \overline{\gamma_{k+1}(i)}\mathrm{E}[q_k^2(i)]l^2 \\
&+\left(\overline{h_{k+1}(i)} + \mu\overline{n_{k+1}(i)}\right)(s_k(i)x + \mathrm{E}[c_k(i)]) + \left(\overline{g_{k+1}(i)} + \mu\overline{f_{k+1}(i)}\right)\mathrm{E}[q_k(i)]l \\
&+\overline{\psi_{k+1}(i)} + \overline{\theta_{k+1}(i)}\mu + \overline{\phi_{k+1}(i)}\mu^2 - \tfrac{1}{2}\left(\overline{w_{k+1}(i)}\mathrm{E}[P_k'(i)]s_k(i)x + \overline{\lambda_{k+1}(i)}\mathrm{E}[q_k(i)P_k'(i)]l\right. \\
&\left.+\left(\overline{h_{k+1}(i)} + \mu\overline{n_{k+1}(i)}\right)\mathrm{E}[P_k'(i)] + \overline{w_{k+1}(i)}\mathrm{E}[c_k(i)P_k'(i)]\right)\mathrm{E}^{-1}[P_k(i)P_k'(i)] \\
&\left(\mathrm{E}[P_k(i)]s_k(i)x + \tfrac{\overline{\lambda_{k+1}(i)}}{\overline{w_{k+1}(i)}}\mathrm{E}[q_k(i)P_k(i)]l + \tfrac{\overline{h_{k+1}(i)}+\mu\overline{n_{k+1}(i)}}{\overline{w_{k+1}(i)}}\mathrm{E}[P_k(i)] + \mathrm{E}[c_k(i)P_k(i)]\right)
\end{aligned}
$$

Collecting the similar items and taking as the form of a polynomial of x, l and μ, we obtain

$$
\begin{aligned}
&V_k(x,l,i) \\
&= \tfrac{1}{2}\overline{w_{k+1}(i)}\,(1 - A_k(i))\,s_k^2(i)x^2 + \overline{\lambda_{k+1}(i)}\,(\mathrm{E}[q_k(i)] - O_k(i))\,s_k(i)xl \\
&+\left(\overline{\gamma_{k+1}(i)}\mathrm{E}[q_k^2] - \tfrac{1}{2}\tfrac{(\overline{\lambda_{k+1}(i)})^2}{\overline{w_{k+1}(i)}}B_k(i)\right)l^2 + \left(\left(\overline{g_{k+1}(i)} + \mu\overline{f_{k+1}(i)}\right)\mathrm{E}[q_k(i)]\right. \\
&\left.+\overline{\lambda_{k+1}(i)}\left(\mathrm{E}[c_k(i)q_k(i)] - J_k(i) - \tfrac{\left(\overline{h_{k+1}(i)}+\mu\overline{n_{k+1}(i)}\right)}{\overline{w_{k+1}(i)}}O_k\right)\right)l \\
&+\left(\left(\overline{h_{k+1}(i)} + \mu\overline{n_{k+1}(i)}\right)(1 - A_k(i)) + \overline{w_{k+1}(i)}D_k(i)\right)s_k(i)x \\
&-\tfrac{1}{2}\tfrac{\left(\overline{h_{k+1}(i)}+\mu\overline{n_{k+1}(i)}\right)^2}{\overline{w_{k+1}(i)}}A_k(i) + \overline{\psi_{k+1}(i)} + \overline{\theta_{k+1}(i)}\mu + \overline{\phi_{k+1}(i)}\mu^2 \\
&+\tfrac{1}{2}\overline{w_{k+1}(i)}M_k(i) + \left(\overline{h_{k+1}(i)} + \mu\overline{n_{k+1}(i)}\right)D_k(i)
\end{aligned}
$$

where $A_k(i),\ B_k(i), O_k(i), J_k(i), D_k(i)$ and $M_k(i)$ are defined by (20). After some calculations, the above formula further gives

