

## MULTISCALE ASYMPTOTIC METHOD FOR STEKLOV EIGENVALUE EQUATIONS IN COMPOSITE MEDIA\*

LIQUN CAO<sup>†</sup>, LEI ZHANG<sup>‡</sup>, WALTER ALLEGRETTO<sup>§</sup>, AND YANPING LIN<sup>¶</sup>

**Abstract.** In this paper we consider the multiscale analysis of a Steklov eigenvalue equation with rapidly oscillating coefficients arising from the modeling of a composite media with a periodic microstructure. There are mainly two new results in the present paper. First, we obtain the convergence rate with  $\varepsilon^{1/2}$  for the multiscale asymptotic expansions of the eigenvalues and the eigenfunctions of the Steklov eigenvalue problem. Second, the boundary layer solution is defined. Numerical simulations are then carried out to validate the above theoretical results.

**Key words.** Steklov eigenvalue problem, multiscale asymptotic expansion, boundary layer solution

**AMS subject classifications.** 65F10, 35B50

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**1. Introduction.** In this paper we discuss the multiscale analysis of the Steklov eigenvalue equation in composite media given by

$$(1.1) \quad \begin{cases} \mathcal{L}_\varepsilon u^\varepsilon = 0 & \text{in } \Omega, \\ u^\varepsilon = 0 & \text{on } \Gamma_0, \\ \sigma_\varepsilon(u^\varepsilon) = \lambda^\varepsilon u^\varepsilon & \text{on } \Gamma_1, \end{cases}$$

where  $\Omega \subset R^n$ ,  $n = 2, 3$ , is a bounded smooth domain or a bounded Lipschitz polygonal convex domain with a periodic microstructure and whose boundary is denoted by  $\Gamma = \overline{\Gamma}_0 \cup \overline{\Gamma}_1$  with  $\Gamma_0 \cap \Gamma_1 = \emptyset$ . Here  $\mathcal{L}_\varepsilon$  denotes a second-order partial differential operator with rapidly oscillating coefficients given by

$$\mathcal{L}_\varepsilon \phi \equiv -\frac{\partial}{\partial x_i} \left( a_{ij} \left( \frac{x}{\varepsilon} \right) \frac{\partial \phi}{\partial x_j} \right) + a_0 \left( \frac{x}{\varepsilon} \right) \phi$$

and

$$\sigma_\varepsilon(\phi) \equiv \nu_i a_{ij} \left( \frac{x}{\varepsilon} \right) \frac{\partial \phi}{\partial x_j},$$

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<sup>†</sup>LSEC, NCMIS, Institute of Computational Mathematics and Scientific/Engineering Computing, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China (clq@lsec.cc.ac.cn).

<sup>‡</sup>Department of Logistics Management, Logistics Academy, Beijing 100858, China, and Institute of Computational Mathematics and Scientific/Engineering Computing, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China (zhanglei@lsec.cc.ac.cn).

<sup>§</sup>Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Alberta T6G 2G1, Canada (wallegre@math.ualberta.ca).

<sup>¶</sup>Department of Applied Mathematics, Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong, China (malin@polyu.edu.hk), and Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Alberta T6G 2G1, Canada (ylin@math.ualberta.ca).

where  $\vec{\nu} = (\nu_1, \dots, \nu_n)$  is the outward unit normal to  $\Gamma_1$  and  $\varepsilon > 0$  is a small period parameter. Here and below we use the Einstein summation convention on repeated indices.

We make the following assumptions:

(A<sub>1</sub>) Let  $\xi = \varepsilon^{-1}x$ , and assume that  $a_{ij}(\xi)$ ,  $a_0(\xi)$  are 1-periodic functions in  $\xi$ .

(A<sub>2</sub>)  $\mathcal{L}_\varepsilon$  is uniformly strongly elliptic; i.e., there is a positive constant  $\gamma_0$  which is independent of  $\varepsilon$  such that

$$a_{ij}\left(\frac{x}{\varepsilon}\right)\eta_i\eta_j \geq \gamma_0|\eta|^2$$

$\forall(\eta_1, \dots, \eta_m) \in R^n$ ,  $|\eta|^2 = \eta_i\eta_i$  and all  $x \in \Omega$ ,  $a_0(\frac{x}{\varepsilon}) \geq 0$ .

(A<sub>3</sub>)  $a_{ij}(\frac{x}{\varepsilon}) = a_{ji}(\frac{x}{\varepsilon})$ .

(A<sub>4</sub>)  $a_{ij}(\frac{x}{\varepsilon})$ ,  $a_0(\frac{x}{\varepsilon}) \in L^\infty(\bar{\Omega})$ .

*Remark 1.1.* For the composite media with a periodic microstructure, conditions (A<sub>1</sub>)–(A<sub>4</sub>) are reasonable.

Problems with an eigenvalue parameter on the boundary appear in many physical situations (see, e.g., [11, 14, 19]). For example, problem (1.1) arises in the separation of variables approach for parabolic or hyperbolic equations with dynamical boundary conditions [19] or in the dynamics of liquids in moving containers [11], i.e., sloshing problems. Other interesting problems include those of the vibrations of a pendulum [1] and those of the eigen oscillations of mechanical systems with boundary conditions containing the frequency [19, 14]. There are many others (see [5] and the references therein).

Courant and Hilbert [15] presented early results on the Steklov eigenvalue problems. Osborn [26] developed a general approximation theory for compact operators. Bramble and Osborn [9] presented a Galerkin method for the approximation of the Steklov problem for a non self-adjoint second-order differential operator. Andreev and Todorov [4] gave the isoparametric finite element approximation of Steklov eigenvalue problems for second-order, self-adjoint, elliptic differential operators. Several eigenvalue problems arising in physics and engineering, as well as their approximations, are presented in [29, 7].

This paper discusses the Steklov eigenvalue problems in composite media. In such cases, the direct accurate numerical computation of the solution is difficult because of the very fine mesh required. We recall that the homogenization method gives the overall behavior by incorporating the fluctuations due to the heterogeneities.

Vanninathan [28] investigated a homogenization method for a spectral problem with Steklov boundary conditions on periodically distributed holes inside the domain  $\Omega$ . For boundary homogenization of the Steklov eigenvalue problems, a number of results have been obtained in different papers, e.g., for vibrating systems with concentrated masses in [24] and for the limiting behavior at low frequencies for vibrating systems with stiff and/or thin heavy bands around curves in [17] and [18]. Ionescu, Onofrei, and Vernescu [20] considered a three-dimensional elastic body with a plane fault under a slip-weakening friction. The fault has  $\varepsilon$ -periodically distributed holes, called (small-scale) barriers. In each  $\varepsilon$ -square of the  $\varepsilon$ -lattice on the fault plane, the friction contact was considered outside an open set  $T_\varepsilon$  (small-scale barrier) of size  $r_\varepsilon < \varepsilon$ , compactly enclosed in the  $\varepsilon$ -square (see Figure 2 of [20]). The asymptotic behavior as  $\varepsilon$  tends to 0 for the friction contact problem was studied and different limit problems were derived. In [10], Bucur and Ionescu discussed the asymptotic behavior of the first eigenvalue as  $\varepsilon \rightarrow 0$ , leaving the limiting behavior of the associated eigenfunction and of the rest of eigenvalues and eigenfunctions as open problems. Pérez

[27] studied the asymptotic behavior of the eigenvalues and the associated eigenfunctions of an  $\varepsilon$ -dependent Steklov type eigenvalue problem posed in a bounded domain  $\Omega$  of  $R^2$ , when  $\varepsilon \rightarrow 0$ .

Numerous simulation results have shown that the numerical accuracy of the homogenization method may not be satisfactory if  $\varepsilon$  is not sufficiently small (see, e.g., [12, 13]). This is the motivation for the multiscale asymptotic methods and the associated numerical algorithms.

In [21, 25], the authors introduced the general theory of spectral properties of a sequence of abstract operators and gave numerous applications in asymptotic analysis of the eigenvalue problems arising in the theory of homogenization except for the Steklov eigenvalue problem. Allaire and Conca [2, 3] investigated the asymptotic behavior of the spectrum of a mathematical model that describes the vibrations of a coupled fluid-solid periodic structure (see, e.g., [14]). They used the Bloch wave homogenization method and two-scale convergence method to prove that in the limit as the period goes to zero, the spectrum is made of three parts: homogenized spectrum, Bloch spectrum, and the so-called boundary layer spectrum. Also they obtained a “completeness” result of all possible asymptotic behaviors of the sequence of eigenvalues in the special cases.

*Remark 1.2.* We observe the general theory of abstract spectral operators of [21, 25] and there are two crucial points. First, Lemma 1.1 of [25, p. 264] plays an important role in asymptotic analysis of the eigenvalues, where the key point is to use that fact that embedding of  $H^1(\Omega) \subset L^2(\Omega)$  is compact. For the Steklov eigenvalue problem, since  $H^1(\Omega)$  is not in  $L^2(\partial\Omega)$  or in  $L^2(\Gamma_1)$ , we cannot directly use Lemma 1.1 of [25, p. 264]. Second, Theorem 1.7 of [25, p. 274] is the foundation for investigating the asymptotic behavior of the eigenfunctions. The basic idea is to transfer the error estimates of the eigenfunctions into those of the corresponding boundary value problems. Since an eigenvalue parameter is on the boundary for the Steklov eigenvalue problem, we cannot directly employ Theorem 1.7 of [25].

There are two main new contributions in the present paper. First, we obtain the convergence rate with  $\varepsilon^{1/2}$  for the multiscale asymptotic expansions of the eigenvalues and the eigenfunctions of the Steklov eigenvalue problem. Second, the boundary layer solution for the Steklov eigenvalue problem will be defined; see (2.42). For a general bounded Lipschitz polygonal convex domain  $\Omega$ , since the corresponding eigenfunctions are not sufficiently smooth, the construction of boundary layer correctors is necessary and important. It should be emphasized that the problem and the definition of the boundary layer spectrum of [3] are essentially different from those of this paper.

The paper is organized as follows. In section 2, we introduce the multiscale asymptotic expansions of the eigenvalues and the eigenfunctions for the Steklov eigenvalue problem (1.1) and define boundary layer solutions. The main convergence results for the multiscale asymptotic expansions (see Theorems 2.1 and 2.2) are derived. In section 3, we give some numerical case studies as validation for the theoretical results.

**2. Multiscale asymptotic method.** In this section, we present the multiscale asymptotic method for the Steklov eigenvalue problem (1.1) and derive the convergence theorem.

Let  $V$  be the closed subspace of  $H^1(\Omega)$  given by

$$V = H^1(\Omega, \Gamma_0) = \{v \in H^1(\Omega), \quad | \quad v = 0 \quad \text{on} \quad \Gamma_0\}.$$

Obviously  $H_0^1(\Omega) \subset V \subset H^1(\Omega)$ . Assume that the space  $L^2(\Gamma_1)$  is equipped with the scalar product  $\langle \phi, \psi \rangle = \int_{\Gamma_1} \phi \psi d\sigma$ . The bilinear form on  $V \times V$  associated with  $\mathcal{L}_\varepsilon$  is

given by

$$a_\varepsilon(\phi, \psi) = \int_\Omega \left( a_{ij} \left( \frac{x}{\varepsilon} \right) \frac{\partial \phi}{\partial x_i} \frac{\partial \psi}{\partial x_j} + a_0 \left( \frac{x}{\varepsilon} \right) \phi \psi \right) dx.$$

Let  $(\lambda^\varepsilon, u^\varepsilon)$  be the exact Steklov eigenpair of problem (1.1) in the weak formulation:

$$(2.1) \quad a_\varepsilon(u^\varepsilon, v) = \lambda^\varepsilon \langle u^\varepsilon, v \rangle \quad \forall v \in V.$$

From the assumptions  $(A_2)$ – $(A_4)$ , we can easily infer that

$$\beta_0 \|v\|_{1,\Omega}^2 \leq a_\varepsilon(v, v), \quad |a_\varepsilon(u, v)| \leq \beta_1 \|u\|_{1,\Omega} \|v\|_{1,\Omega} \quad \forall u, v \in V,$$

where  $\beta_0, \beta_1$  are positive constants independent of  $\varepsilon$ .

Then from the classical theory of abstract elliptic eigenvalue problems (see, e.g., [29], [7]), we can prove the following lemma.

LEMMA 2.1. *Under the assumptions  $(A_1)$ – $(A_4)$ , problem (2.1) has a countable infinite set of eigenvalues, all having finite multiplicity, without a finite accumulation point. If  $\Gamma_0 = \emptyset$ , then it holds that*

$$0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_k \leq \dots \rightarrow \infty.$$

If  $\Gamma_0 \neq \emptyset$ , then it holds that

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k \leq \dots \rightarrow \infty,$$

where each eigenvalue occurs as many times as given by its multiplicity. Furthermore, the orthonormal eigenfunctions  $u_k^\varepsilon$ ,  $k \geq 1$  form the basis of a Hilbert space  $L^2(\bar{\Omega})$ .