$$
\begin{aligned}
&V_k(x,l,i) = \tfrac{1}{2}\overline{w_{k+1}(i)}\,(1 - A_k(i))\,s_k^2(i)x^2 + \overline{\lambda_{k+1}(i)}\,(\mathrm{E}[q_k(i)] - O_k(i))\,s_k(i)xl \\
&+\left(\overline{\gamma_{k+1}(i)}\mathrm{E}[q_k^2(i)] - \tfrac{1}{2}\tfrac{(\overline{\lambda_{k+1}(i)})^2}{\overline{w_{k+1}(i)}}B_k(i)\right)l^2 + \left(\overline{h_{k+1}(i)}\,(1 - A_k(i))\,s_k(i)\right. \\
&\left.+\overline{w_{k+1}(i)}D_k(i)s_k(i)\right)x + \overline{n_{k+1}(i)}\,(1 - A_k(i))\,s_k(i)\mu x + \left(\overline{\phi_{k+1}(i)} - \tfrac{1}{2}\tfrac{(\overline{n_{k+1}(i)})^2}{\overline{w_{k+1}(i)}}A_k(i)\right)\mu^2 \\
&+\left(\overline{g_{k+1}(i)}\mathrm{E}[q_k(i)] + \overline{\lambda_{k+1}(i)}\left(\mathrm{E}[c_k(i)q_k(i)] - J_k(i) - \tfrac{\overline{h_{k+1}(i)}}{\overline{w_{k+1}(i)}}O_k(i)\right)\right)l \\
&+\left(\overline{f_{k+1}(i)}\mathrm{E}[q_k] - \tfrac{\overline{\lambda_{k+1}(i)n_{k+1}(i)}}{\overline{w_{k+1}(i)}}O_k(i)\right)\mu l + \overline{\psi_{k+1}(i)} + \tfrac{1}{2}\overline{w_{k+1}(i)}M_k(i) \\
&+\overline{h_{k+1}(i)}D_k(i) - \tfrac{(\overline{h_{k+1}(i)})^2}{2\overline{w_{k+1}(i)}}A_k + \left(\overline{\theta_{k+1}(i)} + \overline{n_{k+1}(i)}D_k(i) - \tfrac{\overline{h_{k+1}(i)n_{k+1}(i)}}{\overline{w_{k+1}(i)}}A_k(i)\right)\mu
\end{aligned}
$$

Therefore, according to (10)-(19), we have

$$
\begin{aligned}
V_k(x,l,i) = {} & \tfrac{1}{2}w_k(i)x^2 + \lambda_k(i)xl + \gamma_k(i)l^2 + h_k(i)x + g_k(i)l \\
& +\mu n_k(i)x + f_k(i)\mu l + \psi_k(i) + \theta_k(i)\mu + \phi_k(i)\mu^2.
\end{aligned}
$$

This means that (21) also holds for k.

By Principle of Mathematical Induction, (21) holds for all $k = 0, 1, \cdots, T$. Namely, the theorem is proved. □

According to the proof of Theorem 1, we have the following theorem.

Theorem 2. *The optimal strategy for Problem (6) is given by (25).*

In the following, we investigate closed form computational formulas for series $w_k(i)$, $\lambda_k(i)$, $n_k(i)$, $\gamma_k(i)$, $h_k(i)$, $\phi_k(i)$, $f_k(i)$,$g_k(i)$, $\psi_k(i)$ and $\theta_k(i)$. We first derive the expression for $w_k(i)$, $\lambda_k(i)$ and $n_k(i)$. To this end, let

$$\begin{cases} \vec{w}_k = (w_k(1),\ w_k(2), \cdots,\ w_k(m))',\ \vec{\lambda}_k = (\lambda_k(1),\ \lambda_k(2), \cdots,\ \lambda_k(m))', \\ \vec{n}_k = (n_k(1),\ n_k(2), \cdots,\ n_k(m))', \vec{A}_k = (A_k(1), A_k(2), \cdots, A_k(m)), \\ \vec{s}_k = (s_k(1),\ s_k(2), \cdots,\ s_k(m))', \vec{O}_k = (O_k(1), O_k(2), \cdots, O_k(m)), \\ \vec{q}_k^E = (\mathrm{E}[q_k(1)], \mathrm{E}[q_k(2)], \cdots, \mathrm{E}[q_k(m)])'. \end{cases} \tag{26}$$

Then, (10)-(12) can be reformulated as

$$\vec{w}_k = \mathrm{diag}(\mathbf{1}_m - \vec{A}_k)\mathrm{diag}^2(\vec{s}_k)\Pi(k)\vec{w}_{k+1},\ \vec{w}_T = 2\mathbf{1}_m, \tag{27}$$

$$\vec{\lambda}_k = \mathrm{diag}(\vec{q}_k^E - \vec{O}_k)\mathrm{diag}(\vec{s}_k)\Pi(k)\vec{\lambda}_{k+1},\ \vec{\lambda}_T = -2\mathbf{1}_m, \tag{28}$$

$$\vec{n}_k = \mathrm{diag}(\mathbf{1}_m - \vec{A}_k)\mathrm{diag}(\vec{s}_k)\Pi(k)\vec{n}_{k+1},\ \vec{n}_T = 2\mathbf{1}_m, \tag{29}$$

where $\mathbf{1}_m = (1, 1, \cdots, 1)' \in \mathrm{R}^m$, $\mathrm{diag}(y_1, y_2, \cdots, y_m)$ denote a diagonal matrix of order $m \times m$ with diagonal element being $y_1, y_2, \cdots, y_m$, and $\mathrm{diag}^2(\cdot) = (\mathrm{diag}(\cdot))^2$.

For convenience, throughout this paper, we define $\prod\limits_{j=k}^{k-1}(\cdot) = 1$ and $\sum\limits_{i=k}^{k-1}(\cdot) = 0$. The following proposition gives the expression of $\vec{w}_k$, $\vec{\lambda}_k$ and $\vec{n}_k$ for $k = 0, 1, \cdots, T$.

Proposition 3. *For $k = 0, 1, \cdots, T$, we have*

$$\vec{w}_k = 2\left(\prod_{t=k}^{T-1} diag(\mathbf{1}_m - \vec{A}_t)\, diag^2(\vec{s}_t)\Pi(t)\right)\mathbf{1}_m, \tag{30}$$

$$\vec{\lambda}_k = -2\left(\prod_{t=k}^{T-1} diag(\vec{q}_t^E - \vec{O}_t)\, diag(\vec{s}_t)\Pi(t)\right)\mathbf{1}_m, \tag{31}$$

$$\vec{n}_k = 2\left(\prod_{t=k}^{T-1} diag(\mathbf{1}_m - \vec{A}_t)\, diag(\vec{s}_t)\Pi(t)\right)\mathbf{1}_m. \tag{32}$$

Proof. We only prove Formula (30), the other formulas can be proved in the similar way. Mathematical induction methods is used. For $k = T$, it is easy to verify

$$2\left(\prod_{t=T}^{T-1}\mathrm{diag}(I - \vec{A}_t)\mathrm{diag}^2(\vec{s}_t)\Pi(t)\right)\mathbf{1}_m = 2\mathbf{1}_m = \vec{w}_T$$

For $k = T-1$, It follows from (27) that

$$
\begin{aligned}
&\vec{w}_{T-1} = \mathrm{diag}(\mathbf{1}_m - \vec{A}_{T-1})\mathrm{diag}^2(\vec{s}_{T-1})\Pi(T-1)\vec{w}_T \\
&= 2\mathrm{diag}(\mathbf{1}_m - \vec{A}_{T-1})\mathrm{diag}^2(\vec{s}_{T-1})\Pi(T-1)\mathbf{1}_m = \left(\prod_{t=T-1}^{T-1} \mathrm{diag}(\mathbf{1}_m - \vec{A}_t)\mathrm{diag}^2(\vec{s}_t)\Pi(t)\right)\mathbf{1}_m.
\end{aligned}
$$

Therefore, (30) holds true for $k = T, T-1$.

Now suppose that $\vec{w}_{k+1} = 2\left(\prod_{t=k+1}^{T-1} \mathrm{diag}(\mathbf{1}_m - \vec{A}_t)\mathrm{diag}^2(\vec{s}_t)\Pi(t)\right)\mathbf{1}_m$. According to (27), we have

$$
\begin{aligned}
&\vec{w}_k = \mathrm{diag}(\mathbf{1}_m - \vec{A}_k)\mathrm{diag}^2(\vec{s}_k)\Pi(k)\vec{w}_{k+1} \\
&= \mathrm{diag}(\mathbf{1}_m - \vec{A}_k)\mathrm{diag}^2(\vec{s}_k)\Pi(k)2\left(\prod_{t=k+1}^{T-1} \mathrm{diag}(\mathbf{1}_m - \vec{A}_t)\mathrm{diag}^2(\vec{s}_t)\Pi(t)\right)\mathbf{1}_m \\
&= 2\left(\prod_{t=k}^{T-1} \mathrm{diag}(\mathbf{1}_m - \vec{A}_t)\mathrm{diag}^2(\vec{s}_t)\Pi(t)\right)\mathbf{1}_m
\end{aligned}
$$

Therefore, according to mathematical induction, $\vec{w}_k = 2\left(\prod_{t=k}^{T-1} \mathrm{diag}(\mathbf{1}_m - \vec{A}_t)\mathrm{diag}^2(\vec{s}_t)\Pi(t)\right)\mathbf{1}_m$ for all $k = 0, 1, \cdots, T$. Namely, the proposition is proved. $\square$

After obtaining the expression of $\vec{w}_k$, $\vec{\lambda}_k$ and $\vec{n}_k$, then the ith ($i \in \Pi$) component of $\vec{w}_k$, $\vec{\lambda}_k$ and $\vec{n}_k$ are $w_k(i)$, $\lambda_k(i)$ and $n_k(i)$. In the following, we further derive expressions for series $\gamma_k(i)$, $f_k(i)$, $\phi_k(i)$ and $h_k(i)$, $i \in \Pi$. Let