We next seek the multiscale asymptotic expansions of the eigenvalues and the eigenfunctions of problem (1.1). Setting  $\xi = \varepsilon^{-1}x$  and following the terminology of [8],  $x, \xi$  are called “slow” and “fast” variables, respectively. We define

$$(2.2) \quad \begin{aligned} u_{1,k}^\varepsilon(x) &= u_k^0(x) + \varepsilon N_{\alpha_1}(\xi) \frac{\partial u_k^0(x)}{\partial x_{\alpha_1}}, \\ u_{2,k}^\varepsilon(x) &= u_k^0(x) + \varepsilon N_{\alpha_1}(\xi) \frac{\partial u_k^0(x)}{\partial x_{\alpha_1}} + \varepsilon^2 N_{\alpha_1 \alpha_2}(\xi) \frac{\partial^2 u_k^0(x)}{\partial x_{\alpha_1} \partial x_{\alpha_2}}, \quad k \geq 1. \end{aligned}$$

The cell functions  $N_{\alpha_1}(\xi)$ ,  $N_{\alpha_1 \alpha_2}(\xi)$  are defined in turn as

$$(2.3) \quad \begin{cases} \frac{\partial}{\partial \xi_i} \left( a_{ij}(\xi) \frac{\partial N_{\alpha_1}(\xi)}{\partial \xi_j} \right) = -\frac{\partial}{\partial \xi_i} (a_{i\alpha_1}(\xi)), & \xi \in Q, \\ N_{\alpha_1}(\xi) \text{ is 1-periodic in } \xi, \\ \int_Q N_{\alpha_1}(\xi) d\xi = 0, \end{cases}$$

and

$$(2.4) \quad \begin{cases} \frac{\partial}{\partial \xi_i} \left( a_{ij}(\xi) \frac{\partial N_{\alpha_1 \alpha_2}(\xi)}{\partial \xi_j} \right) = -\frac{\partial}{\partial \xi_i} (a_{i\alpha_1}(\xi) N_{\alpha_2}(\xi)) \\ \quad - a_{\alpha_1 j}(\xi) \frac{\partial N_{\alpha_2}(\xi)}{\partial \xi_j} - a_{\alpha_1 \alpha_2}(\xi) + \hat{a}_{\alpha_1 \alpha_2}, & \xi \in Q, \\ N_{\alpha_1 \alpha_2}(\xi) \text{ is 1-periodic in } \xi, \\ \int_Q N_{\alpha_1 \alpha_2}(\xi) d\xi = 0, \end{cases}$$

where  $\hat{a}_{\alpha_1\alpha_2} = \int_Q (a_{\alpha_1\alpha_2}(\xi) + a_{\alpha_1q}(\xi) \frac{\partial N_{\alpha_2}(\xi)}{\partial \xi_q}) d\xi$ ,  $\alpha_1, \alpha_2 = 1, 2, \dots, n$ , and the reference cell  $Q = (0, 1)^n$ .

The homogenized problem associated with the Steklov eigenvalue problem (1.1) is then given by

$$(2.5) \quad \begin{cases} \mathcal{L}u_k^0 \equiv -\frac{\partial}{\partial x_i} \left( \hat{a}_{ij} \frac{\partial u_k^0(x)}{\partial x_j} \right) + \langle a_0 \rangle u_k^0(x) = 0 & \text{in } \Omega, \\ u_k^0(x) = 0 & \text{on } \Gamma_0, \\ \nu_i \hat{a}_{ij} \frac{\partial u_k^0(x)}{\partial x_j} = \lambda_k^{(0)} u_k^0(x) & \text{on } \Gamma_1, \quad k \geq 1, \end{cases}$$

where  $\vec{\nu} = (\nu_1, \dots, \nu_n)$  is the outward unit normal to the boundary  $\Gamma_1$ ,  $(\hat{a}_{ij})$  is the homogenized coefficients matrix, and  $\langle a_0 \rangle = \int_Q a_0(\xi) d\xi$ .

*Remark 2.1.* For convenience, we choose the reference cell  $Q = (0, 1)^n$  in this paper. In fact, for a general case, we refer the reader to [8].

*Remark 2.2.*  $u_{1,k}^\varepsilon(x)$  and  $u_{2,k}^\varepsilon(x)$  are called the first-order and second-order multi-scale asymptotic solutions of the  $k$ th eigenfunction for the Steklov eigenvalue problem (1.1), respectively.

From (2.2), we find

$$(2.6) \quad \sigma_\varepsilon(u_{1,k}^\varepsilon) \equiv \nu_i a_{ij} \left( \frac{x}{\varepsilon} \right) \frac{\partial u_{1,k}^\varepsilon(x)}{\partial x_j} = \left[ \nu_i a_{ij}(\xi) \frac{\partial N_{\alpha_1}(\xi)}{\partial \xi_j} + \nu_i a_{i\alpha_1}(\xi) \right] \frac{\partial u_k^0(x)}{\partial x_{\alpha_1}} + \varepsilon \nu_i a_{ij}(\xi) N_{\alpha_1}(\xi) \frac{\partial^2 u_k^0(x)}{\partial x_j \partial x_{\alpha_1}}, \quad k \geq 1,$$

$$(2.7) \quad \sigma_\varepsilon(u_{2,k}^\varepsilon) \equiv \nu_i a_{ij} \left( \frac{x}{\varepsilon} \right) \frac{\partial u_{2,k}^\varepsilon(x)}{\partial x_j} = \left[ \nu_i a_{ij}(\xi) \frac{\partial N_{\alpha_1}(\xi)}{\partial \xi_j} + \nu_i a_{i\alpha_1}(\xi) \right] \frac{\partial u_k^0(x)}{\partial x_{\alpha_1}} + \varepsilon \left[ \nu_i a_{ij}(\xi) \frac{\partial N_{\alpha_1\alpha_2}(\xi)}{\partial \xi_j} + \nu_i a_{i\alpha_1}(\xi) N_{\alpha_2}(\xi) \right] \frac{\partial^2 u_k^0(x)}{\partial x_{\alpha_1} \partial x_{\alpha_2}} + \varepsilon^2 \nu_i a_{ij}(\xi) N_{\alpha_1\alpha_2}(\xi) \frac{\partial^3 u_k^0(x)}{\partial x_j \partial x_{\alpha_1} \partial x_{\alpha_2}}, \quad k \geq 1,$$

where  $\vec{\nu} = (\nu_1, \dots, \nu_n)$  is the outward unit normal to  $\Gamma_1$ .

We consider the Steklov boundary conditions on  $\Gamma_1$  and assume that

$$(2.8) \quad \sigma_\varepsilon(u_{1,k}^\varepsilon) = \lambda_{1,k}^\varepsilon u_{1,k}^\varepsilon, \quad \sigma_\varepsilon(u_{2,k}^\varepsilon) = \lambda_{2,k}^\varepsilon u_{2,k}^\varepsilon \quad \text{on } \Gamma_1,$$

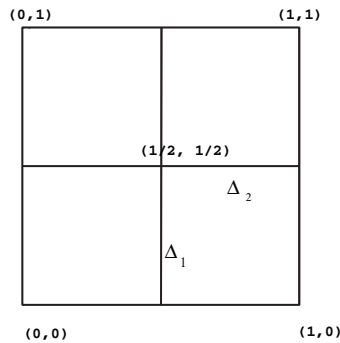
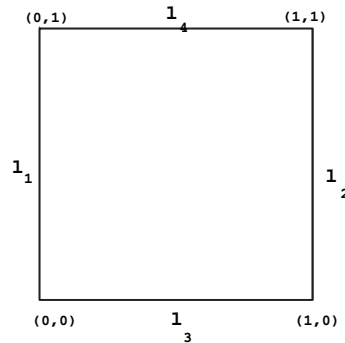
where  $\lambda_{1,k}^\varepsilon = \lambda_k^{(0)} + \varepsilon \lambda_k^{(1)}$ ,  $\lambda_{2,k}^\varepsilon = \lambda_k^{(0)} + \varepsilon \tilde{\lambda}_k^{(1)} + \varepsilon^2 \tilde{\lambda}_k^{(2)}$ ,  $k \geq 1$ .

We seek the higher-order correction terms of the eigenvalues of the Steklov boundary conditions on  $\Gamma_1$ . From (2.8)<sub>1</sub>, we compare coefficients of powers of  $\varepsilon$  and get

$$(2.9) \quad \nu_i a_{ij}(\xi) N_{\alpha_1}(\xi) \frac{\partial^2 u_k^0(x)}{\partial x_j \partial x_{\alpha_1}} = \lambda_k^{(1)} u_k^0(x) + \lambda_k^{(0)} N_{\alpha_1}(\xi) \frac{\partial u_k^0(x)}{\partial x_{\alpha_1}} \quad \text{on } \Gamma_1,$$

where  $\vec{\nu} = (\nu_1, \dots, \nu_n)$  denotes the outward unit normal to  $\Gamma_1$ . We thus have

$$(2.10) \quad \lambda_k^{(1)} = \frac{\left\{ \int_{\Gamma_1} \nu_i a_{ij} \left( \frac{x}{\varepsilon} \right) N_{\alpha_1} \left( \frac{x}{\varepsilon} \right) \frac{\partial^2 u_k^0(x)}{\partial x_j \partial x_{\alpha_1}} u_k^0(x) d\sigma - \lambda_k^{(0)} \int_{\Gamma_1} N_{\alpha_1} \left( \frac{x}{\varepsilon} \right) \frac{\partial u_k^0(x)}{\partial x_{\alpha_1}} u_k^0(x) d\sigma \right\}}{\int_{\Gamma_1} (u_k^0(x))^2 d\sigma}.$$

FIG. 2.1. The symmetry of  $Q$ .FIG. 2.2. The sides of  $Q$ .

From (2.8)<sub>2</sub>, we obtain the first-order and second-order correctors of the eigenvalues of problem (1.1), denoted by  $\tilde{\lambda}_k^{(1)}$  and  $\tilde{\lambda}_k^{(2)}$ , i.e.,

$$(2.11) \quad \begin{aligned} \tilde{\lambda}_k^{(1)} \int_{\Gamma_1} (u_k^0(x))^2 d\sigma &= \int_{\Gamma_1} \left[ \nu_i a_{ij}(\xi) \frac{\partial N_{\alpha_1 \alpha_2}(\xi)}{\partial \xi_j} + \nu_i a_{i\alpha_1}(\xi) N_{\alpha_2}(\xi) \right] \frac{\partial^2 u_k^0(x)}{\partial x_{\alpha_1} \partial x_{\alpha_2}} u_k^0(x) d\sigma \\ &\quad - \lambda_k^{(0)} \int_{\Gamma_1} N_{\alpha_1}(\xi) \frac{\partial u_k^0(x)}{\partial x_{\alpha_1}} u_k^0(x) d\sigma, \quad k \geq 1, \end{aligned}$$

and

$$(2.12) \quad \begin{aligned} \tilde{\lambda}_k^{(2)} \int_{\Gamma_1} (u_k^0(x))^2 d\sigma &= \int_{\Gamma_1} \nu_i a_{ij}(\xi) N_{\alpha_1 \alpha_2}(\xi) \frac{\partial^3 u_k^0(x)}{\partial x_j \partial x_{\alpha_1} \partial x_{\alpha_2}} u_k^0(x) d\sigma \\ &\quad - \lambda_k^{(0)} \int_{\Gamma_1} N_{\alpha_1 \alpha_2}(\xi) \frac{\partial^2 u_k^0(x)}{\partial x_{\alpha_1} \partial x_{\alpha_2}} u_k^0(x) d\sigma \\ &\quad - \tilde{\lambda}_k^{(1)} \int_{\Gamma_1} N_{\alpha_1}(\xi) \frac{\partial u_k^0(x)}{\partial x_{\alpha_1}} u_k^0(x) d\sigma, \quad k \geq 1. \end{aligned}$$

By using (2.10)–(2.12), we study the behavior of higher-order correction terms of the eigenvalues of problem (1.1) in the special cases. To this end, we make the assumptions on the coefficients  $a_{ij}(\xi)$  as follows:

(H<sub>1</sub>)  $a_{ii}(\xi)$ ,  $i = 1, \dots, n$ , are symmetric with respect to the middle hyperplanes  $\Delta_q$ ,  $q = 1, \dots, n$ , of the unit cell  $Q = (0, 1)^n$  as illustrated in Figures 2.1 and 2.2.

(H<sub>2</sub>)  $a_{ij}(\xi)$ ,  $i \neq j$ , are antisymmetric with respect to the middle hyperplanes  $\Delta_q$ ,  $q = 1, \dots, n$ , of the unit cell  $Q = (0, 1)^n$ . In particular, the choice  $a_{ij} = 0$ ,  $i \neq j$ , satisfies this condition.

*Remark 2.3.* The conditions (H<sub>1</sub>) and (H<sub>2</sub>) imply that the composite media satisfy geometric symmetric (or antisymmetric) properties in a periodic microstructure.

*Remark 2.4.* Suppose that  $\Omega$  is the union of entire cells, i.e.,  $\overline{\Omega} = \bigcup_{z \in I_\varepsilon} \varepsilon(z + \overline{Q})$ , where the index set  $I_\varepsilon \subset \mathbb{Z}^n$  such that  $\varepsilon(z + \overline{Q}) \subset \overline{\Omega}$  and  $Q = (0, 1)^n$ . Let  $\lambda_k^{(1)}$ ,  $\tilde{\lambda}_k^{(1)}$ ,  $\tilde{\lambda}_k^{(2)}$ ,  $k \geq 1$ , be the correctors of the  $k$ th eigenvalue as defined in (2.10), (2.11), and (2.12). If (A<sub>1</sub>)–(A<sub>3</sub>) and (H<sub>1</sub>)–(H<sub>2</sub>) are satisfied, then we can prove that  $\lambda_k^{(1)} = 0$ ,  $\tilde{\lambda}_k^{(1)} = \tilde{\lambda}_k^{(2)} = 0$ ,  $k \geq 1$ . We thus get  $|\lambda_k^\varepsilon - \lambda_k^{(0)}| \leq C(k)\varepsilon^3$  for any  $k$ th eigenvalue of problem (1.1) in the special cases. The numerical results presented in section 3 demonstrate this; see Tables 3.2 and 3.5.

Next we give the main convergence theorems for the multiscale asymptotic method. It should be emphasized that we do not need the conditions  $(H_1)$ – $(H_2)$  in the following convergence theorems.