$$
\begin{cases}
\vec{\gamma}_k = (\gamma_k(1), \gamma_k(2), \cdots, \gamma_k(m))', \vec{f}_k = (f_k(1),\ f_k(2), \cdots,\ f_k(m))', \\
\vec{\varphi}_k = (\varphi_k(1),\ \varphi_k(2), \cdots,\ \varphi_k(m))', \vec{h}_k = (h_k(1),\ h_k(2), \cdots,\ h_k(m))', \\
\vec{\Lambda}_k = (\mathrm{E}[q_k^2(1)], \mathrm{E}[q_k^2(2)], \cdots, \mathrm{E}[q_k^2(m)])', \\
\Psi_k(i) = -\frac{1}{2}\frac{(\overline{\lambda_{k+1}(i)})^2}{\overline{w_{k+1}(i)}}B_k(i),\ i \in \Pi, \vec{\Psi}_k = (\Psi_k(1), \Psi_k(2), \cdots, \Psi_k(m))', \\
\Phi_k(i) = -\frac{\overline{\lambda_{k+1}(i)}\,\overline{n_{k+1}(i)}}{\overline{w_{k+1}(i)}}O_k(i),\ i \in \Pi, \vec{\Phi}_k = (\Phi_k(1), \Phi_k(2), \cdots, \Phi_k(m))', \\
\Theta_k(i) = -\frac{1}{2}\frac{(\overline{n_{k+1}(i)})^2}{\overline{w_{k+1}(i)}}A_k(i),\ i \in \Pi, \vec{\Theta}_k = (\Theta_k(1), \Theta_k(2), \cdots, \Theta_k(m))', \\
\vec{\Delta}_k = (\overline{w_{k+1}(1)}D_k(1)s_k(1), \overline{w_{k+1}(2)}D_k(2)s_k(2), \cdots, \overline{w_{k+1}(m)}D_k(m)s_k(m))'.
\end{cases}
\tag{33}
$$

Then formulation (13) and (16) can be reformulated as

$$
\vec{\gamma}_k = \mathrm{diag}(\vec{\Lambda}_k)\Pi(k)\vec{\gamma}_{k+1} + \vec{\Psi}_k, \vec{\gamma}_T = \mathbf{1}_m, \tag{34}
$$

$$
\vec{f}_k = \mathrm{diag}(\vec{q}_k^{E})\Pi(k)\vec{f}_{k+1} + \vec{\Phi}_k, \vec{f}_T = -2\mathbf{1}_m, \tag{35}
$$

$$
\vec{\varphi}_k = \Pi(k)\vec{\varphi}_{k+1} + \vec{\Theta}_k,\ \vec{\varphi}_T = \mathbf{0}_m, \tag{36}
$$

$$
\vec{h}_k = \mathrm{diag}(\mathbf{1}_m - \vec{A}_k)\mathrm{diag}(\vec{s}_k)\Pi(k)\vec{h}_{k+1} + \vec{\Delta}_k, \vec{h}_T = \mathbf{0}_m, \tag{37}
$$

where $\mathbf{0}_m$ is a zero vector or order m.

The following proposition gives expressions for vector series $\vec{\gamma}_k$, $\vec{f}_k$, $\vec{h}_k$ and $\vec{\phi}_k$, namely for series $\gamma_k(i)$, $f_k(i)$, $h_k(i)$ and $\varphi_k(i)$, $i \in \Pi$ and $k = 0, 1, \cdots, T$.

Proposition 4. *for $k = 0, 1, \cdots, T$, we have*

$$\begin{cases} \vec{\gamma}_k = \left(\prod_{t=k}^{T-1} \mathrm{diag}(\vec{\Lambda}_t)\Pi(t)\right) \mathbf{1}_m + \sum_{s=k}^{T-1} \left(\prod_{t=k}^{s-1} \mathrm{diag}(\vec{\Lambda}_t)\Pi(t)\right) \vec{\Psi}_s, \\ \vec{f}_k = -2 \left(\prod_{t=k}^{T-1} \mathrm{diag}(\vec{q}_t^E)\Pi(t)\right) \mathbf{1}_m + \sum_{s=k}^{T-1} \left(\prod_{t=k}^{s-1} \mathrm{diag}(\vec{\Lambda}_t)\Pi(t)\right) \vec{\Phi}_s, \\ \vec{h}_k = \sum_{s=k}^{T-1} \left(\prod_{t=k}^{s-1} \mathrm{diag}(\mathbf{1}_m - \vec{A}_t)\mathrm{diag}(\vec{s}_t)\Pi(t)\right) \vec{\Delta}_s, \vec{\phi}_k = \sum_{s=k}^{T-1} \left(\prod_{t=k}^{s-1} \Pi(t)\right) \vec{\Theta}_s. \end{cases} \tag{38}$$

Proof. we only need to prove the first formula, the other formula can be proved in the similar way. Mathematical induction methods is used to prove the proposition. For $k = T$, it is easy to verify

$$\left(\prod_{t=T}^{T-1} \mathrm{diag}(\vec{\Lambda}_t)\Pi(t)\right) \mathbf{1}_m + \sum_{s=T}^{T-1} \left(\prod_{t=T}^{s-1} \mathrm{diag}(\vec{\Lambda}_t)\Pi(t)\right) \vec{\Psi}_s = \mathbf{1}_m = \vec{\gamma}_T$$

For $k = T-1$, on one hand, we have

$$\begin{aligned} &\left(\prod_{t=T-1}^{T-1} \mathrm{diag}(\vec{\Lambda}_t)\Pi(t)\right) \mathbf{1}_m + \sum_{s=T-1}^{T-1} \left(\prod_{t=T-1}^{s-1} \mathrm{diag}(\vec{\Lambda}_t)\Pi(t)\right) \vec{\Psi}_s \\ &= \mathrm{diag}(\vec{\Lambda}_{T-1})\Pi(T-1)\mathbf{1}_m + \left(\prod_{t=T-1}^{T-2} \mathrm{diag}(\vec{\Lambda}_t)\Pi(t)\right) \vec{\Psi}_{T-1} \\ &= \mathrm{diag}(\vec{\Lambda}_{T-1})\Pi(T-1)\mathbf{1}_m + \vec{\Psi}_{T-1} \end{aligned}$$

On the other hand, by (34), it follows that

$$\vec{\gamma}_{T-1} = \mathrm{diag}(\vec{\Lambda}_{T-1})\Pi(T-1)\vec{\gamma}_T + \vec{\Psi}_{T-1} = \mathrm{diag}(\vec{\Lambda}_{T-1})\Pi(T-1)I + \vec{\Psi}_{T-1}.$$

Now suppose that $\vec{\gamma}_{k+1} = \left(\prod_{t=k+1}^{T-1} \mathrm{diag}(\vec{\Lambda}_t)\Pi(t)\right) \mathbf{1}_m + \sum_{s=k+1}^{T-1} \left(\prod_{t=k+1}^{s-1} \mathrm{diag}(\vec{\Lambda}_t)\Pi(t)\right) \vec{\Psi}_s$. According to (34), we have

$$\begin{aligned} &\vec{\gamma}_k = \mathrm{diag}(\vec{\Lambda}_k)\Pi(k)\vec{\gamma}_{k+1} + \vec{\Psi}_k \\ &= \mathrm{diag}(\vec{\Lambda}_k)\Pi(k) \left[\left(\prod_{t=k+1}^{T-1} \mathrm{diag}(\vec{\Lambda}_t)\Pi(t)\right) \mathbf{1}_m + \sum_{s=k+1}^{T-1} \left(\prod_{t=k+1}^{s-1} \mathrm{diag}(\vec{\Lambda}_t)\Pi(t)\right) \vec{\Psi}_s\right] + \vec{\Psi}_k \\ &= \left(\prod_{t=k}^{T-1} \mathrm{diag}(\vec{\Lambda}_t)\Pi(t)\right) \mathbf{1}_m + \sum_{s=k+1}^{T-1} \left(\prod_{t=k}^{s-1} \mathrm{diag}(\vec{\Lambda}_t)\Pi(t)\right) \vec{\Psi}_s + \left(\prod_{t=k}^{k-1} \mathrm{diag}(\vec{\Lambda}_t)\Pi(t)\right) \vec{\Psi}_k \\ &= \left(\prod_{t=k}^{T-1} \mathrm{diag}(\vec{\Lambda}_t)\Pi(t)\right) \mathbf{1}_m + \sum_{s=k}^{T-1} \left(\prod_{t=k}^{s-1} \mathrm{diag}(\vec{\Lambda}_t)\Pi(t)\right) \vec{\Psi}_s \end{aligned}$$

Therefore, according to mathematical induction, we have

$$\vec{\gamma}_k = \left(\prod_{t=k}^{T-1} \mathrm{diag}(\vec{\Lambda}_t)\Pi(t)\right) \mathbf{1}_m + \sum_{s=k}^{T-1} \left(\prod_{t=k}^{s-1} \mathrm{diag}(\vec{\Lambda}_t)\Pi(t)\right) \vec{\Psi}_s$$

for all $k = 0, 1, \cdots, T$. The proposition is proved. $\square$

After obtaining the expressions for $w_k(i)$, $\lambda_k(i)$, $n_k(i)$, $\gamma_k(i)$, $f_k(i)$, $\phi_k(i)$ and $h_k(i)$, finally, we derive the expressions for $g_k(i)$, $\psi_k(i)$ and $\theta_k(i)$. Let