**THEOREM 2.1.** *Suppose that  $\Omega \subset R^n$ ,  $n = 2, 3$ , is a bounded smooth domain with the boundary  $\partial\Omega \in C^{s+2}$ ,  $s = 1, 2$ . The boundary is denoted by  $\partial\Omega = \bar{\Gamma}_0 \cup \bar{\Gamma}_1$  with  $\Gamma_0 \cap \Gamma_1 = \emptyset$ . Let  $(\lambda_k^\varepsilon, u_k^\varepsilon)$  be the  $k$ th eigenpair of the Steklov eigenvalue problem (1.1), and let  $\lambda_k^{(0)}$  and  $u_{s,k}^\varepsilon$  be respectively the approximate solutions as given in (2.5) and (2.2) associated with  $\lambda_k^\varepsilon, u_k^\varepsilon$ . If the conditions  $(A_1)$ – $(A_4)$  are satisfied, then we have the following estimates:*

$$(2.13) \quad |\lambda_k^\varepsilon - \lambda_k^{(0)}| \leq C(k)\varepsilon^{1/2}, \quad k \geq 1.$$

If the multiplicity of the eigenvalues  $\lambda_k^{(0)}$  is equal to  $t$ , then

$$(2.14) \quad \|\bar{u}_k^\varepsilon - u_{s,k}^\varepsilon\|_{1,\Omega} \leq C_s(k)\varepsilon^{1/2}, \quad s = 1, 2, \quad k \geq 1,$$

where  $\bar{u}_k^\varepsilon$  is a linear combination of the eigenfunctions of problem (1.1) corresponding to  $\lambda_k^\varepsilon, \dots, \lambda_{k+t-1}^\varepsilon$ . In particular, if the eigenvalue  $\lambda_k^{(0)}$  is simple, then

$$(2.15) \quad \|u_k^\varepsilon - u_{s,k}^\varepsilon\|_{1,\Omega} \leq C_s(k)\varepsilon^{1/2}, \quad s = 1, 2, \quad k \geq 1,$$

where  $C(k), C_s(k)$  are constants independent of  $\varepsilon$ .

*Proof.* We first consider the Dirichlet–Neumann boundary value problem as follows:

$$(2.16) \quad \begin{cases} \mathcal{L}_\varepsilon w^\varepsilon = 0 & \text{in } \Omega, \\ w^\varepsilon = 0 & \text{on } \Gamma_0, \\ \nu_i a_{ij} \left(\frac{x}{\varepsilon}\right) \frac{\partial w^\varepsilon}{\partial x_j} = g(x) & \text{on } \Gamma_1. \end{cases}$$

The variational form of (2.16) is to find  $w^\varepsilon \in V = H^1(\Omega, \Gamma_0)$  such that

$$(2.17) \quad a_\varepsilon(w^\varepsilon, v) = \langle g, v \rangle, \quad g \in L^2(\Gamma_1) \quad \forall v \in V.$$

Since the bilinear form  $a_\varepsilon(u, v)$  is  $V$ -elliptic, this problem is uniquely solvable. Moreover, as the boundary  $\partial\Omega \in C^3$ , for  $g \in L^2(\Gamma_1)$  the solution  $w^\varepsilon$  is in  $H^1(\Omega)$ . We define the solution operator  $\mathcal{B}_\varepsilon : L^2(\Gamma_1) \rightarrow H^1(\Omega)$  by  $\mathcal{B}_\varepsilon g = w^\varepsilon$ . Now let us consider the operator  $\mathcal{T}_\varepsilon : L^2(\Gamma_1) \rightarrow L^2(\Gamma_1)$  as the restriction of  $\mathcal{B}_\varepsilon$  on  $\Gamma_1$ .  $\square$

We denote by  $\mu^\varepsilon$  a nonzero eigenvalue of  $\mathcal{T}_\varepsilon$  and by  $z^\varepsilon \in L^2(\Gamma_1)$  the associated eigenfunction, normalized with respect to the  $L^2(\Gamma_1)$ . Then  $\mathcal{T}_\varepsilon z^\varepsilon = \mu^\varepsilon z^\varepsilon$  and

$$(2.18) \quad a_\varepsilon(\mathcal{B}_\varepsilon z^\varepsilon, v) = \langle z^\varepsilon, v \rangle = \frac{1}{\mu^\varepsilon} \langle \mathcal{T}_\varepsilon z^\varepsilon, v \rangle = \frac{1}{\mu^\varepsilon} \langle \mathcal{B}_\varepsilon z^\varepsilon, v \rangle, \quad v \in V.$$

One can verify that  $\mathcal{T}_\varepsilon : L^2(\Gamma_1) \rightarrow L^2(\Gamma_1)$  is a linear self-adjoint compact operator in a Hilbert space  $L^2(\Gamma_1)$ ; see [9, 4].

We first prove Theorem 2.1 for  $s = 1$ . If assume that  $\Gamma_0 = \emptyset$ , from (1.1), (2.8)–(2.14), then we get

$$(2.19) \quad \begin{cases} \mathcal{L}_\varepsilon u_{1,k}^\varepsilon = F_0^\varepsilon, & x \in \Omega, \\ \sigma_\varepsilon(u_{1,k}^\varepsilon) - \lambda_k^{(0)} u_{1,k}^\varepsilon = G_0^\varepsilon, & x \in \partial\Omega, \end{cases}$$

where

$$F_0^\varepsilon = \left[ \hat{a}_{i\alpha_1} - a_{i\alpha_1}(\xi) - a_{ij}(\xi) \frac{\partial N_{\alpha_1}(\xi)}{\partial \xi_j} - \frac{\partial}{\partial \xi_j} (a_{ji}(\xi) N_{\alpha_1}(\xi)) \right] \frac{\partial^2 u_k^0(x)}{\partial x_{\alpha_1} \partial x_i} - \varepsilon a_{ij}(\xi) N_{\alpha_1}(\xi) \frac{\partial^3 u_k^0(x)}{\partial x_i \partial x_j \partial x_{\alpha_1}} + (a_0(\xi) - \langle a_0 \rangle) u_k^0(x) + \varepsilon a_0(\xi) N_{\alpha_1}(\xi) \frac{\partial u_k^0(x)}{\partial x_{\alpha_1}}$$

and

$$G_0^\varepsilon = \nu_i \left[ a_{i\alpha_1}(\xi) + a_{ij}(\xi) \frac{\partial N_{\alpha_1}(\xi)}{\partial \xi_j} - \hat{a}_{i\alpha_1} \right] \frac{\partial u_k^0(x)}{\partial x_{\alpha_1}} + \varepsilon \left\{ \nu_i a_{ij}(\xi) N_{\alpha_1}(\xi) \frac{\partial^2 u_k^0(x)}{\partial x_{\alpha_1} \partial x_j} - \lambda_k^{(0)} N_{\alpha_1}(\xi) \frac{\partial u_k^0(x)}{\partial x_{\alpha_1}} \right\}.$$

Here  $\hat{a}_{i\alpha_1}$  and  $\langle a_0 \rangle$  are as given in (2.12), and  $\vec{\nu} = (\nu_1, \dots, \nu_n)$  is the outward unit normal to  $\partial\Omega$ .

Now we recall cell problems (2.3) and (2.4). Under the assumptions  $(A_1)$ – $(A_4)$ , it can be proved that (see Theorem 1.1 of [22]; also see [6, 23])

$$(2.20) \quad \|N_{\alpha_1}\|_{W^{1,\infty}(Q)} \leq C, \quad \|N_{\alpha_1\alpha_2}\|_{W^{1,\infty}(Q)} \leq C,$$

where  $Q = (0, 1)^n$ ,  $\alpha_1, \alpha_2 = 1, 2, \dots, n$ , and  $C$  is a positive constant independent of  $\varepsilon$ . Since  $\int_Q [\hat{a}_{i\alpha_1} - a_{i\alpha_1}(\xi) - a_{ij}(\xi) \frac{\partial N_{\alpha_1}(\xi)}{\partial \xi_j} - \frac{\partial}{\partial \xi_j} (a_{ji}(\xi) N_{\alpha_1}(\xi))] d\xi = 0$  and  $\int_Q (a_0(\xi) - \langle a_0 \rangle) d\xi = 0$ , using Lemma 1.6 of [25, p. 8], we have

$$(2.21) \quad \|F_0^\varepsilon\|_{0,\Omega} \leq C_1(k)\varepsilon,$$

where  $C_1(k)$  is a constant independent of  $\varepsilon$ .

We set  $\beta^{i\alpha_1}(\xi) = [a_{i\alpha_1}(\xi) + a_{ij}(\xi) \frac{\partial N_{\alpha_1}(\xi)}{\partial \xi_j} - \hat{a}_{i\alpha_1}]$  and check that  $\beta^{i\alpha_1}(\xi)$  satisfies the conditions of Lemma 2.2 of [25, p. 137]. It follows from Lemma 2.2 that

$$(2.22) \quad \|G_0^\varepsilon\|_{0,\partial\Omega} \leq C_1(k)\varepsilon^{1/2},$$

where  $C_1(k)$  is a constant independent of  $\varepsilon$ .

Let  $\psi_k^\varepsilon$  be the weak solution of

$$(2.23) \quad \begin{cases} \mathcal{L}_\varepsilon \psi_k^\varepsilon = F_0^\varepsilon, & x \in \Omega, \\ \sigma_\varepsilon(\psi_k^\varepsilon) = 0, & x \in \partial\Omega, \end{cases}$$

where  $\sigma_\varepsilon(v) = \nu_i a_{ij}(\frac{x}{\varepsilon}) \frac{\partial v(x)}{\partial x_j}$ . We get

$$(2.24) \quad \begin{cases} \mathcal{L}_\varepsilon \tilde{u}_{1,k}^\varepsilon = 0, & x \in \Omega, \\ \sigma_\varepsilon(\tilde{u}_{1,k}^\varepsilon) - \lambda_k^{(0)} \tilde{u}_{1,k}^\varepsilon = G_0^\varepsilon + \lambda_k^{(0)} \psi_k^\varepsilon, & x \in \partial\Omega, \end{cases}$$

where  $\tilde{u}_{1,k}^\varepsilon = u_{1,k}^\varepsilon - \psi_k^\varepsilon$ .

From (2.21) and (2.23), it follows that

$$(2.25) \quad \|\psi_k^\varepsilon\|_{1,\Omega} \leq C_1(k)\varepsilon$$

and

$$(2.26) \quad \|\psi_k^\varepsilon\|_{0,\partial\Omega} \leq C \|\psi_k^\varepsilon\|_{1,\Omega} \leq C_1(k)\varepsilon,$$

where  $C_1(k)$  is a constant independent of  $\varepsilon$ .



The variational form of (2.24) is given by

$$(2.27) \quad a_\varepsilon(\tilde{u}_{1,k}^\varepsilon, v) - \lambda_k^{(0)} \langle \tilde{u}_{1,k}^\varepsilon, v \rangle = \langle G_0^\varepsilon + \lambda_k^{(0)} \psi_k^\varepsilon, v \rangle \quad \forall v \in V,$$

where  $\langle g, v \rangle = \int_{\partial\Omega} g(x)v(x)d\sigma$ .

For  $\tilde{z}_k^\varepsilon \in L^2(\partial\Omega)$ , setting  $\tilde{u}_{1,k}^\varepsilon = \mathcal{B}_\varepsilon \tilde{z}_k^\varepsilon$ , (2.27) can be written as follows:

$$(2.28) \quad a_\varepsilon(\mathcal{B}_\varepsilon \tilde{z}_k^\varepsilon, v) - \lambda_k^{(0)} \langle \mathcal{T}_\varepsilon \tilde{z}_k^\varepsilon, v \rangle = \langle G_0^\varepsilon + \lambda_k^{(0)} \psi_k^\varepsilon, v \rangle,$$

i.e.,

$$(2.29) \quad \langle \tilde{z}_k^\varepsilon, v \rangle - \lambda_k^{(0)} \langle \mathcal{T}_\varepsilon \tilde{z}_k^\varepsilon, v \rangle = \langle G_0^\varepsilon + \lambda_k^{(0)} \psi_k^\varepsilon, v \rangle.$$

If  $\lambda_k^{(0)} \neq 0$ , then we set  $\mu_k^{(0)} = \frac{1}{\lambda_k^{(0)}}$  and get

$$(2.30) \quad \langle \mathcal{T}_\varepsilon \tilde{z}_k^\varepsilon - \mu_k^{(0)} \tilde{z}_k^\varepsilon, v \rangle = \langle \mu_k^{(0)} (G_0^\varepsilon + \lambda_k^{(0)} \psi_k^\varepsilon), v \rangle.$$

By setting  $v = \mathcal{T}_\varepsilon \tilde{z}_k^\varepsilon - \mu_k^{(0)} \tilde{z}_k^\varepsilon \in L^2(\partial\Omega)$  in (2.30) and using (2.22)–(2.26), we derive

$$(2.31) \quad \begin{aligned} \|\mathcal{T}_\varepsilon \tilde{z}_k^\varepsilon - \mu_k^{(0)} \tilde{z}_k^\varepsilon\|_{0,\partial\Omega}^2 &\leq C_1(k)\varepsilon^{1/2} \|v\|_{0,\partial\Omega} + C_1(k)\varepsilon \|v\|_{0,\partial\Omega} \\ &\leq C_1(k)\varepsilon^{1/2} \|v\|_{0,\partial\Omega}, \end{aligned}$$

and consequently

$$(2.32) \quad \|\mathcal{T}_\varepsilon \tilde{z}_k^\varepsilon - \mu_k^{(0)} \tilde{z}_k^\varepsilon\|_{0,\partial\Omega} \leq C_1(k)\varepsilon^{1/2},$$

where  $C_1(k)$  is a constant independent of  $\varepsilon$ .

To apply Lemma 11.2 of [21, p. 340], we set

$$H = L^2(\partial\Omega), \quad A = \mathcal{T}_\varepsilon, \quad u = \tilde{z}_k^\varepsilon, \quad \mu = \mu_k^{(0)}, \quad \beta = C_1(k)\varepsilon^{1/2}.$$

Since  $\mathcal{T}_\varepsilon : L^2(\partial\Omega) \rightarrow L^2(\partial\Omega)$  is a linear self-adjoint compact operator in a Hilbert space  $H = L^2(\partial\Omega)$ , using Lemma 11.2 of [21, p. 340], there exists an eigenvalue  $\mu_{n(k)}^\varepsilon$  of the operator  $\mathcal{T}_\varepsilon$  such that

$$|\mu_{n(k)}^\varepsilon - \mu_k^{(0)}| \leq C(k)\varepsilon^{1/2},$$

i.e.,

$$|(\lambda_{n(k)}^\varepsilon)^{-1} - (\lambda_k^{(0)})^{-1}| \leq C(k)\varepsilon^{1/2}.$$

In order to apply Lemma 1.6 of [25, p. 270], we set  $\mathcal{H}_\varepsilon = \mathcal{H}_0 = L^2(\partial\Omega)$  with the real valued scalar product  $\langle u, v \rangle = \int_{\partial\Omega} uv d\sigma$  and let  $\mathcal{R}_\varepsilon : L^2(\partial\Omega) \rightarrow L^2(\partial\Omega)$  be an identity operator.

We consider the spectral problems for the operators  $\mathcal{T}_\varepsilon$  and  $\mathcal{T}_0$ :

$$(2.33) \quad \begin{aligned} \mathcal{T}_\varepsilon v_k^\varepsilon &= \mu_k^\varepsilon v_k^\varepsilon, \quad k = 1, 2, \dots, \quad v_k^\varepsilon \in L^2(\partial\Omega), \\ \mu_1^\varepsilon &\geq \mu_2^\varepsilon \geq \dots \geq \mu_k^\varepsilon \geq \dots, \\ \langle v_l^\varepsilon, v_m^\varepsilon \rangle &= \delta_{lm}, \end{aligned}$$

$$(2.34) \quad \begin{aligned} \mathcal{T}_0 v_k^0 &= \mu_k^{(0)} v_k^0, \quad k = 1, 2, \dots, \quad v_k^0 \in L^2(\partial\Omega), \\ \mu_1^{(0)} &\geq \mu_2^{(0)} \geq \dots \geq \mu_k^{(0)} \geq \dots, \\ \langle v_l^0, v_m^0 \rangle &= \delta_{lm}, \end{aligned}$$

where  $\mathcal{T}_\varepsilon$  and  $\mathcal{T}_0$  are associated with problem (1.1) and the corresponding homogenized problem (2.5), respectively,  $\delta_{lm}$  is the Kronecker symbol, the eigenvalues  $\mu_k^\varepsilon$  and  $\mu_k^{(0)}$  form decreasing sequences, and each eigenvalue is counted as many times as its multiplicity. Here  $\langle u, v \rangle = \int_{\partial\Omega} uv d\sigma$ .

We can directly verify that the conditions  $(C_1)$ – $(C_4)$  of [25, pp. 266–267] are satisfied. Using Lemma 1.6 of [25, p. 270], we have  $\mu_k^\varepsilon \rightarrow \mu_k^{(0)}$ ,  $k = 1, 2, \dots$ , as  $\varepsilon \rightarrow 0$ . Since  $\mu_k^\varepsilon = \frac{1}{\lambda_k^\varepsilon}$ ,  $\mu_k^{(0)} = \frac{1}{\lambda_k^{(0)}}$ , it leads to  $\lambda_k^\varepsilon \rightarrow \lambda_k^{(0)}$ ,  $k = 1, 2, \dots$ , as  $\varepsilon \rightarrow 0$ . So, for a fixed  $k$ , there is a small neighborhood of point  $\lambda_k^{(0)}$  which contains a eigenvalue  $\lambda_k^\varepsilon$  such that  $\lambda_{n(k)}^\varepsilon = \lambda_k^\varepsilon$ . We thus obtain

$$|\lambda_k^\varepsilon - \lambda_k^{(0)}| \leq C(k)\varepsilon^{1/2}, \quad k = 1, 2, \dots,$$

where  $C(k)$  is a constant independent of  $\varepsilon$ .

By using Lemma 11.2 of [21, p. 340] again, we get

$$(2.35) \quad \|\bar{v}_k^\varepsilon - \tilde{z}_k^\varepsilon\|_{0,\partial\Omega} \leq C_1(k)\varepsilon^{1/2},$$

where  $\bar{v}_k^\varepsilon$  is a linear combination of the eigenfunctions of problem (2.33) corresponding to  $\mu_k^\varepsilon, \dots, \mu_{k+t-1}^\varepsilon$  and  $C_1(k)$  is a constant independent of  $\varepsilon$ .

From (2.17), for any  $g \in L^2(\partial\Omega)$ , we have

$$\beta_0 \|w^\varepsilon\|_{1,\Omega}^2 \leq a_\varepsilon(w^\varepsilon, w^\varepsilon) = \langle g, w^\varepsilon \rangle \leq \|g\|_{0,\partial\Omega} \|w^\varepsilon\|_{1,\Omega},$$

i.e.,

$$(2.36) \quad \|\mathcal{B}_\varepsilon g\|_{1,\Omega} = \|w^\varepsilon\|_{1,\Omega} \leq C \|g\|_{0,\partial\Omega}.$$

We set  $\bar{u}_k^\varepsilon = \mathcal{B}_\varepsilon \bar{v}_k^\varepsilon$ ,  $\tilde{u}_{1,k}^\varepsilon = \mathcal{B}_\varepsilon \tilde{z}_k^\varepsilon$ . Since  $\mathcal{B}_\varepsilon : L^2(\partial\Omega) \rightarrow H^1(\Omega)$  is a bounded linear operator, we obtain

$$\|\bar{u}_k^\varepsilon - \tilde{u}_{1,k}^\varepsilon\|_{1,\Omega} = \|\mathcal{B}_\varepsilon(\bar{v}_k^\varepsilon - \tilde{z}_k^\varepsilon)\|_{1,\Omega} \leq C \|\bar{v}_k^\varepsilon - \tilde{z}_k^\varepsilon\|_{0,\partial\Omega} \leq C_1(k)\varepsilon^{1/2},$$

and consequently

$$\|\bar{u}_k^\varepsilon - u_{1,k}^\varepsilon\|_{1,\Omega} \leq \|\bar{u}_k^\varepsilon - \tilde{u}_{1,k}^\varepsilon\|_{1,\Omega} + \|\psi_k^\varepsilon\|_{1,\Omega} \leq C_1(k)\varepsilon^{1/2} + C_1(k)\varepsilon \leq C_1(k)\varepsilon^{1/2},$$

where  $\bar{u}_k^\varepsilon$  is a linear combination of the eigenfunctions of problem (1.1) corresponding to  $\lambda_k^\varepsilon, \dots, \lambda_{k+t-1}^\varepsilon$ . In particular, if the eigenvalue  $\lambda_k^{(0)}$  of (2.5) is simple, then we can choose  $\bar{u}_k^\varepsilon = c_0 u_{1,k}^\varepsilon$ ,  $c_0 = \text{const}$ , such that

$$\|u_k^\varepsilon - u_{1,k}^\varepsilon\|_{1,\Omega} \leq C_1(k)\varepsilon^{1/2},$$

where  $C_1(k)$  is a constant independent of  $\varepsilon$ .

On the other hand, if we assume that  $\Gamma_0 \neq \emptyset$ , from (1.1), (2.8)–(2.14), then we have

$$(2.37) \quad \begin{cases} \mathcal{L}_\varepsilon u_{1,k}^\varepsilon = F_0^\varepsilon, & x \in \Omega, \\ u_{1,k}^\varepsilon = \phi_\varepsilon(x), & x \in \Gamma_0, \\ \sigma_\varepsilon(u_{1,k}^\varepsilon) - \lambda_k^{(0)} u_{1,k}^\varepsilon = G_0^\varepsilon, & x \in \Gamma_1, \end{cases}$$

where  $F_0^\varepsilon$  and  $G_0^\varepsilon$  have the same forms as in (2.19), where  $\vec{\nu} = (\nu_1, \dots, \nu_n)$  is the outward unit normal to the boundary  $\Gamma_1$ , and  $\phi_\varepsilon(x) = \varepsilon N_{\alpha_1}(\frac{x}{\varepsilon}) \frac{\partial u_k^0(x)}{\partial x_{\alpha_1}}$ .

Let  $\zeta_k^\varepsilon$  be the solution of

$$(2.38) \quad \begin{cases} \mathcal{L}_\varepsilon \zeta_k^\varepsilon = 0, & x \in \Omega, \\ \zeta_k^\varepsilon = \phi_\varepsilon(x), & x \in \Gamma_0, \\ \sigma_\varepsilon(\zeta_k^\varepsilon) = 0, & x \in \Gamma_1. \end{cases}$$

Following the idea of Oleinik, Shamaev, and Yosifian (see [25, pp. 126–127]), we obtain

$$\|\phi_\varepsilon\|_{1/2, \Gamma_0} \leq C_1(k)\varepsilon^{1/2},$$

and consequently

$$(2.39) \quad \|\zeta_k^\varepsilon\|_{1, \Omega} \leq C\|\phi_\varepsilon\|_{1/2, \Gamma_0} \leq C_1(k)\varepsilon^{1/2}.$$

Subtracting (2.38) from (2.37) yields

$$(2.40) \quad \begin{cases} \mathcal{L}_\varepsilon(u_{1,k}^\varepsilon - \zeta_k^\varepsilon) = F_0^\varepsilon, & x \in \Omega, \\ u_{1,k}^\varepsilon - \zeta_k^\varepsilon = 0, & x \in \Gamma_0, \\ \sigma_\varepsilon(u_{1,k}^\varepsilon - \zeta_k^\varepsilon) - \lambda_k^{(0)}(u_{1,k}^\varepsilon - \zeta_k^\varepsilon) = G_0^\varepsilon + \lambda_k^{(0)}\zeta_k^\varepsilon, & x \in \Gamma_1. \end{cases}$$

From (2.40), repeating the process of the proof in the case  $\Gamma_0 = \emptyset$ , we have

$$\begin{aligned} |\lambda_k^\varepsilon - \lambda_k^{(0)}| &\leq C(k)\varepsilon^{1/2}, \\ \|\bar{u}_k^\varepsilon - (u_{1,k}^\varepsilon - \zeta_k^\varepsilon)\|_{1, \Omega} &\leq C_1(k)\varepsilon^{1/2}. \end{aligned}$$

Hence we use (2.39) and obtain

$$\|\bar{u}_k^\varepsilon - u_{1,k}^\varepsilon\|_{1, \Omega} \leq \|\bar{u}_k^\varepsilon - (u_{1,k}^\varepsilon - \zeta_k^\varepsilon)\|_{1, \Omega} + \|\zeta_k^\varepsilon\|_{1, \Omega} \leq C_1(k)\varepsilon^{1/2},$$

where  $C_1(k)$  is a constant independent of  $\varepsilon$ .

It remains to prove the theorem for the case  $s = 2$ . Its main process is the same as the case with  $s = 1$ . The important difference is the formulation of problem (2.19). For simplicity, we assume that  $\Gamma_0 = \emptyset$ . From (1.1), (2.8)–(2.14), we have

$$(2.41) \quad \begin{cases} \mathcal{L}_\varepsilon u_{2,k}^\varepsilon = F^\varepsilon, & x \in \Omega, \\ \sigma_\varepsilon(u_{2,k}^\varepsilon) - \lambda_k^{(0)}u_{2,k}^\varepsilon = G^\varepsilon, & x \in \partial\Omega, \end{cases}$$

where

$$\begin{aligned} F^\varepsilon &= -\varepsilon a_{ij}(\xi)N_{\alpha_1}(\xi)\frac{\partial^3 u_k^0(x)}{\partial x_i \partial x_j \partial x_{\alpha_1}} - \varepsilon a_{ij}(\xi)\frac{\partial N_{\alpha_1 \alpha_2}(\xi)}{\partial \xi_j}\frac{\partial^3 u_k^0(x)}{\partial x_i \partial x_{\alpha_1} \partial x_{\alpha_2}} \\ &\quad - \varepsilon \frac{\partial}{\partial \xi_i}(a_{ij}(\xi)N_{\alpha_1 \alpha_2}(\xi))\frac{\partial^3 u_k^0(x)}{\partial x_j \partial x_{\alpha_1} \partial x_{\alpha_2}} - \varepsilon^2 a_{ij}(\xi)N_{\alpha_1 \alpha_2}(\xi)\frac{\partial^4 u_k^0(x)}{\partial x_i \partial x_j \partial x_{\alpha_1} \partial x_{\alpha_2}} \end{aligned}$$

and

$$\begin{aligned} G^\varepsilon &= \left[ \nu_i a_{ij}(\xi)\frac{\partial N_{\alpha_1}(\xi)}{\partial \xi_j} + \nu_i a_{i\alpha_1}(\xi) - \nu_i \hat{a}_{i\alpha_1} \right] \frac{\partial u_k^0(x)}{\partial x_{\alpha_1}} \\ &\quad + \varepsilon \left[ \nu_i a_{ij}(\xi)\frac{\partial N_{\alpha_1 \alpha_2}(\xi)}{\partial \xi_j} + \nu_i a_{i\alpha_1}(\xi)N_{\alpha_2}(\xi) \right] \frac{\partial^2 u_k^0(x)}{\partial x_{\alpha_1} \partial x_{\alpha_2}} \\ &\quad + \varepsilon^2 \nu_i a_{ij}(\xi)N_{\alpha_1 \alpha_2}(\xi)\frac{\partial^3 u_k^0(x)}{\partial x_j \partial x_{\alpha_1} \partial x_{\alpha_2}} \\ &\quad - \varepsilon \lambda_k^{(0)}N_{\alpha_1}(\xi)\frac{\partial u_k^0(x)}{\partial x_{\alpha_1}} - \varepsilon^2 \lambda_k^{(0)}N_{\alpha_1 \alpha_2}(\xi)\frac{\partial^2 u_k^0(x)}{\partial x_{\alpha_1} \partial x_{\alpha_2}}. \end{aligned}$$

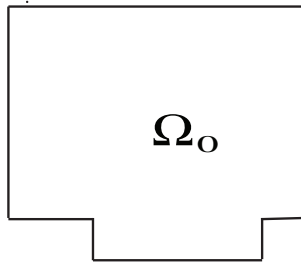


FIG. 2.3. Interior subdomain  $\Omega_0$ .

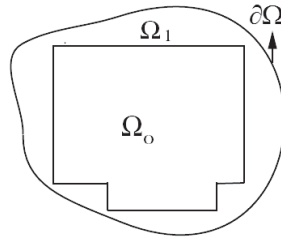


FIG. 2.4. The boundary layer  $\Omega_1$ .