$$\begin{cases} \vec{g}_k = (g_k(1), g_k(2), \cdots, g_k(m))', \vec{\psi}_k = (\psi_k(1), \psi_k(2), \cdots, \psi_k(m))', \\ \vec{\theta}_k = (\theta_k(1), \theta_k(2), \cdots, \theta_k(m))', G(i) = \overline{n_{k+1}(i)} D_k(i) - \frac{\overline{h_{k+1}(i)}\,\overline{n_{k+1}(i)}}{\overline{w_{k+1}(i)}} A_k(i), \\ \vec{G}_k = (G_k(1),\ G_k(2),\ \cdots,\ G_k(m))' \\ F_k(i) = \overline{\lambda_{k+1}(i)} \left(\mathrm{E}[c_k(i) q_k(i)] - J_k(i) - \frac{\overline{h_{k+1}(i)}}{\overline{w_{k+1}(i)}} O_k \right), i \in \Pi, \\ N_k(i) = \frac{1}{2} \overline{w_{k+1}(i)} M_k(i) + \overline{h_{k+1}(i)} D_k(i) - \frac{\overline{(h_{k+1}(i))^2}}{2\overline{w_{k+1}(i)}} A_k(i), i \in \Pi, \\ \vec{F}_k = (F_k(1),\ F_k(2),\ \cdots,\ F_k(m))', \vec{N}_k = (N_k(1),\ N_k(2),\ \cdots,\ N_k(m))'. \end{cases} \tag{39}$$

Then formulations (17)- (19) can be reformulated as

$$\vec{g}_k = \mathrm{diag}(\vec{q}_k^E)\Pi(k)\vec{g}_{k+1} + \vec{F}_k,\ \vec{g}_T = \mathbf{0}_m, \tag{40}$$

$$\vec{\psi}_k = \Pi(k)\vec{\psi}_{k+1} + \vec{N}_k, \vec{\psi}_T = \mathbf{0}_m, \tag{41}$$

$$\vec{\theta}_k = \Pi(k)\vec{\theta}_{k+1} + \vec{G}_k, \vec{\theta}_T = \mathbf{0}_m. \tag{42}$$

The following proposition gives the expressions for vector series $\vec{g}_k$, $\vec{\psi}_k$ and $\vec{\theta}_k$, namely for $g_k(i)$, $\psi_k(i)$ and $\theta_k(i)$, $i \in \Pi$ and $k = 0, 1, \cdots, T$.

Proposition 5. *Fork* $= 0, 1, \cdots, T$, *we have*

$$\vec{g}_k = \sum_{s=k}^{T-1} \left(\prod_{t=k}^{s-1} diag(\vec{q}_t^E)\Pi(t)\right) \vec{F}_s, \vec{\psi}_k = \sum_{s=k}^{T-1} \left(\prod_{t=k}^{s-1} \Pi(t)\right) \vec{N}_s, \vec{\theta}_k = \sum_{s=k}^{T-1} \left(\prod_{t=k}^{s-1} \Pi(t)\right) \vec{G}_s. \tag{43}$$

The proof of Proposition 5 is similar to that of Propositions 3 and 4, therefore we omit its proof.

4. Efficient investment strategy and efficient frontier

It is known from the previous analysis in Section 3 that the optimal value of Problem (5) is

$$U(x_0, l_0, \varsigma_0, \mu) = V_0(x_0, l_0, \varsigma_0) - d^2 - 2\mu d. \tag{44}$$

By Theorem 1, it follows that

$$
\begin{aligned}
&U(x_0, l_0, \varsigma_0, \mu) = V_0(x_0, l_0, \varsigma_0) - d^2 - 2\mu d \\
&= \tfrac{1}{2} w_0(\varsigma_0) x_0^2 + \lambda_0(\varsigma_0) x_0 l_0 + \gamma_0(\varsigma_0) l_0^2 + h_0(\varsigma_0) x_0 + g_0(\varsigma_0) l_0 + \mu n_0(\varsigma_0) x_0 \\
&\quad + f_0(\varsigma_0)\mu l_0 + \psi_0(\varsigma_0) + \theta_0(\varsigma_0)\mu + \varphi_0(\varsigma_0)\mu^2 - d^2 - 2\mu d \\
&= \varphi_0(\varsigma_0)\mu^2 + \left(n_0(\varsigma_0) x_0 + f_0(\varsigma_0) l_0 + \theta_0(\varsigma_0) - 2d\right)\mu + \tfrac{1}{2} w_0(\varsigma_0) x_0^2 \\
&\quad + \lambda_0(\varsigma_0) x_0 l_0 + \gamma_0(\varsigma_0) l_0^2 + h_0(\varsigma_0) x_0 + g_0(\varsigma_0) l_0 + \psi_0(\varsigma_0) - d^2,
\end{aligned} \tag{45}
$$

According to the Lagrange dual principle (see Luenberger (1968)), the optimal value of Problem (3) can be obtained by maximizing $U(x_0, l_0, \varsigma_0, \mu)$ over μ, i.e.,

$$
\mathrm{Var}^*[S_T \,|(x_0, l_0, \varsigma_0)] = \max_{\mu} U(x_0, l_0, \varsigma_0, \mu). \tag{46}
$$

In order to show that an optimal solution exists to Problem (46), we gives the following proposition first.