If assume that  $u_k^{(0)} \in H^4(\Omega)$ , thanks to (2.20), then we can show that  $\|F^\varepsilon\|_{0,\Omega} \leq C_2(k)\varepsilon$ . By using Lemma 2.2 of [25, p. 137] again, we get  $\|G^\varepsilon\|_{0,\partial\Omega} \leq C_2(k)\varepsilon^{1/2}$ , where  $C_2(k)$  is a constant independent of  $\varepsilon$ . Repeating the process of the proof for the case  $s = 1$ , we can complete the proof of Theorem 2.1.

*Remark 2.5.* If  $\Omega$  is a bounded smooth domain in  $R^n$  with the boundary  $\partial\Omega \in C^{s+2}$ , where  $s = 1, 2$ , then one can prove that  $u_k^0 \in H^{s+2}(\Omega)$  (see, e.g., [4, Theorem 1]). However, generally speaking, for a general bounded Lipschitz polygonal convex domain, the condition  $u_k^0 \in H^{s+2}(\Omega)$ ,  $s = 1, 2$ , is invalid. To overcome this difficulty, we need to define the boundary layer correctors. To begin, let us introduce the notation. Let  $\bar{\Omega}_0 = \bigcup_{z \in \hat{I}_\varepsilon} \varepsilon(z + \bar{Q}) \subset \Omega$  as illustrated in Figure 2.3, where the index set  $\hat{I}_\varepsilon = \{z = (z_1, \dots, z_n) \in Z^n, \varepsilon(z + Q) \subset \Omega\}$  and the unit cube  $Q = (0, 1)^n$ . The boundary layer  $\Omega_1 = \Omega \setminus \bar{\Omega}_0$  is as shown in Figure 2.4, where  $\text{dist}(\partial\Omega_0, \partial\Omega) > 2\varepsilon$ .

We define the boundary layer solutions  $u_{s,k}^{\varepsilon,b}(x)$ ,  $s = 1, 2, k \geq 1$ , given by

$$(2.42) \quad \begin{cases} \mathcal{L}_\varepsilon u_{s,k}^{\varepsilon,b} = 0 & \text{in } \Omega_1, \\ u_{s,k}^{\varepsilon,b} = 0 & \text{on } \Gamma_0, \\ u_{s,k}^{\varepsilon,b} = u_{s,k}^\varepsilon & \text{on } \Gamma^*, \\ \sigma_\varepsilon(u_{s,k}^{\varepsilon,b}) = \lambda_k^{(0)} u_{s,k}^{\varepsilon,b} & \text{on } \Gamma_1, \end{cases}$$

where  $\Gamma^* = \partial\Omega_0 \cap \partial\Omega_1$  and the operators  $\mathcal{L}_\varepsilon, \sigma_\varepsilon$  have been defined in section 1.  $\lambda_k^{(0)}$  and  $u_{s,k}^\varepsilon$  are given in (2.5) and (2.2), respectively.

Next we study the existence and uniqueness of solution for the boundary layer equation (2.42). Let  $\eta_k^\varepsilon$  be the unique solution of

$$(2.43) \quad \begin{cases} \mathcal{L}_\varepsilon \eta_k^\varepsilon = 0, & \text{in } \Omega_1 \\ \eta_k^\varepsilon = 0 & \text{on } \Gamma_0, \\ \eta_k^\varepsilon = u_{s,k}^\varepsilon & \text{on } \Gamma^*, \\ \sigma_\varepsilon(\eta_k^\varepsilon) = 0 & \text{on } \Gamma_1. \end{cases}$$

Subtracting (2.43) from (2.42) gives

$$(2.44) \quad \begin{cases} \mathcal{L}_\varepsilon(u_{s,k}^{\varepsilon,b} - \eta_k^\varepsilon) = 0 & \text{in } \Omega_1, \\ u_{s,k}^{\varepsilon,b} - \eta_k^\varepsilon = 0 & \text{on } \Gamma_0 \cup \Gamma^*, \\ \sigma_\varepsilon(u_{s,k}^{\varepsilon,b} - \eta_k^\varepsilon) - \lambda_k^{(0)}(u_{s,k}^{\varepsilon,b} - \eta_k^\varepsilon) = \lambda_k^{(0)} \eta_k^\varepsilon & \text{on } \Gamma_1. \end{cases}$$

Consider the Steklov eigenvalue problem in  $\Omega_1 \subset \Omega$  as follows:

$$(2.45) \quad \begin{cases} \mathcal{L}_\varepsilon w_k^\varepsilon = 0 & \text{in } \Omega_1, \\ w_k^\varepsilon = 0 & \text{on } \Gamma_0 \cup \Gamma^*, \\ \sigma_\varepsilon w_k^\varepsilon = \lambda_k^{\varepsilon,b} w_k^\varepsilon & \text{on } \Gamma_1. \end{cases}$$

Using Lemma 2.1, we can obtain a result similar to that of problem (1.1). Denote by  $\Lambda_d^\varepsilon$  the set of all eigenvalues for problem (2.45). If we assume that

$$(2.46) \quad \lambda_k^{(0)} \notin \Lambda_d^\varepsilon,$$

then (2.44) has one and only one solution by using Fredholm’s alternative theorem, where  $\lambda_k^{(0)} = \lambda_k^{(0)}(\Omega)$  is  $k$ th eigenvalue of problem (2.5). Furthermore, (2.42) has one and only one solution.

Now we show that the assumption (2.46) is true in the specific case. We prove  $\lambda_1^{(0)} \notin \Lambda_d^\varepsilon$ , where  $\lambda_1^{(0)} = \lambda_1^{(0)}(\Omega)$  is the first eigenvalue of problem (2.5).

LEMMA 2.2 (first Krein–Rutman theorem; see [16, p. 188]). *Let  $K$  be a reproducing cone, with interior  $\overset{\circ}{K} \neq \emptyset$ , and let  $B$  be a strongly positive compact operator on  $K$ . Then the spectral radius of  $B$ ,  $r(B)$ , is a simple eigenvalue of  $B$  and  $B^*$ , and their associated eigenvectors belongs to  $\overset{\circ}{K}$  and  $\overset{\circ}{K}^*$ . (More precisely, there exists a unique associated eigenvector in  $\overset{\circ}{K}$  (resp.,  $\overset{\circ}{K}^*$ ) of norm = 1. Furthermore, all other eigenvalues are strictly less in absolute value than  $r(B)$ ).*

PROPOSITION 2.1. *Let  $\lambda_1^\varepsilon = \lambda_1^\varepsilon(\Omega)$ ,  $\lambda_1^{(0)} = \lambda_1^{(0)}(\Omega)$  be the first eigenvalues of (1.1) and (2.5), respectively. If  $|\Gamma_0| > 0$ , where  $|\Gamma_0|$  denotes the Lebesgue measure of  $\Gamma_0$ , then it holds that*

$$(2.47) \quad \lambda_1^{(0)} \notin \Lambda_d^\varepsilon.$$

*Proof.* Given  $\Omega_1 \subset \Omega$  and  $|\Omega \setminus \overline{\Omega_1}| = |\Omega_0| > 0$ , where  $|\Omega_0|$  denotes the Lebesgue measure of a domain  $\Omega_0$ . Denote by  $\lambda_1^\varepsilon(\Omega)$ ,  $\lambda_1^{\varepsilon,b}(\Omega_1)$  the first eigenvalues associated with problems (1.1) and (2.45), respectively. The variational principle implies that  $\lambda_1^\varepsilon(\Omega) \leq \lambda_1^{\varepsilon,b}(\Omega_1)$ . Suppose that  $\lambda_1^\varepsilon(\Omega) = \lambda_1^{\varepsilon,b}(\Omega_1) = \mu$ . Then the eigenfunction corresponding to problem (2.45) with eigenvalue  $\mu$  expanded by zero values on  $(\Omega \setminus \Omega_1)$  is an eigenfunction in  $\Omega$ . This implies that the eigenfunction vanishes at some points of  $\Omega$ . However, if apply the first Krein–Rutman theorem (see Lemma 2.2) to the Steklov eigenvalue problem (1.1) under the assumption  $|\Gamma_0| > 0$ , then we infer that  $u_1^\varepsilon > 0$  in  $\Omega$ . This is contrary to the result that the eigenfunction vanishes at some points of  $\Omega$ . Hence we get  $\lambda_1^\varepsilon(\Omega) < \lambda_1^{\varepsilon,b}(\Omega_1)$ .

Using (2.13) (see Theorem 2.1), we have  $|\lambda_1^\varepsilon(\Omega) - \lambda_1^{(0)}(\Omega)| \leq C\varepsilon^{1/2}$ . If assume that  $\varepsilon > 0$  is sufficiently small, then we can obtain  $\lambda_1^{(0)}(\Omega) < \lambda_1^{\varepsilon,b}(\Omega_1)$ . Therefore the proof of Proposition 2.1 is complete.  $\square$

Remark 2.6. It follows from Proposition 2.1 that  $\lambda_1^{(0)} \notin \Lambda_d^\varepsilon$ . From (2.42)–(2.45), using Fredholm’s alternative theorem, we can conclude that the boundary layer equation (2.42) has one and only one solution  $u_{s,1}^{\varepsilon,b} \in H^1(\Omega_1)$  for any fixed  $s = 1, 2$ .

We define the multiscale asymptotic solution given by

$$(2.48) \quad U_{s,k}^\varepsilon(x) = \begin{cases} u_{s,k}^\varepsilon(x), & x \in \overline{\Omega_0}, \\ u_{s,k}^{\varepsilon,b}(x), & x \in \Omega_1, \end{cases}$$

where  $s = 1, 2$ ,  $k \geq 1$ , and  $u_{s,k}^\varepsilon(x)$  and  $u_{s,k}^{\varepsilon,b}(x)$  are given in (2.2) and (2.42), respectively.

**THEOREM 2.2.** *Suppose that  $\Omega \subset R^n$ ,  $n = 2, 3$ , is a bounded Lipschitz polygonal convex domain, whose boundary is denoted by  $\partial\Omega = \bar{\Gamma}_0 \cup \bar{\Gamma}_1$  with  $\Gamma_0 \cap \Gamma_1 = \emptyset$ . Let  $(\lambda_k^\varepsilon, u_k^\varepsilon)$  be the  $k$ th eigenpair of problem (1.1), and let  $U_{s,k}^\varepsilon(x)$  be the multiscale solutions as defined in (2.48) associated with  $u_k^\varepsilon$ , where  $\Omega_0 \subset\subset \Omega$ ,  $\Omega_1 = \Omega \setminus \bar{\Omega}_0$ , and  $\text{dist}(\partial\Omega_0, \partial\Omega) > 2\varepsilon$ . If the conditions  $(A_1)$ – $(A_4)$  are satisfied, then we have the following estimates:*

$$(2.49) \quad |\lambda_k^\varepsilon - \lambda_k^{(0)}| \leq C(k)\varepsilon^{1/2}, \quad k \geq 1.$$

If the multiplicity of the eigenvalues  $\lambda_k^{(0)}$  is equal to  $t$ , then

$$(2.50) \quad \|\bar{u}_k^\varepsilon - U_{s,k}^\varepsilon\|_{1,\Omega} \leq C_s(k)\varepsilon^{1/2}, \quad s = 1, 2, \quad k \geq 1,$$

where  $\bar{u}_k^\varepsilon$  is a linear combination of the eigenfunctions of problem (1.1) corresponding to  $\lambda_k^\varepsilon, \dots, \lambda_{k+t-1}^\varepsilon$ . In particular, if the eigenvalue  $\lambda_k^{(0)}$  is simple, then

$$(2.51) \quad \|u_k^\varepsilon - U_{s,k}^\varepsilon\|_{1,\Omega} \leq C_s(k)\varepsilon^{1/2}, \quad s = 1, 2, \quad k \geq 1,$$

where  $C_s(k)$  is a constant independent of  $\varepsilon$ .

*Proof.* Generally speaking, for a bounded Lipschitz polygonal convex domain, the condition  $u_k^0 \in H^{s+2}(\Omega)$ ,  $s = 1, 2$ , is invalid, where  $u_k^0$  is the  $k$ th eigenfunction for the homogenized Steklov eigenvalue problem (2.5). In this case, since we cannot get (2.19) and (2.41) in the sense of distributions, the estimates from Theorem 2.1 fail. To this end, we have to define the boundary layer solutions (see (2.42)). How to use the boundary layer solutions to derive the similar results of Theorem 2.1? The key step is to show that

$$\|\bar{u}_k^\varepsilon - u_{s,k}^\varepsilon\|_{1,\Omega_0} \leq C_s(k)\varepsilon^{1/2},$$

where  $\Omega_0 \subset\subset \Omega$ ,  $\bar{u}_k^\varepsilon$  is a linear combination of the eigenfunctions of problem (1.1) corresponding to  $\lambda_k^\varepsilon, \dots, \lambda_{k+t-1}^\varepsilon$ , if  $\lambda_k^{(0)}$  is an eigenvalue of the homogenized Steklov problem (2.5) of multiplicity  $t$ .  $\square$

To begin, we introduce the following subdomains:

$$\begin{aligned} \Omega' &= \{x \in \Omega : \text{dist}(x, \partial\Omega) \geq \varepsilon/2\}, \\ K_\varepsilon &= \{x \in \Omega : \text{dist}(x, \partial\Omega) \leq 2\varepsilon\}, \\ K'_\varepsilon &= \{x \in \Omega : \varepsilon \leq \text{dist}(x, \partial\Omega) \leq 2\varepsilon\}. \end{aligned}$$

It is obvious that  $\Omega_0 \subset\subset \Omega' \subset\subset \Omega$ . We can infer that  $u_k^0 \in H^{s+2}(\Omega')$ . Let us introduce the cutoff function  $m_\varepsilon(x)$  defined by

$$(2.52) \quad \begin{aligned} m_\varepsilon &\in \mathcal{D}(\Omega), \\ m_\varepsilon &= 0 \quad \text{if } \text{dist}(x, \partial\Omega) \leq \varepsilon, \\ m_\varepsilon &= 1 \quad \text{if } \text{dist}(x, \partial\Omega) \geq 2\varepsilon, \\ \varepsilon \left| \frac{\partial m_\varepsilon}{\partial x_i} \right| &\leq C, \quad i = 1, 2, \dots, n. \end{aligned}$$

Set

$$(2.53) \quad \begin{aligned} \theta_{1,k}^\varepsilon(x) &= u_k^0(x) + \varepsilon m_\varepsilon(x) N_{\alpha_1}(\xi) \frac{\partial u_k^0(x)}{\partial x_{\alpha_1}}, \\ \theta_{2,k}^\varepsilon(x) &= u_k^0(x) + m_\varepsilon(x) \left[ \varepsilon N_{\alpha_1}(\xi) \frac{\partial u_k^0(x)}{\partial x_{\alpha_1}} + \varepsilon^2 N_{\alpha_1\alpha_2}(\xi) \frac{\partial^2 u_k^0(x)}{\partial x_{\alpha_1} \partial x_{\alpha_2}} \right]. \end{aligned}$$

Without loss of generality we assume that  $\Gamma_0 = \emptyset$ . For  $\forall \varphi \in H^1(\Omega)$ , from (2.52)–(2.53), (2.3)–(2.5), we can directly get

$$\begin{aligned}
 (2.54) \quad a_\varepsilon(\theta_{1,k}^\varepsilon, \varphi) &= \int_\Omega a_{ij} \left( \frac{x}{\varepsilon} \right) \frac{\partial \theta_{1,k}^\varepsilon(x)}{\partial x_j} \frac{\partial \varphi(x)}{\partial x_i} dx + \int_\Omega a_0 \left( \frac{x}{\varepsilon} \right) \theta_{1,k}^\varepsilon(x) \varphi(x) dx \\
 &= \int_\Omega \hat{a}_{ij} \frac{\partial u_k^0(x)}{\partial x_j} \frac{\partial \varphi(x)}{\partial x_i} dx + \int_\Omega \langle a_0 \rangle u_k^0(x) \varphi(x) dx \\
 &\quad + \int_\Omega m_\varepsilon(x) \left[ a_{ij}(\xi) + a_{ip}(\xi) \frac{\partial N_{\alpha_1}(\xi)}{\partial \xi_p} - \hat{a}_{ij} \right] \frac{\partial u_k^0(x)}{\partial x_j} \frac{\partial \varphi(x)}{\partial x_i} dx \\
 &\quad + \int_\Omega (1 - m_\varepsilon(x)) \left[ a_{ij}(\xi) + a_{ip}(\xi) \frac{\partial N_{\alpha_1}(\xi)}{\partial \xi_p} - \hat{a}_{ij} \right] \frac{\partial u_k^0(x)}{\partial x_j} \frac{\partial \varphi(x)}{\partial x_i} dx \\
 &\quad + \int_\Omega \varepsilon \frac{\partial m_\varepsilon(x)}{\partial x_j} a_{ij}(\xi) N_{\alpha_1}(\xi) \frac{\partial u_k^0(x)}{\partial x_{\alpha_1}} \frac{\partial \varphi(x)}{\partial x_i} dx \\
 &\quad + \int_\Omega (m_\varepsilon(x) - 1) a_{ij}(\xi) \frac{\partial N_{\alpha_1}(\xi)}{\partial \xi_j} \frac{\partial u_k^0(x)}{\partial x_{\alpha_1}} \frac{\partial \varphi(x)}{\partial x_i} dx \\
 &\quad + \varepsilon \int_\Omega m_\varepsilon(x) a_{ij}(\xi) N_{\alpha_1}(\xi) \frac{\partial^2 u_k^0(x)}{\partial x_j \partial x_{\alpha_1}} \frac{\partial \varphi(x)}{\partial x_i} dx \\
 &\quad + \int_\Omega (a_0(\xi) - \langle a_0 \rangle) u_k^0(x) \varphi(x) dx \\
 &\quad + \varepsilon \int_\Omega m_\varepsilon(x) a_0(\xi) N_{\alpha_1}(\xi) \frac{\partial u_k^0(x)}{\partial x_{\alpha_1}} \varphi(x) dx,
 \end{aligned}$$

where  $\hat{a}_{ij}$  and  $\langle a_0 \rangle$  have been given above.

We recall the homogenized Steklov eigenvalue problem (2.5), and its variational form is as follows:

$$(2.55) \quad \int_\Omega \hat{a}_{ij} \frac{\partial u_k^0(x)}{\partial x_j} \frac{\partial \varphi(x)}{\partial x_i} dx + \int_\Omega \langle a_0 \rangle u_k^0(x) \varphi(x) dx = \lambda_k^{(0)} \int_{\partial\Omega} u_k^0(x) \varphi(x) d\sigma.$$

By using the Green formula and the definition of the cutoff function  $m_\varepsilon(x)$ , from (2.3), we observe that

$$\begin{aligned}
 (2.56) \quad &\int_\Omega m_\varepsilon(x) \left[ a_{ij}(\xi) + a_{ip}(\xi) \frac{\partial N_{\alpha_1}(\xi)}{\partial \xi_p} - \hat{a}_{ij} \right] \frac{\partial u_k^0(x)}{\partial x_j} \frac{\partial \varphi(x)}{\partial x_i} dx \\
 &= \int_\Omega \frac{\partial m_\varepsilon(x)}{\partial x_i} \left[ a_{ij}(\xi) + a_{ip}(\xi) \frac{\partial N_{\alpha_1}(\xi)}{\partial \xi_p} - \hat{a}_{ij} \right] \frac{\partial u_k^0(x)}{\partial x_j} \varphi(x) dx \\
 &\quad + \varepsilon^{-1} \int_\Omega m_\varepsilon(x) \frac{\partial}{\partial \xi_i} \left[ a_{ij}(\xi) + a_{ip}(\xi) \frac{\partial N_{\alpha_1}(\xi)}{\partial \xi_p} - \hat{a}_{ij} \right] \frac{\partial u_k^0(x)}{\partial x_j} \varphi(x) dx \\
 &\quad + \int_\Omega m_\varepsilon(x) \left[ a_{ij}(\xi) + a_{ip}(\xi) \frac{\partial N_{\alpha_1}(\xi)}{\partial \xi_p} - \hat{a}_{ij} \right] \frac{\partial^2 u_k^0(x)}{\partial x_i \partial x_j} \varphi(x) dx \\
 &= \int_\Omega \frac{\partial m_\varepsilon(x)}{\partial x_i} \left[ a_{ij}(\xi) + a_{ip}(\xi) \frac{\partial N_{\alpha_1}(\xi)}{\partial \xi_p} - \hat{a}_{ij} \right] \frac{\partial u_k^0(x)}{\partial x_j} \varphi(x) dx \\
 &\quad + \int_\Omega m_\varepsilon(x) \left[ a_{ij}(\xi) + a_{ip}(\xi) \frac{\partial N_{\alpha_1}(\xi)}{\partial \xi_p} - \hat{a}_{ij} \right] \frac{\partial^2 u_k^0(x)}{\partial x_i \partial x_j} \varphi(x) dx.
 \end{aligned}$$

Combining (2.55) and (2.56), we rewrite (2.54) as follows:

$$(2.57) \quad a_\varepsilon(\theta_{1,k}^\varepsilon, \varphi) - \lambda_k^{(0)} \langle \theta_{1,k}^\varepsilon, \varphi \rangle = \mathcal{J}_{1,k}^\varepsilon(\varphi) \quad \forall \varphi \in H^1(\Omega),$$

where  $\langle \theta_{1,k}^\varepsilon, \varphi \rangle = \int_{\partial\Omega} \theta_{1,k}^\varepsilon(x) \varphi(x) d\sigma$ , and  
 (2.58)

$$\begin{aligned} \mathcal{J}_{1,k}^\varepsilon(\varphi) &= \int_{\Omega} \frac{\partial m_\varepsilon(x)}{\partial x_i} \left[ a_{ij}(\xi) + a_{ip}(\xi) \frac{\partial N_{\alpha_1}(\xi)}{\partial \xi_p} - \hat{a}_{ij} \right] \frac{\partial u_k^0(x)}{\partial x_j} \varphi(x) dx \\ &+ \int_{\Omega} m_\varepsilon(x) \left[ a_{ij}(\xi) + a_{ip}(\xi) \frac{\partial N_{\alpha_1}(\xi)}{\partial \xi_p} - \hat{a}_{ij} \right] \frac{\partial^2 u_k^0(x)}{\partial x_i \partial x_j} \varphi(x) dx \\ &+ \int_{\Omega} (1 - m_\varepsilon(x)) \left[ a_{ij}(\xi) + a_{ip}(\xi) \frac{\partial N_{\alpha_1}(\xi)}{\partial \xi_p} - \hat{a}_{ij} \right] \frac{\partial u_k^0(x)}{\partial x_j} \frac{\partial \varphi(x)}{\partial x_i} dx \\ &+ \int_{\Omega} \varepsilon \frac{\partial m_\varepsilon(x)}{\partial x_j} a_{ij}(\xi) N_{\alpha_1}(\xi) \frac{\partial u_k^0(x)}{\partial x_{\alpha_1}} \frac{\partial \varphi(x)}{\partial x_i} dx \\ &+ \int_{\Omega} (m_\varepsilon(x) - 1) a_{ij}(\xi) \frac{\partial N_{\alpha_1}(\xi)}{\partial \xi_j} \frac{\partial u_k^0(x)}{\partial x_{\alpha_1}} \frac{\partial \varphi(x)}{\partial x_i} dx \\ &+ \varepsilon \int_{\Omega} m_\varepsilon(x) a_{ij}(\xi) N_{\alpha_1}(\xi) \frac{\partial^2 u_k^0(x)}{\partial x_j \partial x_{\alpha_1}} \frac{\partial \varphi(x)}{\partial x_i} dx \\ &+ \int_{\Omega} (a_0(\xi) - \langle a_0 \rangle) u_k^0(x) \varphi(x) dx \\ &+ \varepsilon \int_{\Omega} m_\varepsilon(x) a_0(\xi) N_{\alpha_1}(\xi) \frac{\partial u_k^0(x)}{\partial x_{\alpha_1}} \varphi(x) dx. \end{aligned}$$

If we let  $g(x, \xi) = \varepsilon \frac{\partial m_\varepsilon(x)}{\partial x_i} [a_{ij}(\xi) + a_{ip}(\xi) \frac{\partial N_{\alpha_1}(\xi)}{\partial \xi_p} - \hat{a}_{ij}]$ , then all conditions of Lemma 1.6 of [25, p. 8] can be satisfied. By applying Lemmas 1.6 and 1.5 of [25], we get

$$\begin{aligned} (2.59) \quad & \left| \varepsilon^{-1} \int_{\Omega} \varepsilon \frac{\partial m_\varepsilon(x)}{\partial x_i} \left[ a_{ij}(\xi) + a_{ip}(\xi) \frac{\partial N_{\alpha_1}(\xi)}{\partial \xi_p} - \hat{a}_{ij} \right] \frac{\partial u_k^0(x)}{\partial x_j} \varphi(x) dx \right| \\ & \leq C \varepsilon^{-1} \varepsilon \|u_k^0\|_{2, K'_\varepsilon} \|\varphi\|_{1, K'_\varepsilon} \leq C \varepsilon^{1/2} \|u_k^0\|_{3, \Omega'} \|\varphi\|_{1, \Omega}. \end{aligned}$$

Similarly, applying Lemma 1.6 again gives

$$\begin{aligned} (2.60) \quad & \left| \int_{\Omega} m_\varepsilon(x) \left[ a_{ij}(\xi) + a_{ip}(\xi) \frac{\partial N_{\alpha_1}(\xi)}{\partial \xi_p} - \hat{a}_{ij} \right] \frac{\partial^2 u_k^0(x)}{\partial x_i \partial x_j} \varphi(x) dx \right| \\ & \leq C \varepsilon \|u_k^0\|_{3, \Omega'} \|\varphi\|_{1, \Omega}. \end{aligned}$$

From (2.52), we know

$$\begin{aligned} (2.61) \quad & \int_{\Omega} (1 - m_\varepsilon(x)) \left[ a_{ij}(\xi) + a_{ip}(\xi) \frac{\partial N_{\alpha_1}(\xi)}{\partial \xi_p} - \hat{a}_{ij} \right] \frac{\partial u_k^0(x)}{\partial x_j} \frac{\partial \varphi(x)}{\partial x_i} dx \\ & = \int_{K_\varepsilon} (1 - m_\varepsilon(x)) \left[ a_{ij}(\xi) + a_{ip}(\xi) \frac{\partial N_{\alpha_1}(\xi)}{\partial \xi_p} - \hat{a}_{ij} \right] \frac{\partial u_k^0(x)}{\partial x_j} \frac{\partial \varphi(x)}{\partial x_i} dx. \end{aligned}$$

Applying (2.20), we have  $\|(a_{ij}(\xi) + a_{ip}(\xi) \frac{\partial N_{\alpha_1}(\xi)}{\partial \xi_p} - \hat{a}_{ij})\|_{0, \infty, Q} \leq C$ . By using Lemma 1.5 of [25, p. 7], we thus derive

$$\begin{aligned} (2.62) \quad & \int_{\Omega} (1 - m_\varepsilon(x)) \left[ a_{ij}(\xi) + a_{ip}(\xi) \frac{\partial N_{\alpha_1}(\xi)}{\partial \xi_p} - \hat{a}_{ij} \right] \frac{\partial u_k^0(x)}{\partial x_j} \frac{\partial \varphi(x)}{\partial x_i} dx \\ & = \int_{K_\varepsilon} (1 - m_\varepsilon(x)) \left[ a_{ij}(\xi) + a_{ip}(\xi) \frac{\partial N_{\alpha_1}(\xi)}{\partial \xi_p} - \hat{a}_{ij} \right] \frac{\partial u_k^0(x)}{\partial x_j} \frac{\partial \varphi(x)}{\partial x_i} dx \\ & \leq C \|u_k^0\|_{1, K_\varepsilon} \|\varphi\|_{1, K_\varepsilon} \leq C \varepsilon^{1/2} \|u_k^0\|_{2, \Omega} \|\varphi\|_{1, \Omega}. \end{aligned}$$