Proposition 6. *For any $i \in \Pi$ and $k = 0, 1, \cdots, T-1$, we have $\phi_k(i) < 0$.*

PROOF. We prove the proposition by mathematical induction. For $k = T-1$, according to (15), (10)-(11), and Proposition 1, for any $i \in \Pi$, we have

$$
\varphi_{T-1}(i) = -\frac{1}{2} \frac{\left(\overline{n_T(i)}\right)^2}{\overline{w_T(i)}} A_{T-1}(i) = -A_{T-1}(i) < 0.
$$

Now we prove $\varphi_k(i) < 0$ for any $i \in \Pi$ under the assumption that $\varphi_{k+1}(i) < 0$ for any $i \in \Pi$. By Proposition 2, $w_{k+1}(j) > 0$ for any $j \in \Pi$. Since $\pi_{ij}(k) \geq 0$ and $\sum_{j=1}^{S} \pi_{ij}(k) = 1$ for $i, j \in \Pi$, we have $\overline{w_{k+1}(i)} = \sum_{j=1}^{S} w_{k+1}(j)\pi_{ij}(k) > 0$. Obviously, $\left(\overline{n_{k+1}(i)}\right)^2 \geq 0$. Therefore, according to (15) and Proposition 1, we have

$$
\varphi_k(i) = \overline{\varphi_{k+1}(i)} - \frac{1}{2} \frac{\left(\overline{n_{k+1}(i)}\right)^2}{\overline{w_{k+1}(i)}} A_k(i) \leq \overline{\varphi_{k+1}(i)} = \sum_{j=1}^{S} \varphi_{k+1}(j) q_{ij}(k) < 0.
$$

By mathematical induction, the proposition is proved. □

Proposition 6 shows that $\varphi_0(\varsigma_0) < 0$ for all $\varsigma_0 \in \Pi$. By (45), the optimal solution of Problem (46) exists and, by the first-order condition, is given by

$$
\mu^* = -\frac{n_0(\xi_0) x_0 + f_0(\xi_0) l_0 + \theta_0(\xi_0) - 2d}{2\varphi_0(\xi_0)}. \tag{47}
$$

Substituting (47) into (25) and noting that $x = x_k,\ l = l_k$ and $i = \varsigma_k$, we obtain the optimal investment strategy of the original M-V model (4) as follows

$$\begin{aligned} u_k^* = -\mathrm{E}^{-1}[P_k(\varsigma_k)P_k'(\varsigma_k)] & \Big(\mathrm{E}[P_k(\varsigma_k)]s_k(\varsigma_k)x_k + \tfrac{\overline{\lambda_{k+1}(\varsigma_k)}}{\overline{w_{k+1}(\varsigma_k)}}\mathrm{E}[q_k(\varsigma_k)P_k(\varsigma_k)]l_k \\ & + \tfrac{2\varphi_0(\varsigma_0)\overline{h_{k+1}(\varsigma_k)}-(n_0(\varsigma_0)x_0+f_0(\varsigma_0)l_0+\theta_0(\varsigma_0)-2d)\overline{n_{k+1}(\varsigma_k)}}{2\varphi_0(\xi_0)\overline{w_{k+1}(\varsigma_k)}} \\ & \times\mathrm{E}[P_k(\varsigma_k)] + \mathrm{E}[c_k(\varsigma_k)P_k(\varsigma_k)] \Big). \end{aligned} \tag{48}$$

Again substituting (47) into (46), we obtain the optimal value of the original M-V model (4), namely, the minimum variance as follows

$$\begin{aligned} &\mathrm{Var}^*[S_T\,|(x_0,l_0,\varsigma_0)] = \\ &\begin{cases} -\frac{1+\varphi_0(\varsigma_0)}{\varphi_0(\varsigma_0)}\left(d-\frac{n_0(\varsigma_0)x_0+f_0(\varsigma_0)l_0+\theta_0(\varsigma_0)}{2(1+\varphi_0(\xi_0))}\right)^2+\frac{1}{2}w_0(\varsigma_0)x_0^2 \\ \quad +\lambda_0(\varsigma_0)x_0l_0+\gamma_0(\varsigma_0)l_0^2+h_0(\varsigma_0)x_0+g_0(\varsigma_0)l_0+\psi_0(\varsigma_0) \\ -\frac{1}{4(1+\varphi_0(\varsigma_0))}(n_0(\varsigma_0)x_0+f_0(\varsigma_0)l_0+\theta_0(\varsigma_0))^2, & \varphi_0(\varsigma_0)\neq -1, \\ -\left(n_0(\varsigma_0)x_0+f_0(\varsigma_0)l_0+\theta_0(\varsigma_0)\right)d+\frac{(n_0(\varsigma_0)x_0+f_0(\varsigma_0)l_0+\theta_0(\varsigma_0))^2}{4} \\ +\frac{1}{2}w_0(\varsigma_0)x_0^2+\lambda_0(\varsigma_0)x_0l_0+\gamma_0(\varsigma_0)l_0^2+h_0(\varsigma_0)x_0+g_0(\varsigma_0)l_0+\psi_0(\varsigma_0), & \varphi_0(\varsigma_0)=-1. \end{cases} \end{aligned} \tag{49}$$