From (2.52) and (2.20), we can directly prove that

$$(2.63) \quad \begin{aligned} \int_{\Omega} \varepsilon \frac{\partial m_{\varepsilon}(x)}{\partial x_j} a_{ij}(\xi) N_{\alpha_1}(\xi) \frac{\partial u_k^0(x)}{\partial x_{\alpha_1}} \frac{\partial \varphi(x)}{\partial x_i} dx \\ \leq C \|u_k^0\|_{1, K'_\varepsilon} \|\varphi\|_{1, K'_\varepsilon} \leq C \varepsilon^{1/2} \|u_k^0\|_{2, \Omega'} \|\varphi\|_{1, \Omega} \end{aligned}$$

and

$$(2.64) \quad \begin{aligned} \left| \int_{\Omega} (m_{\varepsilon}(x) - 1) a_{ij}(\xi) \frac{\partial N_{\alpha_1}(\xi)}{\partial \xi_j} \frac{\partial u_k^0(x)}{\partial x_{\alpha_1}} \frac{\partial \varphi(x)}{\partial x_i} dx \right| \\ \leq C \|u_k^0\|_{1, K_\varepsilon} \|\varphi\|_{1, K_\varepsilon} \leq C \varepsilon^{1/2} \|u_k^0\|_{2, \Omega} \|\varphi\|_{1, \Omega}. \end{aligned}$$

From (2.52) and (2.20), it is obvious that

$$(2.65) \quad \left| \varepsilon \int_{\Omega} m_{\varepsilon}(x) a_{ij}(\xi) N_{\alpha_1}(\xi) \frac{\partial^2 u_k^0(x)}{\partial x_j \partial x_{\alpha_1}} \frac{\partial \varphi(x)}{\partial x_i} \right| \leq C \varepsilon \|u_k^0\|_{2, \Omega'} \|\varphi\|_{1, \Omega}.$$

Since  $\int_Q (a(\xi) - \langle a \rangle) d\xi = 0$ , it follows from Lemma 1.6 of [25, p. 8] that

$$(2.66) \quad \left| \int_{\Omega} (a_0(\xi) - \langle a_0 \rangle) u_k^0(x) \varphi(x) dx \right| \leq C \varepsilon \|u_k^0\|_{1, \Omega} \|\varphi\|_{1, \Omega}.$$

From (2.52) and (2.20), one can directly show that

$$(2.67) \quad \begin{aligned} \left| \varepsilon \int_{\Omega} m_{\varepsilon}(x) a_0(\xi) N_{\alpha_1}(\xi) \frac{\partial u_k^0(x)}{\partial x_{\alpha_1}} \varphi(x) dx \right| \leq C \varepsilon \|u_k^0\|_{1, \Omega'} \|\varphi\|_{0, \Omega} \\ \leq C \varepsilon \|u_k^0\|_{1, \Omega'} \|\varphi\|_{1, \Omega}. \end{aligned}$$

Combining (2.58)–(2.67) gives

$$(2.68) \quad |\mathcal{J}_{1,k}^{\varepsilon}(\varphi)| \leq C \varepsilon^{1/2} \|\varphi\|_{1, \Omega}.$$

Set  $\theta_{1,k}^{\varepsilon} = \mathcal{B}_{\varepsilon} \zeta_{1,k}^{\varepsilon}$ , where the solution operator  $\mathcal{B}_{\varepsilon}$  has been defined above. Recalling (2.18), (2.54) is written as follows:

$$(2.69) \quad \langle \zeta_{1,k}^{\varepsilon}, \varphi \rangle - \lambda_k^{(0)} \langle \mathcal{T}_{\varepsilon} \zeta_{1,k}^{\varepsilon}, \varphi \rangle = \mathcal{J}_{1,k}^{\varepsilon}(\varphi).$$

Repeating the process of (2.29)–(2.34) and using (2.68), we get

$$(2.70) \quad |\lambda_k^{\varepsilon} - \lambda_k^{(0)}| \leq C_1(k) \varepsilon^{1/2}.$$

If the multiplicity of  $\lambda_k^{(0)}$  is equal to  $t$  and assuming that  $\varepsilon > 0$  is sufficiently small, we can prove that (also see [25, p. 272])

$$(2.71) \quad |\lambda_j^{\varepsilon} - \lambda_k^{(0)}| \leq C_1(j) \varepsilon^{1/2}, \quad j = k, \dots, k + t - 1,$$

where  $\lambda_k^{\varepsilon}, \dots, \lambda_{k+t-1}^{\varepsilon}$  are associated with  $\lambda_k^{(0)} = \dots = \lambda_{k+t-1}^{(0)}$ , respectively.

For  $j = k, \dots, k + t - 1$ , we can verify that

$$(2.72) \quad \begin{aligned} a_{\varepsilon}(u_j^{\varepsilon} - \theta_{1,k}^{\varepsilon}, \varphi) &= a_{\varepsilon}(u_j^{\varepsilon}, \varphi) - a_{\varepsilon}(\theta_{1,k}^{\varepsilon}, \varphi) \\ &= \lambda_j^{\varepsilon} \langle u_j^{\varepsilon}, \varphi \rangle - \lambda_k^{(0)} \langle u_k^0, \varphi \rangle - \mathcal{J}_{1,k}^{\varepsilon}(\varphi). \end{aligned}$$

In (2.72), taking  $\varphi = u_j^\varepsilon - \theta_{1,k}^\varepsilon$  and noting that  $\langle u_j^\varepsilon, \theta_{1,k}^\varepsilon \rangle = \langle u_j^\varepsilon, u_j^0 \rangle$ ,  $\langle u_j^\varepsilon, u_k^\varepsilon \rangle = 1$ ,  $\langle u_k^0, u_k^0 \rangle = 1$ , we can obtain

$$(2.73) \quad a_\varepsilon(u_j^\varepsilon - \theta_{1,k}^\varepsilon, u_j^\varepsilon - \theta_{1,k}^\varepsilon) = (\lambda_j^\varepsilon - \lambda_k^{(0)}) \langle u_j^\varepsilon, u_j^\varepsilon - \theta_{1,k}^\varepsilon \rangle - \mathcal{J}_{1,k}^\varepsilon(u_j^\varepsilon - \theta_{1,k}^\varepsilon).$$

Using (A<sub>2</sub>), (2.68), (2.71), and the trace theorem, we get

$$(2.74) \quad \begin{aligned} \gamma_0 \|u_j^\varepsilon - \theta_{1,k}^\varepsilon\|_{1,\Omega}^2 &\leq a_\varepsilon(u_j^\varepsilon - \theta_{1,k}^\varepsilon, u_j^\varepsilon - \theta_{1,k}^\varepsilon) \\ &\leq C(j)\varepsilon^{1/2} \left\{ \|u_j^\varepsilon\|_{0,\partial\Omega} \|u_j^\varepsilon - \theta_{1,k}^\varepsilon\|_{0,\partial\Omega} + \|u_j^\varepsilon - \theta_{1,k}^\varepsilon\|_{1,\Omega} \right\} \\ &\leq C_1(j)\varepsilon^{1/2} \|u_j^\varepsilon - \theta_{1,k}^\varepsilon\|_{1,\Omega}, \end{aligned}$$

and consequently

$$(2.75) \quad \|u_j^\varepsilon - \theta_{1,k}^\varepsilon\|_{1,\Omega} \leq C_1(j)\varepsilon^{1/2}.$$

We recall (2.2), (2.52), and (2.53). Since  $\Omega_0 \subset\subset \Omega$ ,  $\text{dist}(\partial\Omega_0, \partial\Omega) > 2\varepsilon$ , we have

$$(2.76) \quad \|u_j^\varepsilon - u_{1,k}^\varepsilon\|_{1,\Omega_0} \leq C_1(j)\varepsilon^{1/2},$$

where  $C_1(j)$  is a constant independent of  $\varepsilon$  but dependent of  $j$ .

Let  $\bar{u}_k^\varepsilon$  be a linear combination of the eigenfunctions of problem (1.1) corresponding to  $\lambda_k^\varepsilon, \dots, \lambda_{k+t-1}^\varepsilon$ , i.e.,  $\bar{u}_k^\varepsilon = \sum_{j=k}^{k+t-1} c_j u_j^\varepsilon$ ,  $\sum_{j=k}^{k+t-1} c_j \equiv 1$ . We thus have

$$(2.77) \quad \|\bar{u}_k^\varepsilon - u_{1,k}^\varepsilon\|_{1,\Omega_0} \leq C_1(k)\varepsilon^{1/2},$$

where  $C_1(k)$  is a constant independent of  $\varepsilon$  but dependent of  $k$ .

Following the lines of the proof of (2.76), we can similarly prove that

$$(2.78) \quad \|\bar{u}_k^\varepsilon - u_{2,k}^\varepsilon\|_{1,\Omega_0} \leq C_2(k)\varepsilon^{1/2},$$

where  $C_2(k)$  is a constant independent of  $\varepsilon$  but dependent of  $k$ .

For  $x \in \Omega_1$ , from (1.1) and (2.42), we have

$$(2.79) \quad \begin{cases} \mathcal{L}_\varepsilon(u_k^\varepsilon - u_{s,k}^{\varepsilon,b}) = 0 & \text{in } \Omega_1, \\ u_k^\varepsilon - u_{s,k}^{\varepsilon,b} = 0 & \text{on } \Gamma_0, \\ u_k^\varepsilon - u_{s,k}^{\varepsilon,b} = u_k^\varepsilon - u_{s,k}^\varepsilon & \text{on } \Gamma^*, \\ \sigma_\varepsilon(u_k^\varepsilon - u_{s,k}^{\varepsilon,b}) - \lambda_k^{(0)}(u_k^\varepsilon - u_{s,k}^{\varepsilon,b}) \\ = (\lambda_k^\varepsilon - \lambda_k^{(0)})u_k^\varepsilon & \text{on } \Gamma_1, \end{cases}$$

where  $s = 1, 2$  and  $u_{1,k}^\varepsilon, u_{2,k}^\varepsilon$  are called the first-order and second-order multiscale asymptotic solutions for  $u_k^\varepsilon$ .

Repeating the process of (2.37)–(2.40) and using the trace theorem gives

$$(2.80) \quad \begin{aligned} \|\bar{u}_k^\varepsilon - u_{s,k}^{\varepsilon,b}\|_{1,\Omega_1} &\leq C_s(k) \left\{ \|\bar{u}_k^\varepsilon - u_{s,k}^\varepsilon\|_{1/2,\Gamma^*} + |\lambda_k^\varepsilon - \lambda_k^{(0)}| \right\} \\ &\leq C_s(k) \left\{ \|\bar{u}_k^\varepsilon - u_{s,k}^\varepsilon\|_{1,\Omega_0} + |\lambda_k^\varepsilon - \lambda_k^{(0)}| \right\}, \end{aligned}$$

where  $\bar{u}_k^\varepsilon$  is a linear combination of the eigenfunctions of problem (1.1) corresponding to  $\lambda_k^\varepsilon, \dots, \lambda_{k+t-1}^\varepsilon$  and  $C_s(k)$  is a constant independent of  $\varepsilon$ .

Thanks to (2.70), we obtain

$$(2.81) \quad \|\bar{u}_k^\varepsilon - u_{s,k}^{\varepsilon,b}\|_{1,\Omega_1} \leq C_s(k)\varepsilon^{1/2},$$

where  $s = 1, 2$ .

From (2.75), (2.78), and (2.81), using the triangle inequality, we get

$$(2.82) \quad \|\bar{u}_k^\varepsilon - U_{s,k}^\varepsilon\|_{1,\Omega} \leq \|\bar{u}_k^\varepsilon - u_{s,k}^\varepsilon\|_{1,\Omega_0} + \|\bar{u}_k^\varepsilon - u_{s,k}^{\varepsilon,b}\|_{1,\Omega_1} \leq C_s(k)\varepsilon^{1/2},$$

where  $s = 1, 2$ ,  $C_s(k)$  is a constant independent of  $\varepsilon$  but dependent of  $k$ . Therefore we complete the proof of Theorem 2.2.

*Remark 2.7.* If assume that  $\Omega \subset R^n$ ,  $n = 2, 3$ , is a bounded smooth domain with boundary  $\partial\Omega \in C^{s+2}$ , where  $s = 1, 2$ , then Theorem 2.2 is valid too.

*Remark 2.8.* We recall Theorems 2.1 and 2.2 and their proofs. Since we apply Lemma 11.2 of [21, p. 340], it only can be proved that the convergence rates of the  $k$ th eigenvalue and eigenfunction are both of order  $\varepsilon^{1/2}$ . However, the numerical results presented in section 3 clearly show that the accuracy of the eigenvalue is much better than that of the corresponding eigenfunction.

**3. Numerical tests.** We recall (2.2) and (2.48) and summarize the above theoretical results as follows. The multiscale finite element method for solving the Steklov eigenvalue problems consists of the following parts:

Part I. Compute cell functions  $N_{\alpha_1}(\xi), N_{\alpha_1\alpha_2}(\xi)$  in a reference cell  $Q = (0, 1)^n$ .

Part II. Solve numerically the homogenized Steklov eigenvalue problem (2.5) on the whole domain  $\Omega$  in a coarse mesh.

Part III. Solve directly the boundary layer equation (2.42) in a fine mesh.

Part IV. Calculate numerically the higher-order derivatives  $\frac{\partial^l u_k^0(x)}{\partial x_{\alpha_1} \dots \partial x_{\alpha_l}}$  by using the finite difference method (see [12]), where  $u_k^0(x)$  is the  $k$ th eigenfunction of the homogenized Steklov eigenvalue problem. We remark that one cannot directly compute higher-order derivatives from their finite element solutions.

To validate the developed multiscale algorithm and to confirm the theoretical analysis reported in this paper, we present numerical simulations for the following case studies.