Obviously, according to the definition of variance, for any real number d, we must have $\mathrm{Var}^*[S_T\,|(x_0,l_0,\varsigma_0)] \geq 0$. Therefore, we can exclude the case of $\varphi_0(\varsigma_0) = -1$. Then, given expected terminal surplus level $\mathrm{E}[S_T\,|(x_0,l_0,\varsigma_0)] = d$, the minimum variance should be

$$\begin{aligned} \mathrm{Var}^*[S_T\,|(x_0,l_0,\varsigma_0)] = & -\tfrac{1+\varphi_0(\varsigma_0)}{\varphi_0(\varsigma_0)}\left(d-\tfrac{n_0(\varsigma_0)x_0+f_0(\varsigma_0)l_0+\theta_0(\varsigma_0)}{2(1+\varphi_0(\xi_0))}\right)^2+\tfrac{1}{2}w_0(\varsigma_0)x_0^2 \\ & +\lambda_0(\varsigma_0)x_0l_0+\gamma_0(\varsigma_0)l_0^2+h_0(\varsigma_0)x_0+g_0(\varsigma_0)l_0+\psi_0(\varsigma_0) \\ & -\tfrac{1}{4(1+\varphi_0(\varsigma_0))}(n_0(\varsigma_0)x_0+f_0(\varsigma_0)l_0+\theta_0(\varsigma_0))^2. \end{aligned} \tag{50}$$

Again $\mathrm{Var}^*[S_T] \geq 0$ for any real number d implies $-\frac{1+\varphi_0(\varsigma_0)}{\varphi_0(\varsigma_0)} > 0$. Setting $d = d_{\sigma_{\min}} := \frac{n_0(\varsigma_0)x_0+f_0(\varsigma_0)l_0+\theta_0(\varsigma_0)}{2(1+\varphi_0(\xi_0))}$, we can obtain the global minimum variance

$$\begin{aligned} \mathrm{Var}^*_{\min}[S_T\,|(x_0,l_0,\varsigma_0)] := & \tfrac{1}{2}w_0(\varsigma_0)x_0^2+\lambda_0(\varsigma_0)x_0l_0+\gamma_0(\varsigma_0)l_0^2+h_0(\varsigma_0)x_0 \\ & +g_0(\varsigma_0)l_0+\psi_0(\varsigma_0)-\tfrac{1}{4(1+\varphi_0(\varsigma_0))}(n_0(\varsigma_0)x_0+f_0(\varsigma_0)l_0+\theta_0(\varsigma_0))^2. \end{aligned} \tag{51}$$

So far, we obtain the following results.

Theorem 3. *For given expected terminal surplus* $\mathrm{E}[S_T\,|(x_0,l_0,\varsigma_0)] = d$ $(d \geq d_{\sigma_{\min}})$, *then the efficient investment strategy and the efficient frontier of the multi-period M-V ALM problem (4) in a Markov regime-switching market with stochastic cash flow are given by (48) and (50), respectively.*

5. Numerical illustration

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6. Conclusion

Starting from the actual needs of households, banks, insurance company, pension funds, and so on, we suppose in this paper that the wealth is affected by the uncontrolled cash flow during the investment process. During each period, the cash flow can be negative or positive, representing the capital addition or withdrawal, such as the stochastic income or expenditure for households. Using M-V model, we investigate a multi-period ALM problem with stochastic cash flow in a Markov regime switching market. The returns of assets and liabilities and the amounts of cash flows all depend on the stochastic market state. Using the Lagrange duality principle, matrix theory and the dynamic programming approach, we derive the analytical expressions for the efficient investment strategy and the efficient frontier. There are at least two interesting problems deserving further research:

i) In our model, both the liability and cash flow are uncontrollable and exogenous, how about the case when the liability and cash flow are endogenous and can be controlled by financial instruments and investors' decisions. It is an interesting research topic.

ii) Due to some technical difficulties, we only considered the case where the short-selling of all assets is allowed. Another potential research topic in the future is to extend the results to the case with no-shorting constraint.

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