We consider the Steklov eigenvalue problem in composite media as follows:

$$(3.1) \quad \begin{cases} -\frac{\partial}{\partial x_i} \left( a_{ij} \left( \frac{x}{\varepsilon} \right) \frac{\partial u_k^\varepsilon(x)}{\partial x_j} \right) + a_0 \left( \frac{x}{\varepsilon} \right) u_k^\varepsilon(x) = 0, & x \in \Omega, \\ u_k^\varepsilon(x) = 0, & x \in \Gamma_0, \\ \nu_i a_{ij} \left( \frac{x}{\varepsilon} \right) \frac{\partial u_k^\varepsilon(x)}{\partial x_j} = \lambda_k^\varepsilon u_k^\varepsilon(x), & x \in \Gamma_1, \quad k \geq 1, \end{cases}$$

where  $\Omega \subset R^n$ ,  $n = 2, 3$ , is a bounded Lipschitz polygonal convex domain; the boundary  $\partial\Omega = \bar{\Gamma}_0 \cup \bar{\Gamma}_1$  with  $\Gamma_0 \cap \Gamma_1 = \emptyset$ ; and  $\bar{\nu} = (\nu_1, \dots, \nu_n)$  is the outward unit normal to  $\partial\Omega$ .

*Example 3.1.* In (3.1), assume that  $\Omega = (0, 1)^2$  is a periodic structure as illustrated in Figure 3.1, the reference cell  $Q$  is as shown in Figure 3.2,  $\Gamma_0 = \{(x_1, x_2) \mid 0 < x_1 < 0.2, x_2 = 0.2 - x_1\} \cup \{(x_1, x_2) \mid 0 < x_1 < 0.2, x_2 = 0.8 + x_1\} \cup \{(x_1, x_2) \mid 0.8 < x_1 < 1, x_2 = 1.8 - x_1\} \cup \{(x_1, x_2) \mid 0.8 < x_1 < 1, x_2 = x_1 - 0.8\}$ ,  $\Gamma_1 = \{(x_1, x_2) \mid 0.2 \leq x_1 \leq 0.8, x_2 = 0\} \cup \{(x_1, x_2) \mid x_1 = 1, 0.2 \leq x_2 \leq 0.8\} \cup \{(x_1, x_2) \mid 0.2 \leq x_1 \leq 0.8, x_2 = 1\} \cup \{(x_1, x_2) \mid x_1 = 0, 0.2 \leq x_2 \leq 0.8\}$ , and  $\nu = (\nu_1, \nu_2)$  is the outward unit normal to  $\Gamma_1$ . We take  $\varepsilon = \frac{1}{5}$ .

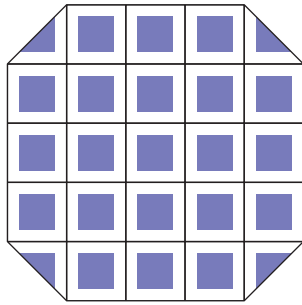
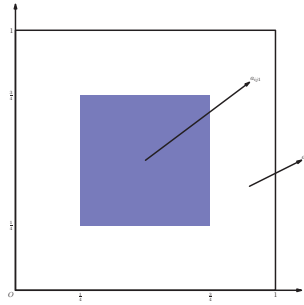
FIG. 3.1. Domain  $\Omega$ .FIG. 3.2. Unit cell  $Q$ .

TABLE 3.1  
Comparison of computational cost in Case 3.1.1.

	Original problem	Cell problem	Homogenized equation	Boundary layer
Elements	18400	800	1150	11200
Nodes	9361	441	616	5880

In (3.1), let  $a_0(\frac{x}{\varepsilon}) = 0$  and  $\delta_{ij}$  be a Kronecker symbol.

Case 3.1.1.  $a_{ij0} = \delta_{ij}$ ,  $a_{ij1} = 0.01\delta_{ij}$ .

Case 3.1.2.  $a_{ij0} = \delta_{ij}$ ,  $a_{ij1} = 0.001\delta_{ij}$ .

In this paper, in order to show the numerical accuracy of the proposed method, we need to know the exact solution of the original Steklov eigenvalue problem (3.1). Since the coefficients of problem (3.1) are discontinuous, it is extremely difficult to seek its exact solution. To overcome this difficulty, we replace the exact solution with the finite element solution in a fine mesh. Now we employ the linear triangular elements to solve the original problem (3.1) in a fine mesh. It should be mentioned that in engineering applications, this step is not necessary and one can directly use the method presented in this paper to solve problem (3.1). We observe four parts of our method and believe that it has competitiveness for numerically solving problem (3.1) in the more complicated three-dimensional structure of composite media.

Here we use the linear triangular elements to compute the cell functions  $N_{\alpha_1}(\xi)$ ,  $N_{\alpha_1\alpha_2}(\xi)$  defined in (2.3)–(2.4) and to solve the homogenized Steklov problem (2.5) and the boundary layer equation (2.42), respectively. The numbers of elements and nodes are listed in Table 3.1.

The numerical results of several eigenvalues and eigenfunctions of the related problems in Example 3.1, Case 3.1.1 are illustrated as in Tables 3.2 and 3.3, respectively. The  $\lambda_k^\varepsilon$ ,  $k = 1, 2, 3, 4$ , are the finite element solutions of four minimal eigenvalues of the original problem (3.1) in a fine mesh, and  $\lambda_k^{(0)}$ ,  $k = 1, 2, 3, 4$ , are the finite element solutions of the corresponding eigenvalues of the homogenized Steklov eigenvalue problem (2.5) in a coarse mesh. The functions  $u_k^\varepsilon(x)$ ,  $k = 1, 2, 3, 4$ , are the finite element solutions of the eigenfunctions associated with four minimal eigenvalues of problem (3.1) in a fine mesh, while  $u_k^0(x)$ ,  $k = 1, 2, 3, 4$ , denote the finite element solutions of the corresponding eigenfunctions for the homogenized Steklov eigenvalue problem (2.5) in a coarse mesh. Finally,  $U_{1,k}^\varepsilon(x)$ ,  $U_{2,k}^\varepsilon(x)$ ,  $k = 1, 2, 3, 4$ , are respectively the first-order and second-order multiscale finite element solutions based on the expansion (2.2). Set  $e_{0,k} = u_k^\varepsilon - u_k^0$ ,  $e_{1,k} = u_k^\varepsilon - U_{1,k}^\varepsilon$ ,  $e_{2,k} = u_k^\varepsilon - U_{2,k}^\varepsilon$ .

*Example 3.2.* In the second case study, we assume that  $\Omega = (0, 1)^3$  is a periodic structure as illustrated in Figure 3.3, and the reference cell  $Q$  is as shown in Figure 3.4.

TABLE 3.2

Comparison of computational results in Case 3.1.1: four minimal eigenvalues.

$k$	Original problem	Homogenized solutions	Relative error
1	1.778715	1.740666	0.021858
2	2.410815	2.397551	0.005532
3	2.410815	2.397551	0.005532
4	2.649853	2.656357	0.002448

TABLE 3.3

Comparison of computational results in Case 3.1.1: eigenfunctions.

$k$	$\frac{\ e_{0,k}\ _{L^2}}{\ u_k^0\ _{L^2}}$	$\frac{\ e_{1,k}\ _{L^2}}{\ U_{1,k}^\varepsilon\ _{L^2}}$	$\frac{\ e_{2,k}\ _{L^2}}{\ U_{2,k}^\varepsilon\ _{L^2}}$	$\frac{\ e_{0,k}\ _{H^1}}{\ u_k^0\ _{H^1}}$	$\frac{\ e_{1,k}\ _{H^1}}{\ U_{1,k}^\varepsilon\ _{H^1}}$	$\frac{\ e_{2,k}\ _{H^1}}{\ U_{2,k}^\varepsilon\ _{H^1}}$
1	0.053723	0.017170	0.017380	0.547313	0.110316	0.111782
2	0.077907	0.017476	0.017273	0.478182	0.058563	0.058872
3	0.078021	0.017343	0.017435	0.471127	0.059898	0.059838
4	0.107685	0.023138	0.024143	0.543290	0.100363	0.107822

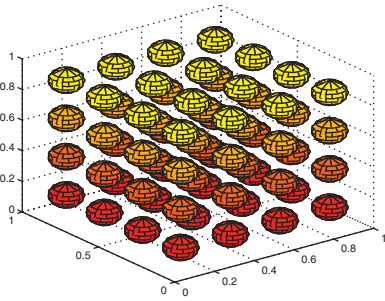


FIG. 3.3. Domain  $\Omega$ .

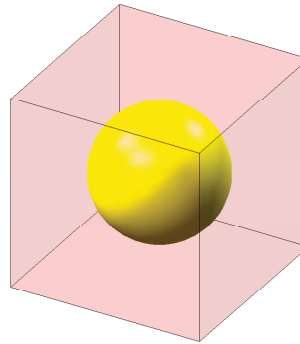


FIG. 3.4. Unit cell  $Q$ .

In (3.1), let  $\Gamma_0 = \emptyset$  and  $a_0(\frac{x}{\varepsilon}) = 0$ , and we recall that  $\delta_{ij}$  is a Kronecker symbol.  $\nu = (\nu_1, \nu_2, \nu_3)$  is the outward unit normal to  $\partial\Omega$ . We take  $\varepsilon = \frac{1}{4}$ .

Case 3.2.1. We set

$$a_{ij}(\xi) = \begin{cases} a_{ij1} = 0.001\delta_{ij} & \text{in } B, \\ a_{ij0} = \delta_{ij} & \text{otherwise,} \end{cases}$$

where the equation of the ellipsoid  $B$  is

$$\frac{(\xi_1 - 0.5)^2}{a^2} + \frac{(\xi_2 - 0.5)^2}{b^2} + \frac{(\xi_3 - 0.5)^2}{c^2} = 1,$$

and  $a = b = c = 0.32$ .

In a standard approach, we first apply the linear tetrahedral elements to solve the original problem (3.1) in a fine mesh. Then we employ linear tetrahedral elements and bilinear cube elements to compute the cell functions  $N_{\alpha_1}(\xi), N_{\alpha_1\alpha_2}(\xi)$ , defined in (2.3) and (2.4), and the homogenized Steklov problem (2.5), respectively. The numbers of elements and nodes are listed in Table 3.4.

The numerical results of several eigenvalues and eigenfunctions of the related problems in Example 3.2 are listed in Tables 3.5 and 3.6, respectively. It should be noted that we use respectively  $u_k^\varepsilon(x), u_{1,k}^\varepsilon(x), u_{2,k}^\varepsilon(x), e_{2,k}, k \geq 1$ , to denote the finite

TABLE 3.4  
Comparison of computational cost in Case 3.2.1.

	Original problem	Cell problem	Homogenized equation
Elements	117504	1836	48000
Nodes	22637	425	9261

TABLE 3.5  
Comparison of computational results in Case 3.2.1: four minimal eigenvalues.

$k$	Original problem	Homogenized solutions	Relative error
1	0.000000	0.000000	0.000000
2	1.068041	1.065780	0.002121
3	1.069354	1.071169	0.001694
4	1.090259	1.091308	0.000961

TABLE 3.6  
Comparison of computational results in Case 3.2.1: eigenfunctions.

$k$	$\frac{\ e_{0,k}\ _{L^2}}{\ u_k^0\ _{L^2}}$	$\frac{\ e_{1,k}\ _{L^2}}{\ u_{1,k}^e\ _{L^2}}$	$\frac{\ e_{2,k}\ _{L^2}}{\ u_{2,k}^e\ _{L^2}}$	$\frac{\ e_{0,k}\ _{H^1}}{\ u_k^0\ _{H^1}}$	$\frac{\ e_{1,k}\ _{H^1}}{\ u_{1,k}^e\ _{H^1}}$	$\frac{\ e_{2,k}\ _{H^1}}{\ u_{2,k}^e\ _{H^1}}$
2	0.005686	0.001033	0.001110	0.077042	0.019482	0.020374
3	0.00742	0.004887	0.004929	0.077742	0.022302	0.024378
4	0.007964	0.005669	0.005684	0.077634	0.022293	0.023908

element solution of the  $k$ th eigenfunction for problem (3.1) in a fine mesh, the first-order multiscale finite element solution, the second-order multiscale finite element solution, and the absolute error of the second-order multiscale finite element solution. It should be emphasized that since a whole domain  $\Omega$  is the union of entire cells, here we do not need to define the boundary layer solution (2.42).

*Remark 3.1.* The numerical results that are illustrated in Tables 3.2 and 3.5, show that the eigenvalues of the homogenized Steklov eigenvalue problem (2.5) in a coarse mesh are close to those of the original Steklov eigenvalue problem (3.1) in a fine mesh. This is an interesting phenomena. It implies that in order to calculate the eigenvalues for the Steklov eigenvalue problem (3.1) in composite media, we only need to compute the associated eigenvalues for the homogenized Steklov eigenvalue problem in a coarse mesh.

*Remark 3.2.* The numerical results illustrated in Tables 3.3 and 3.6 validate the theoretical results of Theorems 2.1 and 2.2. From the numerical simulations for the two case studies, it is seen that in the high-contrast case, the homogenization method fails to provide satisfactory results. The first-order and second-order multiscale approaches, however, do yield the high-accuracy numerical results. In order to obtain the convergence rate with  $\varepsilon^{1/2}$  for the first-order and the second-order multiscale method (see Theorems 2.1 and 2.2), we need to assume that  $u_k^0 \in H^3(\Omega)$  and  $u_k^0 \in H^4(\Omega)$ , respectively. For a general bounded Lipschitz polygonal convex domain, roughly speaking,  $u_k^0 \in H^3(\Omega)$  and  $u_k^0 \in H^4(\Omega)$  are invalid. On the other hand, generally speaking, the multiscale asymptotic expansions (2.2) do not satisfy the boundary conditions of the original Steklov problem (1.1) in a general domain regardless of a bounded smooth one or a bounded Lipschitz polygonal convex one. To overcome the above difficulties, the boundary layer solutions are introduced and the convergence rate with  $\varepsilon^{1/2}$  is derived in this paper (see Theorem 2.2). In a word, the boundary layer correctors are necessary and essential in the high-contrast case and in a bounded Lipschitz polygonal convex domain.

*Remark 3.3.* Finally, we observe the numerical results listed in Tables 3.3 and 3.6. The results with  $s = 1$  and  $s = 2$  are very close. This clearly shows that the first-order asymptotic method is the best choice, and we do not need to use the second-order asymptotic method for the Steklov eigenvalue problem in this paper. However, it should be emphasized that the second-order asymptotic method is necessary and essential for other spectral problems; see [12].

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